Chapter 2
Quantum Systems and Resolvent Algebras

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2.1 Introduction

The conceptual backbone for the modeling of the kinematics of quantum systems is the Heisenberg commutation relations which have found their mathematical expression in various guises. There is an extensive literature analyzing their properties, starting with the seminal paper of Born, Jordan and Heisenberg on the physical foundations and reaching a first mathematical satisfactory formulation in the works of von Neumann and of Weyl.

These canonical systems of operators may all be presented in the following general form: there is a real (finite or infinite dimensional) vector space X equipped with a non-degenerate symplectic form $\sigma : X \times X \to \mathbb{R}$ and a linear map ϕ from X onto the generators of a polynomial *-algebra $\mathcal{P}(X, \sigma)$ of operators satisfying the canonical commutation relations

$$\big[\phi(f), \phi(g)\big] = i\sigma(f, g)\,\mathbf{1}, \quad \phi(f)^* = \phi(f)\,.$$

In the case that X is finite dimensional, one can reinterpret this relation in terms of the familiar quantum mechanical position and momentum operators, and if X consists of Schwartz functions on some manifold one may consider ϕ to be a bosonic quantum field. As is well-known, the operators $\phi(f)$ cannot all be bounded. Moreover, the algebra $\mathcal{P}(X, \sigma)$ does not admit much interesting dynamics acting on it by automorphisms; in fact there are in general only transformations induced

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by polynomial Hamiltonians which leave it invariant [7]. Thus $\mathcal{P}(X, \sigma)$ is not a convenient kinematical algebra in either respect.

The inconveniences of unbounded operators can be evaded by expressing the basic commutation relations in terms of bounded functions of the generators $\phi(f)$. In the approach introduced by Weyl, this is done by considering the C*-algebra generated by the set of unitaries $W(f) \doteq \exp(i\phi(f))$, $f \in X$ (the Weyl operators) satisfying the Weyl relations

$$W(f)W(g) = e^{-i\sigma(f,g)/2}\, W(f+g)\,, \quad W(f)^* = W(-f)\,.$$

This is the familiar Weyl (or CCR) algebra $\mathcal{W}(X, \sigma)$. Yet this algebra still suffers from the fact that its automorphism group does not contain physically significant dynamics [9]. This deficiency can be traced back to the fact that the Weyl algebra is simple, whereas any unital C*-algebra admitting an expedient variety of dynamics must have ideals [4, Sec. 10], cf. also the conclusions.

For finite systems this problem can be solved by proceeding to the twisted group algebra [10] derived from the unitaries $W(f)$, $f \in X$. By the Stone–von Neumann theorem this algebra is isomorphic to $\mathcal{K}(\mathcal{H})$, the compact operators on a separable Hilbert space, for any finite dimensional X. This step solves the problem of dynamics for finite systems, but it cannot be applied as such to infinite systems since there X is not locally compact. Moreover, one pays the price that the original operators, having continuous spectrum, are not affiliated with $\mathcal{K}(\mathcal{H})$. So one forgets the specific properties of the underlying quantum system.

This unsatisfactory situation motivated the formulation of an alternative version of the C*-algebra of canonical commutation relations, given in [4]. Here one considers the C*-algebra generated by the resolvents of the basic canonical operators which are formally given by $R(\lambda, f) \doteq (i\lambda\mathbf{1} - \phi(f))^{-1}$ for $\lambda \in \mathbb{R}\backslash\{0\}$, $f \in X$. All algebraic properties of the operators $\phi(f)$ can be expressed in terms of polynomial relations amongst these resolvents. Hence, in analogy to the Weyl algebra generated by the exponentials, one can abstractly define a unital C*-algebra $\mathcal{R}(X, \sigma)$ generated by the resolvents, called the resolvent algebra.

In accordance with the requirement of admitting sufficient dynamics the resolvent algebras have ideals. Their ideal structure was recently clarified in [1], where it was shown that it depends sensitively on the size of the underlying quantum system. More precisely, the specific nesting of the primitive ideals encodes information about the dimension of the underlying space X. This dimension, if it is finite, is an algebraic invariant which labels the isomorphism classes of the resolvent algebras. Moreover, the primitive ideals are in one-to-one correspondence to the spectrum (dual) of the respective algebra, akin to the case of commutative algebras. The resolvent algebras are postliminal (type I) if the dimension of X is finite and they are still nuclear if X is infinite dimensional. Thus these algebras not only encode specific information about the underlying systems but also have comfortable mathematical properties.

The resolvent algebras already have proved to be useful in several applications to quantum physics such as the representation theory of abelian Lie algebras of derivations [5], the study of constraint systems and of the BRST method in a C*-algebraic setting [4, 6], the treatment of supersymmetric models on non-compact spacetimes and the rigorous construction of corresponding JLOK-cocycles [3]. Their virtues also came to light in the formulation and analysis of the dynamics of finite and infinite quantum systems [2, 4].

