Chapter 3
Dissipativity Analysis and Synthesis of Singular Systems via Delta Operator Method

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Abstract This paper investigates the problems of dissipativity analysis and synthesis for singular systems through delta operator method. First, a sufficient condition is obtained such that a singular delta operator system is admissible and strictly dissipative. Then the existence condition and explicit expression of a state feedback strictly dissipative controller are presented. A numerical example is also provided to demonstrate the effectiveness of the theoretical results.

Keywords Singular delta operator systems · Admissibility · Strict dissipativity · State feedback · Linear matrix inequality

3.1 Introduction

During the past decades, much attention has been paid to singular systems as they can describe many practical systems such as economic systems, electrical networks, highly interconnected large-scale systems, etc. [1]. Many achievements have been made in singular system theory in recent years [1–5, 8–10, 15]. Dissipativity theory is an important part in control theory which has made a positive effect on studying stability and other properties of control systems [12, 13]. There have been some valuable results on dissipativity analysis and dissipative control for singular systems [2, 8, 9]. For example, a necessary and sufficient condition was obtained to ensure an uncertain singular discrete system admissible and strict dissipative [2]. The existence condition and explicit expression of a state feedback strictly dissipative controller were also given in [2]. The results of robust dissipativity analysis and some design method of a robust dissipative controller were presented for singular continuous systems with affine uncertainty in [8, 9], respectively.

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In control theory, most research results adopt the standard shift operator in the study of discrete systems. But there exists a problem that the dynamic response of a discrete system does not converge smoothly to its continuous counterpart when the sampling period tends to zero [11], which is called the numerical ill-condition. In order to avoid the above problem, a delta operator method was proposed in [6]. It was shown that the delta operator requires smaller word length when implementing fixed-point digital control processors than the shift operator does [7]. The delta operator method is also significantly less sensitive than the shift operator method at high sampling rates [14]. Furthermore, the delta operator model can provide a theoretically unified formulation of normal continuous and discrete systems. Most recently, the delta operator method has been introduced to study singular systems and some valuable results have been derived. Dong [3] and Dong et al. [4] studied the problem of admissibility analysis for singular systems via delta operator method. Dong et al. [4] and Mao et al. [10] considered the problem of admissible control for singular delta operator systems. But until now there is no result on dissipativity analysis and control for singular delta operator systems.

In this paper, we consider the problems of dissipativity analysis and synthesis for singular delta operator systems. A sufficient condition is obtained such that a singular delta operator system is admissible and strictly dissipative. Based on the above result, the existence condition and explicit expression of a state feedback strictly dissipative controller are presented. A numerical example is also provided to demonstrate the effectiveness of the theoretical results.

Throughout this paper, the following notations are adopted: δ is the delta operator defined by $\delta x(t) = \dot{x}(t)$ when $h = 0$ and $\delta x(t) = h^{-1}(x(t+h) - x(t))$ when $h \neq 0$, where h is the sampling period. Matrix $P > 0$ (or $P < 0$, respectively) means that P is symmetric and positive definite (or negative definite, respectively). $D_{\text{int}}(a, r)$ is the interior of the region in the complex plane with the center at $(a, 0)$ and the radius r. $\lambda(A, B) = \{z | \det(zA - B) = 0\}$.

3.2 Preliminaries

Consider the following singular delta operator system:

$$\begin{aligned} E\delta x(t_k) &= Ax(t_k) + B_1 w(t_k) \\ z(t_k) &= Cx(t_k) + D_1 w(t_k) \end{aligned} \tag{3.1}$$

where $x(t_k) \in R^n$ is the state, $w(t_k) \in R^p$ is the disturbance input, $z(t_k) \in R^q$ is the controlled output, t_k means the time $t = kh$, and $h > 0$ is the sampling period. $E \in R^{n \times n}$ and $\text{rank}(E) = r < n$, A, B_1, C, D_1 are known real matrices with appropriate dimensions.

Consider the following system:

$$E\delta x(t_k) = Ax(t_k) \tag{3.2}$$

Definition 3.1 [3] The system (3.2) is said to be regular if $\det(\eta E - A)$ is not identically zero. The system (3.2) is said to be causal if $^\circ(\det(\eta E - A)) = \mathrm{rank}(E)$. The system (3.2) is said to be stable if $\lambda(E, A) \subset D_{\mathrm{int}}(-1/h, 1/h)$. The system (3.2) is said to be admissible if it is regular, causal, and stable.

