Chapter 5
Over-Determined and Under-Determined Systems of Time-Varying Linear Equations

Abstract In this chapter, focusing on solving over-determined system of time-varying linear equations, we first propose, generalize, develop, and investigate two ZD models based on two different ZFs. Then, by introducing another two different ZFs, another two ZD models are proposed, generalized, developed, and investigated to solve under-determined system of time-varying linear equations. Computer simulation results with different illustrative examples are presented to further substantiate the efficacy of the proposed ZD models for solving over-determined and under-determined systems of time-varying linear equations.

5.1 Introduction

Solving over-determined and under-determined systems of linear equations is widely encountered in a variety of scientific and engineering research fields [1–7]. As presented in Chap. 4, it has no solution for over-determined system of linear equations; while it has infinitely many solutions for under-determined system of linear equations. These characteristics make it difficult for solving over-determined and under-determined systems of linear equations.

Focusing on solving over-determined and under-determined systems of linear equations, many approaches (including numerical algorithms and neural-dynamics methods) have thus been developed, analyzed, and investigated [8–12]. Note that the problems of over-determined and under-determined systems of linear equations in most of these researches or investigations are static (or termed, time-invariant). This also means that almost all of these methods are theoretically/intrinsically designed for solving over-determined and under-determined systems of time-invariant linear equations. When these methods are exploited directly to solve the (over-determined or under-determined) system of time-varying linear equations, they may be less accurate and effective enough [10, 11].

In this chapter, by following the idea of ZFs, different ZD models are proposed, generalized, developed, and investigated to solve over-determined and

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under-determined systems of time-varying linear equations. Specifically, we first construct two different ZD models based on two ZFs for solving over-determined system of time-varying linear equations. Then, another two different ZD models are constructed for solving under-determined system of time-varying linear equations. Four illustrative examples are provided and computer simulation results further substantiate the efficacy of the proposed ZD models for solving over-determined and under-determined systems of time-varying linear equations.

5.2 ZFs and ZD Models

In this section, by defining different ZFs, different ZD models are proposed for solving the following system of time-varying linear equations:

$$A(t)\mathbf{x}(t) = \mathbf{b}(t) \in \mathbb{R}^m, \ t \in [0, +\infty), \tag{5.1}$$

where $A(t) \in \mathbb{R}^{m \times n}$ with $m \neq n$ is the smoothly time-varying full-rank coefficient matrix, $\mathbf{b}(t) \in \mathbb{R}^m$ is the smoothly time-varying coefficient vector, and $\mathbf{x}(t) \in \mathbb{R}^n$ is the unknown vector that needs to be obtained in an error-free and real-time manner (or termed, the manner of real-time time-varying problem-solving). Note that (5.1) can be viewed as a general time-varying system of m real-valued time-varying linear equations and n real-valued time-varying variables.

To lay a basis for further discussion, the following corollary is presented, with the related proof being generalized from the proof of Theorem 4.1 and being left to interested readers to complete as a topic of exercise.

Corollary 5.1 *Consider a smoothly time-varying full-rank matrix $A(t) \in \mathbb{R}^{m \times n}$ with $m \neq n$. Let $A^+(t) \in \mathbb{R}^{n \times m}$ denote the time-varying Moore–Penrose pseoduinverse of $A(t)$. Then, the time derivative of $A^+(t)$ is formulated as $\dot{A}^+(t) = \mathrm{d}A^+(t)/\mathrm{d}t = -A^+(t)\dot{A}(t)A^+(t)$.*

5.2.1 With m > n (Over-Determined System)

In this subsection, two different ZD models based on two ZFs are developed and investigated for solving over-determined system of time-varying linear equations, i.e., (5.1) with $m > n$. Note that, as mentioned in [6], if there exists at least one choice for the time-varying vector $\mathbf{x}(t)$ which satisfies (5.1) with $m > n$, then the over-determined system of time-varying linear equations is consistent; and if no such time-varying vector exists, then the over-determined system of time-varying linear equations is inconsistent. In this chapter, we only consider the situation of the inconsistent over-determined system of time-varying linear equations. Besides, in the study of the inconsistent over-determined system of time-varying linear equations, the two-norm technique is adopted to zero out the time-varying residual error $A(t)\mathbf{x}(t) - \mathbf{b}(t)$ as possible as we can in this chapter.

The First ZF and ZD Model In order to solve the system of time-varying linear equations (5.1) with $m > n$, the first ZF (i.e., a vector-valued lower-unbounded error function) is defined as follows:

$$\mathbf{e}(t) = A(t)\mathbf{x}(t) - \mathbf{b}(t) \in \mathbb{R}^m. \tag{5.2}$$

With ZF (5.2), by expanding the ZD design formula (4.2), we obtain the following ZD model for solving over-determined system of time-varying linear equations:

$$A^{\mathrm{T}}(t)A(t)\dot{\mathbf{x}}(t) = -A^{\mathrm{T}}(t)\dot{A}(t)\mathbf{x}(t) + A^{\mathrm{T}}(t)\dot{\mathbf{b}}(t) - \gamma A^{\mathrm{T}}(t)(A(t)\mathbf{x}(t) - \mathbf{b}(t)), \tag{5.3}$$

where $\mathbf{x}(t)$, starting from an initial condition $\mathbf{x}(0)$, is the neural state corresponding to an approximate time-varying solution (e.g., a pseudoinverse-type solution) $\mathbf{x}^*(t)$ of (5.1) with $m > n$.