In the present article we give a survey of the basic properties of the resolvent algebras and an outline of recent progress in the construction of dynamics, shedding light on the role of the ideals. The subsequent section contains the formal definition of the resolvent algebras and some comments on their relation to the standard Weyl formulation of the canonical commutation relations. Section 2.3 provides a synopsis of representations of the resolvent algebras and some structural implications and Sect. 2.4 contains the discussion of observables and of dynamics. The article concludes with a brief summary and outlook.

2.2 Definitions and Basic Facts

Let (X, σ) be a real symplectic space; in order to avoid pathologies we make the standing assumption that (X, σ) admits a unitary structure [11]. The pre-resolvent algebra $\mathcal{R}_0(X, \sigma)$ is the universal *-algebra generated by the elements of the set $\{R(\lambda, f) : \lambda \in \mathbb{R}\backslash\{0\},\ f \in X\}$ satisfying the relations

$$R(\lambda, f) - R(\mu, f) = i(\mu - \lambda) R(\lambda, f) R(\mu, f) \tag{2.1}$$

$$R(\lambda, f)^* = R(-\lambda, f) \tag{2.2}$$

$$\big[R(\lambda, f),\ R(\mu, g)\big] = i\sigma(f, g)\ R(\lambda, f)\ R(\mu, g)^2 R(\lambda, f) \tag{2.3}$$

$$\nu\ R(\nu\lambda,\ \nu f) = R(\lambda, f) \tag{2.4}$$

$$\begin{aligned} R(\lambda, f) R(\mu, g) = R(\lambda + \mu, f + g)\big(&R(\lambda, f) + R(\mu, g) \\ &+ i\sigma(f, g) R(\lambda, f)^2 R(\mu, g)\big) \end{aligned} \tag{2.5}$$

$$R(\lambda, 0) = -\tfrac{i}{\lambda}\, \mathbf{1} \tag{2.6}$$

where $\lambda,\ \mu,\ \nu \in \mathbb{R}\backslash\{0\}$ and $f,\ g \in X$, and for (2.5) we require $\lambda + \mu \neq 0$. That is, start with the free unital *-algebra generated by $\{R(\lambda, f) : \lambda \in \mathbb{R}\backslash\{0\},\ f \in X\}$ and factor out by the ideal generated by the relations (2.1) to (2.6) to obtain the *-algebra $\mathcal{R}_0(X, \sigma)$.

Remarks (a) Relations (2.1), (2.2) encode the algebraic properties of the resolvent of some self-adjoint operator, (2.3) amounts to the canonical commutation relations and relations (2.4) to (2.6) correspond to the linearity of the initial map ϕ on X.

(b) The *-algebra $\mathcal{R}_0(X, \sigma)$ is nontrivial, because it has nontrivial representations. For instance, in a Fock representation $(\pi, \mathcal{H})$ one has self-adjoint operators $\phi_\pi(f)$, $f \in X$ satisfying the canonical commutation relations over (X, σ) on a sufficiently big domain in the Hilbert space $\mathcal{H}$ so that one can define $\pi(R(\lambda, f)) \doteq (i\lambda\mathbf{1} - \phi_\pi(f))^{-1}$ to obtain a representation π of $\mathcal{R}_0(X, \sigma)$.

It has been shown in [4, Prop. 3.3] that the following definition is meaningful.

Definition 2.1 Let (X, σ) be a symplectic space. The supremum of operator norms with regard to all cyclic *-representations $(\pi, \mathcal{H})$ of $\mathcal{R}_0(X, \sigma)$

$$\|R\| \doteq \sup_{(\pi,\mathcal{H})} \|\pi(R)\|_{\mathcal{H}}\,, \quad R \in \mathcal{R}_0(X, \sigma)$$

exists and defines a C*-seminorm on $\mathcal{R}_0(X, \sigma)$. The resolvent algebra $\mathcal{R}(X, \sigma)$ is defined as the C*-completion of the quotient algebra $\mathcal{R}_0(X, \sigma)/\ker \|\cdot\|$, where here and in the following the symbol ker denotes the kernel of the respective map.

Of particular interest are representations of the resolvent algebras, such as the Fock representations, where the abstract resolvents characterized by conditions (2.1), (2.2) (sometimes called pseudo-resolvents) are represented by genuine resolvents of self-adjoint operators.

Definition 2.2 A representation $(\pi, \mathcal{H})$ of $\mathcal{R}(X, \sigma)$ is said to be regular if for each $f \in X$ there exists a densely defined self-adjoint operator $\phi_\pi(f)$ such that one has $\pi(R(\lambda, f)) = (i\lambda\mathbf{1} - \phi_\pi(f))^{-1}$, $\lambda \in \mathbb{R}\backslash\{0\}$. (This is equivalent to the condition that all operators $\pi(R(\lambda, f))$ have trivial kernel.)

The following result characterizing regular representations, cf. [4, Thm. 4.10 and Prop. 4.5], is of importance, both in the structural analysis of the resolvent algebras and in their applications. It implies in particular that the resolvent algebras have faithful irreducible representations (e.g. the Fock representations), so their centers are trivial.

