Chapter 13
Time-Varying Complex Matrix Generalized Inverse

Abstract In Chaps. 8 and 9, different ZD models based on different ZFs have been presented and investigated to solve for time-varying matrix (left and right) pseudoinverse in real domain. In this chapter, the ZD approach (i.e., different ZFs leading to different ZD models) is extended and exploited to solve for time-varying matrix generalized inverse (in most cases, the pseudoinverse) in complex domain. Specifically, by introducing five different complex ZFs, five different complex ZD models are proposed, generalized, developed, and investigated for time-varying complex matrix generalized inverse computation. Theoretical results of convergence analysis are presented to show the desirable properties of the complex ZD models. In addition, we discover the link between the proposed complex ZD models and the Getz-Marsden (G-M) dynamic system in complex domain. Computer simulation results further substantiate the efficacy of the proposed complex ZD models based on different complex ZFs on solving for time-varying complex matrix generalized inverse.

13.1 Introduction

As presented in Chaps. 8 and 9, the solution of generalized inverse (in most cases, the pseudoinverse, and also known as Moore-Penrose generalized inverse) is one of the basic problems encountered in a variety of science and engineering fields, e.g., robotics [1], signal processing [2], associative memories [3] and image restoration [4, 5]. Owing to its important roles, numerous efforts have been devoted to the fast solution of generalized inverse matrices. As a result, many algorithms/methods (including those ZD models presented in Chaps. 8 and 9) have been put forward by researchers [6–12] for constant and/or time-varying matrix generalize inverse computation. However, it is worth pointing out that these research works are confined to applying different numerical algorithms or neural dynamics to solving for matrix generalized inverse (or sometimes termed, matrix pseudoinverse) in real domain.

Besides, as presented in Chap. 12, in some situations complex-valued matrices may also occur, when the problem incorporates online frequency domain identification processes, or when the input signals contain both magnitude and phase information [13, 14]. The presence of a complex-valued matrix points to the need for

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efficient online complex matrix inversion/pseudoinversion as well. In general, as for a complex-valued matrix $C \in \mathbb{C}^{m\times n}$, there are two cases, i.e., $m = n$ and $m \neq n$. Then, we have that $C^{-} \in \mathbb{C}^{n\times m}$ is known as the inverse of matrix C with $m = n$, and $C^{+} \in \mathbb{C}^{n\times m}$ is known as the generalized inverse of matrix C with $m \neq n$. Note that the investigations of solving for constant and time-varying complex square matrix inverse (i.e., corresponding to the situation of $m = n$) have been presented in [14, 15] and Chap. 12, respectively. Thus, in this chapter, we focus on solving for the generalized inverse (in most cases, the pseudoinverse) of time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ under the situation of $m \neq n$.

To lay a basis for further discussion, some necessary preliminaries of the time-varying complex matrix generalized inverse are given.

Definition 13.1 For a given time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ with $m \neq n$, if $Z(t) \in \mathbb{C}^{n\times m}$ satisfies at least one of the following four Penrose equations [16, 17]:

$$C(t)Z(t)C(t) = C(t), \quad Z(t)C(t)Z(t) = Z(t),$$
$$(C(t)Z(t))^{\mathrm{H}} = C(t)Z(t), \quad (Z(t)C(t))^{\mathrm{H}} = Z(t)C(t),$$

where superscript $^{\mathrm{H}}$ denotes the conjugate transpose (also called Hermitian transpose) of a complex matrix, $Z(t)$ is called the time-varying complex generalized inverse of $C(t)$. If matrix $Z(t)$ satisfies all of the Penrose equations, then matrix $Z(t)$ is called the pseudoinverse of matrix $C(t)$, which is often denoted by $C^{+}(t)$. Note that the pseudoinverse $C^{+}(t)$ exists and is unique, while the generalized inverse is not unique usually.

In addition, if matrix $C(t)$ is of full-rank at any time instant t, i.e., $\mathrm{rank}(C(t)) = \min\{m, n\}$ with $t \in [0, \infty)$, we have the following theorem to obtain the time-varying pseudoinverse of matrix $C(t)$.

Theorem 13.1 *For a given time-varying matrix $C(t) \in \mathbb{C}^{m\times n}$ with $m \neq n$, if it satisfies that* $\mathrm{rank}(C(t)) = \min\{m, n\}$ *at any time instant t, then the unique time-varying pseudoinverse $C^{+}(t)$is given as follows [17–19]:*

$$C^{+}(t) = \begin{cases} C^{\mathrm{H}}(t)(C(t)C^{\mathrm{H}}(t))^{-1}, & \text{if } m < n, \\ (C^{\mathrm{H}}(t)C(t))^{-1}C^{\mathrm{H}}(t), & \text{if } m > n. \end{cases} \tag{13.1}$$

Besides, as for the unique time-varying pseudoinverse of a full-rank matrix $C(t)$, we have another important theorem as follows (which motivates us to define many more ZFs for time-varying complex matrix generalized inverse).

Theorem 13.2 *For a given time-varying matrix $C(t) \in \mathbb{C}^{m\times n}$ with $m \neq n$, if it satisfies that* $\mathrm{rank}(C(t)) = \min\{m, n\}$ *at any time instant t, then the unique time-varying pseudoinverse $C^{+}(t)$ is also given as follows:*

$$C^{+}(t) = \begin{cases} \lim\limits_{\mu \to 0} (C^{\mathrm{H}}(t)C(t) + \mu I)^{-1} C^{\mathrm{H}}(t), & \textit{if } m < n, \\ \lim\limits_{\mu \to 0} C^{\mathrm{H}}(t)(C(t)C^{\mathrm{H}}(t) + \mu I)^{-1}, & \textit{if } m > n, \end{cases}$$

where $\mu > 0 \in \mathbb{R}$.

Proof It can be generalized from the proof of Theorem 8.2. □

For simplicity, in this chapter, we only consider the smoothly time-varying full-rank complex matrix $C(t) \in \mathbb{C}^{m\times n}$ with $m < n$. This paper aims at finding $Z(t) \in \mathbb{C}^{n\times m}$ such that at least one of the Penrose equations holds true at any time instant $t \in [0, +\infty)$, i.e., obtaining the complex generalized inverse (in most cases, the pseudoinverse) of matrix $C(t)$. Note that, in the case of $m > n$, the complex generalized inverse of matrix $C(t)$ could be obtained in a similar way, and is thus omitted due to similarity and space limitation.

More specifically, focusing on solving for the generalized inverse of time-varying complex matrix $C(t)$ with $m < n$, we propose, generalize, develop, and investigate five different complex ZD models by defining five different complex ZFs. It is then theoretically proved that the proposed complex ZD models (globally) exponentially converge to the theoretical time-varying generalized inverse. Moreover, we discover the link between the proposed complex ZD models and the Getz-Marsden (G-M) dynamic system [20] in the complex domain. Through illustrative computer-simulation examples, the efficacy of the proposed complex ZD models for time-varying complex matrix generalized inverse computation is well substantiated.

13.2 Complex ZFs and ZD Models

In this section, five different complex ZD models based on five different complex ZFs are constructed to solve for the time-varying complex generalized inverse (in most cases, the pseudoinverse). In addition, their excellent convergence performances are analyzed in detail.