The Second ZF and ZD Model To solve over-determined system of time-varying linear equations, i.e., (5.1) with $m > n$, the second ZF is defined as follows:

$$\mathbf{e}(t) = \mathbf{x}(t) - A^+(t)\mathbf{b}(t) \in \mathbb{R}^n. \tag{5.4}$$

where $A^+(t) = (A^{\mathrm{T}}(t)A(t))^{-1}A^{\mathrm{T}}(t)$ denotes the left Moore–Penrose inverse of $A(t)$.

Then, in view of (5.4) and $\dot{A}^+(t) = \mathrm{d}A^+(t)/\mathrm{d}t = -A^+(t)\dot{A}(t)A^+(t)$, we have the following ZD model by expanding ZD design formula (4.2):

$$A^{\mathrm{T}}(t)A(t)\dot{\mathbf{x}}(t) = -A^{\mathrm{T}}(t)\dot{A}(t)A^+(t)\mathbf{b}(t) + A^{\mathrm{T}}(t)\dot{\mathbf{b}}(t) - \gamma A^{\mathrm{T}}(t)\left(A(t)\mathbf{x}(t) - \mathbf{b}(t)\right). \tag{5.5}$$

Thus, based on the second ZF (5.4), the second ZD model (5.5) is obtained for solving over-determined system of time-varying linear equations.

Before closing this subsection of constructing ZD models (5.3) and (5.5), the block diagrams and overall Simulink models corresponding to such two ZD models are shown in Figs. 5.1, 5.2, 5.3 and 5.4, which may be useful for their future implementations on circuit systems. Besides, it is worth pointing out that the over-determined system of time-varying linear equations discussed in this chapter is inconsistent and there does not exist accurate theoretical solution for it. Thus, in the ensuing simulations, the two-norm measure is adopted to show the residual error about the obtained approximate solution of (5.1) with $m > n$, i.e., $\|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$.

5.2.2 With m < n (Under-Determined System)

In this subsection, another two different ZD models based on two ZFs are developed and investigated for solving under-determined system of time-varying linear

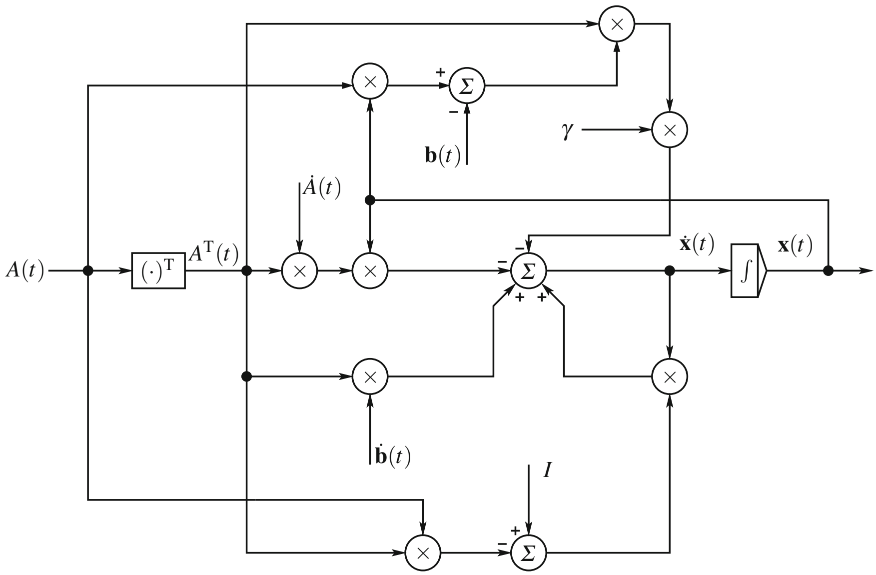

Fig. 5.1 Block diagram of ZD model (5.3) for solving over-determined system of time-varying linear equations, where I is the identity matrix

