# 9 Differential Equations

**1. A Differential Equation** is an equation, in which one or more variables, one or more functions of these variables, and also the derivatives of these functions with respect to these variables occur. The *order* of a differential equation is equal to the order of the highest occurring derivative.
**2. Ordinary and Partial Differential Equations** differ from each other in the number of their independent variables; in the first case there is only one, in the second case there are several.

■ **A:** $\left(\frac{dy}{dx}\right)^2 - xy^5\frac{dy}{dx} + \sin y = 0.$ ■ **B:** $xd^2ydx - dy(dx)^2 = e^y(dy)^3.$ ■ **C:** $\frac{\partial^2 z}{\partial x\partial y} = xyz\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}.$

## 9.1 Ordinary Differential Equations

### 1. General Ordinary Differential Equation of Order $n$

in *implicit form* has the equation

$$F\left[x, y(x), y'(x), \ldots, y^{(n)}(x)\right] = 0. \tag{9.1}$$

If this equation is solved for $y^{(n)}(x)$, then it is the *explicit form* of an ordinary differential equation of order $n$.

### 2. Solution or Integral

of a differential equation is every function satisfying the equation in an interval $a \le x \le b$ which can be also infinite. A solution, which contains $n$ arbitrary constants $c_1, c_2, \ldots, c_n$, is called the *general solution* or *general integral*. If the values of these constants are fixed, a *particular integral* or a *particular solution* is obtained. The value of these constants can be determined by $n$ further conditions. If the values of $y$ and its derivatives up to order $n-1$ are prescribed at one of the endpoints of the interval, then the problem is called an *initial value problem*. If there are given values at both endpoints of the interval, then the problem is called a *boundary value problem*.

■ The differential equation $-y'\sin x + y\cos x = 1$ has the general solution $y = \cos x + c\sin x$. For the condition $c = 0$ one gets the particular solution $y = \cos x$.

### 3. Initial Value Problem

If the $n$ values $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$ are given at $x_0$ for the solution $y = y(x)$ of an $n$-th order ordinary differential equation, then an *initial value problem* is given. The numbers are called the *initial values* or *initial conditions*. They form a system of $n$ equations for the unknown constants $c_1, c_2, \ldots, c_n$ of the general solution of the $n$-th order ordinary differential equation.

■ The harmonic motion of a special elastic spring-mass system can be modeled by the initial value problem $y'' + y = 0$ with $y(0) = y_0, y'(0) = 0$. The solution is $y = y_0\cos x$.

### 4. Boundery Value Problem

If the solution of an ordinary differential equation and/or its derivatives are given at several points of its domain, then these values are called the *boundary conditions*. A differential equation with boundary conditions is called a *boundary value problem*.

■ The bending line of a bar with fixed endpoints and uniform load is described by the differential equation $y'' = x - x^2$ with the boundary conditions $y(0) = 0, y(1) = 0$ $(0 \le x \le 1)$. The solution is $y = \frac{x^3}{6} - \frac{x^4}{12} - \frac{x}{12}$.

### 9.1.1 First-Order Differential Equations

#### 9.1.1.1 Existence Theorems, Direction Field

**1. Existence of a Solution**

In accordance with the Cauchy existence theorem the differential equation

$$y' = f(x, y) \tag{9.2}$$



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has at least one solution in a neighborhood of $x_0$ such that it takes the value $y_0$ at $x = x_0$ if the function $f(x, y)$ is continuous in a neighborhood $G$ of the point $(x_0, y_0)$. For example, $G$ can be selected as the region given by $|x - x_0| < a$ and $|y - y_0| < b$ with some $a$ and $b$.

## 2. Lipschitz Condition

The *Lipschitz condition* with respect to $y$ is satisfied by $f(x, y)$ if

$$|f(x, y_1) - f(x, y_2)| \le N|y_1 - y_2| \tag{9.3}$$

holds for all $(x, y_1)$ and $(x, y_2)$ from $G$, where $N$ is independent of $x$, $y_1$, and $y_2$. If this condition is satisfied, then the differential equation (9.2) has a unique solution through $(x_0, y_0)$. The Lipschitz condition is obviously satisfied if $f(x, y)$ has a bounded partial derivative $\partial f/\partial y$ in this neighborhood. In 9.1.1.4, p. 546 there are examples in which the assumptions of the Cauchy existence theorem are not satisfied.