According to the ZD design formula (12.2), different complex ZFs can lead to different complex ZD models for solving the same time-varying complex-valued problem. Especially, to solve for the time-varying complex generalized inverse, we define the following five different complex ZFs as the fundamental error-monitoring functions:

$$E(t) = Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t) \in \mathbb{C}^{n\times m}, \tag{13.2}$$

$$E(t) = C^{\mathrm{H}}(t)C(t)Z(t) - C^{\mathrm{H}}(t) \in \mathbb{C}^{n\times m}, \tag{13.3}$$

$$E(t) = C(t)Z(t) - I \in \mathbb{C}^{m\times m}, \tag{13.4}$$

$$E(t) = Z(t)C(t) - I \in \mathbb{C}^{n\times n}, \tag{13.5}$$

$$E(t) = C(t) - Z^{+}(t) \in \mathbb{C}^{m\times n}. \tag{13.6}$$

13.2.1 The First Complex ZD Model

Considering complex ZF (13.2), we have the following derivation:

$$\dot{E}(t) = \dot{Z}(t)C(t)C^{\mathrm{H}}(t) + Z(t)\left(\dot{C}(t)C^{\mathrm{H}}(t) + C(t)\dot{C}^{\mathrm{H}}(t)\right) - \dot{C}^{\mathrm{H}}(t).$$

Then, adopting the ZD design formula (12.2), we can derive the corresponding dynamic equation of the first complex ZD model as

$$\begin{aligned}\dot{Z}(t)C(t)C^{\mathrm{H}}(t) = \dot{C}^{\mathrm{H}}(t) - Z(t)\left(\dot{C}(t)C^{\mathrm{H}}(t) + C(t)\dot{C}^{\mathrm{H}}(t)\right)\\ -\gamma\left(Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)\right).\end{aligned} \tag{13.7}$$

In other words, we obtain complex ZD model (13.7) based on complex ZF (13.2) to solve for the time-varying complex generalized inverse (specifically, the pseudoinverse). By following complex ZD model (13.7), the *ij*th neuron's dynamic equation can be presented in the following form:

$$\dot{z}_{ij} = \sum_{l=1}^{m} \dot{z}_{il} a_{lj} - \gamma\left(\sum_{l=1}^{m} z_{il} b_{lj} - c^*_{ji}\right) - \sum_{l=1}^{m} z_{il} d_{lj} + \dot{c}^*_{ji},$$

where c_{ji}, a_{lj}, b_{lj} and d_{lj} denote the corresponding elements of matrices C, $A = I - CC^{\mathrm{H}}$, $B = CC^{\mathrm{H}}$ and $D = \dot{C}C^{\mathrm{H}} + C\dot{C}^{\mathrm{H}}$, respectively, and the operator $*$ denotes complex conjugate. Then, the neuron-connection architecture of complex ZD model (13.7) is depicted in Fig. 13.1, and the specific structure of the ith row of neurons is illustrated in Fig. 13.2. Figures 13.1 and 13.2 well show that complex ZD model (13.7) is a kind of Hopfield-type recurrent neural networks which can be implemented finally on analog circuits such as very large-scale integration [11, 21, 22].

To lay a basis for discussion, an important theorem is presented below.

Theorem 13.3 *For any time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$, we have [23]*

$$\frac{\mathrm{d}C^{\mathrm{H}}(t)}{\mathrm{d}t} = \left(\frac{\mathrm{d}C(t)}{\mathrm{d}t}\right)^{\mathrm{H}},$$

which, via a simpler notation of $\mathrm{d}C^{\mathrm{H}}(t)/\mathrm{d}t$, can be rewritten as $\dot{C}^{\mathrm{H}}(t) = (\dot{C}(t))^{\mathrm{H}}$. Especially, for a scalar complex variable $c(t) \in \mathbb{C}$, we have $\dot{c}^*(t) = (\dot{c}(t))^*$.

For complex ZD model (13.7), we have the following theoretical result on its global exponential convergence performance.

Theorem 13.4 *Given a smoothly time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ (with $m < n$) of full rank, the state matrix $Z(t) \in \mathbb{C}^{n\times m}$ of complex ZD model (13.7), starting from an initial state $Z(0)$, globally and exponentially converges to the theoretical time-varying generalized inverse [specifically, the pseudoinverse $C^{+}(t) \in \mathbb{C}^{n\times m}$] of matrix $C(t)$.*

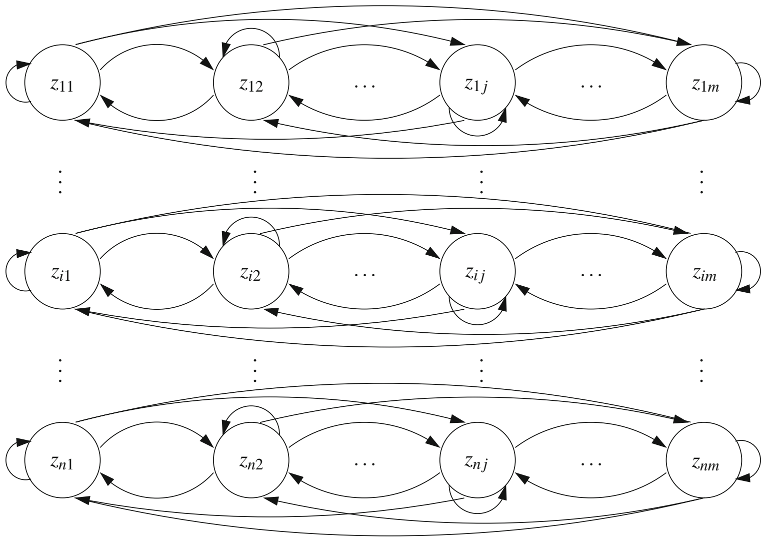

Fig. 13.1 Neuron-connection architecture of complex ZD model (13.7) for time-varying complex generalized inverse computation

Proof Let $\tilde{Z}(t) = Z(t) - C^{+}(t)$ denote the difference between the solution $Z(t)$ generated by complex ZD model (13.7) and the theoretical pseudoinverse $C^{+}(t)$. Following from $C^{+}(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t) = 0$, its time derivative is depicted as

$$\dot{C}^{+}(t)C(t)C^{\mathrm{H}}(t) + C^{+}(t)\left(\dot{C}(t)C^{\mathrm{H}}(t) + C(t)\dot{C}^{\mathrm{H}}(t)\right) - \dot{C}^{\mathrm{H}}(t) = 0.$$

Substituting $C^{+}(t) = Z(t) - \tilde{Z}(t)$ into the above identity, we have

$$\begin{aligned}&\dot{\tilde{Z}}(t)C(t)C^{\mathrm{H}}(t) + \tilde{Z}(t)\left(\dot{C}(t)C^{\mathrm{H}}(t) + C(t)\dot{C}^{\mathrm{H}}(t)\right) = \\ &\dot{Z}(t)C(t)C^{\mathrm{H}}(t) + Z(t)\left(\dot{C}(t)C^{\mathrm{H}}(t) + C(t)\dot{C}^{\mathrm{H}}(t)\right) - \dot{C}^{\mathrm{H}}(t).\end{aligned}$$

