A Note on Incompleteness, Transitivity and Suzumura Consistency

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Abstract Rationality does not require of preferences that they be complete. Nor therefore that they be transitive: Suzumura consistency suffices. This paper examines the implications of these claims for the theory of rational choice. I propose a new choice rule—Strong Maximality—and argue that it better captures rational preference-based choice than other more familiar rules. Suzumura consistency of preferences is shown to be both necessary and sufficient for non-empty strongly maximal choice. Finally conditions on a choice function are stated that are necessary and sufficient for it to be rationalisable in terms of a Suzumura consistent preference relation.

Keywords Choice function • Incomplete preferences • Rationalisability • Suzumura consistency • Transitivity

1 Preference and Choice

This note concerns two questions about reason-based choice: What is required of the agent who makes her choices on the basis of her preferences? What can be inferred about an agent's preferences from the choices she makes? On both of these questions, I have learnt a great deal from Nick Baigent: from his writings, of course, but even more from discussions with him. If I could achieve even a small fraction of the clarity that he does when addressing these topics, I would be very happy indeed.

When thinking about the relation between preference and choice, it is worth distinguishing between the choices that are *permissible* given the agent's preferences, those that are *mandatory* and those that she *actually* makes. Rationality does not generally require that agents have strict preferences over all alternatives, so it is to be expected that these sets of choices will not coincide. For instance if she is indifferent between two alternatives or is unable to compare them it might be permissible for

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her to choose both of them, not mandatory to pick either, while in fact choosing only one of them.

There are two implications of this point. Firstly, preference-based explanations and/or rationalisations are necessarily limited in scope. Invoking someone's preferences will suffice to explain why some choices were not made (i.e. in terms of rational impermissibility) but not typically why some particular choice was made. To take up the slack, explanations must draw on factors other than preference: psychological ones such as the framing of the choice problem or the saliency of particular options, or sociological ones such as the existence of norms or conventions governing choices of the relevant kind. Some work has been done on how to rationalise choice when it has more than one determinant (see, for instance, Baigent [2]), but in general it is an insufficiently studied problem.

Secondly, observations of actual choices will only partially constrain preference attribution. For instance, that someone chooses a banana when an apple is available does not allow one to conclude that the choice of an apple was ruled out by her preferences, only that her preferences ruled the banana in. In this simple observation lies a serious obstacle to the ambition of Revealed Preference theory to give conditions on observed choices sufficient for the existence of a preference relation that rationalises them. For the usual practice of inferring the completeness of the agent's preferences from the fact that she always makes a choice when required to is clearly illegitimate if more than one choice is permitted by her preferences.

The upshot is that the usual focus on the case where an agent has complete preferences is quite unjustified. The aim of this note is therefore to explore the two opening questions without assuming completeness, building on the work of Sen [9], Richter [7] and especially the recent work of Bossert and Suzumura [4, 5]. I argue that when incompleteness of preference is reasonable then rationality does not require full transitivity of preferences. Instead it requires it that they be Suzumura consistent—roughly that there be no cycles of weak preference containing a strong preference. In a similar vein I argue for a choice rule—Strong Maximality—that is roughly intermediate between optimisation and maximisation and show that Suzumura consistency of preference is sufficient to ensure that this choice rule picks a non-empty set of alternatives from any given non-empty set of them. Finally, I investigate the rationalisability of choice functions in terms of Suzumura consistent preferences and strong maximal choice.

1.1 Preference

In the usual fashion we introduce a reflexive binary relation $\succeq$ (called the weak preference relation) on a set of alternatives X, with symmetric part $\approx$ (indifference) and anti-symmetric part $\succ$ (strict preference). In contrast to the way these terms are often used, we do not assume that in general any two alternatives are comparable under these preference relations. Instead we define a comparability relation $\bowtie$ on alternatives by: $\alpha \bowtie \beta$ iff $\alpha \succeq \beta$ or $\beta \succeq \alpha$. When all alternatives are comparable the preference relation is said to be complete. (Hence it is incomplete iff there is a pair of alternatives α and β such that $\alpha \not\bowtie \beta$.)

1.1.1 Transitivity

A number of different forms of transitivity-like properties of preference relations will be of interest. We say that $\succeq$ is:

1. *Transitive* iff for all $\alpha, \beta, \gamma \in X$, $\alpha \succeq \beta$ and $\beta \succeq \gamma$ implies that $\alpha \succeq \gamma$ (and intransitive otherwise)
2. *Incompletely transitive* iff for all $\alpha, \beta, \gamma \in X$, $\alpha \succeq \beta$ and $\beta \succeq \gamma$ implies that $\gamma \not\succ \alpha$
3. *PI-transitive* iff for all $\alpha, \beta, \gamma \in X$, $\alpha \succ \beta$ and $\beta \approx \gamma$ implies that $\alpha \succ \gamma$
4. *Quasi-transitive* iff $\succ$ is transitive

Transitivity implies incomplete transitivity, PI-transitivity and quasi-transitivity. On the other hand, a reflexive relation is transitive if it is either both complete and incompletely transitive or both PI-transitive and quasi-transitive [8, Theorem I.6]. But in general a relation can be incompletely transitive without being PI-transitive or quasi-transitive, and vice versa: they constitute alternative weakenings of transitivity.

The view taken here is that completeness is not a rationality requirement on preference. This is not in itself very controversial. Much more so is something that follows rather naturally from this view, namely that transitivity is too strong a requirement to impose on preferences. The problem is that transitivity imposes comparability even when it is not appropriate to do so. The following example serves to illustrate this point.

Suppose that Ann, Bob and Carol have interval scores in Maths and English as follows:

- (Ann) Maths: 80–90, English: 60–70
- (Bob) Maths: 56–65, English: 66–75
- (Carol) Maths: 75–85, English: 55–65

The teacher decides to rank them in each subject using the heuristic that two students with overlapping intervals scores in a subject should be regarded as on a par in that subject, but one is ranked higher than the other if the lower bound of their interval score is greater than the upper bound of the interval score of the other. So Ann ranks higher than Bob because she is definitely better at Maths and not comparably worse at English, Bob and Carol are ranked the same because each is better at one of the subjects and Ann and Carol are unranked relative to each other because neither is comparably better than the other in either subject.

The teacher's ranking of her students does not satisfy transitivity, but it is not obvious that her ranking is irrational given her inability to discriminate between Ann and Carol on the basis of their performances. It is not that the teacher should not infer that Ann is better than Carol, but rather that she is not rationally *compelled* to do so. This suggest that in situations in which a preference relation is not complete the requirements of rationality (with regard to preferences between pairs of a triple of alternatives) are more appropriately expressed by the condition of incomplete transitivity, than by that of full transitivity.

1.1.2 Consistency

In addition to the basic conditions on preference listed above, which are defined in terms of pairs or triples of alternatives, we are also interested in a number of derived consistency properties of the preference relation that can be defined in terms of these basic ones.

