Chapter 10
Time-Varying Matrix Square Root

Abstract In this chapter, different indefinite ZFs, which lead to different ZD models, are proposed and developed as the error-monitoring functions for time-varying matrix square root finding. Toward the final purpose of field programmable gate array (FPGA) and application-specific integrated circuit (ASIC) realizations, the MATLAB Simulink modeling and verifications of such ZD models are further investigated to solve for time-varying matrix square root. Both theoretical analysis and modeling results substantiate the efficacy of the proposed ZD models for time-varying matrix square root finding.

10.1 Introduction

The problem of solving for matrix square root is considered to be an important special case of nonlinear matrix equation problem, which widely arises in many scientific and engineering fields; e.g., control theory [1], optimization [2], and signal processing [3]. In general, the solution of matrix square root, which can usually be a fundamental part of many solutions, can be achieved via matrix equations solving. Thus, many numerical algorithms/methods have been presented and developed for online solution of matrix square roots [1–7]. However, it may not be efficient enough for most numerical algorithms due to their serial-processing nature performed on digital computers [2, 3]. For large-scale online or real-time applications, the minimal arithmetic operations of such numerical algorithms are usually proportional to the cube of the matrix dimension n, i.e., $O(n^3)$ operations [8]. To remedy the inherent weaknesses of such numerical algorithms, many parallel-processing computational methods, including various dynamic system approaches, have been developed and implemented on specific architectures [9–15]. Note that the aforementioned computational schemes are theoretically/intrinsically designed for solving time-invariant (or termed, static, constant) problems (e.g., time-invariant matrix square root finding) rather than time-varying ones. Thus, these schemes may be less accurate and effective enough, when they are exploited directly to solve for time-varying matrix square root [16–18].

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In this chapter, focusing on time-varying matrix square root finding, we propose, generalize, develop, and investigate eight different ZD models by defining eight different ZFs as the error-monitoring functions and constructing eight first-order differential equations to force the ZFs converge to zero. In addition to the theoretical analysis and results on the convergence characteristics of the proposed ZD models, the MATLAB Simulink modeling and verification examples are investigated with the final purpose of FPGA and ASIC realizations [19]. Moreover, some primary software modeling techniques are investigated to model and simulate such ZD models. The modeling results further substantiate the efficacy of the proposed ZD models based on different ZFs for time-varying square root finding.

10.2 ZFs and ZD Models

In this section, we introduce eight different ZFs and propose the resultant ZD models for time-varying matrix square root finding.

Let us consider the following time-varying matrix square root problem (which can also be viewed as a time-varying nonlinear matrix equation problem) [16–18]:

$$X^2(t) - A(t) = 0 \in \mathbb{R}^{n\times n}, \quad t \in [0, +\infty), \tag{10.1}$$

where $A(t) \in \mathbb{R}^{n\times n}$ denotes a smoothly time-varying positive-definite matrix, which, together with its time derivative $\dot{A}(t)$, is assumed to be known numerically or can be measured accurately. In addition, $X(t) \in \mathbb{R}^{n\times n}$ is the time-varying unknown matrix to be solved for. Our objective in this chapter is to find $X(t)$ so that (10.1) holds true for any $t \geqslant 0$. To lay a basis for further discussion, $A(t)$ is assumed to be nonsingular at any time instant $t \in [0, +\infty)$ in this chapter, and thus the inverse of $A(t)$ [(i.e., $A^{-1}(t)$)] exists and is obtained.

Besides, the following preliminaries [1, 3, 18] are provided as a basis for further discussion on solving (10.1).

Definition 10.1 Given a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$, if matrix $X(t) \in \mathbb{R}^{n\times n}$ satisfies the time-varying nonlinear equation $X^2(t) = A(t)$, then $X(t)$ is a time-varying square root of matrix $A(t)$ [or say, $X(t)$ is a time-varying solution to the presented nonlinear equation (10.1)].

Definition 10.2 Since $X(t)X(t)X^{-1}(t)X^{-1}(t) = I$ (with $I \in \mathbb{R}^{n\times n}$ denoting the identity matrix) and $X(t)X(t) = X^2(t)$, then $X^{-2}(t)$ is defined as $X^{-2}(t) = X^{-1}(t)X^{-1}(t)$; i.e., we have $X^2(t)X^{-2}(t) = I$.

Concept 10.1 (Square-root existence condition) *If a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ is positive-definite (in general sense [18]) at any time instant $t \in [0, +\infty)$, then there exists a time-varying matrix square root $X(t) \in \mathbb{R}^{n\times n}$ for matrix $A(t)$.*

Thus, specifically for solving time-varying matrix square root problem (10.1), in this chapter, we define eight different ZFs as follows:

$$E(t) = X^2(t)A^{-1}(t) - I, \tag{10.2}$$
$$E(t) = A^{-1}(t)X^2(t) - I, \tag{10.3}$$
$$E(t) = X^2(t) - A(t), \tag{10.4}$$
$$E(t) = X^{-2}(t) - A^{-1}(t), \tag{10.5}$$
$$E(t) = X(t) - A(t)X^{-1}(t), \tag{10.6}$$
$$E(t) = X(t) - X^{-1}(t)A(t), \tag{10.7}$$
$$E(t) = X^{-1}(t) - A^{-1}(t)X(t), \tag{10.8}$$
$$E(t) = X^{-1}(t) - X(t)A^{-1}(t). \tag{10.9}$$

Before deriving different ZD models from different ZFs, the following theorem is provided as a basis for further discussion.

Theorem 10.1 *The time derivative of* $X^{-2}(t)$ *[(i.e.,* $\mathrm{d}(X^{-2}(t))/\mathrm{d}t$*] is formulated as*

$$\frac{\mathrm{d}(X^{-2}(t))}{\mathrm{d}t} = -X^{-2}(t)(X(t)\dot{X}(t) + \dot{X}(t)X(t))X^{-2}(t). \tag{10.10}$$

Proof It follows from Definition 10.2 that $X^2(t)X^{-2}(t) = I$. Then, we have

$$\frac{\mathrm{d}(X^2(t)X^{-2}(t))}{\mathrm{d}t} = \frac{\mathrm{d}I}{\mathrm{d}t} = 0.$$

Expanding the left-hand side of the above equation, we thus obtain

$$\frac{\mathrm{d}(X^2(t))}{\mathrm{d}t}X^{-2}(t) + X^2(t)\frac{\mathrm{d}(X^{-2}(t))}{\mathrm{d}t} = 0,$$

which is further rewritten as

$$X^2(t)\frac{\mathrm{d}(X^{-2}(t))}{\mathrm{d}t} = -\frac{\mathrm{d}(X^2(t))}{\mathrm{d}t}X^{-2}(t) = -(X(t)\dot{X}(t) + \dot{X}(t)X(t))X^{-2}(t).$$

Finally, in view of $X^2(t)X^{-2}(t) = I$, we have

$$\frac{\mathrm{d}(X^{-2}(t))}{\mathrm{d}t} = -X^{-2}(t)(X(t)\dot{X}(t) + \dot{X}(t)X(t))X^{-2}(t),$$

which now completes the proof. □

According to the ZD design formula (7.3), different ZFs lead to different ZD models for time-varying matrix square root finding, which is presented as follows.