Using complex ZD model equation (13.7), with $Z(t) = \tilde{Z}(t) + C^{+}(t)$, it follows that $\tilde{Z}(t)$ is the solution to the ensuing dynamics with the initial state $\tilde{Z}(0) = Z(0) - C^{+}(0)$,

$$\dot{\tilde{Z}}(t)C(t)C^{\mathrm{H}}(t) + \tilde{Z}(t)\left(\dot{C}(t)C^{\mathrm{H}}(t) + C(t)\dot{C}^{\mathrm{H}}(t)\right) = -\gamma\tilde{Z}(t)C(t)C^{\mathrm{H}}(t). \quad (13.8)$$

Since $E(t) = \tilde{Z}(t)C(t)C^{\mathrm{H}}(t)$, (13.8) can thus be rewritten as $\dot{E}(t) = -\gamma E(t)$, which is a compact matrix form of the following set of $n \times m$ equations:

$$\dot{e}_{ij}(t) = -\gamma e_{ij}(t),\ \forall i \in \{1, 2, \ldots, n\} \text{ and } j \in \{1, 2, \ldots, m\}. \quad (13.9)$$

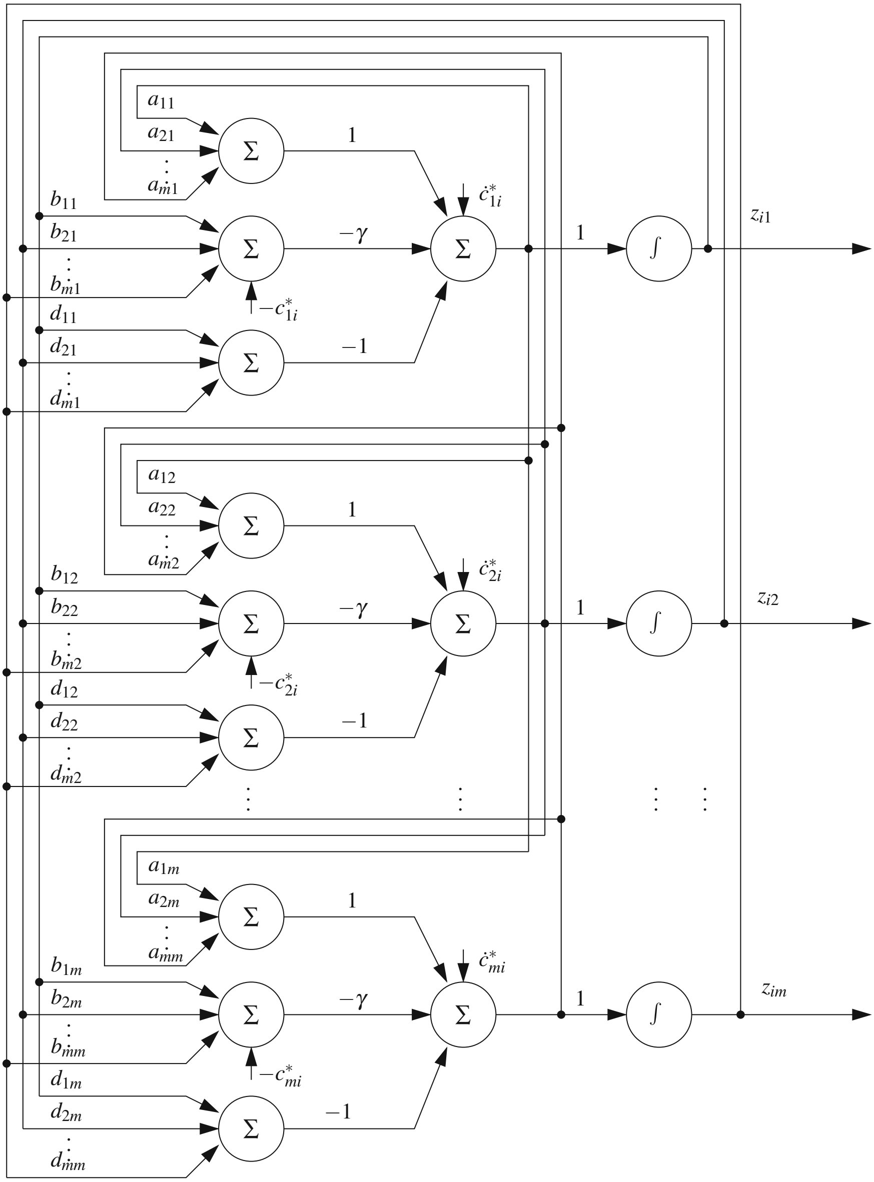

Fig. 13.2 Structure of the ith row of neurons in complex ZD model (13.7) for time-varying complex generalized inverse computation

Evidently, we can define a Lyapunov function candidate $v_{ij} = e_{ij}e_{ij}^*/2 \geqslant 0$ for the ijth subsystem (13.9), which is positive-definite, i.e., $v_{ij} > 0$ for $e_{ij} \neq 0$ and $v_{ij} = 0$ for $e_{ij} = 0$. Then, we have its time derivative

$$\frac{\mathrm{d}v_{ij}(t)}{\mathrm{d}t} = \frac{1}{2}\left(\dot{e}_{ij}e_{ij}^* + e_{ij}\dot{e}_{ij}^*\right).$$

Adopting Theorem 13.3 and (13.9), we obtain

$$\frac{\mathrm{d}v_{ij}(t)}{\mathrm{d}t} = \frac{1}{2}\left((-\gamma e_{ij})e_{ij}^* + e_{ij}(-\gamma e_{ij})^*\right) = -\gamma e_{ij}e_{ij}^*.$$

Apparently, $\dot{v}_{ij}$ is negative-definite, i.e., $\dot{v}_{ij} < 0$ for $e_{ij} \neq 0$ and $\dot{v}_{ij} = 0$ for $e_{ij} = 0$. In addition, if $|e_{ij}| \to \infty$, the Lyapunov function candidate $v_{ij} = |e_{ij}|^2/2 \to \infty$. By the Lyapunov stability theory, $e_{ij}(t)$ globally converges to zero for any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Thus, in view of $E(t) = Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)$ and the nonsingularity of $C(t)C^{\mathrm{H}}(t)$, we have $Z(t) \to C^{\mathrm{H}}(t)(C(t)C^{\mathrm{H}}(t))^{-1} \in \mathbb{C}^{n\times m}$ as $t \to \infty$. Then, in view of $m < n$ and based on Theorem 13.1, the state matrix $Z(t)$ of (13.7) globally converges to the theoretical time-varying generalized inverse [specifically, the pseudoinverse $C^+(t) \in \mathbb{C}^{n\times m}$] starting from a randomly-generated initial state $Z(0)$. Next, we are going to prove the exponential convergence performance of complex ZD model (13.7).

In view of (13.9), we can obtain its analytic solution in the compact matrix form:

$$E(t) = E(0)\exp(-\gamma t).$$

Thus, we further have

$$\|E(t)\|_{\mathrm{F}} = \|E(0)\|_{\mathrm{F}}\exp(-\gamma t).$$

Evidently, as $t \to \infty$, $\|E(t)\|_{\mathrm{F}}$ exponentially converges to 0 with rate γ, which means that, starting from any randomly-generated initial state $Z(0)$, state matrix $Z(t)$ of complex ZD model (13.7) exponentially converges to the theoretical time-varying generalized inverse [specifically, the pseudoinverse $C^+(t)$] with rate $\gamma > 0$. The proof on global and exponential convergence of complex ZD model (13.7) is thus complete. □

For further investigation and illustration, we can also make use of other complex ZFs [i.e., complex ZFs (13.3) through (13.6)] to construct other types of complex ZD models. Thus, it can provide many more models for researchers to choose.