The weak preference relation $\succeq$ will be said to be:

1. *Strongly consistent* iff $\succeq$ is transitive
2. *Suzumura consistent* iff for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in X, \alpha_1 \succeq \alpha_2, \alpha_2 \succeq \alpha_3, \ldots, \alpha_{n-1} \succeq \alpha_n$ implies that $\alpha_n \not\succ \alpha_1$
3. *Weakly consistent* iff $\succ$ is acyclic iff for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in X$, $\alpha_1 \succ \alpha_2, \alpha_2 \succ \alpha_3, \ldots, \alpha_{n-1} \succ \alpha_n$ implies that $\alpha_n \not\succ \alpha_1$

These properties are in descending order of strength: strong consistency implies Suzumura consistency which implies weak consistency. Suzumura consistency strengthens incomplete transitivity, by extending it to arbitrary sets of alternatives.[1] As Bossert and Suzumura [4] point out there are three notable characteristics of Suzumura consistency. Firstly it rules out cycles with at least one strict preference and so preferences that satisfy it are not vulnerable to money pumps. Secondly, Suzumura consistency of a weak preference relation is necessary and sufficient for the existence of a complete and transitive extension of it.[2] And thirdly, any preference relation that is both Suzumura consistent and complete is strongly consistent. So there is good reason to think of Suzumura consistency as being the appropriate consistency condition for incomplete preferences.

2 Preference-Based Choice

Let $\mathcal{C}$ be a choice function on $\wp(X) - \emptyset$: a mapping from non-empty subsets $A \subseteq X$ to subsets $\mathcal{C}(A) \subseteq A$. Intuitively $\mathcal{C}(A)$ is the set of objects from the set A that *could* be chosen: could permissibly be so in normative interpretations, could factually be so in descriptive ones. If $\mathcal{C}(A)$ is always non-empty then it is said to be *decisive*. (Decisiveness is often built into the definition of a choice function, but it will prove more convenient here to make it a separate assumption.)

We are especially interested in the case when a choice function $\mathcal{C}$ can be said to be based on or determined by a preference relation. A natural condition for this being the case is that an object is chosen from a set only if no other object in the set is strictly preferred to it. Formally:

SPBC: (Strict Preference Based Choice)

$$\alpha \succ \beta \Rightarrow \forall(A : \alpha \in A), \beta \notin \mathcal{C}(A)$$

[1] The concept of Suzumura consistency was introduced in Suzumura [11].

[2] See Suzumura [11] for a proof.

SPBC certainly seems necessary for preference-based choice. But is it sufficient? I think not. A further requirement is that two alternatives that are regarded indifferently should always either both be chosen or both not chosen. Formally:

IBC: (Indifference Based Choice)

$$\alpha \approx \beta \Rightarrow \forall (A : \alpha, \beta \in A), \alpha \in \mathcal{C}(A) \Leftrightarrow \beta \in \mathcal{C}(A)$$

SPBC and IBC are not generally sufficient to determine choice because they don't settle the question of how to handle incomparability. Let us therefore consider three possible preference-based rules of choice that do fully determine what may be chosen and consider how they relate to these conditions. To do so it is useful to consider the transitive closure of $\approx$ on $A \subseteq X$, denoted $\hat{\approx}^A$ and defined by, for all $\alpha, \beta, \gamma \in A$: (1) $\alpha \hat{\approx}^A \alpha$, and (2) if $\alpha \hat{\approx}^A \beta$ and $\gamma \approx \beta$ then $\alpha \hat{\approx}^A \gamma$. Note that if $\alpha \hat{\approx}^A \beta$ then there exists a sequence of elements in X, $\alpha_1, \alpha_2, \ldots, \alpha_n$ linking α and β in the sense that $\alpha \approx \alpha_1$, $\alpha_1 \approx \alpha_2$, …, and $\alpha_n \approx \beta$. It follows that $\hat{\approx}^A$ is transitive and symmetric and hence an equivalence relation on A. We call the set of $\beta \in A$ such that $\alpha \hat{\approx}^A \beta$, the indifference class of α in A.

The three rules of interest are the following:

Optimality: An object is chosen from a set if and only if it is weakly preferred to all others in the set. Formally, for all A such that $\alpha \in A$:

$$\alpha \in \mathcal{C}(A) \Leftrightarrow \forall (\beta \in A), \alpha \succeq \beta$$

Maximality: An object is chosen from a set if and only if no alternative in the set is strictly preferred to it. Formally, for all A such that $\alpha \in A$:

$$\alpha \in \mathcal{C}(A) \Leftrightarrow \neg \exists (\beta \in A : \beta \succ \alpha)$$

Strong Maximality: An object α is chosen from a set A iff there is no alternative in A strictly preferred to any alternative in α's indifference class in A. Formally, for all A such that $\alpha \in A$:

$$\alpha \in \mathcal{C}(A) \Leftrightarrow \neg \exists (\beta, \gamma \in A : \alpha \hat{\approx}^A \gamma \text{ and } \beta \succ \gamma)$$

Of these three rules, Optimality is the one that is most commonly taken to express rational preference-based choice (see, for instance, Arrow [1] and Sen [9]). But although Optimality satisfies both SPBC and IBC, it is clearly too strong a condition on *permissible* choice. This is because it implies that if $\alpha \not\bowtie \beta$ then $\mathcal{C}(\{\alpha, \beta\}) = \varnothing$. But even if there are situations in which no choice is permissible (contrary to the usual assumption of decisiveness), this is not a consequence of incomparability. If two alternatives are incomparable it should normally be permissible to choose either of them.

For this reason Maximality is often seen as the more appropriate rule of rational choice when the possibility of incomparability is not ruled out (see Sen [10]).

But Maximality is also not quite right, as the following schematic version of our earlier example shows. Suppose that $\alpha \succ \beta$ and $\beta \approx \gamma$ but $\alpha \not\bowtie \gamma$. Then it would not be unreasonable for $\mathcal{C}(\{\alpha, \gamma\}) = \{\alpha, \gamma\}$ because the two alternatives are incomparable and $\mathcal{C}(\{\beta, \gamma\}) = \{\beta, \gamma\}$ because the two alternatives are equally preferred, but $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\alpha\}$ because β should not be chosen when a strictly preferred alternative—α—is available and γ should not be chosen if β is not, given that $\gamma \approx \beta$. But these choices are inconsistent with Maximality which requires that $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\alpha, \gamma\}$.

The problem with Maximality is that it leads to violations of IBC. Since Maximality requires that $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\alpha, \gamma\}$, it is not the case that β is chosen whenever γ is, even though $\beta \approx \gamma$. So just as admitting the possibility of incompleteness required a shift from Optimality to Maximality, so too recognition of the rational permissibility of incompletely transitive preferences requires a shift from Maximality to Strong Maximality.

Let us consider a reformulation of Strong Maximality that will make its implications clearer. For any $A \subseteq X$ let $\mathbf{A} = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots\}$ be the set of equivalence classes in A induced by the relation $\hat{\approx}^A$. Define a weak preference relation $\geq$ on $\mathbf{A}$ by $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{A}$:

$$\boldsymbol{\alpha} \geq \boldsymbol{\beta} \Leftrightarrow \exists(\alpha \in \boldsymbol{\alpha}, \beta \in \boldsymbol{\beta} : \alpha \succeq \beta)$$

Then choosing from any A in accordance with Strong Maximality is equivalent to choosing the $\geq$-maximal element of the set $\mathbf{A}$ of equivalence classes in A induced by the equivalence relation $\hat{\approx}^A$.