## 3. Direction Field

If the graph of a solution $y = \varphi(x)$ of the differential equation $y' = f(x, y)$ goes through the point $P(x, y)$, then the slope $dy/dx$ of the tangent line of the graph at this point can be determined from the differential equation. So, at every point $(x, y)$ the differential equation defines the slope of the tangent line of the solution passing through the considered point. The collection of these directions **(Fig. 9.1)** forms the *direction field*. An element of the direction field is a point together with the direction associated to it. Integration of a first-order differential equation geometrically means to connect the elements of a direction field into an *integral curve*, whose tangents have the same slopes at all points as the corresponding elements of the direction field.

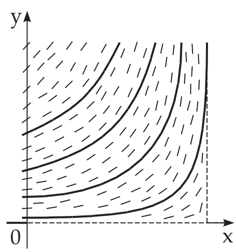


Figure 9.1

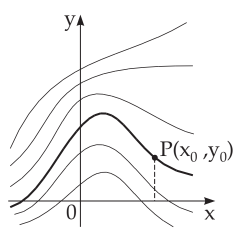


Figure 9.2

## 4. Vertical Directions

If a vertical direction can be found in a direction field, e.a., if the function $f(x, y)$ has a pole, then one can change the role of the independent and dependent variables and consider the differential equation

$$\frac{dx}{dy} = \frac{1}{f(x, y)} \tag{9.4}$$

as an equivalent equation to (9.2). In the region where the conditions of the existence theorems are fulfilled for the differential equations (9.2) or (9.4), there exists a unique integral curve **(Fig. 9.2)** through every point $P(x_0, y_0)$.

## 5. General Solution

The set of all integral curves of (9.2) can be characterized by one parameter and it can be given by the equation

$$F(x, y, C) = 0 \tag{9.5a}$$

of the corresponding one-parameter family of curves. The parameter $C$, an arbitrary constant, can be chosen freely and it is a necessary part of the general solution of every first-order differential equation.

A particular solution $y = \varphi(x)$, which satisfies the condition $y_0 = \varphi(x_0)$, can be obtained from the general solution (9.5a) if $C$ is expressed from the equation

$$F(x_0, y_0, C) = 0. \tag{9.5b}$$

### 9.1.1.2 Important Solution Methods

#### 1. Separation of Variables

If a differential equation can be transformed into the form

$$M(x)N(y)dx + P(x)Q(y)dy = 0, \tag{9.6a}$$

then it can be rewritten as

$$R(x)dx + S(y)dy = 0, \tag{9.6b}$$

where the variables $x$ and $y$ are separated into two terms. To get this form, equation (9.6a) is divided by $P(x)N(y)$. The general solution of (9.6a) is

$$\int \frac{M(x)}{P(x)}dx + \int \frac{Q(y)}{N(y)}dy = C. \tag{9.7}$$

If for some values $x = \overline{x}$ or $y = \overline{y}$, the functions $P(x)$ or $N(y)$ or both are equal to zero, then the constant functions $x = \overline{x}$ or/and $y = \overline{y}$ are also solutions of the differential equation. They are called *singular solutions*.

■ $xdy + ydx = 0$; $\displaystyle\int \frac{dy}{y} + \int \frac{dx}{x} = C$; $\ln|y| + \ln|x| = C = \ln|c|$; $yx = c$. If one allows also $c = 0$ in this final equation, then one has the singular solutions $y \equiv 0$ and $x \equiv 0$.

#### 2. Homogeneous Equations

If $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same order (see 2.18.2.6, **1.**, p. 122), then in the equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{9.8}$$

the variables can be separated by substitution of $u = y/x$.

■ $x(x - y)y' + y^2 = 0$ with $y = u(x)x$, gives $(1 - u)u' + u/x = 0$, then by separation of the variables holds $\displaystyle\int \frac{(1-u)}{u}du = -\int \frac{1}{x}dx$. After integration: $\ln|x| + \ln|u - u| = C = \ln|c|$, $ux = ce^u$, $y = ce^{y/x}$. As can be seen in the preceding paragraph, Separation of Variables, the line $x = 0$ is also an integral curve.

#### 3. Exact Differential Equations

An *exact differential equation* is an equation of the form

$$M(x, y)dx + N(x, y)dy = 0 \quad \text{or} \quad N(x, y)y' + M(x, y) = 0, \tag{9.9a}$$

if there exists a function $\Phi(x, y)$ of two variables such that

$$M(x, y)dx + N(x, y)dy \equiv d\Phi(x, y), \tag{9.9b}$$

i.e., if the left side of (9.9a) is the total differential of a function $\Phi(x, y)$ (see 6.2.2.1, p. 447). If functions $M(x, y)$ and $N(x, y)$ and their first-order partial derivatives are continuous on a connected domain $G$, then the equality

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{9.9c}$$

is a necessary and sufficient condition for equation (9.9a) to be exact. In this case the general solution of (9.9a) is the function

$$\Phi(x, y) = C \quad (C = \text{const}), \tag{9.9d}$$

which can be calculated according to (8.3.4), 8.3.4.4, p. 522 as the integral

$$\Phi(x,y) = \int_{x_0}^{x} M(\xi, y)\, d\xi + \int_{y_0}^{y} N(x_0, \eta)\, d\eta, \tag{9.9e}$$

where $x_0$ and $y_0$ can be chosen arbitrarily from $G$.