13.2.2 *The Second Complex ZD Model*

For complex ZF (13.3), as $m < n$, $C^{\mathrm{H}}(t)C(t) \in \mathbb{C}^{n\times n}$ is singular. Thus, we can add a bias term $\lambda I \in \mathbb{R}^{n\times n}$ to $C^{\mathrm{H}}(t)C(t)$, with $\lambda > 0 \in \mathbb{R}$. This method is known as

Tikhonov regularization method [24]. Then, complex ZF (13.3) is modified as

$$E(t) = \left(C^{\mathrm{H}}(t)C(t) + \lambda I\right) Z(t) - C^{\mathrm{H}}(t). \tag{13.10}$$

For modified ZF (13.10), we have its time derivative

$$\dot{E}(t) = \left(C^{\mathrm{H}}(t)C(t) + \lambda I\right) \dot{Z}(t) + \left(\dot{C}^{\mathrm{H}}(t)C(t) + C^{\mathrm{H}}(t)\dot{C}(t)\right) Z(t) - \dot{C}^{\mathrm{H}}(t).$$

Following the ZD design formula (12.2), we obtain the dynamic equation of the second complex ZD model as

$$\begin{aligned}\left(C^{\mathrm{H}}(t)C(t) + \lambda I\right) \dot{Z}(t) = \dot{C}^{\mathrm{H}}(t) - \left(\dot{C}^{\mathrm{H}}(t)C(t) + C^{\mathrm{H}}(t)\dot{C}(t)\right) Z(t) \\ -\gamma \left(\left(C^{\mathrm{H}}(t)C(t) + \lambda I\right) Z(t) - C^{\mathrm{H}}(t)\right).\end{aligned} \tag{13.11}$$

Note that the parameter λ should be set appropriately small, in other words, λ should be sufficiently close to 0. Similarly, after presenting complex ZD model (13.11) for solving for the time-varying complex generalized inverse (specifically, the pseudoinverse), we come to prove its convergence performance through the following important theorem.

Theorem 13.5 *Given a smoothly time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ (with $m < n$) of full rank, the state matrix $Z(t) \in \mathbb{C}^{n\times m}$ of complex ZD model (13.11), starting from an initial state $Z(0)$, globally and exponentially converges to the theoretical time-varying generalized inverse [specifically, the pseudoinverse $C^{+}(t) \in \mathbb{C}^{n\times m}$] of matrix $C(t)$.*

Proof Since complex ZD model (13.11) is derived using the standard ZD design method similar to the aforementioned first complex ZD model, its modified ZF (13.10) satisfies relation (12.2), which means that $E(t) = (C^{\mathrm{H}}(t)C(t) + \lambda I)Z(t) - C^{\mathrm{H}}(t)$ can globally and exponentially converge to zero from an initial value. That is to say, as $t \to \infty$, we have $Z(t) \to (C^{\mathrm{H}}(t)C(t) + \lambda I)^{-1}C^{\mathrm{H}}(t) \in \mathbb{C}^{n\times m}$. In view of $\lambda \to 0$ and $m < n$, then based on Theorem 13.2, the state matrix $Z(t)$ globally and exponentially converges to the theoretical time-varying generalized inverse, specifically, the pseudoinverse $C^{+}(t)$. The proof on global and exponential convergence performance of complex ZD model (13.11) is thus complete. □

13.2.3 The Third Complex ZD Model

Combining the ZD design formula (12.2) and complex ZF (13.4), we can have

$$\dot{C}(t)Z(t) + C(t)\dot{Z}(t) = -\gamma \left(C(t)Z(t) - I\right),$$

and then

$$C(t)\dot{Z}(t) = -\dot{C}(t)Z(t) - \gamma\,(C(t)Z(t) - I)\,.$$

For the purpose of computation and simulation, by left multiplying $C^{\mathrm{H}}(t)$ both sides of the above equation, we obtain

$$C^{\mathrm{H}}(t)C(t)\dot{Z}(t) = -C^{\mathrm{H}}(t)\dot{C}(t)Z(t) - \gamma\left(C^{\mathrm{H}}(t)C(t)Z(t) - C^{\mathrm{H}}(t)\right). \quad (13.12)$$

Note that, in (13.12), $C^{\mathrm{H}}(t)C(t)$ is singular (in view of $m < n$). Hence, to make (13.12) more computable, we can similarly adopt the Tikhonov regularization method [24], i.e., add a bias term λI with $\lambda \to 0$ to $C^{\mathrm{H}}(t)C(t)$. As a result, the dynamic equation of the third complex ZD model is presented as

$$\begin{aligned}\left(C^{\mathrm{H}}(t)C(t) + \lambda I\right)\dot{Z}(t) = &-C^{\mathrm{H}}(t)\dot{C}(t)Z(t)\\ &- \gamma\left(\left(C^{\mathrm{H}}(t)C(t) + \lambda I\right)Z(t) - C^{\mathrm{H}}(t)\right). \end{aligned} \quad (13.13)$$

That is to say, based on complex ZF (13.4), we obtain complex ZD model (13.13) to solve for the time-varying complex generalized inverse (specifically, the pseudoinverse). Similarly, the important theorem about the convergence performance of complex ZD model (13.13) is given as follows.

Theorem 13.6 *Given a smoothly time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ (with $m < n$) of full rank, the state matrix $Z(t) \in \mathbb{C}^{n\times m}$ of complex ZD model (13.13), starting from an initial state $Z(0)$, globally and exponentially converges to the theoretical time-varying generalized inverse [specifically, the pseudoinverse $C^{+}(t) \in \mathbb{C}^{n\times m}$] of matrix $C(t)$.*

Proof The convergence of complex ZD model (13.13) can be proven in a way similar to the proofs of Theorems 13.4 and 13.5, and thus it is omitted here. □

13.2.4 The Fourth Complex ZD Model

With the ZD design formula (12.2) and complex ZF (13.5), we have

$$\dot{Z}(t)C(t) + Z(t)\dot{C}(t) = -\gamma\,(Z(t)C(t) - I)\,,$$

and then

$$\dot{Z}(t)C(t) = -Z(t)\dot{C}(t) - \gamma\,(Z(t)C(t) - I)\,. \quad (13.14)$$

Similarly, to make (13.14) more computable, we right multiply $C^{\mathrm{H}}(t)$ both sides of (13.14), and obtain the dynamic equation of the fourth complex ZD model as

$$\dot{Z}(t)C(t)C^{\mathrm{H}}(t) = -Z(t)\dot{C}(t)C^{\mathrm{H}}(t) - \gamma\left(Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)\right). \quad (13.15)$$

Thus, based on complex ZF (13.5), we have complex ZD model (13.15) to solve for the time-varying complex generalized inverse (specifically, pseudoinverse). Similar to the previous ZD models, we have the following important result about the convergence performance of complex ZD model (13.15). That is, given a smoothly time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ (with $m < n$) of full rank, the state matrix $Z(t) \in \mathbb{C}^{n\times m}$ of complex ZD model (13.15), starting from an initial state $Z(0)$, globally and exponentially converges to the theoretical time-varying generalized inverse, specifically, the pseudoinverse $C^{+}(t) \in \mathbb{C}^{n\times m}$.