Now it might be objected that adopting Strong Maximality as a principle of rational choice is tantamount to smuggling transitivity of indifference back in via the equivalence classes under $\hat{\approx}^A$. But there is another way for formulating the rule which should serve to alleviate this worry. Let us define a sequence of choice functions $\langle \bar{\mathcal{C}}^{\tau}_{\succeq}(A) \rangle_{\tau=1}^{\infty}$ as follows[3]:

1. $\bar{\mathcal{C}}^{0}_{\succeq}(A) = \{\alpha \in A : \exists \beta \in A \text{ such that } \beta \succ \alpha\}$
2. $\bar{\mathcal{C}}^{\tau}_{\succeq}(A) = \{\alpha \in A : \exists \beta \in A \text{ such that } \beta \approx \alpha \text{ and } \beta \in \bar{\mathcal{C}}^{\tau-1}_{\succeq}(A)\}$

Then we define the set of impermissible alternatives by:

$$\bar{\mathcal{C}}_{\succeq}(A) = \bigcup_{\tau=0}^{\infty} \bar{\mathcal{C}}^{\tau}_{\succeq}(A)$$

Intuitively $\bar{\mathcal{C}}_{\succeq}(A)$ is the set of alternatives in A that must not be chosen. Then Strong Maximality is equivalent to the rule of choosing any alternative that is not impermissible, i.e. to the rule:

Non-Elimination: $\alpha \in \mathcal{C}(A) \Leftrightarrow \alpha \notin \bar{\mathcal{C}}(A)$

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[3] I am grateful to an anonymous referee for suggesting this formulation.

To apply this rule it suffices that the agent iteratively eliminates alternatives from her choice set by removing any dominated alternatives; then checking if any alternatives that are left are indifferent to any eliminated ones and, if so, removing them as well; then checking if any alternatives that are left are indifferent to any eliminated ones, and so on.

2.1 Properties of Preference-Based Choice

Each of the three choice rules under examination expresses a view on the relationship between preference and choice. To examine what these are and how they differ for the three choice rules, let us denote the choice function determined by the weak preference relation $\succeq$ together with Maximality, Optimality or Strong Maximality by $\mathcal{C}^{Max}_{\succeq}$, $\mathcal{C}^{Op}_{\succeq}$ and $\mathcal{C}^{SM}_{\succeq}$ respectively, where these are defined as follows. For any $A \subseteq X$:

$$\mathcal{C}^{Op}_{\succeq}(A) = \{\alpha \in A : \forall \beta \in A, \alpha \succeq \beta\}$$

$$\mathcal{C}^{Max}_{\succeq}(A) = \{\alpha \in A : \forall \beta \in A, \beta \not\succ \alpha\}$$

$$\mathcal{C}^{SM}_{\succeq}(A) = \{\alpha \in A : \forall(\gamma \in A : \gamma \gtrapprox^{A} \alpha), \neg\exists(\beta \in A : \beta \succ \gamma)\}$$

For the rest of this section, I will drop the subscript on the choice function as the preference relation is fixed throughout the discussion.

The first thing to note is that the set of permissible choices according to $\mathcal{C}^{Max}$ is always at least as large as those determined by $\mathcal{C}^{Op}$ or $\mathcal{C}^{SM}$. Furthermore when the preference relation is Suzumura consistent then the set of choices that are permissible according to Strong Maximality contain those that are permissible according to Optimality (as well as being contained by those determined by Maximality). On the other hand when the preference relation is complete $\mathcal{C}^{Op}$ coincides with $\mathcal{C}^{Max}$ and when it is transitive, $\mathcal{C}^{SM}$ coincides with $\mathcal{C}^{Max}$. These relationships are summarised in Fig. 1, where arrows indicate implications given the indicated conditions, and proven below as Theorem 1.

It is well known that for finite sets of alternatives $\mathcal{C}^{Op}$ is decisive iff $\succeq$ is complete and weakly consistent and that $\mathcal{C}^{Max}$ is decisive iff $\succeq$ is weakly consistent. Theorem 2 below establishes a corresponding result for choices that are strongly maximal, namely that a choice function based on Strong Maximality is decisive iff the underlying preference relation is Suzumura consistent. The main significance of this result for our argument is that Suzumura consistency is thereby shown to be both necessary and sufficient for decisive, strongly maximal choice.

Theorem 1

1. *$\mathcal{C}^{Op} \subseteq \mathcal{C}^{Max}$ and $\mathcal{C}^{SM} \subseteq \mathcal{C}^{Max}$*
2. *If $\succeq$ is complete then $\mathcal{C}^{Op} = \mathcal{C}^{Max}$*

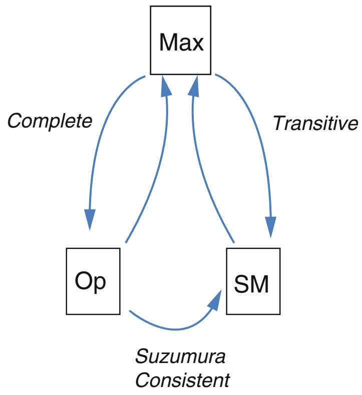

Fig. 1 Relations between choice rules

3. *If* $\succeq$ *is transitive, then* $\mathcal{C}^{SM} = \mathcal{C}^{Max}$
4. *If* $\succeq$ *is Suzumura consistent, then* $\mathcal{C}^{Op} \subseteq \mathcal{C}^{SM}$.
5. *If* $\succeq$ *is complete and Suzumura consistent then* $\mathcal{C}^{SM} = \mathcal{C}^{Op} = \mathcal{C}^{Max}$

Proof

(1) Suppose $\alpha \in \mathcal{C}^{Op}(A)$. Then $\forall \beta \in A, \alpha \succeq \beta$. But then $\forall \beta \in A$, $\beta \not\succ \alpha$. So $\alpha \in \mathcal{C}^{Max}(A)$. Similarly suppose $\alpha \in \mathcal{C}^{SM}(A)$. Now by the symmetry of indifference $\alpha \approx \alpha$, so it follows that $\neg\exists\beta \in A$ such that $\beta \succ \alpha$). So $\alpha \in \mathcal{C}^{Max}(A)$.

(2) Suppose that $\succeq$ is complete. Then for any $\beta \in A$ if $\beta \not\succ \alpha$ then $\alpha \succeq \beta$. Hence if $\alpha \in \mathcal{C}^{Op}(A)$ then $\alpha \in \mathcal{C}^{Max}(A)$. So $\mathcal{C}^{Op} = \mathcal{C}^{Max}$.

(3) Suppose that $\succeq$ is transitive but that there exists $\alpha \in A$ such that $\alpha \in \mathcal{C}^{Max}(A)$ but $\alpha \notin \mathcal{C}^{SM}(A)$. Now if $\alpha \notin \mathcal{C}^{SM}(A)$ then there exists $\beta, \gamma \in A$ such that $\alpha \hat{\approx}^A \gamma$ and $\beta \succ \gamma$. By transitivity, if $\alpha \hat{\approx}^A \gamma$ then $\alpha \approx \gamma$ and so by transitivity again, $\beta \succeq \alpha$. But if $\alpha \in \mathcal{C}^{Max}(A)$ then $\beta \not\succ \alpha$. So $\beta \approx \alpha$ and by transitivity, $\beta \approx \gamma$. Hence, contrary to assumption, $\beta \not\succ \gamma$. It follows that if $\alpha \in \mathcal{C}^{Max}(A)$ then $\alpha \in \mathcal{C}^{SM}(A)$ and hence that $\mathcal{C}^{SM} = \mathcal{C}^{Max}$.

(4) Suppose that $\succeq$ is Suzumura consistent and that $\alpha \in \mathcal{C}^{Op}(A)$. Then $\forall \beta \in A$, $\alpha \succeq \beta$. Let $\gamma \in A$ be such that $\alpha \hat{\approx}^A \gamma$. Then there exists a sequence of elements in A, $\alpha_1, \alpha_2, \ldots, \alpha_n$ linking γ, α and β in the sense that $\gamma \approx \alpha_1, \alpha_1 \approx \alpha_2, \ldots$, $\alpha_n \approx \alpha$ and $\alpha \succeq \beta$. Hence by Suzumura consistency $\beta \not\succ \gamma$. It follows that $\alpha \in \mathcal{C}^{SM}(A)$.

