Single-Profile Choice Functions and Variable Societies: Characterizing Approval Voting

Hanji Wu, Yongsheng Xu, and Zhen Zhong

Abstract We study approval voting in a setting with a fixed profile of individuals' choices and variable societies. Four properties each linking choices made by a group of individuals to choices by its various subgroups are introduced, and are used for characterizing approval voting.

Keywords Approval voting • Choice function • Variable societies

1 Introduction

Approval voting is an important voting method that has been used in many contexts and works as follows: when a group of individuals deciding on several alternatives and assuming that each chooses his 'approved' ones, the alternatives that get the most 'votes' among the available alternatives emerge as the winners.

Since the introduction of approval voting (see [2]), there has been a number of axiomatic studies on its behavior. It is fair to say that all the axiomatic studies in the literature are based on multi profiles of preferences or choice functions with either a fixed society or variable societies. See Xu [9] for a survey on axiomatizations of approval voting in the literature.

In this paper, we take a different approach from the existing ones to study approval voting. In our framework, we work with a fixed profile of individuals' choices while allow various societies to be formed. A similar framework has been employed by Xu and Zhong [10] to study simple majority rule. Approval voting is thus investigated from a perspective of linking the society's choices with choices made by its various sub-societies. It is then natural to see how choices made by various sub-societies can be linked to the choice by the society as a whole. In particular, we can ask questions like the following: when the choices of two

H. Wu • Y. Xu (✉)
Department of Economics, Georgia State University, Atlanta, GA, USA
e-mail: wuhanji@yahoo.com; yxu3@gsu.edu

Z. Zhong
Research Institute of Finance and Banking, the People's Bank of China, Beijing 100800, China
e-mail: zhongzhen96@gmail.com

C. Binder et al. (eds.), *Individual and Collective Choice and Social Welfare*,
Studies in Choice and Welfare, DOI 10.1007/978-3-662-46439-7_9

disjoint sub-societies have some alternatives in common, would those common alternatives continue to be chosen by the society joined by the two sub-societies? what happens to the choices by the society joined by the two disjoint sub-societies when their choices have nothing in common? We show that approval voting can be characterized by the following properties (see formal definitions of these properties in Sect. 3): (1) a society consisting of one individual should reflect this individual's choices, (2) when the choices of two disjoint sub-societies have some alternatives in common, the choices of the society joined by the two sub-societies should be given by those commonly chosen alternatives of the two sub-societies, (3) when an alternative is not chosen by two disjoint sub-societies, this alternative should not be a chosen by the society formed by the two sub-societies, and (4) when an individual's choices have nothing in common with the choices of a sub-society and when they form a new society, the choices of the new society should include those alternatives chosen by the former sub-society. In a sense, approval voting is characterized by two types of properties: (1) how the choices of a society consisting of just one individual are linked to the choices of this individual, and (2) how the choices of a society formed by two distinct societies are linked to the respective choices of the two societies.

The remainder of the paper is organized as follows. In Sect. 2, we introduce the basic notation and definitions. Section 3 presents a set of properties and axiomatic derivation of approval voting. The paper is concluded in Sect. 4 by offering some brief remarks.

2 Notation and Definitions

Let there be $n \geq 2$ individuals, and let $N = \{1, \cdots, n\}$ denote the set of individuals in the society. A is to denote a set of finite alternatives with two or more alternative. Throughout this paper, we assume that A is given and fixed.

For each $i \in N$, $C_i(A)$ stands for individual i's choice set over A. It is assumed that $C_i(A) \subseteq A$ and $C_i(A) \neq \emptyset$ for all $i \in N$. For each $i \in N$, $C_i(A)$ is interpreted as the alternatives approved by individual i from the set A.

Non-empty subsets of N are denoted by $S, T, \cdots$, and are called *coalitions*. For any coalition S, $\#S$ denotes the cardinality of S. The set of all non-empty coalitions is to be denoted by $\mathcal{K}$.

Let $\alpha^N(A) \equiv \{C_1(A), \cdots, C_i(A), \ldots, C_n(A)\}$ denote a *profile* of individuals' choices over A. In this paper, we consider $\alpha^N(A)$ as **fixed**. For any coalition $S \in \mathcal{K}$, let $\alpha^S(A)$ denote the set $\{C_i(A) \in \alpha^N(A) : i \in S\}$.