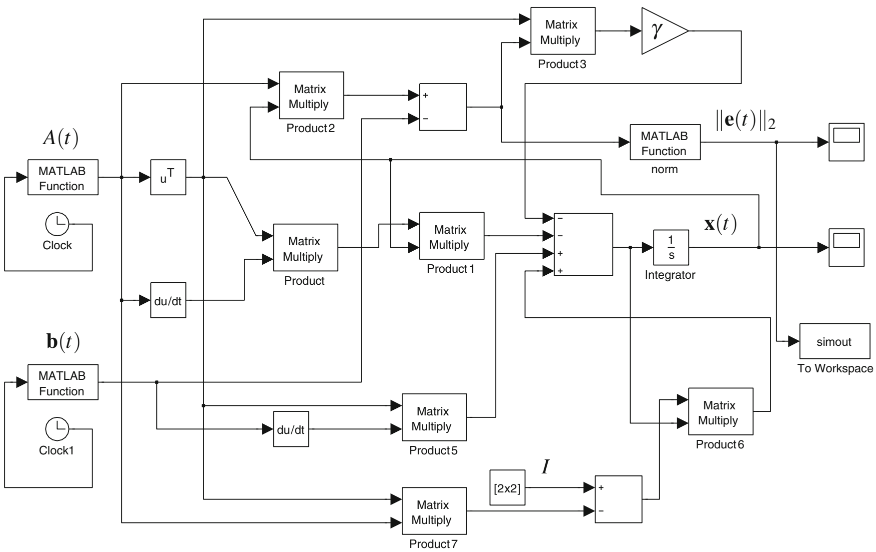

Fig. 5.2 Overall Simulink model of ZD (5.3) for solving over-determined system of time-varying linear equations

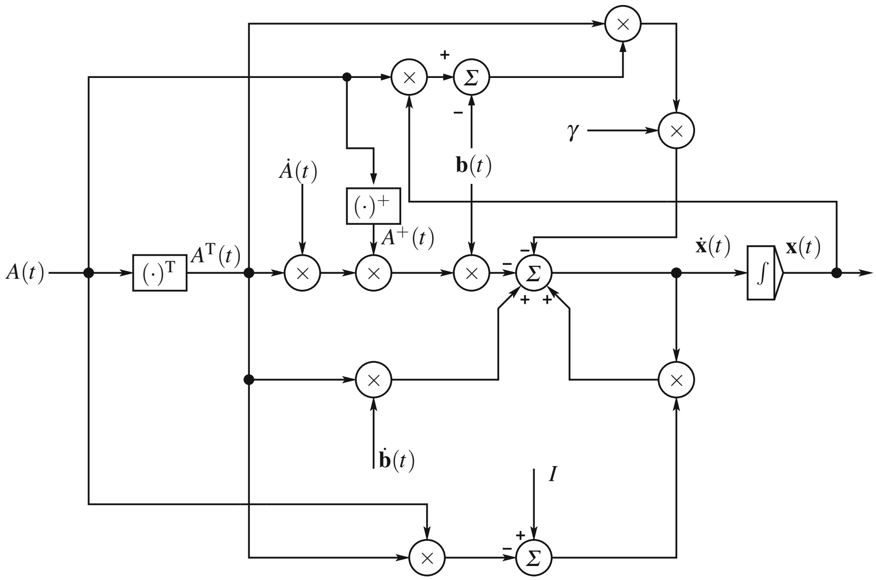

Fig. 5.3 Block diagram of ZD model (5.5) for solving over-determined system of time-varying linear equations

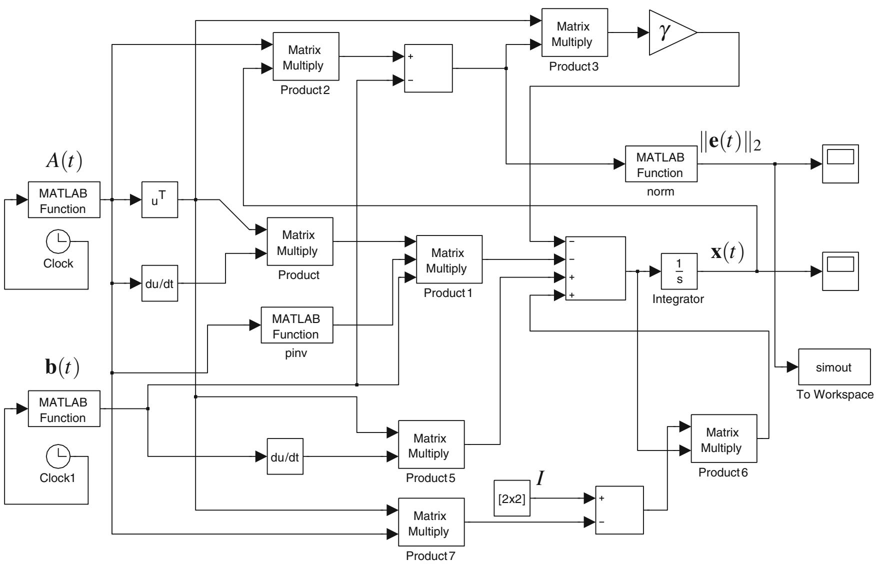

Fig. 5.4 Overall Simulink model of ZD (5.5) for solving over-determined system of time-varying linear equations

equations, i.e., (5.1) with $m < n$. Since $m < n$, there exist multiple or even an infinite number of solutions to (5.1).