(5) Follows from 2, 3 and 4. ■

Theorem 2 *Suppose that the set of alternatives X is finite. Then:*

1. $\mathcal{C}^{Op}$ *is decisive iff* $\succeq$ *is complete and weakly consistent*
2. $\mathcal{C}^{Max}$ *is decisive iff* $\succeq$ *is weakly consistent*
3. $\mathcal{C}^{SM}$ *is decisive iff* $\succeq$ *is Suzumura consistent*

Proof (2) Suppose $\succeq$ is not weakly consistent. Then there exists $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq X$, such that $\alpha_0 \succeq \alpha_1, \alpha_1 \succeq \alpha_2, \ldots, \alpha_{n-1} \succeq \alpha_n$ and $\alpha_n \succ \alpha_0$. But then for $n \geq i \geq 1$, $\alpha_i \notin \mathcal{C}^{Max}(A)$ because $\alpha_{i-1} \succ \alpha_i$. And

$\alpha_0 \notin \mathcal{C}^{Max}(A)$ because $\alpha_n \succ \alpha_0$. So $\mathcal{C}^{Max}(A) = \varnothing$. Hence $\mathcal{C}^{Max}$ is not decisive. For the converse see Kreps [6].

(1) Suppose $\succeq$ is either not weakly consistent or incomplete. Suppose it is not weakly consistent. Then since by Theorem 1(1), $\mathcal{C}^{Op} \subseteq \mathcal{C}^{Max}$ it follows from 2. that $\mathcal{C}^{Op}(A) \neq \varnothing$. Now suppose that $\succeq$ is incomplete. Then there exists $\alpha, \beta \in X$ such that $\alpha \not\bowtie \beta$ and hence such that $\mathcal{C}^{Op}(\{\alpha, \beta\}) = \varnothing$. So $\mathcal{C}^{Op}$ is not decisive. The converse follows from 2. and Theorem 1(2).

(3) Suppose that $\mathcal{C}^{SM}$ is decisive but that $\succeq$ is not Suzumura consistent, i.e. for some set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ it is the case that $\alpha_1 \succeq \alpha_2, \alpha_2 \succeq \alpha_3, \ldots, \alpha_{n-1} \succeq \alpha_n$ but that $\alpha_n \succ \alpha_1$. We prove by induction on i that it then follows that for all $\alpha_i \in A$, $\alpha_i \notin \mathcal{C}^{SM}(A)$ and hence that $\mathcal{C}^{SM}(A) = \varnothing$. First Strong Maximality implies that $\alpha_1 \notin \mathcal{C}^{SM}(A)$ because $\alpha_n \succ \alpha_1$. Now assume that for some $k > 1$, $\alpha_k \notin \mathcal{C}^{SM}(A)$. Then there exists some α_j and $\alpha_{j'}$ such that $\alpha_k \hat{\approx}^A \alpha_j$ but $\alpha_{j'} \succ \alpha_j$. Now consider α_{k+1}. Either $\alpha_k \succ \alpha_{k+1}$ in which case it follows by Strong Maximality that $\alpha_{k+1} \notin \mathcal{C}^{SM}(A)$. Or $\alpha_k \approx \alpha_{k+1}$, in which case $\alpha_k \hat{\approx}^A \alpha_{k+1}$. But then $\alpha_{k+1} \hat{\approx}^A \alpha_j$ and so by Strong Maximality $\alpha_{k+1} \notin \mathcal{C}^{SM}(A)$. But this implies that $\mathcal{C}^{SM}(A) = \varnothing$ in contradiction to the assumption of decisiveness. So $\succeq$ must be Suzumura consistent.

For the other direction, suppose that $\succeq$ is Suzumura consistent, but that for some set $A, \mathcal{C}^{SM}(A) = \varnothing$. If $\mathcal{C}^{SM}(A) = \varnothing$ then $\mathcal{C}^{Max}_{\geq}(\mathbf{A}) = \varnothing$ and so by (2), $>$ is cyclic i.e. there exist subsets of A such that $\mathbf{A}_1 > \mathbf{A}_2, \mathbf{A}_2 > \mathbf{A}_3 \ldots, \mathbf{A}_{n-1} > \mathbf{A}_n$, and $\mathbf{A}_n > \mathbf{A}_1$. So by definition there exists $\alpha_1, \alpha'_1, \alpha_2, \ldots, \alpha'_n, \alpha_n \in A$ such that $\alpha_1 \hat{\approx}^A \alpha'_1, \ldots, \alpha'_n \hat{\approx}^A \alpha_n$ and $\alpha'_1 \succ \alpha_2, \alpha'_2 \succ \alpha_3, \ldots, \alpha'_{n-1} \succ \alpha_n$, but $\alpha'_n \succ \alpha_1$. But by Suzumura consistency, if $\alpha_1 \hat{\approx}^A \alpha'_1, \alpha'_1 \succ \alpha_2$, $\alpha_2 \hat{\approx}^A \alpha'_2, \alpha'_2 \succ \alpha_3, \ldots, \alpha'_{n-1} \succ \alpha_n$ and $\alpha_n \hat{\approx}^A \alpha'_n$ then $\alpha'_n \not\succ \alpha_1$. So $\mathcal{C}^{Max}_{\geq}(\mathbf{A}) \neq \varnothing$. Hence $\mathcal{C}^{SM}$ is decisive. ■

2.2 Properties of Choice Functions

What features of choice functions are induced by our choice rules? The following properties—Sen's alpha, beta and gamma conditions—have figured prominently in the existing literature. Let $\alpha \in A$ and $\gamma \in C$. Then:

Set Contraction: If $\alpha \in \mathcal{C}(B)$ and $A \subseteq B$ then $\alpha \in \mathcal{C}(A)$
Set Expansion: If $\alpha, \beta \in \mathcal{C}(B)$, $B \subseteq A$ and $\beta \in \mathcal{C}(A)$, then $\alpha \in \mathcal{C}(A)$
Set Union: If $\alpha \in \mathcal{C}(A)$ and $\alpha \in \mathcal{C}(B)$, then $\alpha \in \mathcal{C}(A \cup B)$

It is well known that Optimality-based choice will satisfy both Set Contraction and Set Expansion so long as the underlying weak preference relation is weakly consistent (see Sen [9]). In fact choice based on weakly consistent preferences will satisfy Set Contraction given any one of the three choice rules under examination. Set Expansion on the other hand need not be satisfied by maximal or strongly maximal choice. This is as it should be. Suppose, for example, that the agent cannot compare α and β, but that no alternative in A is preferred to either. So both are permissible choices. Now suppose that $B = A \cup \{\gamma\}$ and that $\gamma \succ \alpha$ but $\gamma \not\vartriangleright\!\!\!\vartriangleleft \beta$. Then β is a still permissible choice but not α. So Set Expansion is violated.

More interesting perhaps is that $\mathcal{C}^{SM}$, unlike the other two rules, does not satisfy Set Union, for Sen [9] has shown that satisfaction of this condition along with Set Contraction is essentially equivalent to the choice function being binary in composition. Instead it is both necessary and sufficient that strongly maximal choices be based on PI-transitive preferences for $\mathcal{C}^{SM}$ to satisfy Set Union.

If $\mathcal{C}^{SM}$ does not generally satisfy Set Union, what properties are characteristic of it? Two weaker principles than Set Union turn out to be significant. The first is a very weak consequence of Set Union.

