Chapter 10
Overview of Collisional Stellar Dynamics

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10.1 Introduction

Here I discuss collisional stellar dynamics in the absence of gas; elsewhere in this volume we will also consider the complications added by considering gas and hydrodynamics (see Clarke Chap. 3). Before delving into the dynamics, let me make an essential overarching point. When considering an image of a beautiful cluster, such as M35 (Chap. 8), it is critical to recognize that this is not a representation of a static situation. Select any star—its current location is not where the star was not so long ago nor is it where it's going to be not so long from now. If you wish to understand and study the dynamics of star-forming regions, it is crucial to see them as dynamic, changing places, not only globally but also at the level of individual stars. Every object in the system was somewhere else before, to within the timescales that we will discuss next. This basic concept that the young stars are moving is fundamental, and one ignores it at peril.

As a simple framework for us to consider, I propose to separate the evolution of star-forming regions and star clusters into three time intervals. The first interval is while the stars are embedded in their natal molecular gas, when the gas mass and the stellar mass are comparable: the gas contributes meaningfully to the global gravitational potential for a timescale of a few Myr. Of course, in reality the system evolution depends on stellar dynamics, gas dynamics, magnetic fields, radiative transfer and many other variables (see Clarke Chaps. 2 and 4). The next interval, which we do not discuss in much detail here, is the period when the system is no longer embedded but is still losing significant mass due to the evolution of the massive stars. The evolution of the system at that point is very sensitive to the initial mass function, and occurs on a timescale set by the lifetimes of massive stars, of order 10–100 Myr. The third period is what I refer to as pure stellar dynamics, i.e.

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the dynamics of point sources. Of course, stars are not point sources, and one of the fascinating topics of recent cluster dynamical work concerns stellar collisions. But here I will introduce point-source stellar dynamics, since these processes will fold back into our discussions of the dynamical evolution of star-forming regions.

Very little that I present here is new; my goal is to clearly provide a working knowledge of the key concepts. There are many fine books that will allow you to delve deeper into stellar dynamics, for example Spitzer (1988) and Binney and Tremaine (2008).

10.2 Timescales

It is important to learn to think in terms of relevant timescales. They determine the relative significance of varied processes, and thereby provide important insights on how different systems are going to evolve. For dynamical systems the key timescales are the dynamical timescale, the two-body relaxation timescale, the evolution timescale, and the age of the system.[1]

The dynamical time scale t_{d} is essentially a crossing time. Thus one straightforward approach to its evaluation is taking twice the radius r of the system divided by the typical velocity dispersion about the systemic velocity of the system σ_{v}. For a system in virial equilibrium, this becomes:

$$t_{\mathrm{d}} \equiv \frac{2r}{\sigma_{\mathrm{v}}} = \left(\frac{8r^3}{Gnm}\right)^{1/2} \simeq \frac{1}{\sqrt{G\rho}}, \qquad (10.1)$$

where n is the number density within r, m is the average particle mass, and ρ is the mass density within r. The latter expression is an extremely useful tool for estimation. Whilst we are on the subject of useful tools, it is worth remembering that if one uses units of $1\,\mathrm{M}_\odot$, 1 pc and 1 Myr, then the gravitational constant G is simply 1/233. Finally, note that a velocity of $1\,\mathrm{km\,s^{-1}}$ translates to a distance of approximately 1 pc over a timescale of 1 Myr. Among other things, in the study of star-forming regions this allows you to consider, in an approximately quantitative way, my first point: that each star was and will be someplace else on dynamical timescales.

Now, returning to the physical meaning of the dynamical time scale, essentially t_{d} is the time in which a particle responds to the global gravitational potential. For a system in dynamical equilibrium, such as an older star cluster, it is essentially the orbital time. The dynamical timescale is sometimes called a mixing time, reflecting that it is also the timescale on which a system starts to lose non-equilibrium internal structures. And if a system is not bound, it is an estimate of the dissolution time. Roughly speaking, t_{d} is the timescale on which a system that is no longer bound is going to expand, dissolve, or disappear. For all of these reasons, the dynamical

[1] For completeness we also introduce the free-fall time, the timescale for a system to collapse under its own gravitational potential. For a uniform density sphere of any radius without any internal support, $t_{\mathrm{ff}} \sim 1/\sqrt{\rho}$.

time scale is perhaps the most important timescale for the evolution of star-forming regions.