Note that argument t [e.g., t in $X(t)$] is omitted in the following derivation of the ZD models for ease of presentation.

- Let us consider the ZD design formula (7.3), ZF (10.2), and Eq. (7.11) (see also Theorem 7.1). Then, we have

$$(\dot{X}X + X\dot{X})A^{-1} - X^2(A^{-1}\dot{A}A^{-1}) = -\gamma(X^2A^{-1} - I).$$

Thus, based on ZF (10.2), we obtain the following dynamic equation (i.e., a first-order matrix-valued differential equation) of a ZD model for time-varying matrix square root finding:

$$\dot{X}X + X\dot{X} = X^2A^{-1}\dot{A} - \gamma(X^2 - A). \tag{10.11}$$

In order to display ZD model (10.11) better, we can get its block diagram. Before doing this, we transform such a ZD model into the following explicit form:

$$\dot{X} = \dot{X}(I - X) - X\dot{X} + X^2A^{-1}\dot{A} - \gamma(X^2 - A).$$

Therefore, we have the resultant block diagram of ZD model (10.11), which is shown in Fig. 10.1, and the modeling of ZD model (10.11) can also be done in this manner.

- Considering the ZD design formula (7.3), ZF (10.3), and Eq. (7.11), then we have

$$-A^{-1}\dot{A}A^{-1}X^2 + A^{-1}(\dot{X}X + X\dot{X}) = -\gamma(A^{-1}X^2 - I),$$

which is reformulated as

$$\dot{X}X + X\dot{X} = \dot{A}A^{-1}X^2 - \gamma(X^2 - A). \tag{10.12}$$

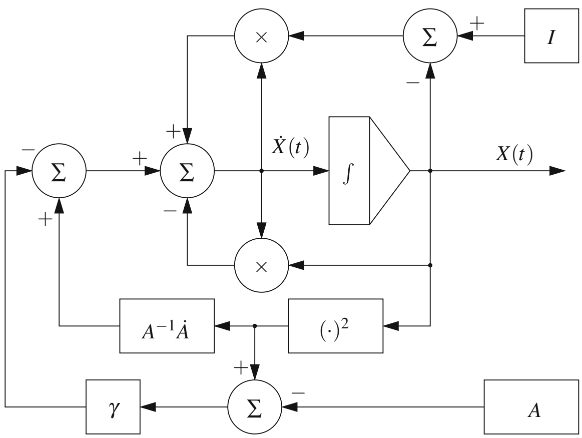

Fig. 10.1 Block diagram of ZD model (10.11) for time-varying matrix square root finding

That is, we obtain another ZD model (10.12), which is based on ZF (10.3), for time-varying matrix square root finding. In addition, the explicit form of ZD model (10.12) is

$$\dot{X} = (I - X)\dot{X} - \dot{X}X + \dot{A}A^{-1}X^2 - \gamma(X^2 - A).$$

It is worth pointing out that, by comparing the explicit form of the ZD model (10.11) with that of ZD model (10.12), we can see that the differences between these two explicit forms lie in the first three terms of the right-hand sides [for comparison, in the explicit form of (10.16), we have $\dot{X}(I - X)$, $X\dot{X}$, and $X^2A^{-1}\dot{A}$]. Due to the similarity, the block diagram of ZD model (10.12) is omitted and is left to interested readers to complete as a topic of exercise.

- With the ZD design formula (7.3) and ZF (10.4) exploited, the following ZD model is established for time-varying matrix square root finding:

$$X\dot{X} + \dot{X}X = -\gamma(X^2 - A) + \dot{A}. \tag{10.13}$$

Thus, ZD model (10.13) based on ZF (10.4) for time-varying matrix square root finding is obtained, and its explicit form is formulated as

$$\dot{X} = (I - X)\dot{X} - \dot{X}X + \dot{A} - \gamma(X^2 - A).$$

Note that, as compared to ZD models (10.11) and (10.12), ZD model (10.13) is viewed as a simplified one (i.e., with less model structure). The block diagram of ZD model (10.13) can thus be generalized from that of (10.11) or (10.12), and is omitted here due to the similarity (but is also left to interested readers to complete as a topic of exercise).

- With the ZD design formula (7.3), ZF (10.5), and Eqs. (7.11) and (10.10) exploited, we have

$$-X^{-2}(X\dot{X} + \dot{X}X)X^{-2} + A^{-1}\dot{A}A^{-1} = -\gamma(X^{-2} - A^{-1}),$$

which is reformulated as

$$X\dot{X} + \dot{X}X = X^2A^{-1}\dot{A}A^{-1}X^2 + \gamma(X^2 - X^2A^{-1}X^2). \tag{10.14}$$

Thus, we obtain another ZD model (10.14) based on ZF (10.5) for time-varying matrix square root finding. To depict the block diagram of ZD model (10.14), we transform such a ZD model into the following explicit form:

$$\dot{X} = \dot{X}(I - X) - X\dot{X} + X^2A^{-1}\dot{A}A^{-1}X^2 + \gamma(X^2 - X^2A^{-1}X^2).$$

Therefore, we have the resultant block diagram of ZD model (10.14) in Fig. 10.2.

- Considering the ZD design formula (7.3), ZF (10.6), and Eq. (7.10) (see also Theorem 7.1), we have

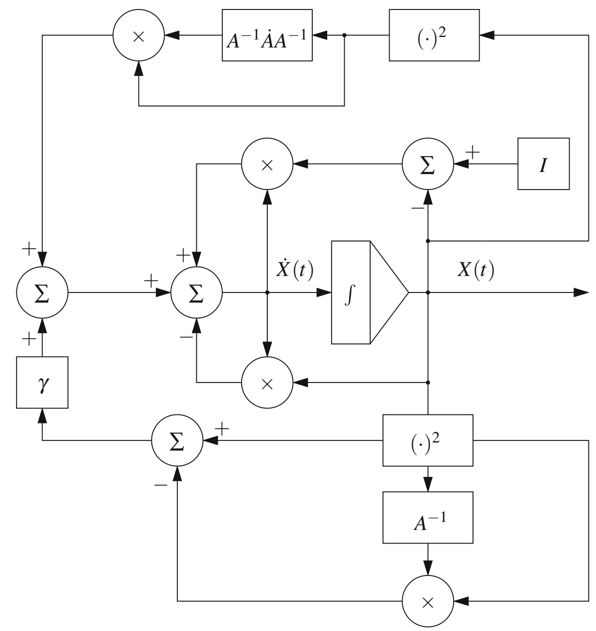

Fig. 10.2 Block diagram of ZD model (10.14) for time-varying matrix square root finding

$$\dot{X} - \dot{A}X^{-1} + AX^{-1}\dot{X}X^{-1} = -\gamma(X - AX^{-1}),$$

which is rewritten as

$$A^{-1}\dot{X} - A^{-1}\dot{A}X^{-1} + X^{-1}\dot{X}X^{-1} = -\gamma A^{-1}(X - AX^{-1}).$$