Element Union: If $\forall\alpha \in A, \alpha \in \mathcal{C}(\{\alpha, \beta\})$ then for some $\alpha^* \in A, \alpha^* \in \mathcal{C}(A \cup \{\beta\})$

To state the second condition, we need to introduce a choice theoretic analogue of the notion of the indifference class of an alternative α in some set A—called α's choice equivalence class. Intuitively the choice equivalence class of α is the set of elements that are chosen whenever α is, in any set containing both. To state it formally, we first define a sequence of functions $\langle\tilde{\mathcal{C}}^\tau(A, \alpha)\rangle_{\tau=1}^{\infty}$, induced by a given choice function $\mathcal{C}$, as follows:

1. $\tilde{\mathcal{C}}^0(A) = \{\alpha\}$
2. $\tilde{\mathcal{C}}^\tau(A) = \{\beta \in A$: For some $\gamma \in \tilde{\mathcal{C}}^{\tau-1}(A, \alpha)$, $\beta \in \mathcal{C}(B) \Leftrightarrow \gamma \in \mathcal{C}(B)$ for all $B \subseteq X$ such that $\beta, \gamma \in B\}$

Then we define α's *choice equivalence class* in A induced by $\mathcal{C}$, as follows:

$$\tilde{\mathcal{C}}(A, \alpha) = \bigcup_{\tau=0}^{\infty} \tilde{\mathcal{C}}^\tau(A)$$

Note that if $\beta \stackrel{\approx}{\approx}^A \alpha$ then $\beta \in \tilde{C}^{SM}(A, \alpha)$. For if $\beta \stackrel{\approx}{\approx}^A \alpha$, then there exists a sequence of elements in A, α_1, α_2, ..., α_n linking α and β in the sense that $\alpha = \alpha_1, \alpha_1 \approx \alpha_2, \ldots, \alpha_{n-1} \approx \alpha_n$, and $\alpha_n = \beta$. And for all α_i in this sequence, $\alpha_i \in \mathcal{C}^{SM}(B) \Leftrightarrow \alpha_{i+1} \in \mathcal{C}^{SM}(B)$ for all $B \subseteq X$ such that $\alpha_i, \alpha_{i+1} \in B$. Since $\alpha \in \tilde{C}^{SM}(A, \alpha)$, it follows that $\alpha_2 \in \tilde{C}^{SM}(A, \alpha)$ and hence $\alpha_3 \in \tilde{C}^{SM}(A, \alpha)$ and hence ... $\beta \in \tilde{C}^{SM}(A, \alpha)$.

Now we can state the final condition of interest:

Equivalence Class Union If $\tilde{\mathcal{C}}(A \cup B, \alpha) \subseteq \mathcal{C}(A)$ and $\tilde{\mathcal{C}}(A \cup B, \alpha) \subseteq \mathcal{C}(B)$, then $\tilde{\mathcal{C}}(A \cup B, \alpha) \subseteq \mathcal{C}(A \cup B)$

This condition, like Element Union, *is* satisfied by choice in accordance with Strong Maximality.

Theorem 3

1. $C^{Op}_{\succsim}/\mathcal{C}^{Max}/\mathcal{C}^{SM}$ *all satisfy Set Contraction*
2. $\mathcal{C}^{Op}$ *and* $\mathcal{C}^{Max}$ *satisfy Set Union, but* $\mathcal{C}^{SM}$ *need not.*

3. If $\mathcal{C}^{SM}$ is decisive then $\mathcal{C}^{SM}$ satisfies Set Union iff $\succeq$ is PI-transitive
4. $\mathcal{C}^{SM}$ satisfies Element Union and Equivalence Class Union

Proof

(1) Suppose $B \subseteq A$ and $\alpha \in \mathcal{C}^{SM}(A)$. Then $\forall \beta, \gamma \in A$ such that $\alpha \hat{\approx}^A \gamma$, $\beta \not\succ \gamma$. Hence $\forall \beta, \gamma \in B$ such that $\alpha \hat{\approx}^A \gamma$, $\beta \not\succ \gamma$. So $\alpha \in \mathcal{C}^{SM}(B)$. Similarly for $\mathcal{C}^{Op}$ and $\mathcal{C}^{Max}$.
(2) Suppose that $\alpha \in \mathcal{C}^{Max}(A)$ and $\alpha \in \mathcal{C}^{Max}(B)$. Then $\forall \beta \in A$, $\beta \not\succ \alpha$ and $\forall \beta \in B$, $\beta \not\succ \alpha$. Hence $\forall \beta \in A \cup B$, $\beta \not\succ \alpha$. It follows that $\alpha \in \mathcal{C}^{Max}(A \cup B)$. Similarly for $\mathcal{C}^{Op}$. However consider a case in which $\alpha \not\bowtie \beta$, $\alpha \approx \gamma$ and $\beta \succ \gamma$. Then $\mathcal{C}^{SM}(\{\alpha, \beta\}) = \{\alpha, \beta\}$, $\mathcal{C}^{SM}(\{\alpha, \gamma\}) = \{\alpha, \gamma\}$ but $\alpha \notin \mathcal{C}^{SM}(\{\alpha, \beta, \gamma\})$ because $\alpha \hat{\approx}^A \gamma$ and $\beta \succ \gamma$.
(3) Suppose $\succeq$ is PI-transitive, that $\alpha \in \mathcal{C}^{SM}(A)$ and that $\alpha \in \mathcal{C}^{SM}(B)$, but that $\alpha \notin \mathcal{C}^{SM}(A \cup B)$. Then there exists $\beta, \gamma \in A \cup B$ such that $\alpha \hat{\approx}^A \gamma$ and $\beta \succ \gamma$. But then by repeated applications of PI-transitivity it follows that $\beta \succ \alpha$. Hence $\alpha \notin \mathcal{C}^{SM}(A)$ or $\alpha \notin \mathcal{C}^{SM}(B)$, depending on whether $\beta \in A$ or $\beta \in B$. Now suppose that $\succeq$ is not PI-transitive. Then there exists $\alpha, \beta, \gamma \in X$ such that $\alpha \succ \beta$ and $\beta \approx \gamma$ but $\alpha \not\succ \gamma$. Then either $\gamma \succ \alpha$, $\alpha \approx \gamma$ or $\alpha \not\bowtie \gamma$. Suppose $\gamma \succ \alpha$ or $\alpha \approx \gamma$. Then $\mathcal{C}^{SM}(\{\alpha, \beta, \gamma\}) = \emptyset$, contrary to the assumption that $\mathcal{C}^{SM}$ is decisive. So suppose that $\alpha \not\bowtie \gamma$. Then $\gamma \in \mathcal{C}^{SM}(\{\alpha, \gamma\})$ and $\gamma \in \mathcal{C}^{SM}(\{\beta, \gamma\})$, but $\gamma \notin \mathcal{C}^{SM}(\{\alpha, \beta, \gamma\})$ because $\alpha \succ \beta$ and $\gamma \approx \beta$. So Set Union is violated.
(4) Suppose that $\forall \alpha \in A$, $\alpha \in \mathcal{C}(\{\alpha, \beta)\})$. Then if $\mathcal{C}^{SM}(A \cup \{\beta\}) = \{\beta\}$, there must exist some $\alpha^* \in A$, such that $\beta \succ \alpha^*$ and hence, contrary to supposition, $\alpha^* \notin \mathcal{C}^{SM}(\{\alpha^*, \beta)\})$. So Element Union is satisfied. Now suppose that $\tilde{\mathcal{C}}^{SM}(A \cup B, \alpha) \subseteq \mathcal{C}^{SM}(A)$ and $\tilde{\mathcal{C}}^{SM}(A \cup B, \alpha) \subseteq \mathcal{C}^{SM}(B)$. Suppose that $\tilde{\mathcal{C}}^{SM}(A \cup B, \alpha) \not\subseteq \tilde{\mathcal{C}}^{SM}(A \cup B)$. Then in particular, $\alpha \notin \mathcal{C}^{SM}(A \cup B)$. Then there exists $\gamma, \delta \in A \cup B$ such that $\alpha \hat{\approx}^{A \cup B} \gamma$ and $\beta \succ \gamma$. But if $\alpha \hat{\approx}^{A \cup B} \gamma$ then $\gamma \in \tilde{\mathcal{C}}^{SM}(A \cup B, \alpha)$. So $\gamma \in A \cap B$ since $\gamma \in \mathcal{C}^{SM}(A)$ and $\gamma \in \mathcal{C}^{SM}(B)$. But $\delta \in A$ or $\delta \in B$. So $\gamma \notin \mathcal{C}^{SM}(A)$ or $\gamma \notin \mathcal{C}^{SM}(B)$, in contradiction to what we have just established. It follows that $\tilde{\mathcal{C}}^{SM}(A \cup B, \alpha) \subseteq \mathcal{C}^{SM}(A \cup B)$. ■