13.2.5 The Fifth Complex ZD Model

Before constructing the fifth complex ZD model, an important corollary (being an extension of Corollary 5.1 from the real domain to the complex domain) is presented here to lay a basis for discussion.

Corollary 13.1 *For a given time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ (with $m < n$) and its time-varying pseudoinverse $C^{+}(t)$, we approximately have $\dot{C}^{+}(t) = -C^{+}(t)\dot{C}(t)C^{+}(t)$.*

Proof It can also be generalized from the proof of Theorem 4.1 in Chap. 4. □

Then, based on the ZD design formula (12.2) and complex ZF (13.6), we can have

$$\dot{C}(t) - \dot{Z}^{+}(t) = -\gamma\left(C(t) - Z^{+}(t)\right).$$

By adopting Corollary 13.1, the above equation can be further rewritten as

$$\begin{aligned} \dot{C}(t) + Z^{+}(t)\dot{Z}(t)Z^{+}(t) &= -\gamma\left(C(t) - Z^{+}(t)\right), \\ Z^{+}(t)\dot{Z}(t)Z^{+}(t) &= -\dot{C}(t) - \gamma\left(C(t) - Z^{+}(t)\right). \end{aligned} \quad (13.16)$$

Reformulating (13.16), we have the following dynamic equation of the new complex ZD model aiming at solving for the time-varying complex generalized inverse:

$$\dot{Z}(t) = -Z(t)\dot{C}(t)Z(t) - \gamma\left(Z(t)C(t)Z(t) - Z(t)\right), \quad (13.17)$$

Thus, we obtain complex ZD model (13.17) based on complex ZF (13.6). Note that complex ZD model (13.17) is also the Getz and Marsden (G-M) dynamic system [20] for the time-varying complex generalized inverse computation. In other words, the

G-M dynamic system could be generalized to the case of complex matrices and is a special case of the complex ZD models. This is quite a novel result beyond our previous work. Following the literature [20], we have the theoretical results of the convergence performance of complex ZD model (13.17). That is, given a smoothly time-varying complex matrix $C(t) \in \mathbb{C}^{m\times n}$ of full rank, if initial state $Z(0)$ satisfies $\|Z(0) - C^{+}(0)\|_{\mathrm{F}} \leqslant \beta < \infty$ and $\beta \in \mathbb{R}$ is sufficiently small, then $C(t)Z(t) - I \to 0$ as $t \to \infty$, i.e., the state matrix $Z(t) \in \mathbb{C}^{n\times m}$ of complex ZD model (13.17) exponentially converges to the theoretical time-varying generalized inverse of matrix $C(t)$. It is worth noting that the initial condition of complex ZD model (13.17) should be chosen as $Z(0) \approx C^{+}(0)$ [i.e., $Z(0)$ should be sufficiently close to $C^{+}(0)$].

In summary, we have constructed five different complex ZD models (13.7), (13.11), (13.13), (13.15), and (13.17) by defining five different complex ZFs [i.e., complex ZFs (13.2)–(13.6)] to solve for time-varying complex generalized inverse (in most cases, the pseudoinverse). For readers' convenience and also for comparison purpose, we summarize these complex ZFs and the corresponding complex ZD models in Table 13.1.

Remark 13.1 Five complex ZFs have been elaborately constructed to obtain five different complex ZD models. There exist clear differences among such complex ZD models. Specifically, the dynamic equations, model complexities and convergence performance differ from each other. For instance, the fifth complex ZD model (13.17) has the simplest network structure which can be more readily implemented, whereas the first complex ZD model (13.7) has better global convergence performance. Therefore, in practical applications, the practitioner could find and choose the most suitable complex ZF and the corresponding complex ZD model in accordance with specific request.

Table 13.1 Different complex ZFs resulting in different complex ZD models for time-varying complex generalized inverse (in most cases, the pseudoinverse) computation

Complex ZF	Complex ZD model
(13.2)	$\dot{Z}(t)C(t)C^{\mathrm{H}}(t) = -\gamma(Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)) - Z(t)(\dot{C}(t)C^{\mathrm{H}}(t) + C(t)\dot{C}^{\mathrm{H}}(t)) + \dot{C}^{\mathrm{H}}(t)$
(13.3)	$(C^{\mathrm{H}}(t)C(t) + \lambda I)\dot{Z}(t) = \dot{C}^{\mathrm{H}}(t) - (\dot{C}^{\mathrm{H}}(t)C(t) + C^{\mathrm{H}}(t)\dot{C}(t))Z(t) - \gamma((C^{\mathrm{H}}(t)C(t) + \lambda I)Z(t) - C^{\mathrm{H}}(t))$
(13.4)	$(C^{\mathrm{H}}(t)C(t) + \lambda I)\dot{Z}(t) = -C^{\mathrm{H}}(t)\dot{C}(t)Z(t) - \gamma((C^{\mathrm{H}}(t)C(t) + \lambda I)Z(t) - C^{\mathrm{H}}(t))$
(13.5)	$\dot{Z}(t)C(t)C^{\mathrm{H}}(t) = -Z(t)\dot{C}(t)C^{\mathrm{H}}(t) - \gamma(Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t))$
(13.6)	$\dot{Z}(t) = -Z(t)\dot{C}(t)Z(t) - \gamma(Z(t)C(t)Z(t) - Z(t))$

13.3 Illustrative Examples

In this section, the related simulation techniques are presented, and four illustrative examples are given to substantiate the efficacy of the proposed complex ZD models [i.e., (13.7), (13.11), (13.13), (13.15), and (13.17)] on solving for time-varying complex generalized inverse (in most cases, the pseudoinverse).

Kronecker product and vectorization In the previous sections, we have developed five complex ZD models (13.7), (13.11), (13.13), (13.15), and (13.17) for time-varying complex generalized inverse computation. Note that all the proposed complex ZD models are described in matrix form, which cannot be directly simulated. Thus, the Kronecker product and vectorization techniques [11, 25] are needed to transform such matrix-form differential equations to vector-form differential equations for simulative purposes. Note that, for presentation convenience, $B(t) = C^{\mathrm{H}}(t)C(t) + \lambda I$ is introduced [for complex ZD models (13.11) and (13.13)].

- For complex ZD model (13.7), based on the Kronecker product (denoted by the symbol of "⊗") and vectorization techniques, we can transform such a complex ZD model into the following vector-form differential equation:

$$\begin{aligned}\left((CC^{\mathrm{H}})^{\mathrm{T}} \otimes I\right) \mathrm{vec}(\dot{Z}) = {} & \mathrm{vec}(\dot{C}^{\mathrm{H}}) - \left((\dot{C}C^{\mathrm{H}})^{\mathrm{T}} \otimes I + (C\dot{C}^{\mathrm{H}})^{\mathrm{T}} \otimes I\right) \mathrm{vec}(Z) \\ & -\gamma \left(\left((CC^{\mathrm{H}})^{\mathrm{T}} \otimes I\right) \mathrm{vec}(Z) - \mathrm{vec}(C^{\mathrm{H}})\right).\end{aligned}$$

- In view of $B(t) = C^{\mathrm{H}}(t)C(t) + \lambda I$, complex ZD model (13.11) is rewritten as

$$B(t)\dot{Z}(t) = \dot{C}^{\mathrm{H}}(t) - \left(\dot{C}^{\mathrm{H}}(t)C(t) + C^{\mathrm{H}}(t)\dot{C}(t)\right) Z(t) - \gamma \left(B(t)Z(t) - C^{\mathrm{H}}(t)\right).$$