Proposition 2.3 *Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{R}(X, \sigma)$.*

(a) If $(\pi, \mathcal{H})$ is regular it is also faithful, i.e. $\|\pi(R)\|_{\mathcal{H}} = \|R\|$ for $R \in \mathcal{R}(X, \sigma)$.
(b) If $(\pi, \mathcal{H})$ is faithful and the weak closure of $\pi(\mathcal{R}(X, \sigma))$ is a factor, then $(\pi, \mathcal{H})$ is regular.

The regular representations of the resolvent algebras are in one-to-one correspondence with the regular representations of the Weyl-algebras, cf. [4, Cor. 4.4]. (Recall that a representation $(\pi, \mathcal{H})$ of $\mathcal{W}(X, \sigma)$ is regular if the maps $\nu \in \mathbb{R} \mapsto \pi(W(\nu f))$ are strong operator continuous for all $f \in X$.) In fact one has the following result.

Proposition 2.4 *Let (X, σ) be a symplectic space and*

(a) let $(\pi, \mathcal{H})$ be a regular representation of the resolvent algebra $\mathcal{R}(X, \sigma)$ with associated self-adjoint operators $\phi_\pi(f)$ defined above. The exponentials

$W_\pi(f) \doteq \exp(i\phi_\pi(f))$, $f \in X$ satisfy the Weyl relations and thus define a regular representation of the Weyl algebra $\mathcal{W}(X,\sigma)$ on $\mathcal{H}$;

(b) let $(\pi,\mathcal{H})$ be a regular representation of the Weyl algebra $\mathcal{W}(X,\sigma)$ and let $\phi_\pi(f)$ be the self-adjoint generators of the Weyl operators. The resolvents $R_\pi(\lambda, f) = (i\lambda\mathbf{1} - \phi_\pi(f))^{-1}$ with $\lambda \in \mathbb{R}\backslash\{0\}$, $f \in X$ satisfy relations (2.1) to (2.6) and thus define a regular representation of the resolvent algebra $\mathcal{R}(X,\sigma)$ on $\mathcal{H}$.

Whilst this proposition establishes the existence of a bijection between the regular representations of $\mathcal{R}(X,\sigma)$ and those of $\mathcal{W}(X,\sigma)$, there is no such map between the non-regular representations of the two algebras. In order to substantiate this point consider for fixed nonzero $f \in X$ the two commutative subalgebras $C^*\{R(1,sf) : s \in \mathbb{R}\} \subset \mathcal{R}(X,\sigma)$ and $C^*\{W(sf) : s \in \mathbb{R}\} \subset \mathcal{W}(X,\sigma)$. These algebras are isomorphic respectively to the continuous functions on the one point compactification of $\mathbb{R}$, and the continuous functions on the Bohr compactification of $\mathbb{R}$. Now the point measures on the compactifications having support in the complement of $\mathbb{R}$ produce non-regular states (after extending to the full C*-algebras by Hahn–Banach) and there are many more of these for the Bohr compactification than for the one point compactification of $\mathbb{R}$. Proceeding to the GNS-representations it is apparent that the Weyl algebra has substantially more non-regular representations than the resolvent algebra.

2.3 Ideals and Dimension

Further insight into the algebraic properties of the resolvent algebras is obtained by a study of its irreducible representations. In case of finite dimensional symplectic spaces these representations have been completely classified [4, Prop. 4.7], cf. also [1].

Theorem 3.1 *Let (X,σ) be a finite dimensional symplectic space and let $(\pi,\mathcal{H})$ be an irreducible representation of $\mathcal{R}(X,\sigma)$. Depending on the representation, the space X decomposes as follows, cf. Fig. 2.1.*

(a) There is a unique subspace $X_R \subset X$ such that there are self-adjoint operators $\phi_\pi(f_R)$ satisfying $\pi(R(\lambda, f_R)) = (i\lambda\mathbf{1} - \phi_\pi(f_R))^{-1}$ for $\lambda \in \mathbb{R}\backslash\{0\}$, $f_R \in X_R$.

(b) Let $X_T \doteq \{f \in X_R : \sigma(f,g) = 0$ for all $g \in X_R\}$. Then ϕ_π restricts on X_T to a linear functional $\varphi : X_T \to \mathbb{R}$ such that $\pi(R(\lambda, f_T)) = (i\lambda - \varphi(f_T))^{-1}\mathbf{1}$ for $f_T \in X_T$, $\lambda \in \mathbb{R}\backslash\{0\}$.

(c) For $f_S \in X_S \doteq X\backslash X_R$ and $\lambda \in \mathbb{R}\backslash\{0\}$ one has $\pi(R(\lambda, f_S)) = 0$.

Conversely, given subspaces $X_T \subset X_R \subset X$ and a linear functional $\varphi : X_T \to \mathbb{R}$ there exists a corresponding irreducible representation $(\pi,\mathcal{H})$ of $\mathcal{R}(X,\sigma)$, unique up to equivalence, with the preceding three properties.