3 Rationalisability

A question that naturally arises is whether, and under what conditions, the choices that are formally represented by a choice function can be rationalised or explained in terms of an underlying preference relation that, together with some choice rule, determines it. To tackle it, let us say that a choice function $\mathcal{C}$ is *rationalisable* by a consistent weak preference relation $\succeq$ iff $\mathcal{C}$ is generated by $\succeq$ together with a given choice rule R, i.e. iff $\mathcal{C} = \mathcal{C}^R_{\succeq}$. This definition of rationalisability contains two unspecified parameters: the type of consistency to be required of preference and the type of choice rule to be used in the determination of the choice function.

Different kinds of rationalisability will be associated with different values for these parameters and a particular choice function may be rationalisable relative to some combination of consistency property and choice rule but not another. Here we will require that the preference relation be at least weakly consistent in order to speak of rationalisation, and differentiate between O-, M- and SM-rationalisations of a choice function in accordance with the choice rule that determines it.

3.1 Revealed Preference

In the literature on revealed preference the question of rationalisability is typically approached by defining the weak preference relation $\dot{\succeq}_{\mathcal{C}}$ 'revealed' by a choice function $\mathcal{C}$ in the following way:

Revealed Preference: $\alpha \dot{\succeq}_{\mathcal{C}} \beta \Leftrightarrow \exists A \subseteq X$ such that $\alpha, \beta \in A$ and $\alpha \in \mathcal{C}(A)$

It is then possible to ask what properties of the revealed preference relation $\dot{\succeq}_{\mathcal{C}}$ are implied by various assumed properties of the choice function $\mathcal{C}$. It is well known for instance that if $\mathcal{C}$ satisfies both Set Contraction and Set Expansion then $\dot{\succeq}_{\mathcal{C}}$ so defined is both complete and transitive (see Sen [9, Theorem II]). In this case, as we learnt from Theorem 1(5), our three choice rules coincide and so it is reasonable to speak without further qualification of the revealed preference relation $\dot{\succeq}_{\mathcal{C}}$ as rationalising or explaining the choices represented by $\mathcal{C}$. But when either transitivity and completeness fails for the choice function then this neat relationship breaks down. Indeed in the absence of grounds for presuming completeness, the underlying conception of revealed preference becomes much less compelling.

The fundamental problem with the usual definition of the revealed preference relation is that it does not allow for any distinction between an attitude of indifference between two alternatives and an inability to compare them. Indeed the effect of Revealed Preference is to collapse the two since it entails that $\alpha \dot{\approx}_{\mathcal{C}} \beta \Leftrightarrow \exists A, B \subseteq X$ such that $\alpha, \beta \in A \cap B$, $\alpha \in \mathcal{C}(A)$ and $\beta \in \mathcal{C}(B)$ and so ascribes to the agent an attitude of indifference between any two alternatives that can permissibly be chosen from some set containing both—in particular to any alternatives α, β such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha, \beta\}$—irrespective of whether they are comparable or not.

To allow for incomparability we need to build a revealed weak preference relation up from its component revealed strict preference and indifference relations. I suggest that the following definitions encode the correct way to do so from a choice function $\mathcal{C}$.

RSP: $\alpha \succ_{\mathcal{C}} \beta \Leftrightarrow \forall (A : \alpha \in A), \beta \notin \mathcal{C}(A)$
RI: $\alpha \approx_{\mathcal{C}} \beta \Leftrightarrow \forall (A : \alpha, \beta \in A), \alpha \in \mathcal{C}(A) \Leftrightarrow \beta \in \mathcal{C}(A)$
RWP: $\alpha \succeq_{\mathcal{C}} \beta \Leftrightarrow \alpha \succ_{\mathcal{C}} \beta$ or $\alpha \approx_{\mathcal{C}} \beta$

RSP strengthens SPBC into a biconditional that mandates the inference that one prospect is strictly preferred to another iff the latter is never chosen when the former is available. Note that RSP implies that $\beta \notin \mathcal{C}(\{\alpha, \beta\})$ if $\alpha \succ_{\mathcal{C}} \beta$. The converse is only true however if $\mathcal{C}$ satisfies Set Contraction. Put more positively, if a choice function satisfies Set Contraction then the revealed strict preference relation based on it is binary in composition.

RI similarly strengthens IBC into a biconditional, but the inference it mandates is more controversial; namely that two alternatives are indifferent iff they are either both chosen or both not. The intuition underlying RI is that what distinguishes indifference between two alternatives from their incomparability is that in the former case (indifference) no third alternative should be strictly preferred to one, but not the other, of the pair, while in the latter case (incomparability) such a third alternative could exist. The problem is that if the set of alternatives is sufficiently sparse such a third alternative might not in fact exist and then RI would mandate an inference of indifference when the case is actually one of incomparability. On the other hand, when the underlying set of alternatives contains, for every pair of alternatives α and β, a third alternative α^+, that is comparably better than α, or alternative β^- that is comparably worse than β, then RI will be applicable.

RWP defines $\succeq_{\mathcal{C}}$ in terms of the relations of strict preference, $\succ_{\mathcal{C}}$, and indifference, $\approx_{\mathcal{C}}$, that are revealed by the choice function $\mathcal{C}$ in accordance with RSP and RI. So defined $\succeq_{\mathcal{C}}$ is not necessarily complete, since it can be the case that there are sets A and B such that $\alpha, \beta \in A, B$ but $\alpha \in \mathcal{C}(A)$ and $\beta \in \mathcal{C}(B)$. This would arise when α and β are incomparable and A and B contain elements that respectively dominate β and α but not the other. Furthermore, although $\approx_{\mathcal{C}}$ must be symmetric and $\succeq_{\mathcal{C}}$ reflexive, in the absence of any further assumptions about $\mathcal{C}$ it is not assured that $\succeq_{\mathcal{C}}$ is a weak preference relation, nor that $\succ_{\mathcal{C}}$ and $\approx_{\mathcal{C}}$ are its symmetric and anti-symmetric parts. For this we must assume that $\mathcal{C}$ is decisive.