Being similar to the above design procedure, in order to solve under-determined system of time-varying linear equations, we define the following two ZFs:

$$\mathbf{e}(t) = A(t)\mathbf{x}(t) - \mathbf{b}(t) \in \mathbb{R}^m, \tag{5.6}$$

$$\mathbf{e}(t) = \mathbf{x}(t) - A^{+}(t)\mathbf{b}(t) \in \mathbb{R}^n, \tag{5.7}$$

where $A^{+}(t) = A^{\mathrm{T}}(t)(A(t)A^{\mathrm{T}}(t))^{-1}$ denotes the right Moore–Penrose inverse of $A(t)$.

On the one hand, with ZF (5.6), by expanding the ZD design formula (4.2), we obtain the following ZD model for solving under-determined system of time-varying linear equations:

$$A(t)\dot{\mathbf{x}}(t) = -\dot{A}(t)\mathbf{x}(t) + \dot{\mathbf{b}}(t) - \gamma(A(t)\mathbf{x}(t) - \mathbf{b}(t)). \tag{5.8}$$

On the other hand, with ZF (5.7), by expanding the ZD design formula (4.2), we obtain another ZD model for solving under-determined system of time-varying linear equations as follows:

$$\dot{\mathbf{x}}(t) = -A^{+}(t)\dot{A}(t)A^{+}(t)\mathbf{b}(t) + A^{+}(t)\dot{\mathbf{b}}(t) - \gamma\left(\mathbf{x}(t) - A^{+}(t)\mathbf{b}(t)\right). \tag{5.9}$$

In summary, based on two different ZFs (5.6) and (5.7), two different ZD models (5.8) and (5.9) have been developed for solving under-determined system of time-varying linear equations, i.e., (5.1) with $m < n$. Note that the block diagrams and overall Simulink models corresponding to such two ZD models are left to interested readers to complete as a topic of exercise (since they are similar to those shown in Figs. 5.1, 5.2, 5.3, and 5.4).

5.3 Illustrative Examples

In this section, two illustrative examples are first simulated and analyzed for comparisons between the proposed ZD models (5.3) and (5.5) for solving over-determined system of time-varying linear equations. Then, another two illustrative examples are provided for substantiating the efficacy of the proposed ZD models (5.8) and (5.9) for solving under-determined system of time-varying linear equations.

Example 5.1 In the first example, the following smoothly time-varying coefficient matrix $A(t)$ and coefficient vector $\mathbf{b}(t)$ of (5.1) with $m = 3$ and $n = 2$ are designed to test the proposed ZD models (5.3) and (5.5):

$$A(t) = \begin{bmatrix} \sin(3t) & \cos(3t) \\ -\cos(3t) & \sin(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix} \in \mathbb{R}^{3\times 2} \text{ and } \mathbf{b}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ -\sin(t) \end{bmatrix} \in \mathbb{R}^3.$$

The corresponding simulation results are shown in Figs. 5.5, 5.6, and 5.7.

Specifically, in the time period [0, 10]s, the state trajectories of the two elements $x_1(t)$ and $x_2(t)$ of $\mathbf{x}(t) = [x_1(t)\ x_2(t)]^{\mathrm{T}}$ synthesized by the proposed ZD models (5.3) and (5.5) with $\gamma = 1$ are illustrated in Fig. 5.5. It is seen that all simulated state trajectories (denoted by solid curves) starting from ten randomly-generated initial states $\mathbf{x}(0) \in [-1.5, 1.5]^2$ can relatively fast converge to the pseudoinverse-type solution $x^*(t) = A^+(t)b(t)$ which is exploited and shown for comparison and denoted by dash-dotted curves. Furthermore, Fig. 5.6 shows the residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.3) and (5.5) with $\gamma = 1$. As seen from Fig. 5.6, the residual errors of both ZD models cannot converge to zero. This phenomenon actually reflects and confirms that, for solving such an inconsistent

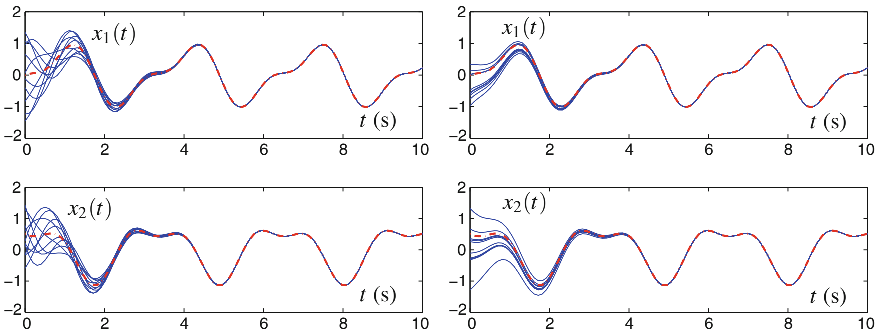

Fig. 5.5 State trajectories of ZD models (5.3) and (5.5) with $\gamma = 1$ for solving over-determined system of time-varying linear equations involved in Example 5.1, where the *dash-dotted* curves correspond to the pseudoinverse-type solution $\mathbf{x}^*(t) = A^+(t)\mathbf{b}(t)$