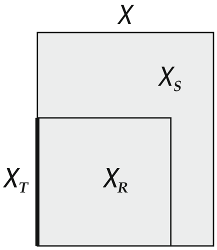

Fig. 2.1 Decomposition of X fixed by an irreducible representation

This result may be regarded as an extension of the Stone–von Neumann uniqueness theorem for regular representations of the CCR algebra. It shows that the only obstruction to regularity is the possibility that some of the underlying canonical operators are infinite and the corresponding resolvents vanish. This happens in particular if there are some canonically conjugate operators having sharp (non-fluctuating) values in a representation, as is the case for constraint systems [4, Prop. 8.1]. But, in contrast to the Weyl algebras, the non-regular representations of the resolvent algebras only depend on the values of these canonical operators. So the abundance of different singular representations of the Weyl algebras shrink to a manageable family on the resolvent algebras.

The preceding theorem is the key to the structural analysis of the resolvent algebras for symplectic spaces of arbitrary finite dimension. We recall in this context that the primitive ideals of a C*-algebra are the (possibly zero) kernels of irreducible representations and that the spectrum of the algebra is the set of unitary equivalence classes of irreducible representations. The following result has been established in [1].

Theorem 3.2 *Let (X, σ) be a finite dimensional symplectic space.*

(a) The mapping $\hat{\pi} \mapsto \ker \hat{\pi}$ from the elements $\hat{\pi}$ of the spectrum (dual) of the resolvent algebra $\mathcal{R}(X, \sigma)$ to its primitive ideals $\ker \hat{\pi}$ is a bijection.
(b) Let $L \doteq \sup\{l \in \mathbb{N} : \ker \hat{\pi}_1 \subset \ker \hat{\pi}_2 \cdots \subset \ker \hat{\pi}_l\}$ be the maximal length of proper inclusions of primitive ideals of $\mathcal{R}(X, \sigma)$. Then $L = \dim(X)/2 + 1$.

Remarks Property (a) is a remarkable feature of the resolvent algebras, shared with the abelian C*-algebras. It rarely holds for non-commutative algebras and also fails if X is infinite dimensional. The quantity L defined in (b) is an algebraic invariant, so this result shows that the dimension $\dim(X)$ of the underlying systems is algebraically encoded in the resolvent algebras. As a matter of fact, L is a complete algebraic invariant of resolvent algebras in the finite dimensional case.

As indicated above, there is an algebraic difference between the resolvent algebras for finite dimensional X and those where X has infinite dimension. A further difference is seen through the minimal (nonzero) ideals [1].

Proposition 3.3 *Let (X, σ) be a symplectic space of arbitrary dimension and let $\mathcal{I} \subset \mathcal{R}(X, \sigma)$ be the intersection of all nonzero ideals of $\mathcal{R}(X, \sigma)$.*

(a) If $\dim(X) < \infty$ *then* $\mathcal{I}$ *is isomorphic to the C*-algebra* $\mathcal{K}(\mathcal{H})$ *of compact operators. Moreover, in any irreducible regular representation* $(\pi, \mathcal{H})$ *one has* $\pi(\mathcal{I}) = \mathcal{K}(\mathcal{H})$.

(b) If $\dim(X) = \infty$ *then* $\mathcal{I} = \{0\}$. *In fact, there exists no nonzero minimal ideal of* $\mathcal{R}(X, \sigma)$ *in this case.*

If (X, σ) is infinite dimensional the resolvent algebra $\mathcal{R}(X, \sigma)$ is the C*-inductive limit of the net of its subalgebras $\mathcal{R}(Y, \sigma)$ where $Y \subset X$ ranges over all finite dimensional non-degenerate subspaces of X, cf. [4, Thm. 4.9]. This fact in combination with the first part of the preceding result is a key ingredient in the construction of dynamics, see below. It also enters in the proof of the following statement [1].

Proposition 3.4 *Let* (X, σ) *be a symplectic space of arbitrary dimension.*

(a) $\mathcal{R}(X, \sigma)$ *is a nuclear C*-algebra,*

(b) $\mathcal{R}(X, \sigma)$ *is a postliminal (type I) C*-algebra if and only if* $\dim(X) < \infty$.

Recall that a C*-algebra is said to be postliminal (type I) if all of its irreducible representations contain the compact operators and that postliminal C*-algebras as well as their C*-inductive limits are nuclear, i.e. their tensor product with any other C*-algebra is unique. It should be noted, however, that the resolvent algebras are not separable [4, Thm. 5.3]. With this remark we conclude our outline of pertinent algebraic properties of the resolvent algebras.

2.4 Observables and Dynamics

The main virtue of the resolvent algebras consists of the fact that it includes many observables of physical interest and admits non-trivial dynamics. In order to illustrate this important feature we discuss in detail a familiar example of a finite quantum system and comment on infinite systems at the end of this section.