Theorem 4 *Suppose that $\mathcal{C}$ is decisive and that $\succeq_{\mathcal{C}}$ is defined from $\mathcal{C}$ in accordance with RSP, RI and RWP. Then $\succeq_{\mathcal{C}}$ is a weakly consistent weak preference relation with symmetric and anti-symmetric parts $\succ_{\mathcal{C}}$ and $\approx_{\mathcal{C}}$.*

Proof RI implies the symmetry of $\approx_{\mathcal{C}}$ and, together with RWP, the reflexivity of $\succeq_{\mathcal{C}}$. Note firstly that it is not possible that both $\alpha \succ_{\mathcal{C}} \beta$ and that $\alpha \approx_{\mathcal{C}} \beta$. For if $\alpha \succ_{\mathcal{C}} \beta$, then by RSP $\beta \notin \mathcal{C}(\{\alpha, \beta\})$. So by decisiveness $\alpha \in \mathcal{C}(\{\alpha, \beta\})$ and hence by RI $\alpha \not\approx_{\mathcal{C}} \beta$. Similarly if $\alpha \approx_{\mathcal{C}} \beta$ then by RI and decisiveness $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha, \beta\}$. So by RSP, $\alpha \not\succ_{\mathcal{C}} \beta$. To establish the anti-symmetry of $\succ_{\mathcal{C}}$, let $\Gamma := \{A \subseteq X : \alpha, \beta \in A\}$. Suppose that $\alpha \succ_{\mathcal{C}} \beta$ so that by RSP, $\forall A \in \Gamma, \beta \notin \mathcal{C}(A)$. Then $\beta \notin \mathcal{C}(\{\alpha, \beta)\}$ and hence by decisiveness, $\alpha \in \mathcal{C}(\{\alpha, \beta)\}$. So it is not the case that $\forall A \in \Gamma, \alpha \notin \mathcal{C}(A)$, i.e. $\beta \not\succ_{\mathcal{C}} \alpha$. Finally suppose that, contrary to hypothesis, $\succeq_{\mathcal{C}}$ is not weakly consistent. Then there exists a sequence of alternatives $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that, $\alpha_1 \succ_{\mathcal{C}} \alpha_2, \alpha_2 \succ_{\mathcal{C}} \alpha_3, \ldots, \alpha_{n-1} \succ_{\mathcal{C}} \alpha_n$ and $\alpha_n \succ_{\mathcal{C}} \alpha_1$. Then by RSP, $\mathcal{C}(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}) = \varnothing$ contrary to decisiveness. So $\succeq_{\mathcal{C}}$ must be weakly consistent. ■

3.2 Conditions for Rationalisability

Let us now turn to the question of whether it is possible in general to rationalise an arbitrary choice function $\mathcal{C}$ in terms of the revealed weak preference relation $\succeq_C$ defined by RWP. As is to be expected, without some restrictions on $\mathcal{C}$ and/or the set of alternatives, the answer is negative for each of the three types of rationalisability under consideration.

1. *O-rationalisability*: Consider $\mathcal{C}$ and set of alternatives $\{\alpha, \beta, \gamma\}$ such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha, \beta\}$ but $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\beta, \gamma\}$. Then by RWP, $\alpha \not\succeq_C \beta$ and $\beta \not\succeq_C \alpha$. So $\mathcal{C}^{Opt}_{\succeq_C}(\{\alpha, \beta\}) = \varnothing \neq \mathcal{C}(\{\alpha, \beta\})$.
2. *M-rationalisability*: Consider $\mathcal{C}$ and set of alternatives $\{\alpha, \beta, \gamma\}$ such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha\}$, $\mathcal{C}(\{\beta, \gamma\}) = \{\beta, \gamma\}$, $\mathcal{C}(\{\alpha, \gamma\}) = \{\alpha, \gamma\}$ but $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\alpha\}$. Then by RWP, $\alpha \succ_C \beta$, $\beta \approx_C \gamma$ but $\gamma \not\bowtie_C \alpha$. So $\mathcal{C}^{Max}_{\succeq_C}(\{\alpha, \beta, \gamma\}) = \{\alpha, \gamma\} \neq \mathcal{C}(\{\alpha, \beta, \gamma\})$.
3. *SM-rationalisability*: Consider $\mathcal{C}$ and set of alternatives $\{\alpha, \beta, \gamma\}$ such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha\}$, $\mathcal{C}(\{\beta, \gamma\}) = \{\beta\}$, and $\mathcal{C}(\{\alpha, \gamma\}) = \{\gamma\}$. So by RWP, $\alpha \not\bowtie_C \beta$, $\beta \not\bowtie_C \gamma$ and $\gamma \not\bowtie_C \alpha$. But then $\mathcal{C}^{SM}_{\succeq_C}(\{\alpha, \beta\}) = \{\alpha, \beta\} \neq \mathcal{C}(\{\alpha, \beta\})$.

What conditions on $\mathcal{C}$ are sufficient to ensure rationalisability? Our earlier observation that satisfaction of Set Contraction and Set Expansion is sufficient for O-rationalisability extends to both M- and SM-rationalisability: this is a consequence of Theorem 1(5). This result is of marginal interest however since these conditions are very restrictive and indeed suffice to ensure the completeness of the revealed preference relation.

It is possible to do better. Blair et al. [3] prove that a choice function satisfies Set Union and Set Contraction iff there exists a weakly consistent preference weak relation that rationalises it. Since both conditions are also implied by Maximality, this theorem provides the required characterisation of consistent maximal choice. Below, in Theorem 5, we establish that the weak preference relation defined by RWP, RSP and RI is just such a rationalising relation.

Set Contraction and Set Union are in fact also jointly sufficient for a SM-rationalisation, but in this case it does not give us the characterisation that we seek since Set Union is not necessary for preference-based strongly maximal choice. What is required, it turns out, are the two weaker conditions we introduced: Element Union and Equivalence Class Union. In Theorem 6 below we show that it is sufficient that the choice function be decisive and satisfy these two conditions along with Set Contraction, for it to have a Suzumura consistent SM-rationalisation. This gives us the characterisation of consistent, strongly maximal choice that we want, namely that *it is necessary and sufficient that a choice function be decisive and satisfy Set Contraction, Element Union and Equivalence Class Union that it be SM-rationalisable by a Suzumura consistent weak preference relation*. This is proved below as a corollary of Theorem 6. (In the proofs that follow, we omit the subscripts from the relations induced by the choice function $\mathcal{C}$.)

Theorem 5 *Suppose that $\mathcal{C}$ is a decisive choice function satisfying Set Contraction and Set Union. Let $\succeq$ be defined from $\mathbf{C}$ by RWP, RSP and RI. Then $\succeq$ is a weakly consistent weak preference relation that M-rationalises $\mathcal{C}$.*

Proof Suppose that $\alpha \notin \mathcal{C}^{M}_{\succeq}(A)$. Then by RSP and RI there exists $\beta \in A$ such that $\forall(B : \alpha \in B)$, $\alpha \notin \mathcal{C}(B)$. So in particular, $\alpha \notin \mathcal{C}(A)$. Now suppose that $\alpha \in \mathcal{C}^{M}_{\succeq}(A)$. Then by RSP there does not exist any $\beta \in A$ such that $\forall(B \subset X : \beta \in B)$, $\alpha \notin \mathcal{C}(B)$. Hence for all $\beta_i \in A$, there exists a set $B_i \subseteq X$ such that $\beta \in B_i$ and $\alpha \in \mathcal{C}(B_i)$. But then by Set Union, $\alpha \in \mathcal{C}(\cup B_i) = \mathcal{C}(A)$. The weak consistency of $\succeq$ then follows from Theorem 2(2). ■