  Therefore, similar to (13.7), we obtain the vector form of (13.11) as follows:

$$\begin{aligned}(I \otimes B)\, \mathrm{vec}(\dot{Z}) = {} & \mathrm{vec}(\dot{C}^{\mathrm{H}}) - \left((I \otimes \dot{C}^{\mathrm{H}}C) + (I \otimes C^{\mathrm{H}}\dot{C})\right) \mathrm{vec}(Z) \\ & -\gamma \left((I \otimes B)\, \mathrm{vec}(Z) - \mathrm{vec}(C^{\mathrm{H}})\right).\end{aligned}$$

- Similarly, for complex ZD model (13.13), we can have its vector form as

$$(I \otimes B)\, \mathrm{vec}(\dot{Z}) = -(I \otimes C^{\mathrm{H}}\dot{C})\, \mathrm{vec}(Z) - \gamma \left((I \otimes B)\, \mathrm{vec}(Z) - \mathrm{vec}(C^{\mathrm{H}})\right).$$

- Considering complex ZD model (13.15), we similarly have its vector form as

$$\begin{aligned}\left((CC^{\mathrm{H}})^{\mathrm{T}} \otimes I\right) \mathrm{vec}(\dot{Z}) = {} & -\left((\dot{C}C^{\mathrm{H}})^{\mathrm{T}} \otimes I\right) \mathrm{vec}(Z) \\ & -\gamma \left(\left((CC^{\mathrm{H}})^{\mathrm{T}} \otimes I\right) \mathrm{vec}(Z) - \mathrm{vec}(C^{\mathrm{H}})\right).\end{aligned}$$

- For complex ZD model (13.17), we can also obtain its vector form as

$$\mathrm{vec}(\dot{Z}) = -(I \otimes Z\dot{C})\, \mathrm{vec}(Z) - \gamma \left((I \otimes ZC)\, \mathrm{vec}(Z) - \mathrm{vec}(Z)\right).$$

Besides, it is worth pointing out here that, in MATLAB, the Kronecker product can be realized by using the routine "kron" [i.e., "$Z \otimes C$" is realized by the code "`kron(Z,C)`"], and "vec(Z)" is realized by the code "`reshape(Z,n*m,1)`".

Therefore, based on the aforementioned vectorization technique, the following four computer simulation examples are illustrated to substantiate the efficacy of the proposed complex ZD models (13.7), (13.11), (13.13), (13.15), and (13.17) on solving for time-varying complex generalized inverse.

Example 13.1 In this example, we consider the time-varying full-rank complex matrix $C(t)$ as follows:

$$C(t) = \begin{bmatrix} i\sin(3t) & i\cos(3t) & -i\sin(3t) \\ -i\cos(3t) & i\sin(3t) & i\cos(3t) \end{bmatrix} \in \mathbb{C}^{2\times 3}. \tag{13.18}$$

For checking the correctness of the ZD solution, according to (13.1), we can obtain the theoretical time-varying pseudoinverse of matrix $C(t)$ in (13.18) as

$$C^{+}(t) = \begin{bmatrix} -0.5i\sin(3t) & 0.5i\cos(3t) \\ -i\cos(3t) & -i\sin(3t) \\ 0.5i\sin(3t) & -0.5i\cos(3t) \end{bmatrix} \in \mathbb{C}^{3\times 2}.$$

Since we have obtained theoretical pseudoinverse $C^{+}(t)$, we can use it as an analytic theoretical solution to verify the correctness of the solution synthesized by complex ZD model (13.7). As illustrated in Fig. 13.3, starting from a randomly-generated initial state $Z(0) \in \mathbb{C}^{3\times 2}$, the state matrix $Z(t) \in \mathbb{C}^{3\times 2}$ of complex ZD model (13.7) with $\gamma = 100$ can converge to the theoretical pseudoinverse $C^{+}(t)$ rapidly and accurately within a rather short time. In addition, we show the residual errors $\|E(t)\| = \|Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)\|_{\mathrm{F}}$ synthesized by the proposed complex ZD model (13.7) starting from 10 randomly-generated initial states. From Fig. 13.4, we can further find that the residual errors of (13.7) all diminish to zero within around 0.06 s. These simulation results demonstrate the efficacy of complex ZD model (13.7) on solving for time-varying complex generalized inverse (specifically, the pseudoinverse).

Example 13.2 In this example, we verify the efficacy of the proposed complex ZD models (13.11) and (13.13) use a more general complex matrix. Let us consider the following time-varying complex matrix:

$$C(t) = \begin{bmatrix} \exp(4it) & i\exp(4it) & \exp(-4it) \\ i\exp(4it) & \exp(4it) & i\exp(4it) \end{bmatrix} \in \mathbb{C}^{2\times 3}. \tag{13.19}$$

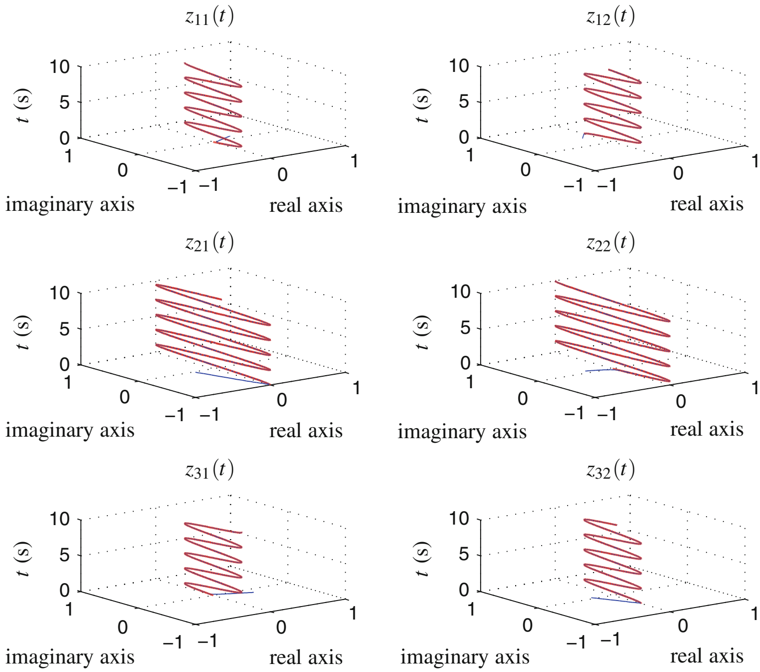

Fig. 13.3 State trajectories of complex ZD model (13.7) with $\gamma = 100$, where *dash-dotted curves* denote the theoretical time-varying pseudoinverse $C^{+}(t)$ in Example 13.1 and *solid curves* denote the solution computed by complex ZD model (13.7)

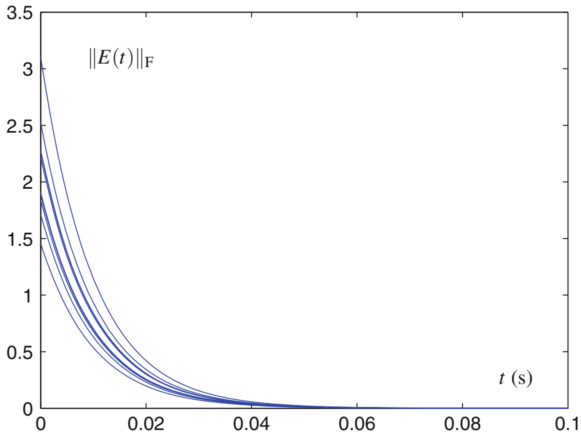

Fig. 13.4 Residual errors $\|E(t)\|_{\mathrm{F}} = \|Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)\|_{\mathrm{F}}$ synthesized by complex ZD model (13.7) with $\gamma = 100$ for the time-varying pseudoinverse of matrix $C(t)$ in (13.18)