■ Examples will be given later.

## 4. Integrating Factor

A function $\mu(x,y)$ is called an *integrating factor* or a *multiplier* if the equation

$$M dx + N dy = 0 \tag{9.10a}$$

multiplied by $\mu(x,y)$ becomes an exact differential equation. The integrating factor satisfies the differential equation

$$N \frac{\partial \ln \mu}{\partial x} - M \frac{\partial \ln \mu}{\partial y} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}. \tag{9.10b}$$

Every particular solution $\mu$ of this equation is an integrating factor. To give a general solution of this partial differential equation is much more complicated than to solve the original equation, so usually one is looking for the solution $\mu(x,y)$ in a special form, e.g., $\mu(x), \mu(y), \mu(xy)$ or $\mu(x^2+y^2)$.

■ To solve the differential equation $(x^2+y)\,dx - x\,dy = 0$, the equation for the integrating factor is $-x\dfrac{\partial \ln \mu}{\partial x} - (x^2+y)\dfrac{\partial \ln \mu}{\partial y} = 2$. An integrating factor which is independent of $y$ must satisfy $x\dfrac{\partial \ln \mu}{\partial x} = -2$, so $\mu = \dfrac{1}{x^2}$. Multiplication of the given differential equation by $\mu$ yields $\left(1 + \dfrac{y}{x^2}\right) dx - \dfrac{1}{x} dy = 0$. The general solution according to (9.9e) with the selection of $x_0 = 1, y_0 = 0$ is then:
$\Phi(x,y) \equiv \displaystyle\int_1^x \left(1 + \frac{y}{\xi^2}\right) d\xi - \int_0^y d\eta = C$ or $x - \dfrac{y}{x} = C_1$.

## 5. First-Order Linear Differential Equations

A *first-order linear differential equation* has the form

$$y' + P(x)y = Q(x), \tag{9.11a}$$

where the unknown function and its derivative occur only in first degree, and $P(x)$ and $Q(x)$ are given functions. If $P(x)$ and $Q(x)$ are continuous functions on a finite, closed interval, then the differential equation satisfies the conditions of the *Picard-Lindelöf theorem* (see 12.2.2.4,**4.**, p. 668) in this region. An integrating factor is here

$$\mu = \exp\left(\int P\, dx\right), \tag{9.11b}$$

the general solution is

$$y = \exp\left(-\int P\, dx\right)\left[\int Q \exp\left(\int P\, dx\right) dx + C\right]. \tag{9.11c}$$

Replacing the indefinite integrals by definite ones with lower bound $x_0$ and upper bound $x$ in this formula, then for the solution $y(x_0) = C$ (see 8.2.1.2, **1.**, p. 495). If $y_1$ is any particular solution of the differential equation, then the general solution of the differential equation is given by the formula

$$y = y_1 + C \exp\left(-\int P\, dx\right). \tag{9.11d}$$

If $y_1(x)$ and $y_2(x)$ are two linearly independent particular solutions (see 9.1.2.3, **2.**, p. 553), then one can get the general solution without any integration as

$$y = y_1 + C(y_2 - y_1). \tag{9.11e}$$

■ To solve the differential equation $y' - y\tan x = \cos x$ with the initial condition $x_0 = 0$, $y_0 = 0$. Calculating $\exp\left(-\int_0^x \tan x\,dx\right) = \cos x$ one gets the solution according to (9.11c):

$$y = \frac{1}{\cos x}\int_0^x \cos^2 x\,dx = \frac{1}{\cos x}\left[\frac{\sin x\cos x + x}{2}\right] = \frac{\sin x}{2} + \frac{x}{2\cos x}.$$

## 6. Bernoulli Differential Equations

The *Bernoulli differential equation* is an equation of the form

$$y' + P(x)y = Q(x)y^n \quad (n \neq 0, n \neq 1), \tag{9.12}$$

which can be reduced to a linear differential equation if it is divided by $y^n$ and the new variable $z = y^{-n+1}$ is introduced.