Let (X, σ) be a finite dimensional symplectic space, i.e. $\dim(X) = 2N$ for some $N \in \mathbb{N}$. Since regular representations of the resolvent algebras are faithful, cf. Proposition 2.3, it suffices to consider any regular irreducible representation $(\pi_0, \mathcal{H}_0)$ of $\mathcal{R}(X, \sigma)$ (which is unique up to equivalence). Choosing some symplectic basis $f_k, g_k \in X$ and putting $P_k \doteq \phi_{\pi_0}(f_k)$, $Q_k \doteq \phi_{\pi_0}(g_k)$, $k = 1, \dots N$ we identify the self-adjoint operators fixed by the corresponding resolvents with the momentum and position operators of N particles in one spatial dimension.

The (self-adjoint) quadratic Hamiltonian

$$H_0 \doteq \sum_{k=1}^{N} \left(\tfrac{1}{2m_k} P_k^2 + \tfrac{m_k \omega_k^2}{2} Q_k^2\right)$$

describes the free, respectively oscillatory motion of these particles, where m_k are the particle masses and $\omega_k \geq 0$ the frequencies of oscillation, $k = 1, \dots N$. The interaction of the particles is described by the operator

$$V \doteq \sum_{1 \leq k < l \leq N} V_{kl}(Q_k - Q_l)$$

where we assume for simplicity that the potentials V_{kl} are real and continuous, vanish at infinity, but are arbitrary otherwise. Since V is bounded, the Hamiltonian $H \doteq H_0 + V$ is self-adjoint on the domain of H_0 and its resolvents are well defined.

Proposition 4.1 *Let H be the Hamiltonian defined above. Then*

$$(i\mu\mathbf{1} - H)^{-1} \in \pi_0(\mathcal{R}(X, \sigma)), \quad \mu \in \mathbb{R}\backslash\{0\}.$$

Remark Since π_0 is faithful its inverse $\pi_0^{-1} : \pi_0(\mathcal{R}(X, \sigma)) \to \mathcal{R}(X, \sigma)$ exists, so this result shows that H is affiliated with the resolvent algebra. Note that this is neither true for the Weyl algebra $\mathcal{W}(X, \sigma)$ nor for the corresponding twisted group algebra $\mathcal{K}(\mathcal{H})$ if one of the frequencies ω_k vanishes. Thus $\mathcal{R}(X, \sigma)$ contains many more observables of physical interest than these conventional algebras.

Proof Let $X_k \subset X$ be the two-dimensional subspaces spanned by the symplectic pairs (f_k, g_k), let $\sigma_k \doteq \sigma \upharpoonright X_k \times X_k$ and let $(\pi_k, \mathcal{H}_k)$ be regular irreducible representations of $\mathcal{R}(X_k, \sigma_k)$, $k = 1, \dots N$. Then $\pi_0 \doteq \pi_1 \otimes \cdots \otimes \pi_N$ defines an irreducible representation of the C*-tensor product $\mathcal{R}(X_1, \sigma_1) \otimes \cdots \otimes \mathcal{R}(X_N, \sigma_N)$ on the Hilbert space $\mathcal{H}_0 \doteq \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. It extends by regularity to the Weyl algebra $\mathcal{W}(X, \sigma) \simeq \mathcal{W}(X_1, \sigma_1) \otimes \cdots \otimes \mathcal{W}(X_N, \sigma_N)$ and hence to a regular representation of $\mathcal{R}(X, \sigma)$, cf. Proposition 2.4.

One has $H_{0k} \doteq (i\mu\mathbf{1} - \frac{1}{2m_k} P_k^2 - \frac{m_k \omega_k^2}{2} Q_k^2)^{-1} \in \pi_k(\mathcal{R}(X_k, \sigma_k))$, $k = 1, \dots N$, disregarding tensor factors of $\mathbf{1}$. If $\omega_k > 0$ this follows from the fact that the resolvent of the harmonic oscillator Hamiltonian is a compact operator and hence belongs to the compact ideal of $\pi_k(\mathcal{R}(X_k, \sigma_k))$, cf. Proposition 3.3. If $\omega_k = 0$ one resorts to the fact that the abelian C*-algebra generated by the resolvents $(i\lambda\mathbf{1} - P_k)^{-1}$, $\lambda \in \mathbb{R}\backslash\{0\}$ coincides with $C_0(P_k)$, the algebra of all continuous functions of P_k vanishing at infinity. Hence $C_0(P_k) \subset \pi_k(\mathcal{R}(X_k, \sigma_k))$. Since $(i\mu\mathbf{1} - \frac{1}{2m_k} P_k^2)^{-1} \in C_0(P_k)$ the preceding statement holds also for $\omega_k = 0$.