Then, we further have

$$XA^{-1}\dot{X}X - XA^{-1}\dot{A} + \dot{X} = -\gamma XA^{-1}(X - AX^{-1})X,$$

which is finally formulated as

$$\dot{X} = -XA^{-1}\dot{X}X + XA^{-1}\dot{A} - \gamma XA^{-1}(X^2 - A). \quad (10.15)$$

Thus, based on ZF (10.6), ZD model (10.15) is obtained for time-varying matrix square root finding, of which the block diagram is shown in Fig. 10.3.

- Similar to the derivation of ZD model (10.15), based on ZF (10.7), we have

$$X\dot{X}A^{-1}X - \dot{A}A^{-1}X + \dot{X} = -\gamma X(X - X^{-1}A)A^{-1}X,$$

$$\dot{X} = -X\dot{X}A^{-1}X + \dot{A}A^{-1}X - \gamma(X^2 - A)A^{-1}X. \quad (10.16)$$

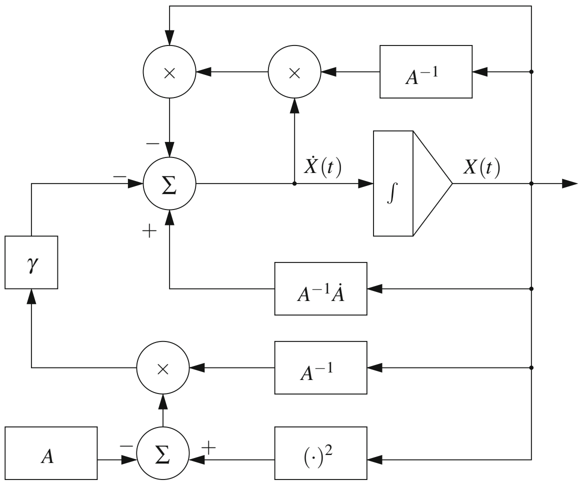

Fig. 10.3 Block diagram of ZD model (10.15) for time-varying matrix square root finding

Thus, we obtain ZD model (10.16) based on ZF (10.7) for time-varying matrix square root finding. Note that, like the situation of ZD model (10.11) and ZD model (10.12), the block diagram of ZD model (10.16) is omitted for its similarity to that of ZD model (10.15) (and is left to interested readers to complete as a topic of exercise).

- By using the ZD design formula (7.3), ZF (10.8), and Eqs. (7.10) and (7.11), we have

$$-X^{-1}\dot{X}X^{-1} + A^{-1}\dot{A}A^{-1}X - A^{-1}\dot{X} = -\gamma(X^{-1} - A^{-1}X),$$

and then

$$\dot{X} - XA^{-1}\dot{A}A^{-1}X^2 + XA^{-1}\dot{X}X = \gamma(X - XA^{-1}X^2),$$

which is finally formulated as

$$\dot{X} = XA^{-1}\dot{A}A^{-1}X^2 - XA^{-1}\dot{X}X - \gamma XA^{-1}(X^2 - A). \tag{10.17}$$

Therefore, based on ZF (10.8), ZD model (10.17) is obtained for time-varying matrix square root finding, of which the block diagram is shown in Fig. 10.4.

- Similar to the derivation of ZD model (10.17), based on ZF (10.9), we have

$$\dot{X} - X^2A^{-1}\dot{A}A^{-1}X + X\dot{X}A^{-1}X = \gamma(X - X^2A^{-1}X),$$

which is reformulated as

$$\dot{X} = X^2A^{-1}\dot{A}A^{-1}X - X\dot{X}A^{-1}X - \gamma(X^2 - A)A^{-1}X. \tag{10.18}$$

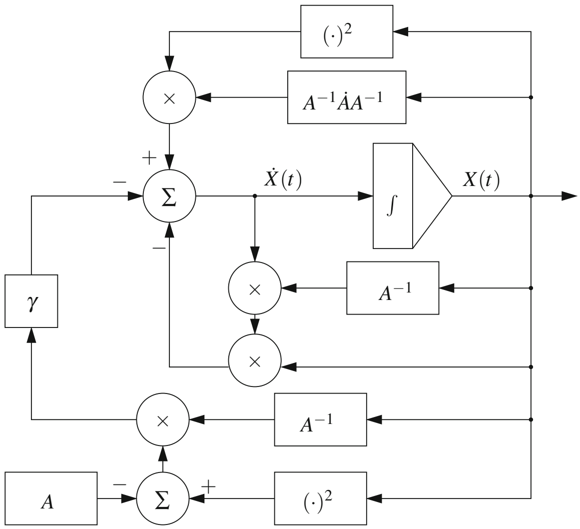

Fig. 10.4 Block diagram of ZD model (10.17) for time-varying matrix square root finding

Table 10.1 Different ZFs resulting in different ZD models for time-varying matrix square root finding

ZF	ZD model
(10.2)	$\dot{X}X + X\dot{X} = X^2A^{-1}\dot{A} - \gamma(X^2 - A)$
(10.3)	$\dot{X}X + X\dot{X} = \dot{A}A^{-1}X^2 - \gamma(X^2 - A)$
(10.4)	$X\dot{X} + \dot{X}X = \dot{A} - \gamma(X^2 - A)$
(10.5)	$X\dot{X} + \dot{X}X = X^2A^{-1}\dot{A}A^{-1}X^2 + \gamma(X^2 - X^2A^{-1}X^2)$
(10.6)	$\dot{X} = -XA^{-1}\dot{X}X + XA^{-1}\dot{A} - \gamma XA^{-1}(X^2 - A)$
(10.7)	$\dot{X} = -X\dot{X}A^{-1}X + \dot{A}A^{-1}X - \gamma(X^2 - A)A^{-1}X$
(10.8)	$\dot{X} = XA^{-1}\dot{A}A^{-1}X^2 - XA^{-1}\dot{X}X - \gamma XA^{-1}(X^2 - A)$
(10.9)	$\dot{X} = X^2A^{-1}\dot{A}A^{-1}X - X\dot{X}A^{-1}X - \gamma(X^2 - A)A^{-1}X$

Thus, we obtain ZD model (10.18) based on ZF (10.9) for time-varying matrix square root finding. Note that the block diagram of ZD model (10.18) is omitted for its similarity to that of ZD model (10.17).