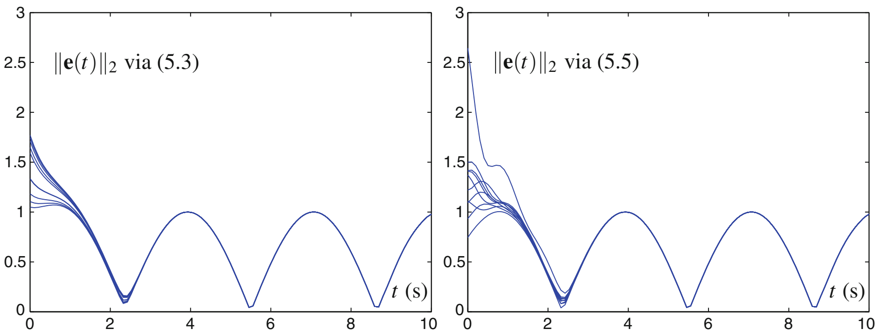

Fig. 5.6 Residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.3) and (5.5) with $\gamma = 1$ for solving over-determined system of time-varying linear equations involved in Example 5.1

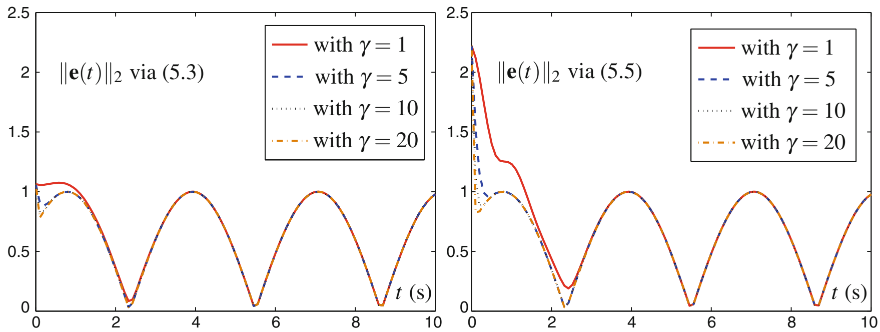

Fig. 5.7 Residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.3) and (5.5) using different γ values for solving over-determined system of time-varying linear equations involved in Example 5.1

over-determined system of time-varying linear equations, we cannot find a time-varying solution that satisfies all of the inconsistent equations simultaneously all the time.

Moreover, as seen from Fig. 5.7, ZD models (5.3) and (5.5) using different values of γ are investigated. Synthesized by the proposed ZD models, the residual errors (with $\gamma = 1$, 5, 10, and 20) cannot converge to zero either, and the reason has been explained in the preceding paragraph. Besides, as shown in Fig. 5.7, the residual errors with larger γ value converge faster than those with smaller γ value, showing that γ plays an important role in such ZD models.

Example 5.2 In the second example, the following time-varying coefficients of (5.1) with $m = 5$ and $n = 4$ are designed to test the proposed ZD models (5.3) and (5.5):

$$A(t) = \begin{bmatrix} a_1(t) & a_2(t) & a_3(t) & a_4(t) \\ a_1(t) & -a_2(t) & a_3(t) & a_4(t) \\ a_1(t) & a_2(t) & -a_3(t) & a_4(t) \\ a_1(t) & a_2(t) & a_3(t) & -a_4(t) \\ a_1(t) & a_2(t) & a_3(t) & a_4(t) \end{bmatrix} \in \mathbb{R}^{5\times 4} \text{ and } \mathbf{b}(t) = \begin{bmatrix} 2\sin(t) \\ 3\cos(2t) \\ 4\sin(2t) \\ 3\cos(t) \\ \sin(2t) \end{bmatrix} \in \mathbb{R}^5.$$

where $a_1(t) = 4 - \sin(t)$, $a_2(t) = 2 + \cos(2t)$, $a_3(t) = 3 - \sin(2t)$ and $a_4(t) = 2 + \cos(t)$. The corresponding simulation results are shown in Figs. 5.8, 5.9, and 5.10.

Specifically, in the time period [0, 10]s, the state trajectories of the four elements $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ of $\mathbf{x}(t) = [x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)]^{\mathrm{T}}$ synthesized by ZD models (5.3) and (5.5) with $\gamma = 1$ are illustrated in Fig. 5.8. Starting from ten randomly-generated initial states $\mathbf{x}(0) \in [-1.5, 1.5]^4$, all state trajectories (denoted by solid curves) can also relatively fast converge to the pseudoinverse-type solution $\mathbf{x}^*(t) = A^+(t)\mathbf{b}(t)$ (denoted by dash-dotted curves again).