Lemma 3.1 [3] *The system (3.2) is admissible if and only if there exist matrices $P > 0$ and F satisfying $hA^TPA + A^TPE + E^TPA + FG^TA + A^TGF^T < 0$, where G is any matrix of full column rank and satisfies $E^TG = 0$.*

The energy supply function of the system (3.1) is defined by $E(w, z, T) = <z, Qz>_T + 2<z, Sw>_T + <w, Rw>_T$, where T is a nonnegative integer; Q, S, and R are known real matrices with Q and R symmetric. $<u, v>_T$ is defined as $<u, v>_T = \sum_{k=0}^{T} u(t_k)^T v(t_k)$.

Definition 3.2 [2] The system (3.1) is said to be strictly (Q, S, R) dissipative if for some scalar $\alpha > 0$ and under zero initial state $x(0) = 0$, the following inequality holds

$$E(w, z, T) \geq \alpha <w, w>_T, \quad \forall T \geq 0 \tag{3.3}$$

In order to include H_∞ performance (where $Q = -I, S = 0, R = \gamma^2 I$) and passivity (where $Q = 0, S = I, R = 0$) as special cases of the above strict (Q, S, R) dissipativity, we make the following assumption:

Assumption 3.1 $Q \leq 0$.

3.3 Dissipativity Analysis

The purpose of this section is to present some conditions for the system (3.1) to be admissible (when $w(t_k) = 0$) and strictly dissipative (when $w(t_k) \neq 0$). The dissipativity analysis result is given in the following theorem:

Theorem 3.1 *Let the matrices Q, S, and R be given with Q and R symmetric and Assumption 3.1 holds. Then the system (3.1) is admissible and strictly (Q, S, R) dissipative if there exist matrices $P > 0$ and F satisfying*

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{21}^T & C^TQ_1^T \\ \Sigma_{21} & \Sigma_{22} & D_1^TQ_1^T \\ Q_1C & Q_1D_1 & -I \end{bmatrix} < 0 \tag{3.4}$$

where Q_1 *is any matrix satisfying* $Q_1^T Q_1 = -Q$, *G is any matrix of full column rank and satisfies* $E^T G = 0$, $\Sigma_{11} = hA^T PA + A^T PE + E^T PA + FG^T A + A^T GF^T$, $\Sigma_{21} = hB_1^T PA + B_1^T PE + B_1^T GF^T - S^T C$, $\Sigma_{22} = hB_1^T PB_1 - D_1^T S - S^T D_1 - R$.

Proof Assume that the inequality (3.4) holds. From (3.4) it is easy to obtain

$$hA^T PA + A^T PE + E^T PA + FG^T A + A^T GF^T < 0 \tag{3.5}$$

Then from Lemma 3.1 we have that the system (3.1) is admissible.
From $E^T G = 0$ we have

$$0 = 2\delta x(t_k)^T E^T GF^T x(t_k) = \begin{bmatrix} x(t_k)^T & w(t_k)^T \end{bmatrix} \Omega_1 \begin{bmatrix} x(t_k) \\ w(t_k) \end{bmatrix}$$

where

$$\Omega_1 = \begin{bmatrix} A^T GF^T + FG^T A & FG^T B_1 \\ B_1^T GF^T & 0 \end{bmatrix}$$

Let $V(x(t_k)) = (x(t_k))^T E^T PEx(t_k)$ and then we can derive $V(x(t_k)) \geq 0$ for any $k \geq 0$ from $P > 0$. Then we have