It follows from (13.1) that the theoretical time-varying pseudoinverse of matrix $C(t)$ in (13.19) is

$$C^{+}(t)=\begin{bmatrix} \frac{3}{8}\exp(-4it)-\frac{1}{8}\exp(4it) & -\frac{3}{8}i\exp(-4it)+\frac{1}{8}i\exp(-12it) \\ -\frac{3}{8}i\exp(-4it)-\frac{1}{8}i\exp(4it) & \frac{3}{8}\exp(-4it)+\frac{1}{8}\exp(-12it) \\ \frac{1}{4}\exp(4it) & -\frac{1}{4}i\exp(-4it) \end{bmatrix}\in\mathbb{C}^{3\times 2}.$$

Note that, in this example, the complex matrix $C(t)$ in (13.19) is more general since its elements have both real and imaginary parts. Furthermore, the variation frequency of such a complex matrix is greater than that of Example 13.1. Figures 13.5 and 13.6, respectively, illustrate the neural state $Z(t)$ of complex ZD models (13.11) and (13.13) by using $\gamma = 100$ and $\lambda = 10^{-3}$, with the residual errors $\|E(t)\|_{\mathrm{F}} = \|Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)\|_{\mathrm{F}}$ shown in Fig. 13.7. As seen from Figs. 13.5 and 13.6, starting from a randomly-generated initial state $Z(0)$, the neural states $Z(t)$ of complex ZD models (13.11) and (13.13) both converge to the theoretical time-varying pseudoinverse $C^{+}(t)$. In addition, from Fig. 13.7, we can see that residual errors $\|E(t)\|_{\mathrm{F}}$ of (13.11) and (13.13) all converge to zero. Therefore, the efficacy of complex ZD models (13.11) and (13.13) on solving for the time-varying complex generalized inverse (specifically, the pseudoinverse) is also substantiated.

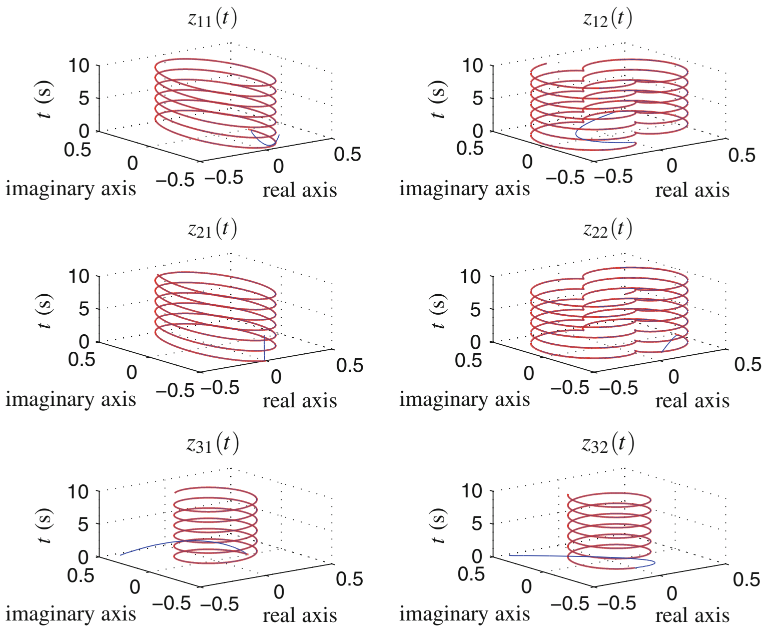

Fig. 13.5 State trajectories of complex ZD model (13.11) with $\gamma = 100$ and $\lambda = 10^{-3}$, where *dash-dotted curves* denote the theoretical time-varying pseudoinverse $C^{+}(t)$ in Example 13.2 and *solid curves* denote the solution computed by complex ZD model (13.11)

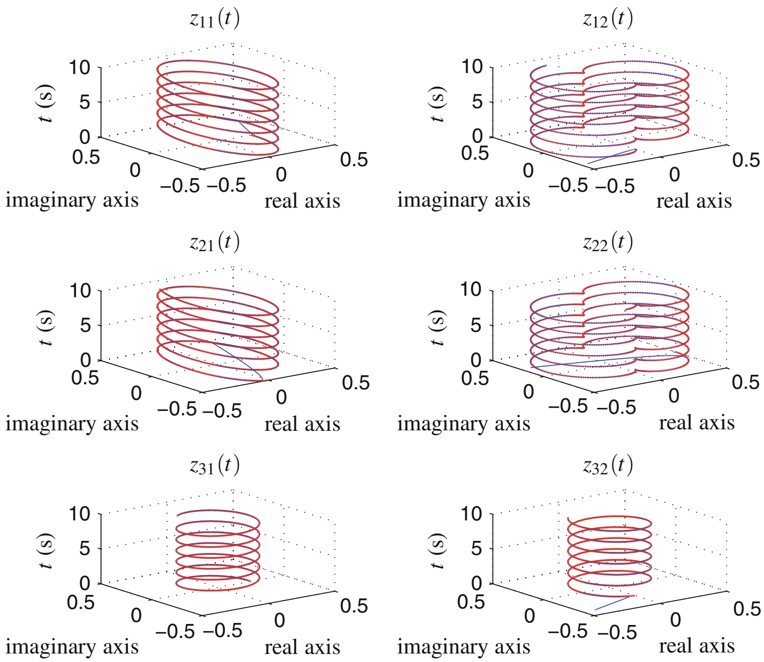

Fig. 13.6 State trajectories of complex ZD model (13.13) with $\gamma = 100$ and $\lambda = 10^{-3}$, where *dash-dotted curves* denote the theoretical time-varying pseudoinverse $C^{+}(t)$ in Example 13.2 and *solid curves* denote the solution computed by complex ZD model (13.13)

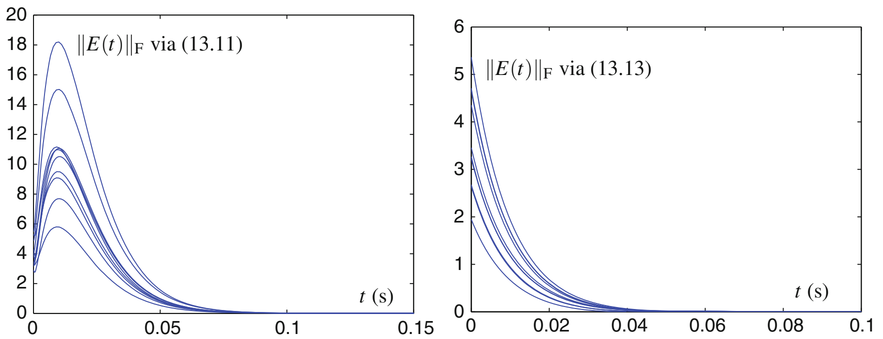

Fig. 13.7 Residual errors $\|E(t)\|_{\mathrm{F}} = \|Z(t)C(t)C^{\mathrm{H}}(t) - C^{\mathrm{H}}(t)\|_{\mathrm{F}}$ synthesized by complex ZD models (13.11) and (13.13) with $\gamma = 100$ and $\lambda = 10^{-3}$ for the time-varying pseudoinverse of matrix $C(t)$ in (13.19)