■ Solution of the differential equation $y' - \frac{4y}{x} = x\sqrt{y}$. Since $n = 1/2$, dividing by $\sqrt{y}$ and introducing the new variable $z = \sqrt{y}$ leads to the equation $\frac{dz}{dx} - \frac{2z}{x} = \frac{x}{2}$. By using the formulas for the solution of a linear differential equation there is $\exp(\int P\,dx) = \frac{1}{x^2}$ and $z = x^2\left[\int \frac{x}{2}\frac{1}{x^2}\,dx + C\right] = x^2\left[\frac{1}{2}\ln|x| + C\right]$. So, finally, $y = x^4\left(\frac{1}{2}\ln|x| + C\right)^2$.

## 7. Riccati Differential Equations

The *Riccati differential equation*

$$y' = P(x)y^2 + Q(x)y + R(x), \tag{9.13a}$$

usually cannot be solved by elementary integration, i.e., not by using a final number of successive elementary integrations. However it is possible to transform it by suitable substitutions into differential equations for which solutions often can be found.

**Method 1:** By the substitution

$$y = \frac{u(x)}{P(x)} + \beta(x) \tag{9.13b}$$

the Riccati differential equation can be transformed into the *normal form*

$$\frac{du}{dx} = u^2 + R_0(x) \quad (9.13c) \qquad \text{with} \quad R_0(x) = P^2\beta^2 + QP\beta + PR - P\beta'. \tag{9.13d}$$

Therefore $\beta(x)$ is determined so that terms with the factor $u(x)$ disappear.
If a particular solution $u_1(x)$ of (9.13c) is known, which can be found, e.g., by a suitable approach, then by the help of the substitution

$$u = \frac{1}{z(x)} + u_1(x) \tag{9.13e}$$

(9.13c) is to be transformed into the linear differential equation for $z(x)$:

$$z' + 2u_1(x)z - 1 = 0. \tag{9.13f}$$

From the solution of (9.13f) the solution of (9.13a) is obtained by using (9.13e) and (9.13b).

**Method 2:** By the substitution

$$y = -\frac{v'}{P(x)v(x)} \tag{9.13g}$$

(9.13a) is transformed into a linear differential equation of second order (see 9.1.2.6,**1.**, p. 560):

$$Pv'' - (P' + PQ)v' + P^2Rv = 0. \tag{9.13h}$$

■ To solve the differential equation $y' + y^2 + \frac{1}{x}y - \frac{4}{x^2} = 0$, i.e. for $P = -1, Q = -\frac{1}{x}, R = \frac{4}{x^2}$.

**Method 1:** One gets $\beta(x) = -\frac{1}{2x}$ and with the help of $y = -u(x) - \frac{1}{2x}$ one gets the normal form $u' = u^2 - \frac{15}{4x^2}$. Particular solutions of the normal form can be got, e.g., with the approach $u = \frac{a}{x}$: $u_1(x) = \frac{3}{2x}$, $u_2(x) = -\frac{5}{2x}$. After substituting $u = \frac{1}{z(x)} + \frac{3}{2x}$ the differential equation $z' + \frac{3}{x}z + 1 = 0$ follows with the solution $z(x) = -\frac{x}{4} + \frac{K}{x^3} = \frac{4K - x^4}{4x^3}$ ($K$ const). The inverse transformation gives $y = \frac{2x^4 + 2C}{x^5 - Cx}$ $(C = 4K)$.

**Method 2:** According to (9.13h) the Euler differential equation $x^2v'' + xv' - 4v = 0$ is obtained with the general solution $v(x) = C_1x^2 + C_2\frac{1}{x^2}$ (see ■ concerning the Euler differential equation, p. 557). One of the constants $C_1$ and $C_2$ can be chosen freely, e.g. $C_2 = -1$ then from (9.13h) follows $y = \frac{2x^4 + 2C_1}{x^5 - C_1x}$.

### 9.1.1.3 Implicit Differential Equations

#### 1. Solution in Parametric Form

Given a differential equation in implicit form

$$F(x, y, y') = 0. \tag{9.14}$$

There are $n$ integral curves passing through a point $P(x_0, y_0)$ if the following conditions hold:

**a)** The equation $F(x_0, y_0, p) = 0$ $(p = dy/dx)$ has $n$ real roots $p_1, \ldots, p_n$ at the point $P(x_0, y_0)$.

**b)** The function $F(x, y, p)$ and its first partial derivatives are continuous at $x = x_0$, $y = y_0$, $p = p_i$; furthermore $\partial F/\partial p \neq 0$.