An *aggregation rule* f assigns, for each $\alpha^S(A) \in \bigcup_{T \in \mathcal{K}} \alpha^T(A)$, a non-empty choice set over A: $C(S, A) = f(\alpha^S(A))$, where $\emptyset \neq C(S, A) \subseteq A$ is called the choice set of the coalition over the set A.

For each coalition S, let $N(x, S, A) \equiv \#\{i \in S : x \in C_i(A)$ for some $i \in S\}$. An aggregation rule f is said to be *Approval Voting* if and only if, for all coalition S, $x \in C(S, A) \Leftrightarrow N(x, S, A) \geq N(y, S, A)$ for all $y \in A$.

3 Axioms and a Characterization Result

We first consider the following axioms that are to be imposed on an aggregation rule.

Self Determination (SD): For all $i \in N$, $C(\{i\}, A) = C_i(A)$.
Monotonicity (M): For all coalitions $S \in \mathcal{K}$ and all $i \in N \setminus S$, if $[C_i(A) \cap C(S, A) \neq \emptyset]$ then $C(S \cup \{i\}, A) = C(S, A) \cap C_i(A)$.
Unanimous Rejection (UR): For all coalition $S \in \mathcal{K}$, all individual $i \in N \setminus S$, and all $x \in A$, if $[x \notin C_i(A)$ and $x \notin C(S, A)]$ then $x \notin C(S \cup \{i\}, A)$.
Positive Association (PA): For all coalitions $S \in \mathcal{K}$ and all $i \in N \setminus S$, if $[C_i(A) \cap C(S, A) = \emptyset]$ then $C(S, A) \subseteq C(S \cup \{i\}, A)$.

(SD) is fairly straightforward and requires that, when a society consists of a single individual i, the choice set of this society coincides with this single individual's choice set. It thus reflects the idea of self determination.

(M) says that, when an individual i is added to a coalition S to form a new society $S \cup \{i\}$, if some alternative happens to be chosen by both the coalition S and the individual i, then the choice set of the new society, $S \cup \{i\}$, consists exactly of those alternatives that are chosen by both S and i. (M) thus reflects the idea that unanimity between a coalition and an individual should be respected. Stronger versions of (M) known as Reinforcement or Consistency have been proposed by several authors including Fine and Fine, Smith and Young[3, 4, 8, 11, 12] for different contexts.

(UR) says that if an alternative is not chosen by a coalition S and by an individual i, then this alternative cannot be in the choice set of the coalition formed by S and i. This axiom reflects again the idea of respecting unanimous choices made by individuals and coalitions.

Finally, (PA) states that if a society is formed by adding a new member to a coalition and the choice set by this individual has nothing in common with the choice set of the coalition, then the choice set of the new society must supersede the choices of the existing coalition. To a certain degree, (PA) gives a 'favorable' treatment to the choices of an existing coalition when a new member is added to this coalition when forming a new coalition.

With the help of the above axioms, we now state and prove our result, a characterization of approval voting in our framework.

Theorem 1 *An aggregation rule f is approval voting if and only if it satisfies (SD), (M), (UR) and (PA).*

Proof First, it can be checked easily that approval voting satisfies (SD), (M) and (UR). We now show that approval voting satisfies (PA) as well. Let $S \in \mathcal{K}$ and $i \in N \setminus S$, and suppose that $C_i(A) \cap C(S, A) = \emptyset$. We need to show that, if the aggregation rule is approval voting, then $C(S, A) \subseteq C(S \cup \{i\}, A)$. Since $x \in C(S, A)$, we have $N(x, S, A) \geq N(y, S, A)$ for all $y \in A$ and $N(x, S, A) \geq 1$. Note that $C_i(A) \cap C(S, A) = \emptyset$. It then follows that $N(x, S \cup \{i\}, A) \geq N(y, S \cup \{i\}, A)$

for all $y \in A$ implying that $x \in C(S \cup \{i\}, A)$. Therefore, (PA) is satisfied by approval voting.