The two-body relaxation time goes back to Chandrasekhar and Spitzer, and there have been many derivations and expressions of it over the years, such as:

$$t_{\mathrm{r}} = \frac{1}{25}\sqrt{\frac{Nr^3}{Gm}}\frac{1}{\log_{10}(\frac{N}{2})}. \tag{10.2}$$

Note that here N is the number of stars in the system, not a number density. So, critically, in virial equilibrium the relaxation time increases with the number of stars. Physically, the relaxation time is the average time to transfer significant energy of orbital motion between two stars. Formally it is often defined as the time over which the cumulative gravitational perturbations due to all particles in a system change the energy of a body by an amount roughly comparable to its orbital energy. This is actually a divergent integral; the effects of all distant stars are more important than nearby stars.[2] Practically speaking, most dynamical systems have natural physical limits on the integral. Two-body relaxation establishes a near-Maxwellian[3] velocity distribution, which I emphasise here because in multi-mass systems a Maxwellian implies particle velocity distributions dependent on the masses of the particles. Two-body relaxation is an energy equipartition process, and in equilibrium yields mass segregation in dynamical systems. In terms of the overall evolution of a dynamical system, two-body relaxation is the timescale for energy flow. Thus the relaxation time is the equivalent of a thermal timescale, such as dictates the time for energy to flow from the core to the surface in a star. Essentially the same energy flow occurs within star clusters, except it is the transfer of orbital energy of stars.

The evolution timescale is the time for secular evolution of the cluster properties such as mass and structure. Physically this is the time for a global change in the energy structure of a cluster. I will show shortly that the energy flows due to two-body relaxation lead to systemic energy changes, for example requiring that cores have to collapse and particles have to escape. Roughly speaking, a star cluster loses 1 % of its stars every relaxation time, so that in of order 100 relaxation times, a cluster disappears except for the last remaining tight binary at the centre. Thus an evolution timescale of $100t_{\mathrm{r}}$ can be defined by the evaporation time for the cluster.

The final timescale is the age of the system, which is not defined by dynamics but is critical for dynamical analyses. Indeed, it is the addition of age to the mix that makes the topic of this volume so interesting: we focus here on systems where the dynamical times, or the crossing times, and the ages are comparable, a few Myr for both. So we can not presume that these systems are well-mixed. In fact, we can presume that they are not well-mixed in most cases! Further many well-studied nearby star-forming regions contain a modest number of stars and richer events are often composed of

[2] Note that this statement is not true in the case of binary stars, where the encounters are tidal in nature and go as r^3, where r is roughly the impact parameter. We will return to this later.

[3] 'Near-Maxwellian' in part because a cluster has an escape velocity that limits the maximum extent of velocity distributions.

smaller sub-groups with significant sub-structure. For $N \lesssim 100$, the dynamical and relaxation times are comparable in near-virial systems. In other words, a few-body system relaxes on the same timescale that its constituents cross the system. And so it is quite possible that we are going to be discussing systems where the crossing times, the relaxation times, and the ages are all of the same timescale. From a theoretical and observational point of view, this comparability of the timescales makes the dynamics of young stellar systems both demanding and interesting.

10.3 Violent Relaxation

A challenge for early stellar dynamical theory was that the two-body relaxation times for galaxies and the most massive globular clusters are very long, indeed longer than the age of the Universe. And yet the structures of these systems appeared dynamically relaxed, with at least Gaussian velocity distributions. And so Donald Lynden-Bell and several of his colleagues in the early 1960s developed the concept of violent relaxation, which is going to be very important to our discussion here. The essential idea of violent relaxation is that the energy of a particle as it orbits a cluster is only conserved if the global potential is fixed. If the potential is varying, then the energy of the particle will also vary, quite independent of any two-body interactions. And so the essence of violent relaxation is that, if a system is not in equilibrium, then $\mathrm{d}E/\mathrm{d}t$ over a particle orbit is not equal to zero because the system structure—and thus the gravitational potential—is going through large macroscopic changes. In this situation, the specific energy change of a star over its orbit, in other words per mass, is comparable to the change in the global potential. The violent relaxation time is defined as the time where change in the specific energy of this star is comparable to its specific energy. (Very similar to the definition for two-body relaxation.) But the change in the potential is really the same as the change in the mass distribution, and for systems far out of equilibrium changes in mass distributions happen on dynamical timescales, or crossing times. So roughly speaking we can expect violent relaxation of a system to occur on a dynamical timescale (or a free-fall time). It can happen very fast, much faster than a two-body relaxation time.