$$\begin{aligned} J &= \delta V(x(t_k)) - z(t_k)^T Qz(t_k) - 2z(t_k)^T Sw(t_k) - w(t_k)^T Rw(t_k) \\ &= (E\delta x(t_k))^T PEx(t_k) + x(t_k)^T E^T PE\delta x(t_k) + h(E\delta x(t_k))^T PE\delta x(t_k) \\ &\quad - z(t_k)^T Qz(t_k) - 2z(t_k)^T Sw(t_k) - w(t_k)^T Rw(t_k) \\ &= \begin{bmatrix} x(t_k)^T & w(t_k)^T \end{bmatrix} \Omega_2 \begin{bmatrix} x(t_k) \\ w(t_k) \end{bmatrix} \\ &= \begin{bmatrix} x(t_k)^T & w(t_k)^T \end{bmatrix} (\Omega_1 + \Omega_2) \begin{bmatrix} x(t_k) \\ w(t_k) \end{bmatrix} \end{aligned}$$

where

$$\Omega_2 = \begin{bmatrix} E^T PA + A^T PE + hA^T PA - C^T QC & E^T PB_1 + hA^T PB_1 - C^T S - C^T QD_1 \\ B_1^T PE + hB_1^T PA - S^T C - D_1^T QC & \Sigma_{22} - D_1^T QD_1 \end{bmatrix}$$

When (3.4) holds, from Schur complement we have that (3.4) is equivalent to $\Omega_1 + \Omega_2 < 0$ which means $J < 0$. In this case, a sufficiently small scalar $\alpha > 0$ can always be found such that $J + \alpha w(t_k)^T w(t_k) \leq 0$. Sum the above inequality up from 0 to T, and notice that $h > 0, x(0) = 0$, $V(x(t_T)) \geq 0$, we can obtain

$$h^{-1} V(x(t_T)) - E(w, z, T) + \alpha < w, w >_T \leq 0$$

Therefore from the above inequality and Definition 3.2 we have that the system (3.1) is strictly (Q, S, R) dissipative. This completes the proof.

3.4 Dissipative Control

Consider the following singular delta operator system with control input

$$\begin{aligned} E\delta x(t_k) &= Ax(t_k) + B_1 w(t_k) + B_2 u(t_k) \\ z(t_k) &= Cx(t_k) + D_1 w(t_k) + D_2 u(t_k) \end{aligned} \tag{3.6}$$

where $u(t_k) \in R^m$ is the control input, B_2, D_2 are known real matrices with appropriate dimensions, the other notations are the same as those in (3.1).

The purpose of this section is to design a state feedback controller

$$u(t_k) = Kx(t_k) \tag{3.7}$$

for the system (3.6), such that the resulting closed-loop system

$$\begin{aligned} E\delta x(t_k) &= A_c x(t_k) + B_1 w(t_k) \\ z(t_k) &= C_c x(t_k) + D_1 w(t_k) \end{aligned} \tag{3.8}$$

is admissible and strictly (Q, S, R) dissipative, and in this case the controller (3.7) is said to be a strictly dissipative controller for the system (3.6), where K is the controller gain matrix to be designed and $A_c = A + B_2 K, C_c = C + D_2 K$.

The dissipativity synthesis result is given in the following theorem.

Theorem 3.2 *Let the matrices Q, S, and R be given with Q and R symmetric and Assumption* 3.1 *hold. Then there exists a state feedback strictly dissipative controller* (3.7) *for the system* (3.6) *if there exist matrices $P > 0, F$ and a scalar $\varepsilon \geq 0$ satisfying*

$$\begin{bmatrix} \Sigma_{11} - W^T Z^{-1} W & \Sigma_{21}^T & C^T Q_1^T \\ \Sigma_{21} & \Sigma_{22} & D_1^T Q_1^T \\ Q_1 C & Q_1 D_1 & -I \end{bmatrix} < 0 \tag{3.9}$$

where $Q_1, G, \Sigma_{11}, \Sigma_{21}, \Sigma_{22}$ are the same as those in Theorem 3.1, and $M = \Sigma_{22} - D_1^T Q D_1, N = hB_1^T P B_2 - S^T D_2 - D_2^T Q D_1, L = \Sigma_{21} - D_1^T QC, Z = hB_2^T PB_2 - D_2^T QD_2 - N^T M^{-1} N + \varepsilon I > 0$, $W = hB_2^T PA + B_2^T PE - B_2^T GF^T - D_2^T QC - N^T M^{-1} L$. In this case, the gain matrix K of the controller (3.7) *can be designed as*