Next, we show that, if an aggregation rule f satisfies (SD), (M), (UR) and (PA), then it must be approval voting. Let f satisfy (SD), (M), (UR) and (PA). Let $\alpha^N(A) \equiv \{C_1(A), \cdots, C_i(A), \ldots, C_n(A)\}$ be given. We shall use mathematical induction (on the number of individuals in a coalition) to show that,

$$\text{for all } S \in \mathcal{K},\ C(S, A) = \{x \in A : N(x, S, A) \geq N(y, S, A)\ \forall y \in A\} \qquad (*)$$

To begin with, note that, for all $i \in N$, by (SD), $C(\{i\}, A) = C_i(A)$ follows easily. Thus, (*) holds for any coalition $S \in \mathcal{K}$ with $\#S = 1$.

Suppose (*) holds for any coalition $S \in \mathcal{K}$ with $n > \#S = k \geq 1$. We next show that (*) holds for any $S \in \mathcal{K}$ with $\#S = k + 1$. Let $T \in \mathcal{K}$ be such that $T = S \cup \{j\}$, $j \in N \setminus S$, and $n > \#S = k \geq 1$. In what follows, we show that $C(T, A) = \{x \in A : N(x, T, A) \geq N(y, T, A)$ for all $y \in A\}$. Let $C^*(T, A) = \{x \in A : N(x, T, A) \geq N(y, T, A)$ for all $y \in A\}$.

Our first task is to show that $C(T, A) \subseteq C^*(T, A)$. Suppose to the contrary that $C(T, A) \nsubseteq C^*(T, A)$. Then, there exists $a \in A$ such that $[a \in C(T, A)$ and $a \notin C^*(T, A)]$. Since $a \notin C^*(T, A)$ and $C^*(T, A) \neq \emptyset$, it must be the case that $N(a, T, A) < N(z, T, A)$ for some $z \in C^*(T, A)$. It then follows that, for some $p \in T$, $z \in C_p(A)$ and $a \notin C_p(A)$. Consider the coalition $S' = T \setminus \{p\}$. Note that $\#S' = k$ and $C(S', A) = C^*(S', A)$ from the induction hypothesis. We consider two cases: (i) $a \in C(S', A)$ and $a \notin C(S', A)$. Case (i), $a \in C(S', A)$. Note that $[N(a, T, A) < N(z, T, A)$, $z \in C^*(T, A)$, $z \in C_p(A)$ and $a \notin C_p(A)]$. It then follows that $N(a, S', A) = N(z, S', A)$. Consequently, $z \in C(S', A)$. Noting that $C(S', A) \cap C_p(A) \neq \emptyset$, by (M), $C(T, A) = C(S', A) \cap C_p(A)$, implying that $a \notin C(T, A)$, a contradiction. Case (ii), $a \notin C(S', A)$. Note that $T = S' \cap \{p\}$ and $a \notin C_p(A)$. By (UR), $a \notin C(T, A)$, another contradiction. Therefore, $C(T, A) \subseteq C^*(T, A)$.

To complete the proof, we show that $C^*(T, A) \subseteq C(T, A)$. Let $x \in C^*(T, A)$. We consider two cases: case (i), $[x \in C_i(T)$ for all $i \in T]$; and case (ii), $x \notin C_q(A)$ for some $q \in T$. Case (i), $[x \in C_i(T)$ for all $i \in T]$. From induction hypothesis, $x \in C(S, A)$ where $S = T \setminus \{p\}$ and $p \in T$. Note that $x \in C_p(A)$. By (M), it then follows that $x \in C(T, A)$. Case (ii), $x \notin C_q(A)$ for some $q \in T$. Consider the coalition $S' = T \setminus \{q\}$. Note that $x \in C^*(T, A)$. It must be the case that $x \in C^*(S', A)$ implying that $N(x, S', A) \geq N(y, S', A)$ for all $y \in A$ and $x \in C(S', A)$, which follows from the induction hypothesis. Note that it must be true that $C(S', A) \cap C_q(A) = \emptyset$. This is because, if $C(S', A) \cap C_q(A) \neq \emptyset$, then, for some $z \in C(S', A)$, $z \in C_q(A)$, and consequently, $N(z, T, A) > N(x, T, A)$ follows from $x \notin C_q(A)$. Since $C(S', A) \cap C_q(A) = \emptyset$, by PA, $C(S', A) \subseteq C(T, A)$. Note that $x \in C(S', A)$. We then obtain that $x \in C(T, A)$. Therefore, $C^*(T, A) \subseteq C(T, A)$.