As is well known $C_0(\mathbb{R}_+^N) = C_0(\mathbb{R}_+) \overbrace{\otimes \cdots \otimes}^{N} C_0(\mathbb{R}_+)$ and it is also clear that $u_1, \dots, u_N \mapsto (i\mu - u_1 \cdots - u_N)^{-1}$ is an element of $C_0(\mathbb{R}_+^N)$. Since the resolvents of the positive self-adjoint operators H_{0k} generate the abelian C*-algebras $C_0(H_{0k})$, $k = 1, \dots, N$, it follows from continuous functional calculus that $(i\mu\mathbf{1} - H_0)^{-1} = \big(i\mu\mathbf{1} - H_{01} \cdots - H_{0N}\big)^{-1} \in C_0(H_{01}) \otimes \cdots \otimes C_0(H_{0N}) \subset \pi_0(\mathcal{R}(X, \sigma))$.

Similarly, for the interaction potentials one uses the fact that the abelian C*-algebras generated by the resolvents $(i\lambda\mathbf{1} - (Q_k - Q_l))^{-1}$, $\lambda \in \mathbb{R}\backslash\{0\}$ coincide with $C_0(Q_k - Q_l)$. So as $V_{kl} \in C_0(\mathbb{R})$, one also has that

$$V = \sum_{1 \le k < l \le N} V_{kl}(Q_k - Q_l) \in \pi_0(\mathcal{R}(X,\sigma)).$$

In summary one gets $(1 - (i\mu\mathbf{1} - H_0)^{-1} V) \in \pi_0(\mathcal{R}(X,\sigma))$. Its inverse exists if $|\mu| > \|V\|$ and $(i\mu\mathbf{1} - H)^{-1} = (1 - (i\mu\mathbf{1} - H_0)^{-1} V)^{-1} (i\mu\mathbf{1} - H_0)^{-1} \in \pi_0(\mathcal{R}(X,\sigma))$ for such μ. The statement for arbitrary $\mu \in \mathbb{R}\backslash\{0\}$ then follows from the resolvent equation for H, completing the proof.

As a matter of fact, the preceding proposition holds for a much larger class of interaction potentials, including discontinuous ones. It does not hold, however, for certain physically inappropriate Hamiltonians such as that of the anti-harmonic oscillator [4, Prop. 6.3]. The characterization of all Hamiltonians which are affiliated with resolvent algebras is an interesting open problem.

We turn now to the analysis of the dynamics induced by the Hamiltonians given above. The exponentials of the quadratic Hamiltonians H_0 induce symplectic transformations, so one has $(\mathrm{Ad}\, e^{itH_0})(\pi_0(\mathcal{R}(X,\sigma))) = \pi_0(\mathcal{R}(X,\sigma))$ for $t \in \mathbb{R}$. For the proof that the resolvent algebra is also stable under the adjoint action of the interacting dynamics the crucial step consists of showing that the cocycles $\Gamma(t) = e^{itH} e^{-itH_0}$ are elements of $\pi_0(\mathcal{R}(X,\sigma))$. Putting $V(t) = (\mathrm{Ad}\, e^{itH_0})(V)$ one can present the cocycles in the familiar form of a Dyson series

$$\Gamma(t) = 1 + \sum_{n=1}^{\infty} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, V(t_n) \cdots V(t_1)$$

and this series converges absolutely in norm since the operators $V(t)$ are uniformly bounded. Moreover, the functions $t \mapsto V(t)$ have values in the algebra $\pi_0(\mathcal{R}(X,\sigma))$; but since they are only continuous in the strong operator topology it is not clear from the outset that their integrals, defined in this topology, are still contained in this algebra. Here again the specific structure of the resolvent algebra matters. It allows to establish the desired result.

Proposition 4.2 *Let H be the Hamiltonian defined above. Then*

$$(\mathrm{Ad}\, e^{itH})(\pi_0(\mathcal{R}(X,\sigma))) = \pi_0(\mathcal{R}(X,\sigma)), \quad t \in \mathbb{R}.$$

Remark Since π_0 is faithful it follows from this result that $\alpha_t \doteq \pi_0^{-1}(\mathrm{Ad}\, e^{itH})\pi_0$, $t \in \mathbb{R}$ defines a one-parameter group of automorphisms of $\mathcal{R}(X,\sigma)$. It should be noted, however, that its action is not continuous in the strong (pointwise norm) topology of $\mathcal{R}(X,\sigma)$.

Proof Let $k, l \in 1, \ldots, N$ be different numbers, let (f_k, g_k) and (f_l, g_l) be symplectic pairs as in the previous proof and let $X_{kl} \subset X$ be the space spanned by $h_{kl}(t) \doteq ((\cos\omega_k t)\, g_k - (\cos\omega_l t)\, g_l + (\sin\omega_k t)/m_k\omega_k\, f_k - (\sin\omega_l t)/m_l\omega_l\, f_l)$, $t \in \mathbb{R}$, where we stipulate $(\sin\omega t)/\omega = t$ if $\omega = 0$. This space is non-degenerate and, depending on the masses and frequencies, either two or four dimensional. We put $\sigma_{kl} \doteq \sigma \upharpoonright X_{kl} \times X_{kl}$. Let $V_{kl}(t) \doteq (\mathrm{Ad}\, e^{itH_0})(V_{kl}(Q_k - Q_l))$, where $V_{kl}(Q_k - Q_l)$ is any one of the two-body potentials contributing to V. Then, for any $t \in \mathbb{R}$,

$$V_{kl}(t) = V_{kl}((\cos \omega_k t) Q_k - (\cos \omega_l t) Q_l + (\sin \omega_k t)/m_k \omega_k P_k - (\sin \omega_l t)/m_l \omega_l \, P_l) \\ \in \pi_0(\mathcal{R}(X_{kl}, \sigma_{kl}))\,.$$