Importantly, velocity distributions and spatial distributions are independent of mass after violent relaxation, just as the orbital properties of Jupiter in the Sun's potential do not depend on the mass of Jupiter to high significance. Thus violent relaxation sets up Gaussian-like velocity distributions, not Maxwellian velocity distributions, and the consequent core-halo structures are independent of mass. Finally, velocity anisotropy, for example from formation conditions or initial collapse phases of a system, tend to be preserved.

Violent relaxation approaches but never achieves equilibrium, because as a system approaches equilibrium the changes in the gravitational potential become ever smaller and the violent relaxation timescale stretches out. So the system never actually reaches complete equilibrium. Finally, the nature of violent relaxation in the context of large (gaseous) sub-structures and merging clumps has been an active subject in this field for the last few years (see Clarke Chap. 6). Again, star-forming regions bring fascinating subtleties and new understanding to classical stellar dynamics.

10.4 Energy Equipartition and Mass Segregation

Two-body relaxation due to gravitational interactions between particles is an energy-equipartition process producing a near-Maxwellian velocity distribution. Thus the kinetic energy distributions of particles vary inversely with their masses, and equivalently the velocity distributions of particles vary as the inverse square root of the masses, or:

$$f(E_{\rm star}) \propto e^{E/\sigma^2} \Rightarrow f(E[v_i]) \propto e^{-\frac{1 m_i v_i^2}{2\sigma^2}} \Rightarrow \frac{< v_i^2 >}{< v_j^2 >} = \frac{m_j}{m_i} \tag{10.3}$$

Thus, for example, the equipartition timescale for massive particles is very short (which is essentially a result of dynamical friction being very efficient). If there are $10\,M_\odot$ stars in a system comprised predominantly of solar-mass stars, then the equipartition time for the $10\,M_\odot$ stars is a factor of 10 shorter than the nominal relaxation time. The structural consequence of energy equipartition is mass segregation, or the greater central concentration of higher-mass stars. Mass segregation is often seen in well-relaxed star clusters, such as the old open star cluster M67 (see Chap. 8) but only down to some stellar mass.

This question of energy equipartition, and thus mass segregation, timescales is more complex in very young clusters. There is no doubt that the Trapezium Cluster shows mass segregation, most notably through the presence of the four massive Trapezium stars at its centre. Accurate mass-dependent equipartition timescales and histories are critical to understanding whether or not the Trapezium stars formed *in situ* at the centre, or whether they fell to the centre very quickly through dynamical processes. To date, dynamical studies of that question have failed to reach a consensus.

10.5 Evolution of Dynamical Systems: Some Fundamental Physics

This section is dedicated to Lyman Spitzer, who provided to me as a young student this beautiful demonstration of fundamental physical thinking.

The probability of a star being in a particular state of energy E_i is, as in statistical mechanics:

$$P_i = C g_i e^{-kE_i}, \tag{10.4}$$

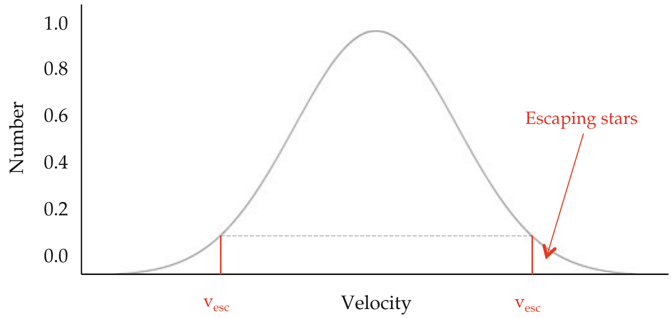

Fig. 10.1 Schematic representation of a thermal velocity distribution, the location of the escape velocity in a virialised system, and the fraction of stars (0.74 %) that escape per relaxation time

where g_i is the amount of phase space available, k is the Boltzmann constant and C is a normalisation factor for the distribution. So how do we maximise that probability?