In summary, we obtain eight different types of ZD models [i.e., ZD models (10.11)–(10.18)] for time-varying matrix square root finding, which correspond to eight different types of ZFs [i.e., ZFs (10.2)–(10.9)]. For readers' convenience and also for comparison, such eight different ZNN models corresponding to eight different ZFs are listed in Table 10.1.

10.3 Theoretical Results and Analyses

In this section, theoretical results and analyses are presented, which show the convergence performance of the proposed ZD models (10.11)–(10.18) on solving for time-varying matrix square root.

Theorem 10.2 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = X^2(t)A^{-1}(t) - I \in \mathbb{R}^{n\times n}$ of ZD model (10.11), converges to zero [which implies that state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.11) converges to the theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].*

Proof From the compact form of the ZD design formula $\dot{E}(t) = -\gamma E(t)$, a set of n^2 decoupled differential equations can be written equivalently as follows:

$$\dot{e}_{ij}(t) = -\gamma e_{ij}(t), \tag{10.19}$$

for any $i \in \{1, 2, 3, \cdots, n\}$ and $j \in \{1, 2, 3, \cdots, n\}$. Thus, to analyze the equivalent ijth subsystem (10.19), we define a Lyapunov function candidate $v_{ij}(t) = e_{ij}^2(t)/2 \geqslant 0$ with its time derivative being

$$\frac{\mathrm{d}v_{ij}(t)}{\mathrm{d}t} = e_{ij}(t)\dot{e}_{ij}(t) = -\gamma e_{ij}^2(t) \leqslant 0,$$

which guarantees the final negative-definiteness of $\dot{v}_{ij}$ (i.e., $\dot{v}_{ij} < 0$ for $e_{ij} \neq 0$ while $\dot{v}_{ij} = 0$ for $e_{ij} = 0$ only). By Lyapunov theory [20, 21], the equilibrium point $e_{ij} = 0$ of (10.19) is asymptotically stable, i.e., $e_{ij}(t)$ converges to zero, for any $i \in \{1, 2, 3, \cdots, n\}$ and $j \in \{1, 2, 3, \cdots, n\}$. In other words, the matrix-valued error function $E(t) = [e_{ij}(t)] \in \mathbb{R}^{n\times n}$ is convergent to zero. In addition, we have $E(t) = X^2(t)A^{-1}(t) - I$, or equivalently, $X^2(t)A^{-1}(t) = I + E(t)$. Since $E(t) \to 0$ as $t \to +\infty$, we have $X^2(t)A^{-1}(t) \to I$ and thus $X^2(t) \to A(t)$ [(i.e., $X(t) \to X^*(t)$] as $t \to +\infty$. That is, the state matrix $X(t)$ of ZD model (10.11) can converge to the theoretical time-varying matrix square root $X^*(t)$ of matrix $A(t)$.

Furthermore, when the state matrix $X(t)$ of (10.11) starts from a randomly-generated positive-definite diagonal initial state-matrix $X(0)$, it can converge to the positive-definite time-varying matrix square root $A^{1/2}(t)$ [i.e., a form of $X^*(t)$]. This can be proven by the contradiction as follows. Suppose that the state matrix $X(t)$ starting from a positive-definite diagonal initial state-matrix $X(0)$ converges to the negative-definite time-varying matrix square root $-A^{1/2}(t)$ [i.e., the other form of $X^*(t)$], then such a state matrix $X(t)$ must pass through at least one 0-eigenvalue, which leads to the contradiction that the left- and right-hand sides of the ZD model (10.11) cannot hold. So, starting from a randomly-generated positive-definite diagonal initial state-matrix $X(0)$, the state matrix $X(t)$ of ZD model (10.11) converges to the positive-definite time-varying matrix square root $A^{1/2}(t)$. Similarly, it can also be proved that, starting from a randomly-generated negative-definite diagonal initial state-matrix $X(0)$, the state matrix $X(t)$ of ZD model (10.11) converges to the negative-definite time-varying matrix square root $-A^{1/2}(t)$ [i.e., another form of $X^*(t)$]. The proof is thus complete. □

As for the other seven ZD models (10.12)–(10.18), we also have the following convergence results, with the related proofs being generalized from the proof of Theorem 10.2 and being left to interested readers to complete as a topic of exercise.

Corollary 10.1 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = A^{-1}(t)X^2(t) - I \in \mathbb{R}^{n\times n}$ of ZD model (10.12), converges to zero [which implies that the state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.12) converges to theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].*

Corollary 10.2 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = X^2(t) - A(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.13), converges to zero [which implies that the state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.13) converges to theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].*

Corollary 10.3 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = X^{-2}(t) - A^{-1}(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.14), converges to zero [which implies that the state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.14) converges to theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].*

Corollary 10.4 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = X(t) - A(t)X^{-1}(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.15), converges to zero [which implies that the state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.15) converges to theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].*

Corollary 10.5 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = X(t) - X^{-1}(t)A(t) \in$*

$\mathbb{R}^{n\times n}$ of ZD model (10.16), converges to zero [which implies that the state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.16) converges to theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].

Corollary 10.6 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = X^{-1}(t) - A^{-1}(t)X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.17), converges to zero [which implies that the state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.17) converges to theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].*

Corollary 10.7 *Consider a smoothly time-varying matrix $A(t) \in \mathbb{R}^{n\times n}$ involved in nonlinear equation (10.1), which satisfies the square-root existence condition. Starting from a randomly-generated positive-definite (or negative-definite) diagonal initial state-matrix $X(0) \in \mathbb{R}^{n\times n}$, the error function $E(t) = X^{-1}(t) - X(t)A^{-1}(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.18), converges to zero [which implies that the state matrix $X(t) \in \mathbb{R}^{n\times n}$ of ZD model (10.18) converges to theoretical positive-definite (or negative-definite) time-varying matrix square root $X^*(t)$ of matrix $A(t)$].*

10.4 MATLAB Simulink Modeling

According to the aforementioned explicit forms and the presented block diagrams of the ZD models (10.11), (10.14), (10.15), and (10.17) shown in Figs. 10.1, 10.2, 10.3, and 10.4, the corresponding MATLAB Simulink modeling of such ZD models [i.e., ZD models (10.11), (10.14), (10.15), and (10.17)] is investigated and presented in this section for possible circuits implementation and also for the final purpose of FPGA and ASIC realizations.

10.4.1 Simulink Blocks

MATLAB Simulink contains a comprehensive block library including sinks, sources, linear, and nonlinear components, as well as connectors. The blocks generally used to construct ZD models (10.11), (10.14), (10.15), and (10.17) are discussed as follows.