Now the function $s_1, \dots s_d \mapsto V_{kl}(s_1) \cdots V_{kl}(s_d)$ is continuous in the strong operator topology and, for almost all $s_1, \dots s_d$, an element of the compact ideal of $\pi_0(\mathcal{R}(X_{kl}, \sigma_{kl}))$, provided $d \geq \dim(X_{kl})$. The latter assertion follows from the fact that $V_{kl}(s)$ is, for given s, an element of the abelian C*-algebra generated by the resolvents $\pi_0(R(\lambda, h_{kl}(s)))$, $\lambda \in \mathbb{R}\backslash\{0\}$ and that the compact ideal coincides with the principal ideal of $\pi_0(\mathcal{R}(X_{kl}, \sigma_{kl}))$ generated by $\pi_0(R(\lambda_1, h_1) \cdots R(\lambda_d, h_d))$ for any choice of $\lambda_1, \dots \lambda_d \in \mathbb{R}\backslash\{0\}$ and of elements $h_1, \dots h_d \in X_{kl}$ which span X_{kl} [2]. It is then clear that $\left(\int_0^t ds\, V_{kl}(s)\right)^d = \int_0^t ds_1 \cdots \int_0^t ds_d \; V_{kl}(s_1) \cdots V_{kl}(s_d)$ is contained in the compact ideal of $\pi_0(\mathcal{R}(X_{kl}, \sigma_{kl}))$. But this is then also true for the operator $\int_0^t ds\, V_{kl}(s)$ since it is self-adjoint. As k, l were arbitrary this implies $\int_0^t dt_1 V(t_1) \in \pi_0(\mathcal{R}(X, \sigma))$.

That all other terms in the Dyson series are elements of $\pi_0(\mathcal{R}(X, \sigma))$ is seen by induction. Let $I_n(t) \doteq \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \; V(t_n) \cdots V(t_1) \in \pi_0(\mathcal{R}(X, \sigma))$, $t \in \mathbb{R}$; then $I_{n+1}(t) = \int_0^t dt_1 I_n(t_1) V(t_1)$, where the integrals are defined in the strong operator topology. Now $t \mapsto I_n(t)$ is continuous in norm, hence $I_{n+1}(t)$ can be approximated according to

$$I_{n+1}(t) = \lim_{J \to \infty} \sum_{j=1}^{J} I_n(jt/J) \int_{(j-1)t/J}^{jt/J} dt_1 V(t_1)\,,$$

where the limit exists in the norm topology. Since each term in this sum is an element of $\pi_0(\mathcal{R}(X, \sigma))$ according to the induction hypothesis it follows that $I_{n+1}(t) \in \pi_0(\mathcal{R}(X, \sigma))$. Because of the convergence of the Dyson series this implies $\Gamma(t) \in \pi_0(\mathcal{R}(X, \sigma))$, $t \in \mathbb{R}$, completing the proof of the statement.

Having illustrated the virtues of the resolvent algebras for finite systems we discuss now the situation for infinite systems. There the results are far from being complete, though promising. For the sake of concreteness we consider an infinite dimensional symplectic space (X, σ) with a countable symplectic basis $f_k, g_k \in X$, $k \in \mathbb{Z}$. Similarly to the case of finite systems one can analyze the observables and dynamics associated with $\mathcal{R}(X, \sigma)$ in any convenient faithful representation $(\pi_0, \mathcal{H}_0)$, such as the Fock representation.

As before, we identify the self-adjoint operators fixed by the resolvents with the momentum and position operators of particles, $P_k \doteq \phi_{\pi_0}(f_k)$, $Q_k \doteq \phi_{\pi_0}(g_k)$, $k \in \mathbb{Z}$. In view of Haag's Theorem [8] it does not come as a surprise that global observables, such as Hamiltonians having a unique ground state or the particle number operator are no longer affiliated with the resolvent algebras of such infinite systems. In fact, one has the following general result [2].