And this is where I get to say that gravity is the coolest force in the Universe, because it has this wonderful property of having negative heat capacity, with even more wonderful implications. For a gravitational system, there are two ways that one can maximise the above probability. First, one can increase the binding energy. In other words, you can make E_i very, very negative (with point masses one can in principle make them infinitely negative). Second, one can maximise the phase space that is available. But there is no upper limit on the spatial coordinates; the particle distribution can extend to infinity in principle. And so how do you maximise the probability in a gravitational system? You can not. There is no maximum entropy of a gravitational system given finite mass and energy. And so bound gravitational systems necessarily evolve continuously; there is no equilibrium state. Furthermore, in its attempt to maximise entropy and minimise energy the system does two things at the same time. It collapses the core to zero size but finite mass, while taking the rest of the cluster mass to infinity. In other words, any gravitational system is going to collapse at its core and spew outward the rest of the material. It does not matter if it is a star. It does not matter if it is a cluster. Ultimately, it does not matter if it is a galaxy. Inevitably, the evolution of gravitational systems is characterised by collapsing cores and expanding halos.

10.6 Evolution of Dynamical Systems: Two-Body Processes

The mechanism of energy exchange by which these inevitable gravitational events happen within stellar systems is two-body relaxation. We know the outcome will be escaping stars and collapsing cores. Let us consider escaping stars—and cluster evaporation—first.

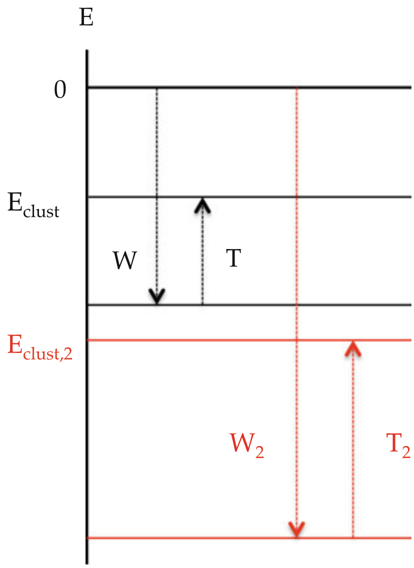

Fig. 10.2 Schematic energy level diagram for a star cluster in virial equilibrium. *Black* energy levels represent an initial state, *red* energy levels represent a subsequent state after the loss of energy (e.g. due to stellar evaporation)

The key concept of this is shown in Fig. 10.1, which goes back to Ambartsumian and Spitzer (independently). Two-body relaxation seeks to establish a Maxwellian velocity distribution. But a gravitational system has an escape velocity.[4] As a consequence, the tail of the velocity distribution will be lost (roughly on a dynamical timescale). Two-body processes continuously re-populate that tail on a relaxation timescale, and the process continues until almost all stars have escaped. That is cluster evaporation.

A rough estimate of the evaporation timescale is straightforward to calculate. The distribution beyond the escape velocity turns out to be about 1 % of the total population. Every relaxation time the cluster loses that much mass, so roughly speaking the evolution evaporation time for the cluster is about 100 times the two-body relaxation time.

Critically for the cluster, the escaping stars carry away energy and mass. Necessarily, this loss of energy leads to collapse of the cluster. (Which by now should come as no surprise.) Consider an energy level diagram for a cluster in virial equilibrium (see Fig. 10.2). The energy of the bound cluster (E_{clust}) is of course negative; the associated ratio of the kinetic (T) and potential energy (W) reflects the virial theorem. Now if escaping stars carry away energy, lowering the cluster energy ($E_{clust,2}$), then the cluster reconfigures itself on a dynamical timescale to re-establish the virial balance. As a result two things happen. One, the potential energy (W_2) becomes deeper. In other words, the cluster (more specifically, the core) contracts. And second, the kinetic energy (T_2) has actually increased. In other words, as you

[4]This discussion is intentionally simplified. The escape velocity differs throughout a cluster, as will the local Maxwellian velocity distribution when the cluster is not isothermal.

remove energy from a gravitational system, it heats up! And that of course is negative heat capacity.

This is not a comfortable situation for a cluster. As energy is removed from that cluster, the core heats up, which leads to yet more energy flow outward. This is the runaway that we predicted from our intuitive physical analysis. Now we know the driving process—two-body relaxation—and the evolutionary timescale—of order $100t_r$.