- The *MATLAB Fcn* block can be employed to (1) generate matrix $A(t)$ using the *Clock* block as its input, or (2) compute the matrix norm.
- The *Constant* block, which outputs a constant specified by its parameter "Constant value", can be used to generate the identity matrix.
- The *Gain* block can be used to scale the neural network convergence, e.g., as a scaling parameter γ to scale the convergence rate of neural dynamics.

- The *Math Function* block can perform various common mathematical operations, and is used in this chapter for generating the inverse of a matrix.
- The *Product* block, specified as the standard matrix-wise product mode, can be used to multiply the matrices involved in the neural-dynamics models.
- The *Integrator* block makes continuous-time integration on the input signals. For instance, in the Example 10.1 discussed in the ensuing section, we set its "Initial condition" as "$diag(2 * rand(3, 1))$" in order to generate a diagonal positive-definite initial state-matrix $X(0)$ with its diagonal elements randomly distributed in $[0, 2]$.

By interconnecting these basic Simulink function blocks and setting appropriate block parameters, the overall modeling of ZD models (10.11), (10.14), (10.15), and (10.17) can then be built up readily for time-varying matrix square roots finding, with the corresponding Simulink models shown in Figs. 10.5, 10.6, 10.7, and 10.8.

10.4.2 Parameter Settings

After showing the overall Simulink models of the proposed ZD models in Figs. 10.5, 10.6, 10.7 and 10.8, we discuss changing some of the default modeling environment options. The options setting can be done by using the "Configuration

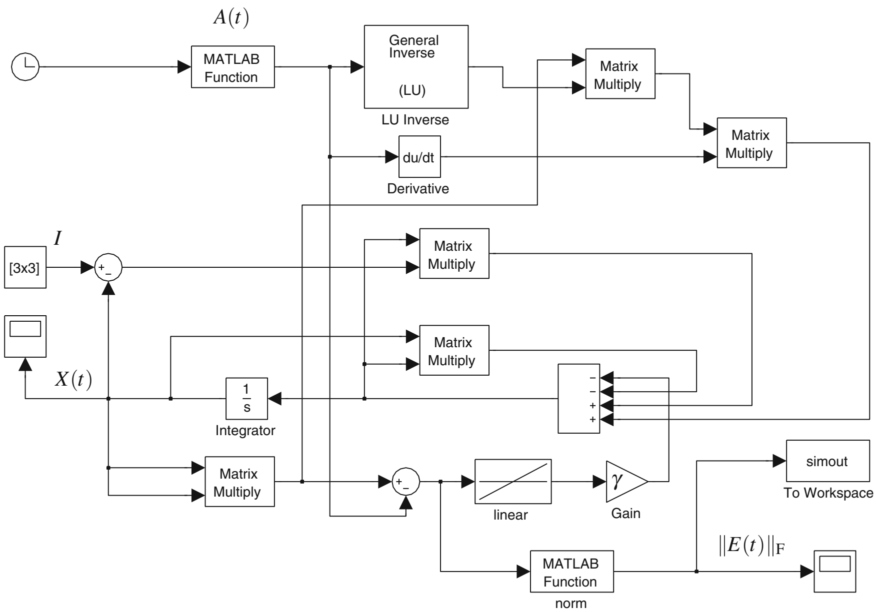

Fig. 10.5 Simulink modeling of ZD model (10.11) for time-varying matrix square root finding

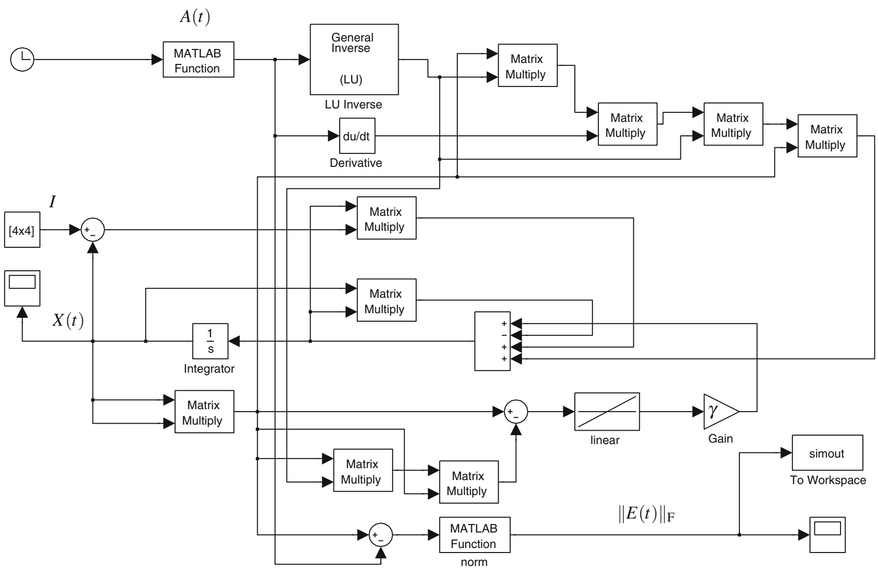

Fig. 10.6 Simulink modeling of ZD model (10.14) for time-varying matrix square root finding

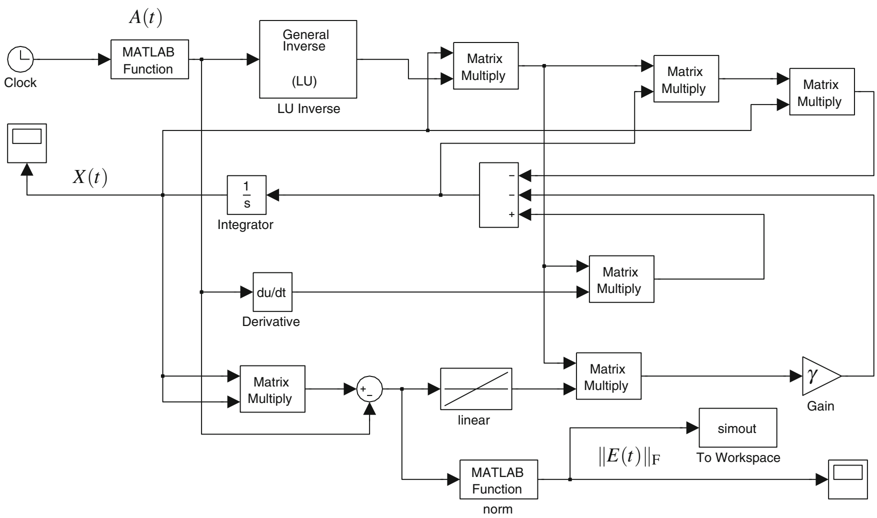

Fig. 10.7 Simulink modeling of ZD model (10.15) for time-varying matrix square root finding