$$K = -Z^{-1} W \tag{3.10}$$

Proof Assume that the inequality (3.9) holds. From Theorem 3.1, we have that system (3.8) is admissible and strictly (Q, S, R) dissipative, if there exist matrices $P > 0$ and F satisfying the following inequality:

$$\begin{bmatrix} \Pi_{11} & \Pi_{21}^T & C_c^T Q_1^T \\ \Pi_{21} & \Sigma_{22} & D_1^T Q_1^T \\ Q_1 C_c & Q_1 D_1 & -I \end{bmatrix} < 0 \tag{3.11}$$

where Π_{11}, Π_{21} are obtained from Σ_{11}, Σ_{21} by replacing the matrices A, C with A_c, C_c, respectively.

From Schur complement, the inequality (3.11) is equivalent to

$$\begin{bmatrix} \Pi_{11} & \Pi_{21}^T \\ \Pi_{21} & \Sigma_{22} \end{bmatrix} - \begin{bmatrix} C_c^T \\ D_1^T \end{bmatrix} Q [C_c \quad D_1] < 0 \tag{3.12}$$

Again by Schur complement, we know that (3.12) is equivalent to $M < 0$ and

$$H = \Pi_{11} - C_c^T Q C_c - (\Pi_{21}^T - C_c^T Q D_1) M^{-1} (\Pi_{21} - D_1^T Q C_c) < 0 \tag{3.13}$$

The inequality (3.13) is also the same as

$$H = H_1 + W^T K + K^T W + K^T (Z - \varepsilon I) K < 0 \tag{3.14}$$

where $H_1 = \Sigma_{11} - C^T Q C - L^T M^{-1} L$.

Similarly, from Schur complement, we can obtain that the inequality (3.9) is equivalent to $H_1 - W^T Z^{-1} W < 0$, which together with $\varepsilon \geq 0$ and (3.10) gives

$$\begin{aligned} H &\leq H_1 + W^T K + K^T W + K^T Z K \\ &= H_1 - W^T Z^{-1} W + (K^T + W^T Z^{-1}) Z (K + Z^{-1} W) = H_1 - W^T Z^{-1} W < 0 \end{aligned}$$

Then the inequality (3.14) (i.e. (3.11)) holds. Thus from Theorem 3.1 we have that the closed-loop system (3.8) is admissible and strictly (Q, S, R) dissipative and the controller (3.7) is indeed a strictly dissipative controller of the system (3.6). This completes the proof.

3.5 Example

Consider the system (3.6) with the following parameter matrices

$$E = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 \\ 1 & 7 \end{bmatrix}, B_1 = \begin{bmatrix} -0.4 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$C = [\,-2 \quad 1\,], D_1 = 1,\ D_2 = -6,\ h = 0.1$$

Let $Q = -1, S = 0.4, R = 2$ and $G = [\,-2 \quad 1\,]^T$. First we solve the inequality (3.4) and cannot find a feasible solution. Thus from Theorem 3.1 we know that the above system is not admissible and strictly dissipative. Next we want to design a state feedback strictly dissipative controller for the above system. Select

$$P = \begin{bmatrix} 1 & -0.01 \\ -0.01 & 0.02 \end{bmatrix},\ F = \begin{bmatrix} 0.1 \\ -1 \end{bmatrix},\ \varepsilon = 0$$

and then we can obtain that the inequality (3.9) indeed holds from

$$\begin{bmatrix} -5.839 & 0.8871 & 0.5614 & -2 \\ 2.8871 & -1.5647 & -1.1832 & 1 \\ 0.5614 & -1.1832 & -2.7812 & 1 \\ -2 & 1 & 1 & -1 \end{bmatrix} < 0$$

Then by Theorem 3.2, there exists a strictly dissipative controller (3.7) for the system (3.1) and it can be designed as $u(t_k) = [\,-0.2816 \quad 0.1763\,]x(t_k)$.

3.6 Conclusion

In this paper, the problems of dissipativity analysis and synthesis have been considered for singular systems via delta operator method. A sufficient condition about dissipativity analysis has been presented for singular delta operator systems. Based on the above result, the existence condition and design method of a state feedback strictly dissipative controller have also been derived. An example is also provided to illustrate the effectiveness of the obtained results.

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