If the original equation can be solved with respect to $y'$, then it yields $n$ equations of the explicit forms discussed above. Solving these equations one gets $n$ families of integral curves. If the equation can be written in the form $x = \varphi(y, y')$ or $y = \psi(x, y')$, then putting $y' = p$ and considering $p$ as an auxiliary variable, after differentiation with respect to $y$ or $x$ one obtains an equation for $dp/dy$ or $dp/dx$ which is solved with respect to the derivative. A solution of this equation together with the original equation (9.14) determines a desired solution in parametric form.

■ To get the solution of the differential equation $x = yy' + y'^2$, one substitutes $y' = p$ and gets $x = py + p^2$. Differentiation with respect to $y$ and substituting $\frac{dx}{dy} = \frac{1}{p}$ results in $\frac{1}{p} = p + (y + 2p)\frac{dp}{dy}$ or $\frac{dy}{dp} - \frac{py}{1 - p^2} = \frac{2p^2}{1 - p^2}$. Solving this equation for $y$ one obtains $y = -p + \frac{C + \arcsin p}{\sqrt{1 - p^2}}$ ($C$ const). Substitution into the initial equation gives the solution for $x$ in parametric form.

#### 2. Lagrange Differential Equation

The *Lagrange differential equation* is the equation

$$a(y')x + b(y')y + c(y') = 0. \tag{9.15a}$$

The solution can be determined by the method given above. If for $p = p_0$ holds

$$a(p) + b(p)p = 0, \tag{9.15b}$$

then

$$a(p_0)x + b(p_0)y + c(p_0) = 0 \tag{9.15c}$$

is a singular solution of (9.15a).

### 3. Clairaut Differential Equation

The *Clairaut differential equation* is the special case of the Lagrange differential equation if

$$a(p) + b(p)p \equiv 0, \tag{9.16a}$$

and so it can be transformed into the form

$$y = y'x + f(y'). \tag{9.16b}$$

The general solution is

$$y = Cx + f(C). \tag{9.16c}$$

Besides the general solution, the Clairaut differential equation also has a singular solution, which can be obtained by eliminating the constant $C$ from the equations

$$y = Cx + f(C) \quad (9.16d) \qquad \text{and} \quad 0 = x + f'(C), \tag{9.16e}$$

The second equation can be obtained by differentiating the first one with respect to $C$. Geometrically, the singular solution is the envelope (see 3.6.1.7, p. 255) of the solution family of lines **(Fig. 9.3)**.

■ Solution of the differential equation $y = xy' + y'^2$. The general solution is $y = Cx + C^2$ . The singular solution one gets with the help of the equation $x + 2C = 0$ to eliminate $C$, and hence $x^2 + 4y = 0$. **Fig. 9.3** shows this case.

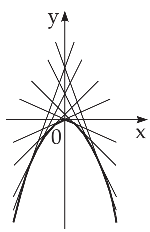

Figure 9.3

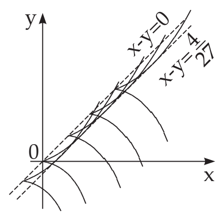

Figure 9.4

## 9.1.1.4 Singular Integrals and Singular Points

### 1. Singular element

An element $(x_0, y_0, y_0')$ is called a *singular element* of the differential equation, if in addition to the differential equation

$$F(x, y, y') = 0 \tag{9.17a}$$

it also satisfies the equation

$$\frac{\partial F}{\partial y'} = 0. \tag{9.17b}$$

### 2. Singular Integral

An integral curve from singular elements is called a *singular integral curve*; the equation

$$\varphi(x, y) = 0 \tag{9.17c}$$

of a singular integral curve is called a *singular integral*. The envelopes of the integral curves are singular integral curves **(Fig. 9.3)**; they consist of the singular elements.

The uniqueness of the solution (see 9.1.1.1, **1.**, p. 540) usually fails at the points of a singular integral curve.

### 3. Determination of Singular Integrals

Usually one cannot obtain singular integrals for any values of the arbitrary constants of the general solution. To determine the singular solution of a differential equation (9.17a) with $p = y'$ one has to introduce the equation

$$\frac{\partial F}{\partial p} = 0 \tag{9.17d}$$

and to eliminate $p$. If the obtained relation is a solution of the given differential equation, then it is a singular solution. The equation of this solution should be transformed into a form which does not contain multiple-valued functions, in particular no radicals where the complex values should also be considered.

*Radicals* are expressions obtained by nesting algebraic equations (see 2.2.1, p. 62). If the equation of the family of integral curves is known, i.e., the general solution of the given differential equation is known, then one can determine the envelope of the family of curves, the singular integral, with the methods of differential geometry (see 3.6.1.7, p. 255).