Parameters" dialog box in the MATLAB Simulink environment [18]. Some important parameter settings have to be specified as follows:

- Starting time (e.g., 0.0) and Stop time (e.g., 8.0);
- Solver (i.e., integrator algorithm): "ode45 (Dormand-Prince)";

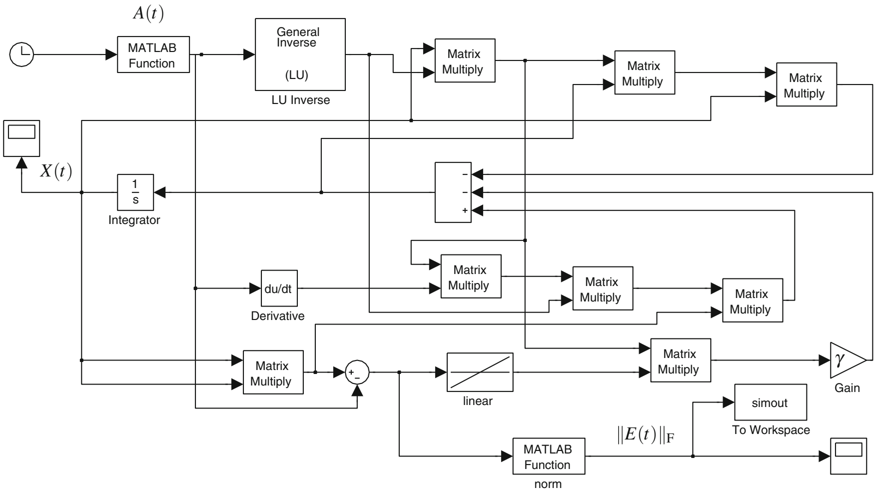

Fig. 10.8 Simulink modeling of ZD model (10.17) for time-varying matrix square root finding

- Max step size: “0.2”, and Min step size: “auto”;
- Initial step size: “auto”;
- Relative tolerance: “1e-6” (i.e., 10^{-6});
- Absolute tolerance:“auto”.

In addition, the check box in front of “States” of the option “Data Import/Export” should be selected, which is for the purpose of better displaying the ZD modeling results and is associated with the “StopFcn” code (of “Callbacks” in the dialog box entitled “Model Properties” which is started from the “File” pull-down menu).

10.5 Illustrative Examples

In the previous sections, different ZD models based on different ZFs have been proposed and developed for time-varying matrix square root finding, together with corresponding theoretical results. Based on the above-presented overall Simulink models depicted in Figs. 10.5, 10.6, 10.7, and 10.8, the ensuing illustrative examples are investigated to substantiate the efficacy of the proposed ZD models. Note that the representative ZD models (10.11), (10.14), (10.15), and (10.17) are chosen and modeled to solve for time-varying matrix square root.

Example 10.1 Let us consider nonlinear equation (10.1) with the following symmetric positive-definite time-varying matrix $A(t) \in \mathbb{R}^{3\times3}$:

$$A(t) = \begin{bmatrix} 5+\sin^2(t) & 4\sin(t)+\exp(-2t) & 4+\exp(-2t)\sin(t) \\ 4\sin(t)+\exp(-2t) & 4+\sin^2(t)+\exp(-4t) & \sin(t)+4\exp(-2t) \\ 4+\exp(-2t)\sin(t) & \sin(t)+4\exp(-2t) & 5+\exp(-4t) \end{bmatrix}. \tag{10.20}$$

For such a matrix, the theoretical time-varying square root $X^*(t)$ is

$$X^*(t) = \begin{bmatrix} 2 & \sin(t) & 1 \\ \sin(t) & 2 & \exp(-2t) \\ 1 & \exp(-2t) & 2 \end{bmatrix} \in \mathbb{R}^{3\times 3},$$

which is given for comparison purposes, i.e., to check the correctness of the neural dynamics solutions.

The proposed ZD models (10.11), (10.14), (10.15), and (10.17) are exploited to solve this problem, and the corresponding modeling results based on the above Simulink models are illustrated in Figs. 10.9, 10.10, 10.11, and 10.12. As shown in the left graph of Fig. 10.9, with design parameter $\gamma = 10$, the state matrix $X(t)$ of the proposed ZD model (10.11) denoted by solid curves converges to the theoretical time-varying solution $X^*(t)$ denoted by dash-dotted curves. In addition, to further investigate the convergence performance of ZD model (10.11), we monitor the residual error $\|E(t)\|_\mathrm{F} = \|X^2(t) - A(t)\|_\mathrm{F}$ during the solving process. As seen from the right graph of Fig. 10.9, by applying ZD model (10.11) to solve for time-varying matrix square root, the residual error converges to zero within around 1 s. For other ZD models [i.e., (10.14), (10.15), and (10.17)], we have the same observations/conclusions, which are shown in Figs. 10.10 and 10.11 (as well as the related modeling results which are omitted due to the similarity).

In addition, it is worth pointing out that the convergence performance of the proposed ZD models can be improved by increasing the value of γ. As an illustrative example, the convergence of residual error $\|E(t)\|_\mathrm{F}$ of ZD model (10.11) with different γ values is shown in Fig. 10.12. As seen from the figure, the convergence time of ZD model (10.11) can be expedited from around 8 s to 0.08 s and to 0.008 s, as the γ value is increased from 1 to 100 and to 1000, respectively. This result shows that ZD model (10.11) has an exponential convergence property, which can be expedited effectively by increasing the value of γ. Note that, for other ZD models [i.e., (10.14), (10.15), and (10.17)], we have the same conclusions by observing the related modeling results, which are omitted here due to the results' similarity. Being a topic of exercise, the corresponding modeling verifications of ZD models (10.14), (10.15), and (10.17) are left for interested readers.

In summary, the above modeling results (i.e., Figs. 10.9, 10.10, 10.11, and 10.12) have substantiated well the efficacy of the proposed ZD models (10.11), (10.14), (10.15), and (10.17) for time-varying matrix square root finding.