Lemma 4.3 *Let (X, σ) be an infinite dimensional symplectic space, let $(\pi_0, \mathcal{H}_0)$ be a faithful irreducible representation of $\mathcal{R}(X, \sigma)$ and let N be a (possibly unbounded) self-adjoint operator on $\mathcal{H}_0$ with an isolated eigenvalue of finite multiplicity. Then $(i\mu\mathbf{1} - N)^{-1} \notin \pi_0(\mathcal{R}(X, \sigma))$ for $\mu \in \mathbb{R}\backslash\{0\}$, i.e. N is not affiliated with $\mathcal{R}(X, \sigma)$.*

Observables corresponding to finite subsystems of the infinite system are still affiliated with $\mathcal{R}(X, \sigma)$. Relevant examples are the partial Hamiltonians of the form given above,

$$H_\Lambda \doteq \sum_{k \in \Lambda} (\tfrac{1}{2m_k} P_k^2 + \tfrac{m_k \omega_k^2}{2} Q_k^2) \; + \sum_{k,l \,\in \Lambda} V_{kl}(Q_k - Q_l)\,,$$

where $\Lambda \subset \mathbb{Z}$ is any finite set. By exactly the same arguments as in the proof of Proposition 4.1 one can show that any such H_Λ is affiliated with $\mathcal{R}(X, \sigma)$. Clearly, these Hamiltonians may have isolated eigenvalues, but these have infinite multiplicity. By the preceding arguments one can also show that the resolvent algebra is stable under the time evolution induced by the partial Hamiltonians. Moreover, for suitable potentials the evolution converges to some global dynamics in the limit $\Lambda \nearrow \mathbb{Z}$. The precise results are as follows.

Proposition 4.4 *Let H_Λ, $\Lambda \subset \mathbb{Z}$ be the partial Hamiltonians introduced above, where V_{kl} are continuous functions tending to 0 at infinity, $k, l \in \mathbb{Z}$.*

(a) Then $(\mathrm{Ad}\, e^{itH_\Lambda})\,(\pi_0(\mathcal{R}(X, \sigma))) = \pi_0(\mathcal{R}(X, \sigma))$, $t \in \mathbb{R}$.
(b) Let C, D be positive constants such that $\|V_{kl}\| \le C$ and $V_{kl} = 0$ for $|k - l| \ge D$, $k, l \in \mathbb{Z}$. Then $\lim_{\Lambda \nearrow \mathbb{Z}} (\mathrm{Ad}\, e^{itH_\Lambda})$, $t \in \mathbb{R}$ exists pointwise on $\pi_0(\mathcal{R}(X, \sigma))$ in the norm topology.

A proof of this statement is given in [2]. It generalizes the results on a class of models describing particles which are confined to the points of a one-dimensional lattice by a harmonic pinning potential and interact with their nearest neighbors [4]. In the present more general form it also has applications to other models of physical interest. These results provide evidence to the effect that the resolvent algebras are an expedient framework also for the discussion of the dynamics of infinite systems. Yet a full assessment of their power for the treatment of such systems requires further analysis.

2.5 Conclusions

In the present survey we have outlined some recent structural results and instructive applications of the theory of resolvent algebras. These algebras are built from the resolvents of the canonical operators in quantum theory and their algebraic relations encode the basic kinematical features of quantum systems just as well as the Weyl algebras. But, as we have shown, the novel approach cures several shortcomings of this traditional algebraic setting.

The resolvent algebras comply with the condition that kinematical algebras of quantum systems must have ideals if they are to carry various dynamics of physical interest. This requirement can easily be inferred from the preceding arguments in case of a single particle: there the cocycles $\Gamma(t) = e^{itH}e^{-itH_0}$ appearing in the interaction picture have the property that the differences $(\Gamma(t) - 1)$ are compact operators for generic interaction potentials. Hence $(e^{itH}We^{-itH} - e^{itH_0}We^{-itH_0})$ is a compact operator for any choice of bounded operator W. It is then clear that any unital C*-algebra which is stable under the action of these dynamics must contain compact operators and consequently have ideals.

The resolvent algebras, respectively their subalgebras corresponding to finite subsystems, contain these ideals from the outset. As we have demonstrated by several physically significant examples, the ideals play a substantial role in the construction of dynamics of finite and infinite quantum systems. For they accommodate the terms in the Dyson expansion of the cocycles resulting from the interaction picture and thereby entail the stability of the resolvent algebras under the action of the perturbed dynamics. In order to cover a wider class of models it would, however, be desirable to invent some more direct argument, avoiding this expansion and the ensuing questions of convergence.

The ideals of the resolvent algebras also play a prominent role in their classification. The nesting of primitive ideals encodes precise information about the size of the underlying quantum system, i.e. its dimension. It is a complete algebraic invariant in the finite dimensional case. There is also a sharp algebraic distinction between finite and infinite quantum systems in terms of their minimal ideals. In either case the resolvent algebras have comfortable algebraic properties: they are nuclear, thereby allowing to form unambiguously tensor products with other algebras which plays a role in the discussion of coupled systems.

In company with the resolvents of the canonical operators all their continuous functions vanishing at infinity are contained in the resolvent algebras. This feature ensures, as we have shown, that many operators of physical interest are affiliated with the resolvent algebras. It also implies that these algebras contain multiplicative mollifiers for unbounded operators which appear in the algebraic treatment of supersymmetric models [3] or of constraint systems [4, 6]. Thus the resolvent algebras provide in many respects a natural and convenient mathematical setting for the discussion of finite and infinite quantum systems.

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