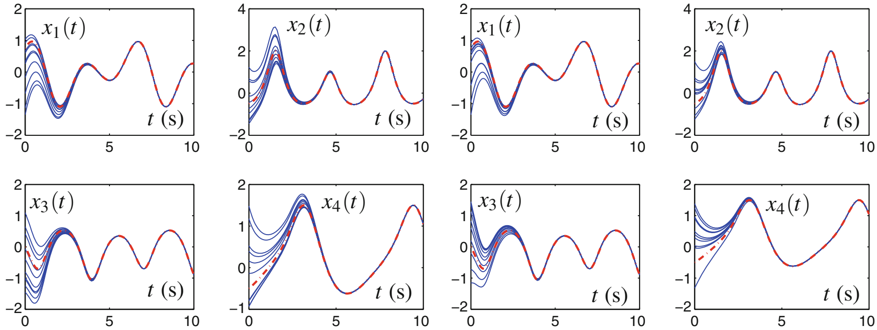

Fig. 5.8 State trajectories of ZD models (5.3) and (5.5) with $\gamma = 1$ for solving over-determined system of time-varying linear equations involved in Example 5.1

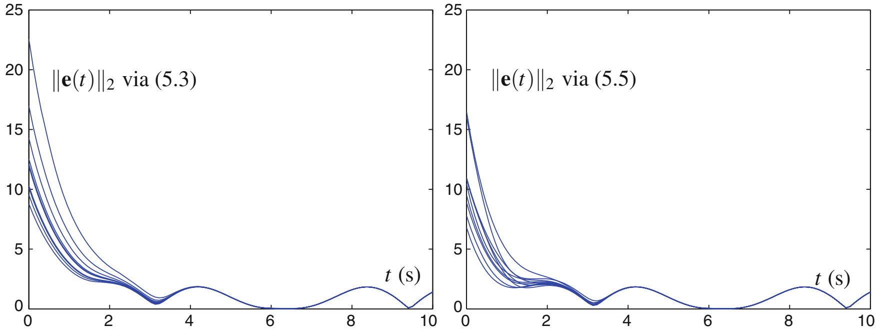

Fig. 5.9 Residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.3) and (5.5) with $\gamma = 1$ for solving time-varying over-determined system of linear equations involved in Example 5.1

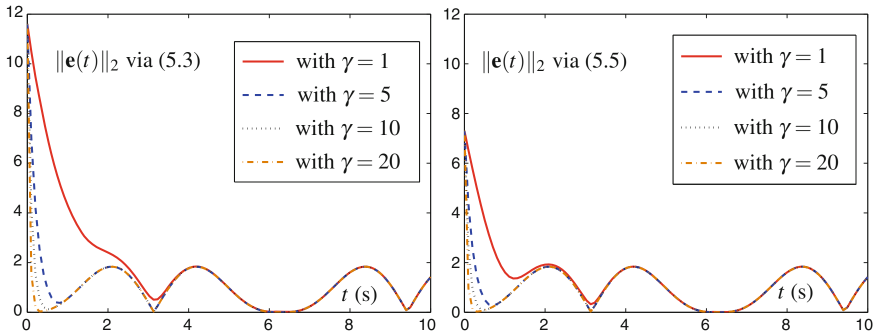

Fig. 5.10 Residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.3) and (5.5) using different γ values for solving over-determined system of time-varying linear equations involved in Example 5.2

Furthermore, Fig. 5.9 shows the residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.3) and (5.5) with $\gamma = 1$. As seen from the figure, the residual errors of ZD models (5.3) and (5.5) cannot converge to zero either. The reason is explained before; i.e., we cannot find a time-varying solution which can satisfy all of the inconsistent equations simultaneously.

Moreover, as seen from Fig. 5.10, ZD models (5.3) and (5.5) using different values of γ are investigated. We confirmedly observe that the residual errors with larger γ value converge faster than those with smaller γ value (showing again the important role of γ for the proposed ZD models).

In summary, the simulation results of the above two examples have substantiated the efficacy of the proposed ZD models (5.3) and (5.5) (derived from two different ZFs) for solving over-determined system of time-varying linear equations.

Example 5.3 In the third example, the following smoothly time-varying coefficient matrix $A(t)$ and coefficient vector $\mathbf{b}(t)$ of (5.1) with $m = 2$ and $n = 3$ are designed to test the proposed ZD models (5.8) and (5.9):

$$A(t) = \begin{bmatrix} \sin(0.6t) & \cos(0.6t) & -\sin(0.6t) \\ -\cos(0.6t) & \sin(0.6t) & \cos(0.6t) \end{bmatrix} \in \mathbb{R}^{2\times 3} \text{ and } \mathbf{b}(t) = \begin{bmatrix} 1.5\cos(t) \\ \sin(2t) \end{bmatrix} \in \mathbb{R}^2.$$

The corresponding simulation results are shown in Figs. 5.11, 5.12, and 5.13.