Example 10.2 In order to further investigate the efficacy of the proposed ZD models for larger dimension matrices, let us consider nonlinear equation (10.1) with the following time-varying circulant matrix $A(t)$:

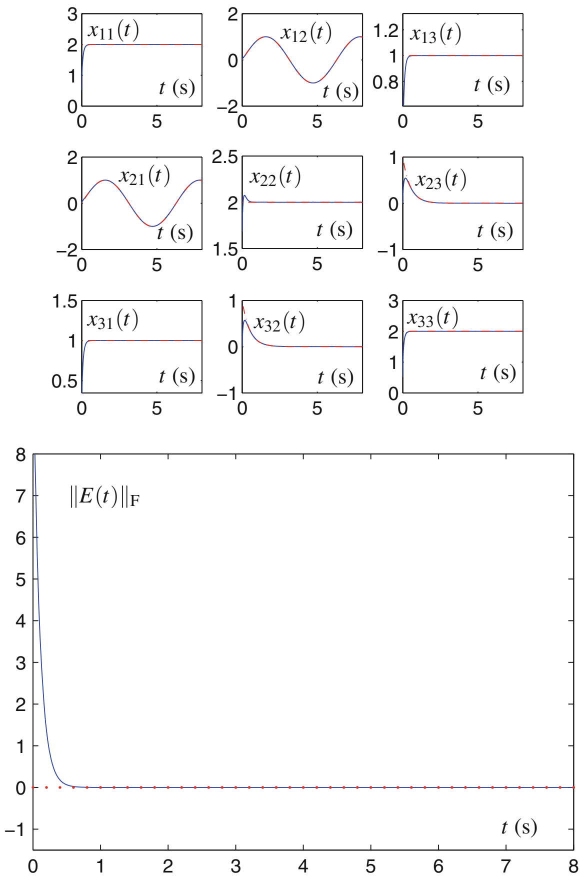

Fig. 10.9 Convergence performance of ZD model (10.11) with $\gamma = 10$ for finding the square root of time-varying matrix $A(t)$ in (10.20)

$$A(t) = \begin{bmatrix} a_0(t) & a_1(t) & a_2(t) & \cdots & a_{n-1}(t) \\ a_{n-1}(t) & a_0(t) & a_1(t) & \cdots & a_{n-2}(t) \\ a_{n-2}(t) & a_{n-1}(t) & a_0(t) & \cdots & a_{n-3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1(t) & a_2(t) & a_3(t) & \cdots & a_0(t) \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{10.21}$$

in which, without loss of generality, we choose $a_0(t) = n$ and $a_i(t) = \sin(it)/i$ for $i = 1, 2, \cdots, n-1$. In this example, we choose $n = 6$. Evidently, the circulant matrix $A(t)$ is strictly diagonally dominant for any time instant $t \geqslant 0$.

Figure 10.13 shows the modeling results by using the proposed ZD models (10.11), (10.14), (10.15), and (10.17) with $\gamma = 10$ to find the time-varying matrix square root of the above circulant matrix $A(t)$. As seen from the figure, residual errors $\|E(t)\|_\mathrm{F}$ of such ZD models all converge to zero, which implies that their corresponding state matrices always converge to the theoretical time-varying square root of $A(t)$. These

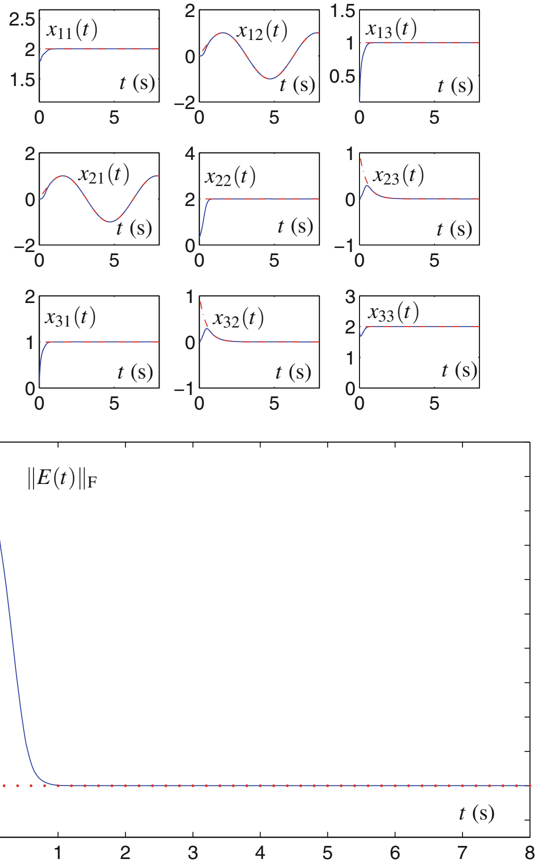

Fig. 10.10 Convergence performance of ZD model (10.14) with $\gamma = 10$ for finding the square root of time-varying matrix $A(t)$ in (10.20)

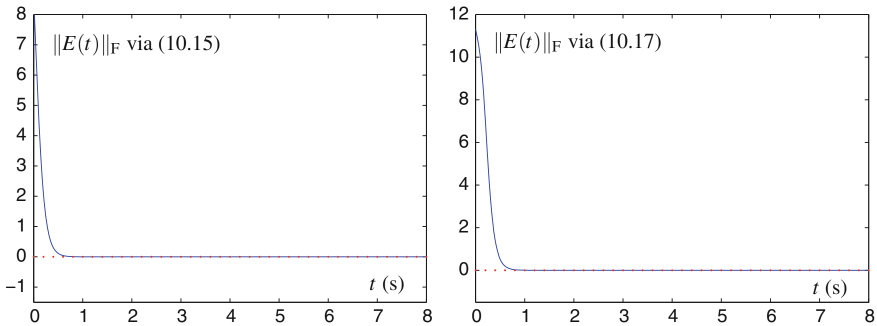

Fig. 10.11 Residual errors $\|E(t)\|_F$ synthesized by ZD models (10.15) and (10.17) with $\gamma = 10$ for finding the square root of time-varying matrix $A(t)$ in (10.20)

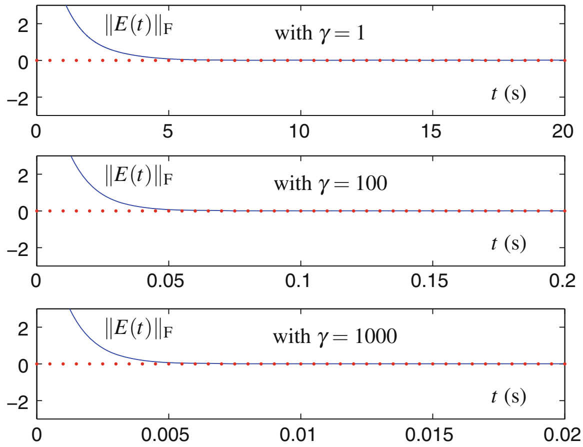

Fig. 10.12 Residual errors $\|E(t)\|_F$ synthesized by ZD model (10.11) with different values of γ (i.e., $\gamma = 1$, 100 and 1000) for finding the square root of time-varying matrix $A(t)$ in (10.20)