■ Solution of the differential equation $x - y - \frac{4}{9}y'^2 + \frac{8}{27}y'^3 = 0$. Substituting $y' = p$, the calculation of the additional equation with (9.17d) yields $-\frac{8}{9}p + \frac{8}{9}p^2 = 0$. Elimination of $p$ results in equation a) $x - y = 0$ and b) $x - y = \frac{4}{27}$, where a) is not a solution, b) is a solution, a special case of the general solution $(y - C)^2 = (x - C)^3$. The integral curves of a) and b) are shown in **Fig. 9.4**.

## 4. Singular Points of a Differential Equation

Singular points of a differential equation are the points where the right side of the differential equation

$$y' = f(x, y) \tag{9.18a}$$

is not defined. This is the case, e.g., in the differential equations of the following forms:

### 1. Differential Equation with a Fraction of Linear Functions

$$\frac{dy}{dx} = \frac{ax + by}{cx + ey} \qquad (ae - bc \neq 0) \tag{9.18b}$$

has an *isolated singular point* at $(0, 0)$, since the assumptions of the existence theorem are fulfilled almost at every point arbitrarily close to $(0, 0)$ but not at this point itself. The conditions are not fulfilled at the points where $cx + ey = 0$. One can force the fulfillment of the conditions at these points exchanging the role of the variables and considering the equation

$$\frac{dx}{dy} = \frac{cx + ey}{ax + by}\,. \tag{9.18c}$$

The behavior of the integral curve in the neighborhood of a singular point depends on the roots of the *characteristic equation*

$$\lambda^2 - (b + c)\lambda + bc - ae = 0. \tag{9.18d}$$

The following cases can be distinguished:

**Case 1:** If the roots are real and they have the same sign, then the singular point is a *branch point.* The integral curves in a neighborhood of the singular point pass through it and if the roots of the characteristic equation do not coincide, they have a common tangent except for one. If the roots coincide, then either all integral curves have the same tangent, or there is a unique integral curve passing through the singular point in each direction.

■ **A:** For the differential equation $\frac{dy}{dx} = \frac{2y}{x}$ the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$, $\lambda_1 = 2$, $\lambda_2 = 1$. The integral curves have the equation $y = C\,x^2$ **(Fig. 9.5)**. The general solution also contains the line $x = 0$ considering the form $x^2 = C_1\,y$.

■ **B:** The characteristic equation for $\frac{dy}{dx} = \frac{x + y}{x}$ is $\lambda^2 - 2\lambda + 1 = 0$, $\lambda_1 = \lambda_2 = 1$. The integral curves are $y = x\ln|x| + Cx$ **(Fig. 9.6)**. The singular point is a so-called *node*.

■ **C:** The characteristic equation for $\dfrac{dy}{dx} = \dfrac{y}{x}$ is $\lambda^2 - 2\lambda + 1 = 0$, $\lambda_1 = \lambda_2 = 1$. The integral curves are $y = C\,x$ **(Fig. 9.7)**. The singular point is a so-called *ray point*.

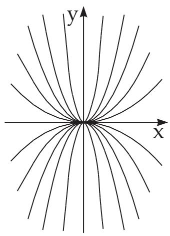


Figure 9.5

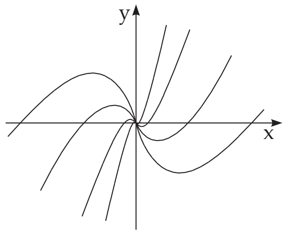


Figure 9.6

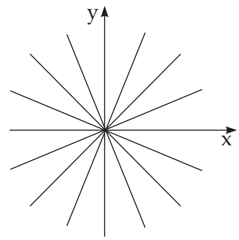


Figure 9.7

**Case 2:** If the roots are real and they have different signs, the singular point is a *saddle point*, and two of the integral curves pass through it.

■ **D:** The characteristic equation for $\dfrac{dy}{dx} = -\dfrac{y}{x}$ is $\lambda^2 - 1 = 0$, $\lambda_1 = +1$, $\lambda_2 = -1$. The integral curves are $x\,y = C$ **(Fig. 9.8)**. For $C = 0$ the particular solutions $x = 0$, $y = 0$ hold.

**Case 3:** If the roots are conjugate complex numbers with a non-zero real part ($\mathrm{Re}(\lambda) \neq 0$), then the singular point is a *spiral point* which is also called a *focal point*, and the integral curves wind about this singular point.

■ **E:** The characteristic equation for $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ is $\lambda^2 - 2\lambda + 2 = 0$, $\lambda_1 = 1 + \mathrm{i}$, $\lambda_2 = 1 - \mathrm{i}$. The integral curves in polar coordinates are $r = C\,e^{\varphi}$ **(Fig. 9.9)**.