Specifically, in the time period $t \in [0, 10]$s, state trajectories of the elements $x_1(t)$, $x_2(t)$, and $x_3(t)$ (denoted by solid curves) synthesized by ZD models (5.8) and (5.9) with $\gamma = 1$ are illustrated in Fig. 5.11. Evidently, starting from ten randomly-generated initial states $\mathbf{x}(0) \in [-2, 2]^3$, some of the simulated state trajectories synthesized by ZD model (5.8) (e.g., $x_1(t)$ in the left graph of Fig. 5.11) do not converge to the trajectories of the referenced theoretical solution $\mathbf{x}^*(t) = A^+(t)\mathbf{b}(t)$ (denoted by dash-dotted curves), but run in parallel with the theoretical-solution trajectories. The reason is that there are multiple time-varying solutions satisfying

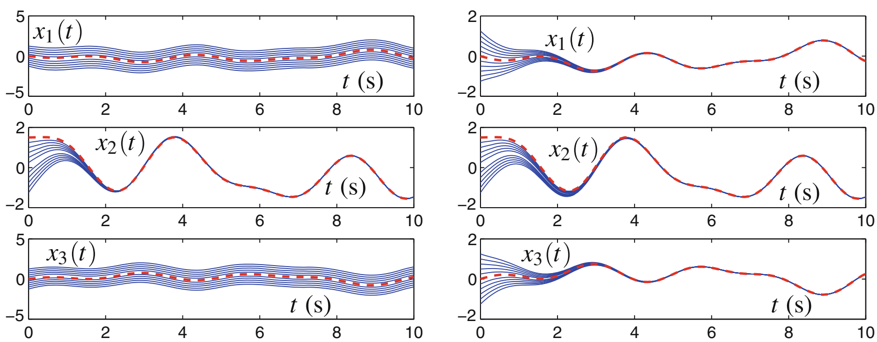

Fig. 5.11 State trajectories of ZD models (5.8) and (5.9) with $\gamma = 1$ for solving under-determined system of time-varying linear equations involved in Example 5.3

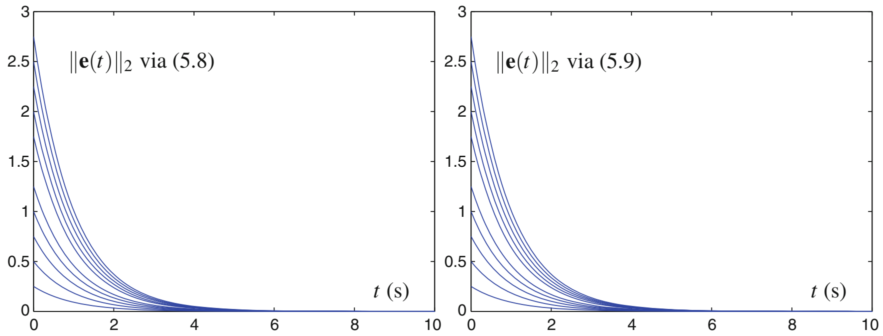

Fig. 5.12 Residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.8) and (5.9) with $\gamma = 1$ for solving under-determined system of time-varying linear equations involved in Example 5.3

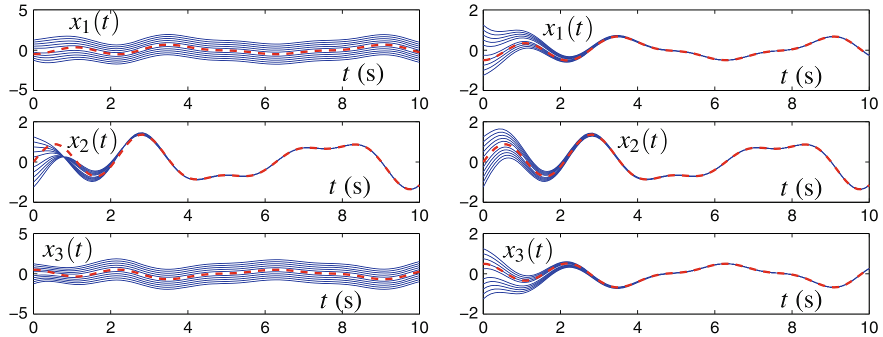

Fig. 5.13 State trajectories of ZD models (5.8) and (5.9) with $\gamma = 1$ for solving under-determined system of time-varying linear equations involved in Example 5.4

the under-determined system of time-varying linear equations with different initial states $\mathbf{x}(0)$ used. In contrast, other simulated state trajectories of ZD model (5.8) and all simulated state trajectories of ZD model (5.9), starting from randomly-generated initial states, relatively fast converge to the trajectories of the referenced theoretical solution $\mathbf{x}^*(t) = A^+(t)\mathbf{b}(t)$, as shown in Fig. 5.11.

Furthermore, Fig. 5.12 shows the residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.8) and (5.9) with $\gamma = 1$ used. As seen from the figure, the residual errors $\|\mathbf{e}(t)\|_2$ fast converge to zero. Note that the simulation results synthesized by ZD models (5.8) and (5.9) using different γ values are similar to those shown in Figs. 5.7 and 5.10 (and thus are omitted due to results similarity). That is, the residual errors with larger γ value converge faster than those with smaller γ value, showing the important role of γ for the proposed ZD models (5.8) and (5.9).