Example 13.3 In this example, we consider a more complicated situation of the time-varying complex generalized inverse (specifically, the pseudoinverse), which is the pseudoinverse of the following time-varying full-rank complex matrix:

$$C(t) = \begin{bmatrix} c_{11}(t) & c_{12}(t) & c_{13}(t) & \cdots & c_{1n}(t) \\ c_{21}(t) & c_{22}(t) & c_{23}(t) & \cdots & c_{2n}(t) \\ c_{31}(t) & c_{32}(t) & c_{33}(t) & \cdots & c_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1}(t) & c_{m2}(t) & c_{m3}(t) & \cdots & c_{mn}(t) \end{bmatrix} \in \mathbb{C}^{m\times n}, \quad (13.20)$$

where $m < n$. Thereinto,

$$c_{mn}(t) = \begin{cases} \exp(it), & \text{if } m = n, \\ n + \exp(-it), & \text{if } m > n, \\ m + \exp(-it), & \text{if } m < n. \end{cases}$$

In this example, due to the complexity of matrix $C(t)$ in (13.20) (with large dimensions, i.e., $m = 8$ and $n = 9$), the analytical theoretical pseudoinverse solution is difficult to be obtained. Therefore, we only present the convergence performance of the residual errors $\|E(t)\|_\text{F} = \|Z(t)C(t)C^\text{H}(t) - C^\text{H}(t)\|_\text{F}$ synthesized by complex ZD model (13.15). The simulation results are shown in Fig. 13.8. As seen from the figure, starting from 10 randomly-generated initial states, the residual errors $\|E(t)\|_\text{F}$ synthesized by complex ZD model (13.15) with $\gamma = 100$ can diminish to 0 within a short time (also about 0.06 s), which means that the corresponding solutions $Z(t)$ converge to the theoretical time-varying pseudoinverse of complex matrix matrix $C(t)$ in (13.20) rapidly and accurately. Thus, the efficacy of the proposed complex ZD model (13.15) on solving for the more complicated time-varying complex generalized

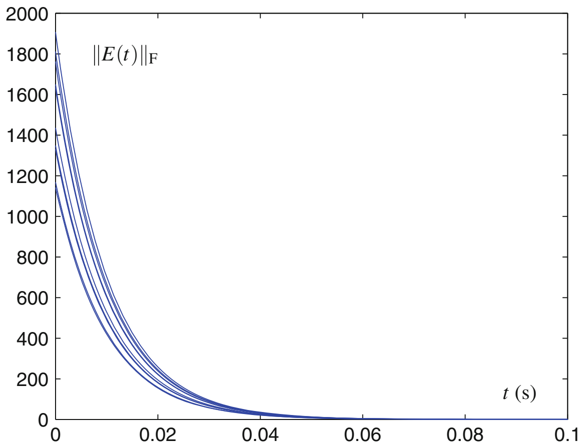

Fig. 13.8 Residual errors $\|E(t)\|_\text{F} = \|Z(t)C(t)C^\text{H}(t) - C^\text{H}(t)\|_\text{F}$ synthesized by complex ZD model (13.15) with $\gamma = 100$ for the time-varying pseudoinverse of matrix $C(t)$ in (13.20)

inverse (specifically, the pseudoinverse) is substantiated evidently. Note that, based on the above three examples, we can conclude that the convergence time of the proposed complex ZD models does not increase as the matrix dimension increases.

Example 13.4 In this example, we investigate the important effect of the design parameter γ for the convergent rate of the proposed complex ZD models. For illustrative purpose, we only exploit complex ZD model (13.17) to solve for the generalized inverse of complex matrix $C(t)$ in (13.19) (see also Example 13.2).

Note that, for complex ZD model (13.17), the initial state $Z(0)$ should be sufficiently close to $C^{+}(0)$. Moreover, $C^{+}(0)$ can be obtained from $C^{+}(t)$ presented in Example 13.2 by setting $t = 0$. As displayed in Fig. 13.9, we can clearly find that the residual errors $\|E(t)\|_{\mathrm{F}} = \|C(t)Z(t) - I\|_{\mathrm{F}}$ synthesized by complex ZD model (13.17) are decreasing faster as the value of design parameter γ increases (i.e., with $\gamma = 100, 200$ and 500). That is, with $\gamma = 100, 200$ and 500, the convergence time of the residual errors $\|E(t)\|_{\mathrm{F}}$ diminishes from about 0.06 to 0.03 s, and even to 0.01 s. Note that the simulative results using other complex ZD models [i.e., (13.7), (13.11), (13.13) and (13.15)] are similar to those shown in Fig. 13.9, and are thus omitted due to results similarity. Being a topic of exercise, the corresponding simulative verifications of such four complex ZD models are left for interested readers. Thus, the efficacy of complex ZD model (13.17) is demonstrated. Meanwhile, we can draw the conclusion that the superior convergence performance of the proposed complex ZD models can be achieved by choosing a larger value of design parameter γ.

In summary, from the above four illustrative examples, we have substantiated the efficacy of the proposed complex ZD models (13.7), (13.11), (13.13), (13.15), and (13.17) on solving for time-varying complex generalized inverse (in most cases, the pseudoinverse). Besides, the important role of the design parameter γ in such complex ZD models has also been discussed and illustrated.

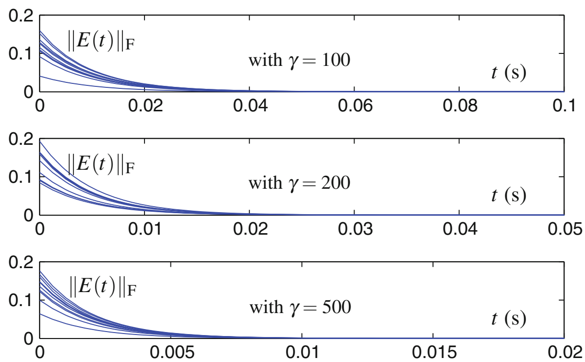

Fig. 13.9 Comparison on residual errors $\|E(t)\|_{\mathrm{F}} = \|C(t)Z(t) - I\|_{\mathrm{F}}$ synthesized by complex ZD model (13.17) with $\gamma = 100$, 200 and 500 for the time-varying pseudoinverse of matrix $C(t)$ in (13.19)

13.4 Summary

In this chapter, by defining different complex ZFs [i.e., (13.2)–(13.6)], five different complex ZD models [i.e., (13.7), (13.11), (13.13), (13.15), and (13.17)] have been proposed, generalized, developed and investigated for time-varying complex generalized inverse (in most cases, the pseudoinverse). Based on the complex ZF and the ZD design method, the complex ZD model has fully utilized the first-order time-derivative information of the time-varying complex matrix and has achieved the global convergence performance. In addition, the relationship between the proposed complex ZD models and the G-M dynamic system for time-varying complex generalized inverse computation has been discovered and presented. Moreover, through four illustrative examples, the efficacy of the proposed complex ZD models has been substantiated evidently.

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