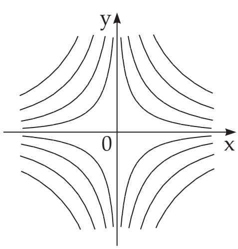


Figure 9.8

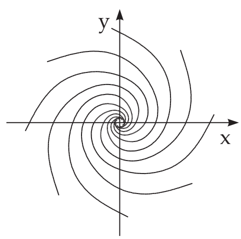


Figure 9.9

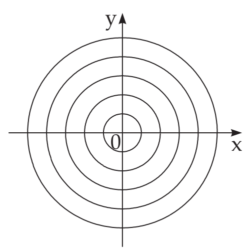


Figure 9.10

**Case 4:** If the roots are pure imaginary numbers, then the singular point is a *central point*, or *center*, which is surrounded by the closed integral curves.

■ **F:** The characteristic equation for $\dfrac{dy}{dx} = -\dfrac{x}{y}$ is $\lambda^2 + 1 = 0$, $\lambda_1 = \mathrm{i}$, $\lambda_2 = -\mathrm{i}$. The integral curves are $x^2 + y^2 = C$ **(Fig. 9.10)**.

**2. Differential Equation with the Ratio of Two Arbitrary Functions**

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} \tag{9.19a}$$

has the singular points for the values of the variables where

$$P(x,y) = Q(x,y) = 0. \tag{9.19b}$$

If $P$ and $Q$ are continuous functions and they have continuous partial derivatives, (9.19a) can be written in the form

$$\frac{dy}{dx} = \frac{a(x-x_0) + b(y-y_0) + P_1(x,y)}{c(x-x_0) + e(y-y_0) + Q_1(x,y)}. \tag{9.19c}$$

Here $x_0$ and $y_0$ are the coordinates of the singular point and $P_1(x,y)$ and $Q_1(x,y)$ are infinitesimals of a higher order than the distance of the point $(x,y)$ from the singular point $(x_0,y_0)$. With these assumptions the type of a singular point of the given differential equation is the same as that of the *approximate equation* obtained by omitting the terms $P_1$ and $Q_1$, with the following exceptions:

**a)** If the singular point of the approximate equation is a center, the singular point of the original equation is either a center or a focal point.

**b)** If $a\,e - b\,c = 0$, i.e., $\frac{a}{c} = \frac{b}{e}$ or $a = c = 0$ or $a = b = 0$, then the type of singular point should be determined by examining the terms of higher order.

### 9.1.1.5 Approximation Methods for Solution of First-Order Differential Equations

#### 1. Successive Approximation Method of Picard

The integration of the differential equation

$$y' = f(x,y) \tag{9.20a}$$

with the initial condition $y = y_0$ for $x = x_0$ results in the fixed-point problem

$$y = y_0 + \int\limits_{x_0}^{x} f(x,y)\,dx. \tag{9.20b}$$

Substituting another function $y_1(x)$ instead of $y$ into the right-hand side of (9.20b), then the result will be a new function $y_2(x)$, which is different from $y_1(x)$, if $y_1(x)$ is not already a solution of (9.20a). Substituting $y_2(x)$ instead of $y$ into the right-hand side of (9.20b) gives a function $y_3(x)$. If the conditions of the existence theorem are fulfilled (see 9.1.1.1, **1.**, p. 540), the sequence of functions $y_1, y_2, y_3, \ldots$ converges to the desired solution in a certain interval containing the point $x_0$.
This *Picard method of successive approximation* is an *iteration method* (see 19.1.1, p. 949).

■ Solve the differential equation $y' = e^x - y^2$ with initial values $x_0 = 0$, $y_0 = 0$. Rewriting the equation in integral form and using the successive approximation method with an initial approximation $y_0(x) \equiv 0$ gives: $y_1 = \int_0^x e^x\,dx = e^x - 1$, $\quad y_2 = \int_0^x \left[e^x - (e^x-1)^2\right] dx = 3e^x - \frac{1}{2}e^{2x} - x - \frac{5}{2}$, etc.