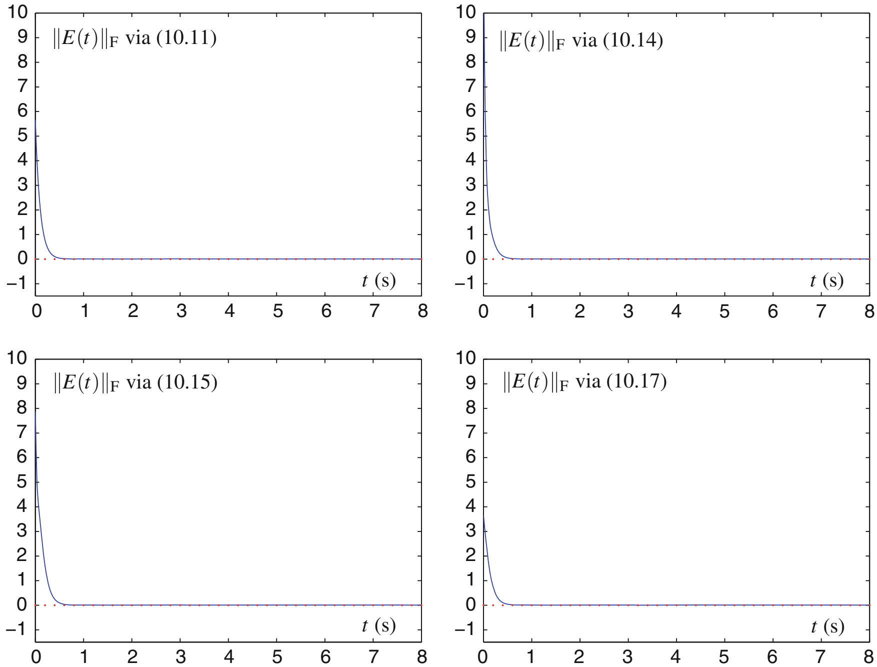

Fig. 10.13 Residual errors $\|E(t)\|_F$ synthesized by ZD models (10.11), (10.14), (10.15), and (10.17) with $\gamma = 10$ for finding the square root of time-varying circulant matrix $A(t)$ in (10.21)

results substantiate again the efficacy of the proposed ZD models (10.11), (10.14), (10.15), and (10.17) for time-varying matrix square root finding.

In summary, the above modeling results have shown the efficacy of the proposed ZD models (10.11), (10.14), (10.15), and (10.17) based on different ZFs for solving the time-varying matrix square root problem (10.1); and they have also confirmed the theoretical analysis and results given in Sect. 10.3. Besides, it is worth mentioning that the other ZD models [i.e., (10.12), (10.13), (10.16), and (10.18)] are also effectively exploited for time-varying matrix square root finding. The corresponding modeling verifications of such ZD models are left to interested readers to complete as a topic of exercise.

10.6 Summary

In this chapter, to solve for time-varying matrix square root, based on different ZFs (10.2)–(10.9), different ZD models shown in Table 10.1 [i.e., (10.11)–(10.18)] have been proposed, generalized, developed, and investigated in the form of the first-order matrix-valued differential equations. In addition, theoretical analysis and results have been given to show the convergence performance of such eight different ZD models. For possible hardware implementation based on electronic circuits, the MATLAB Simulink modeling of the proposed ZD models has been presented as well. Through illustrative computer-modeling examples, the efficacy of the proposed ZD models has been further substantiated for time-varying matrix square root finding [with the problem formulation depicted in (10.1)].

References

1. Higham NJ (1997) Stable iterations for the matrix square root. Numer Algorithms 15(2):227–242
2. Long JH, Hu XY, Zhang L (2007) Newton's method with exact line search for the square root of a matrix. In: Proceedings of international symposium on nonlinear dynamics, pp 1–5
3. Hasan MA, Hasan AA, Rahman S (2000) Fixed point iterations for computing square roots and the matrix sign function of complex matrices. In: Proceedings of 39th IEEE conference on decision and control, pp 4253–4258
4. Yao PF (1998) On the nonnegative square root of the fourth derivative operators. J Differ Equ 148(2):318–333
5. Johnson CR, Okubo K, Ream R (2001) Uniqueness of matrix square roots and an application. Linear Algebra Appl 323(1–3):51–60
6. Higham NJ (2002) Accuracy and stability of numerical algorithms, society for industrial and applied mathematics. SIAM, England
7. Higham NJ (2008) Functions of matrices: theory and computation, society for industrial and applied mathematics. SIAM, England

8. Zhang Y, Leithead WE, Leith DJ (2005) Time-series Gaussian process regression based on Toeplitz computation of $O(N^2)$ operations and $O(N)$-level storage. In: Proceedings of the 44th IEEE conference on decision and control and European control conference, pp 3711–3716
9. Babaei S, Geranmayeh A, Seyyedsalehi SA (2012) Towards designing modular recurrent neural networks in learning protein secondary structures. Expert Syst Appl 39(6):6263–6274
10. Zhang Y, Jiang D, Wang J (2002) A recurrent neural network for solving Sylvester equation with time-varying coefficients. IEEE Trans Neural Netw 13(5):1053–1063
11. Ghazali R, Hussain AJ, Liatsis P (2011) Dynamic ridge polynomial neural network: forecasting the univariate non-stationary and stationary trading signals. Expert Syst Appl 38(4):3765–3776
12. Zhang Y, Ge SS (2005) Design and analysis of a general recurrent neural network model for time-varying matrix inversion. IEEE Trans Neural Netw 16(6):1477–1490
13. Zhang Y, Ma W, Chen K, Li P (2007) MATLAB simulation of Zhang neural networks for time-varying Sylvester equation solving. In: Proceedings of the international conference on information computing and automation, pp 392–395
14. Wang J (1997) Recurrent neural networks for computing pseudoinverses of rank-deficient matrices. SIAM J Sci Comput 18(5):1479–1493
15. Übeyli ED (2010) Recurrent neural networks employing Lyapunov exponents for analysis of ECG signals. Expert Syst Appl 37(2):1192–1199
16. Zhang Y, Yang Y, Cai B, Guo D (2012) Zhang neural network and its application to Newton iteration for matrix square root estimation. Neural Comput Appl 21(3):453–460
17. Zhang Y, Yang Y (2008) Simulation and comparison of Zhang neural network and gradient neural network solving for time-varying matrix square roots. In: Proceedings of the 2nd international symposium on intelligent information technology application, pp 966–970
18. Zhang Y, Li W, Guo D, Ke Z (2013) Different Zhang functions leading to different ZNN models illustrated via time-varying matrix square roots finding. Expert Syst Appl 40(11):4393–4403
19. Shanblatt MA, Foulds B (2005) A Simulink-to-FPGA implementation tool for enhanced design flow. In: Proceedings of the IEEE international conference on microelectronic systems education, pp 89–90
20. Thieme HR (2011) Global stability of the endemic equilibrium in infinite dimension: Lyapunov functions and positive operators. J Differ Equ 250(9):3772–3801
21. Zhang Y, Yi C (2011) Zhang neural networks and neural-dynamic method. Nova Science Publishers, New York