Thus, we have shown that $C(T, A) \subseteq C^*(T, A)$ and $C^*(T, A) \subseteq C(T, A)$. Therefore, $C(T, A) = C^*(T, A)$. Thus, (*) is established. This completes the proof. $\diamond$

Proposition 1 *The axioms figured in Theorem 1 are independent.*

Proof Let $x_0 \in A$ be given and let $C^*(\cdot, A)$ be the choice set given by approval voting. Consider the following aggregation rules:

f_1: for all $S \in \mathcal{K}$, $C_1(S, A) = A$
f_2: for all $S \in \mathcal{K}$, $C_2(S, A) = \bigcup_{i \in S} C_i(A)$
f_3: for all $S \in \mathcal{K}$, $C_3(S, A) = \begin{cases} A, & \text{if } C_i(A) \cap C_j(A) = \emptyset \text{ for all distinct } i, j \in S \\ C^*(S, A), & \text{if otherwise} \end{cases}$
f_4: for all $S \in \mathcal{K}$, if $S = \{i\}$ for some $i \in$ then $C_4(S, A) = C_i(A)$, and if $\#S \geq 2$, then ($C_4(S, A) = \{x_0\}$ if [$C_i(A) \cap C_j(A) = \emptyset$ for all distinct $i, j \in S$ and $x_0 \in C_i(A)$ for some $i \in S$], and $C_4(S, A) = C^*(S, A)$ if otherwise).

It can be checked that f_1 satisfies (M), (UR) and (PA) while violates (SD), f_2 satisfies (SD), (UR) and (PA) but violates (M), f_3 satisfies (SD), (M) and (PA) while violates (UR), and f_4 satisfies (SD), (M) and (UR) but violates (PA). $\diamond$

4 Concluding Remarks

In this paper, we have developed an alternative framework to study approval voting axiomatically. The main feature of our framework is that we work with a single-profile choice functions and variable societies. In such a framework, we have studied approval voting from the perspective that links the choices of a society to the choices of its sub-societies. To put our contribution in perspective, we locate our contribution to the literature by grouping various characterizations of approval voting into the following categories:

1. Variable societies and multi profile of preferences: Fishburn [5, 6] , Sertel [7].
2. Fixed society and multi profile of choice functions: Baigent and Xu [1].
3. Variable societies and a single profile of choice functions: this paper.

Acknowledgements This paper is dedicated to Nick Baigent. Y. Xu would like to thank Nick Baigent for his mentoring and support. An earlier version of the paper was presented at the CEPET Workshop in Honour of Nick Baigent, Udine, Italy, June 2–4, 2010, and we would like to thank the audience for questions and comments. We are also grateful to two referees for their helpful comments.

References

1. Baigent N, Xu Y (1991) Independent, necessary and sufficient conditions for approval voting. Math Soc Sci 21:21–29
2. Brams S, Fishburn P (1978) Approval voting. Am Polit Sci Rev 72(3):831–847
3. Fine B, Fine K (1974) Social choice and individual ranking I. Rev Econ Stud 41:303–322
4. Fine B, Fine K (1974) Social choice and individual ranking II. Rev Econ Stud 41:459–475

5. Fishburn P (1978) Axioms for approval voting: direct proof. J Econ Theory 19:180–185
6. Fishburn P (1978) Symmetric and consistent aggregation with dichotomous preferences. In: Laffont J (ed) Aggregation and revelation of preferences. North- Holland, Amsterdam
7. Sertel M (1988) Characterizing approval voting. J Econ Theory 45:207–211
8. Smith J (1973) Aggregation of preferences with variable electorate. Econometrica 41:1027–1041
9. Xu Y (2010) Axiomatizations of approval voting, Chap 5. In: Laslier J-F, Remzi Sanver M (eds) Handbook on approval voting. Springer, New York
10. Xu Y, Zhong Z (2010) Single profile of preferences with variable societies: a characterization of simple majority rule. Econ Lett 107:119–121
11. Young HP (1974) An axiomatization of Borda's rule. J Econ Theory 9:43–52
12. Young HP (1975) Social choice scoring functions. SIAM J App Math 28:824–838