There are a multitude of excellent papers examining these processes and timescales in more detail. One particularly worth noting again goes back again to Donald Lynden-Bell, who showed that this core collapse can be an unstable process. Consider a star in the halo of a cluster—the escape velocity from the halo is smaller than the core, and if the star is to be bound it cannot have a large velocity. On the other hand, stars in the core can have much higher velocity dispersions. So that means that the halo has to be colder than the core, and energy has to flow from the core to the halo. Thus the inner region contracts and heats up. The halo, which is not self-gravitating because it responds to the core potential, receives that energy and also heats up. But if the inner region heats up more than the outer region, you have a thermal runaway. For idealised conditions, this instability depends on the central concentration of the core. If it is a highly centrally concentrated core, the gravothermal catastrophe (one of the finest phrases in astrophysics!) occurs, with core collapse on the order of 15 relaxation times. In this situation, the core collapse happens much faster because of the energy draw that the halo places on the core.

There is one more instability, known as the mass-segregation instability, which may again be relevant to the Trapezium. Consider a system which has only a few stars that are much more massive than the typical star, such that these massive stars are not dictating the gravitational potential. These stars will migrate to the centre of the potential very quickly as a result of energy equipartition. Once there, they form their own self-gravitating dynamical system in the core. Schematically, the cluster potential is broad with a localised dip at its deepest point. As the rest of the cluster keeps removing energy from that small self-gravitating core of massive stars, the core has to collapse rapidly to provide the energy. And so mass segregation can actually accelerate the gravothermal catastrophe, because all of the energy that the halo is demanding from the core is being drawn from just a few massive stars at the centre.

So how do star clusters solve this fundamental gravitational problem, in all its forms? Well, we have seen this problem before in stellar evolution. And how do stars solve this problem? Well, they solve the problem at least temporarily by providing a fuel source. Instead of providing the energy demanded by the envelopes of the stars from the gravitational energy of the stellar core, nature provides an energy source—nuclear fusion. Clusters do the same thing in principle, but they do it with binaries. We will return to this later.

10.7 Evolution of Dynamical Systems: Cluster Dissolution

The final topic that I will touch on is cluster dissolution. We know that clusters have limited lifetimes. For example, in Chap. 8, I showed that the number of clusters has dropped significantly by 800 Myr or so, much sooner than the age of the Galactic disc. This almost certainly is the result of internal processes such as evaporation, likely accelerated by the Galactic tidal field and tidal impact encounters with molecular clouds.

However, for our purposes we are more concerned with the loss of the binding energy of the gas in which the clusters form than about evaporation (except for the smallest-N systems). The loss of the natal gas is certainly going to happen, and the response will be on a dynamical timescale.

If you consider a simple virialised system and you remove a fraction ε of the mass rapidly compared to the dynamical time, then it is very easy to compare the final and initial radii r of the cluster:

$$\frac{r_f}{r_i} = \frac{1-\varepsilon}{1-2\varepsilon} = \frac{\eta}{2\eta - 1}. \tag{10.5}$$

Often the fraction of gas lost, ε, is rewritten in terms the star formation efficiency, η, the fraction of gas that gets converted into stars. As is well-known from introductory physics, if more than half of the total mass is lost, the system becomes unbound. But this is only true if the mass loss happens rapidly. If it happens slowly with respect to the dynamical timescale, then the system will adiabatically expand:

$$\frac{r_f}{r_i} = \frac{1}{1-\varepsilon} = \frac{1}{\eta} = \frac{m_i}{m_f}. \tag{10.6}$$

In principle all of the mass could be lost and the system could expand to infinity without becoming unbound. If the mass-loss is slow compared to a dynamical time, then the ratio of the final mass to the initial mass is the inverse of the star formation efficiency. Thus, unless the star formation efficiency is zero, in other words unless all of the gas is converted to stars, which is not a terribly interesting scenario dynamically, the system remains bound, but it does expand and it gets very large. Of course in reality during expansion the system runs into physical issues with its environment within the molecular cloud. My main point here is that large mass-loss does not necessarily imply that a system is going to fly apart on a dynamical timescale.

This analytic analysis was carried out by several of us in the early 1980s, from which we concluded that the existence of bound clusters implied at least some locations of high star formation efficiency. Charlie Lada and his colleagues ran N-body models and added an important nuance to the discussion, that cores are more tightly bound than are clusters globally. Thus even high fractions of rapid mass-loss may leave behind bound cores that ultimately become lower mass star clusters.

References

Binney, J. & Tremaine, S. 2008, Galactic Dynamics: Second Edition, ed. J. Binney & S. Tremaine, Princeton University Press

Spitzer, Jr., L. 1988, Dynamical Evolution of Globular Clusters, ed. L. Spitzer Jr., Princeton University Press