Example 5.4 In the fourth example, the following smoothly time-varying coefficient matrix $A(t)$ and coefficient vector $\mathbf{b}(t)$ of (5.1) with $m = 2$ and $n = 3$ are designed to test the proposed ZD models (5.8) and (5.9):

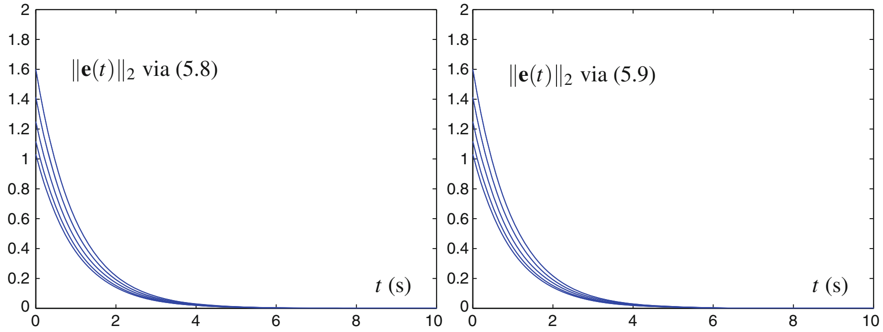

Fig. 5.14 Residual errors $\|\mathbf{e}(t)\|_2 = \|A(t)\mathbf{x}(t) - \mathbf{b}(t)\|_2$ of ZD models (5.8) and (5.9) with $\gamma = 1$ for solving under-determined system of time-varying linear equations involved in Example 5.4

$$A(t) = \begin{bmatrix} \sin(2t) & \cos(2t) & -\sin(2t) \\ -\cos(2t) & \sin(2t) & \cos(2t) \end{bmatrix} \in \mathbb{R}^{2\times 3} \text{ and } \mathbf{b}(t) = \begin{bmatrix} \sin(0.5t) \\ \cos(t) \end{bmatrix} \in \mathbb{R}^2.$$

The corresponding simulation results are shown in Figs. 5.13 and 5.14, where phenomena are similar to those in Example 5.3. That is, corresponding to $\mathbf{x}(t)$ in Fig. 5.13, the residual errors of ZD models (5.8) and (5.9) in Fig. 5.14 all converge to zero. Note that, being a topic of exercise, the related simulative verifications of ZD models (5.8) and (5.9) using different values of γ are left for interested readers.

In summary, the simulation results of the above two illustrative examples have substantiated the efficacy of the proposed ZD models (5.8) and (5.9) for solving under-determined system of time-varying linear equations.

5.4 Summary

In this chapter, by introducing different ZFs [i.e., (5.2), (5.4), (5.6), and (5.7)], different ZD models [i.e., (5.3), (5.5), (5.8), and (5.9)] have been proposed, generalized, developed, and investigated to solve over-determined and under-determined systems of time-varying linear equations (5.1). With different illustrative examples, computer simulation results have further substantiated the efficacy of the proposed ZD models for solving over-determined and under-determined systems of time-varying linear equations.

References

1. Siciliano B, Khatib O (2008) Springer handbook of robotics. Springer, Heidelberg
2. Siciliano B, Sciavicco L, Villani L, Oriolo G (2009) Robotics: modelling, planning and control. Springer, London
3. Zhang Y, Zhang Z (2013) Repetitive motion planning and control of redundant robot manipulators. Springer, New York
4. Brannigan M (1982) The strict Chebyshev solution of overdetermined systems of linear equations with rank deficient matrix. Numer Math 40(3):307–318
5. Grava T (2002) Riemann-Hilbert problem for the small dispersion limit of the KdV equation and linear overdetermined systems of Euler-Poisson-Darboux type. Commun Pure Appl Math 55(4):395–430
6. Cadzow JA (2002) Minimum ℓ_1, ℓ_2, and ℓ_∞ norm approximate solutions to an overdetermined system of linear equations. Dig Signal Process 12(4):524–560
7. Godunov SK (2008) On approximations for overdetermined hyperbolic equations. In: Proceedings of the 12th international conference on hyperbolic problems: theory, numerics, applications, pp 19–33
8. Rosen JB, Park H, Glick J, Zhang L (2000) Accurate solution to overdetermined linear equations with errors using L_1 norm minimization. Comput Optim Appl 17(2–3):329–341
9. Rathinavelan T, Wonpil IM (2007) Explicit treatment of force contribution from alignment tensor using overdetermined linear equations and its application in NMR structure determination. J Comput Chem 28(11):1858–1864
10. Zhang Y, Li W, Guo D, Mu B, Zheng H (2013) Different Zhang functions leading to various ZNN models illustrated via solving the time-varying overdetermined system of linear equations. In: Proceedings of international conference on information science and technology, pp 771–776
11. Zhang Y, Wang Y, Jin L, Mu B, Zheng H (2013) Different ZFs leading to various ZNN models illustrated via online solution of time-varying underdetermined systems of linear equations with robotic application. Lect Notes Comput Sci 7952:481–488
12. Mathews JH, Fink KD (2004) Numerical methods using MATLAB. Prentice Hall, New Jersey