Lemma 1 *Suppose that choice function $\mathcal{C}$ and indifference relation $\approx$ are related in accordance with RI. Then for all $\gamma \in X$:*

$$\gamma \hat{\approx}^{A} \alpha \Leftrightarrow \gamma \in \tilde{\mathcal{C}}(A, \alpha)$$

Proof Suppose that $\gamma \hat{\approx}^{A} \alpha$. Then then there exists a sequence of elements in A, α_1, α_2, ..., α_n linking α and γ in the sense that $\alpha = \alpha_1$, $\alpha_1 \approx \alpha_2$, ..., and $\alpha_n = \gamma$. Hence by RI, for all α_i in the sequence, $\alpha_i \in \mathcal{C}(B) \Leftrightarrow \alpha_{i+1} \in \mathcal{C}(B)$ for all $B \subseteq X$ such that $\alpha_i, \alpha_{i+1} \in B$. Now $\alpha \in \tilde{C}(A, \alpha)$. And so since $\alpha_1 \in \tilde{C}(A, \alpha)$, it follows by the definition of $\tilde{\mathcal{C}}$ that $\alpha_2 \in \tilde{C}(A, \alpha)$ and hence ... $\alpha_n \in \tilde{C}(A, \alpha)$. We conclude that $\gamma \in \tilde{C}(A, \alpha)$.

We establish the other direction by proving by induction that if $\gamma \in \tilde{\mathcal{C}}^{\tau}(A, \alpha)$ then $\gamma \hat{\approx}^{A} \alpha$ for all $\tau \geq 0$. Suppose that $\tau = 0$. Then $\gamma = \alpha$ and so it follows from the symmetry of $\approx$ that $\gamma \hat{\approx}^{A} \alpha$. Next assume true for $\tau = k$, i.e. if $\gamma \in \tilde{\mathcal{C}}^{k}(A, \alpha)$ then $\gamma \hat{\approx}^{A} \alpha$. Now we prove the hypothesis for $\tau = k + 1$. Suppose that $\gamma \in \tilde{\mathcal{C}}^{k+1}(A, \alpha)$. Then there exists $\beta \in \tilde{\mathcal{C}}^{k}(A, \alpha)$ such that $\beta \in \mathcal{C}(B) \Leftrightarrow \gamma \in \mathcal{C}(B)$ for all $B \subseteq X$ such that $\beta, \gamma \in B$. Hence by RI, $\gamma \approx \beta$. But by assumption $\beta \hat{\approx}^{A} \alpha$. So $\gamma \hat{\approx}^{A} \alpha$. ■

Corollary 1 *If $\gamma \hat{\approx}^{A} \alpha$ then $\tilde{\mathcal{C}}(A, \gamma) = \tilde{\mathcal{C}}(A, \alpha)$*

Theorem 6 *Suppose that $\mathcal{C}$ is a decisive choice function satisfying Set Contraction, Element Union and Equivalence Class Union. Let $\succeq$ be defined from $\mathbf{C}$ by RWP, RSP and RI. Then $\succeq$ is a Suzumura consistent weak preference relation that SM-rationalises $\mathcal{C}$.*

Proof Suppose that $\alpha \notin \mathcal{C}^{SM}_{\succeq}(A)$. Then by RSP and RI there exists $\beta, \gamma \in A$ such that $\beta \succ \gamma$ and $\gamma \hat{\approx}^{A} \alpha$. This implies that there exists $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n \in A$ such that $\gamma = \alpha_1, \alpha_1 \approx_C \alpha_2, \ldots, \alpha_{n-1} \approx_C \alpha_n$, and $\alpha_n = \alpha$. Now since $\beta \in A$, we know that $\gamma \notin \mathcal{C}(A)$. So by RI, $\alpha_2 \notin \mathcal{C}(A)$. Hence by RI, $\alpha_3 \notin \mathcal{C}(A)$ .... and hence by RI, $\alpha \notin \mathcal{C}(A)$.

Now suppose that $\alpha \in \mathcal{C}^{SM}_{\succeq}(A)$. Let $\Gamma = \{\gamma \in A : \gamma \hat{\approx}^{A} \alpha\}$. Then $\forall \gamma \in \Gamma$, there exists no $\delta \in A$ such that $\beta \succ \gamma$. Hence also for all such γ there exists no $\gamma', \delta' \in A$ such that $\gamma \hat{\approx}^{A} \gamma'$ and $\delta' \succ \gamma'$. So $\forall \gamma \in \Gamma$, $\gamma \in \mathcal{C}^{SM}_{\succeq}(A)$. Now by RSP, $\forall \gamma \in \Gamma$, there does not exist any $\beta \in A$ such that $\forall(B \subseteq X : \beta \in B)$, $\gamma \notin \mathcal{C}(B)$. Hence for all $\beta \in A$, there exists a set $B \subseteq X$ such that $\beta \in B$ and $\gamma \in \mathcal{C}(B)$. But then by Set Contraction, $\gamma \in \mathcal{C}(\{\bigcirc, \beta\})$. So by Element Union, there exist $\gamma^* \in \Gamma$

such that $\gamma^* \in \mathcal{C}(\Gamma \cup \{\beta\})$. Now by Lemma 1, since $\approx$ is constructed from $\mathcal{C}$ in accordance with RI, $\tilde{\mathcal{C}}(A, \gamma^*) = \tilde{\mathcal{C}}(A, \alpha) = \Gamma$. Hence $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(\Gamma \cup \{\beta\})$ for all $\beta \in A$. Hence by Equivalence Class Union, $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(A)$. But then it follows from the fact that $\alpha \in \tilde{\mathcal{C}}(A, \alpha)$, that $\alpha \in \mathcal{C}(A)$. So $\mathcal{C} = \mathcal{C}^{SM}_{\succeq}$. Finally the Suzumura consistency of $\succeq$ follows from Theorem 2(3). ■

Corollary 2 *$\mathcal{C}$ is a decisive choice function satisfying Set Contraction, Element Union and Equivalence Class Union iff there exists a Suzumura consistent weak preference relation $\succeq$ that SM-rationalises $\mathcal{C}$.*

Proof Follows from Theorems 6 and 3(1) and (4). ■

4 Conclusion

When preferences are incomplete, as they often are, they will not suffice to determine a unique choice from all sets of alternatives. Nonetheless, it is useful to know what choices an agent's preferences permit her to make. In this paper I have proposed a new choice rule—Strong Maximality—and argued that it better characterises rational preference-based choice than the more familiar rules of Maximality and Optimality. Only Strong Maximality respects both the requirement that an alternative never be chosen when something strictly preferred to it is available (PBC) and the requirement that two alternatives that are comparably indifferent to one another must either both be chosen or both not chosen from any set containing both (IBC).

When preferences are transitive, Strong Maximality will yield the same prescriptions as Maximality, when preferences are also complete both will coincide with the prescriptions of Optimality. But just as recognition of the rational permissibility of incompleteness motivates a move from Optimality to Maximality, so the recognition that transitivity is too strong a requirement motivates a move from Maximality to Strong Maximality.

Strong Maximality is closely linked to the requirement that preferences be Suzumura consistent; in particular Suzumura consistency is both necessary and sufficient for decisive strongly maximal choice. These two concepts are thus mutually supportive in the same way as are the concepts of maximal choice and weak consistency and the concepts of optimal choice and transitivity. And just as weak consistency is too weak and transitivity too strong, so too is maximal choice too permissive and optimal choice too demanding. Strong Maximality and Suzumura consistency are, like small bear's porridge, just right.

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