#### 2. Solution by Series Expansion

The Taylor series expansion of the solution of a differential equation (see 7.3.3.3, **1.**, p. 471) can be given in the form

$$y = y_0 + (x-x_0)y_0{}' + \frac{(x-x_0)^2}{2}y_0{}'' + \cdots + \frac{(x-x_0)^n}{n!}y_0{}^{(n)} + \cdots \tag{9.21}$$

if the values $y_0{}'$, $y_0{}''$, $\ldots$, $y_0{}^{(n)}, \ldots$ of all derivatives of the solution function are known at the initial value $x_0$ of the independent variable. The values of the derivatives can be determined by successively differentiating the original equation and substituting the initial conditions. If the differential equation can be differentiated infinitely many times, the obtained series will be convergent in a certain neighborhood of the initial value of the independent variable. This method can be used also for $n$-th order differential equations.

**Remark**: The above result is the Taylor series of the function, which may not represent the function itself (see 7.3.3.3, **1.**, p. 471).

It is often useful to substitute the solution by an infinite series with unknown coefficients, and to determine them by comparing coefficients.

■ **A:** To solve the differential equation $y' = e^x - y^2$, $x_0 = 0$, $y_0 = 0$ one can consider the series $y = a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$. Substituting this into the equation considering the formula (7.88), p. 470 for the square of the series gives

$$a_1 + 2a_2x + 3a_3x^2 + \cdots + [{a_1}^2x^2 + 2a_1a_2x^3 + ({a_2}^2 + 2a_1a_3)x^4 + \cdots] = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

Comparing coefficients gives: $a_1 = 1$, $2a_2 = 1$, $3a_3 + {a_1}^2 = \frac{1}{2}$, $4a_4 + 2a_1a_2 = \frac{1}{6}$, etc. Solving these equations successively and substituting the coefficient values into the series representation yields $y = x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{5}{24}x^4 + \cdots$.

■ **B:** The same differential equation with the same initial conditions can also be solved in the following way: Substituting $x = 0$ into the equation, gives ${y_0}' = 1$ and successive differentiation yields $y'' = e^x - 2yy'$, ${y_0}'' = 1$, $y''' = e^x - 2y'^2 - 2yy''$, ${y_0}''' = -1$, $y^{(4)} = e^x - 6y'y'' - 2yy'''$, ${y_0}^{(4)} = -5$, *etc.* From the Taylor theorem (see 7.3.3.3, **1.**, p. 471) follows the solution $y = x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{5x^4}{4!} + \cdots$.

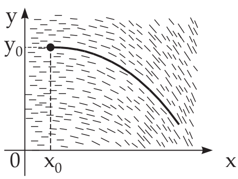


Figure 9.11

### 3. Graphical Solution of Differential Equations

The graphical integration of a differential equation is a method, which is based on the direction field (see 9.1.1.1, **3.**, p. 541). The integral curve in **Fig. 9.11** is represented by a broken line which starts at the given initial point and is composed of short line segments. The directions of the line segments are always the same as the direction of the direction field at the starting point of the line segment. This is also the endpoint of the previous line segment.

### 4. Numerical Solution of Differential Equations

The numerical solutions of differential equations will be discussed in detail in 19.4, p. 969. Numerical methods are used to determine a solution of a differential equation, if the equation $y' = f(x, y)$ does not belong to the special cases discussed above whose analytic solutions are known, or if the function $f(x, y)$ is too complicated. This can happen if $f(x, y)$ is non-linear in $y$.

## 9.1.2 Differential Equations of Higher Order and Systems of Differential Equations

### 9.1.2.1 Basic Results

### 1. Existence of a Solution

**1. Reduction to a System of Differential Equations** Every explicit $n$-th order differential equation

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \tag{9.22a}$$

by introducing the new variables

$$y_1 = y', \quad y_2 = y'', \ldots, y_{n-1} = y^{(n-1)} \tag{9.22b}$$

can be reduced to a system of $n$ first-order differential equations

$$\frac{dy}{dx} = y_1, \quad \frac{dy_1}{dx} = y_2, \ldots, \frac{dy_{n-1}}{dx} = f(x, y, y_1, \ldots, y_{n-1}). \tag{9.22c}$$

**2. Existence of a System of Solutions** The system of $n$ differential equations

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \ldots, y_n) \qquad (i = 1, 2, \ldots, n), \tag{9.23a}$$

which is more general than system (9.22c), has a unique system of solutions

$$y_i = y_i(x) \quad (i = 1, 2, \ldots, n), \tag{9.23b}$$

which is defined in an interval $x_0 - h \le x \le x_0 + h$ and for $x = x_0$ takes the previously given initial values $y_i(x_0) = y_i{}^0$ $(i = 1, 2, \ldots, n)$, if the functions $f_i(x, y_1, y_2, \ldots, y_n)$ are continuous with respect to all variables and satisfy the following Lipschitz condition.

**3. Lipschitz condition** For the values $x, y_i$ and $y_i + \Delta y_i$, which are in a certain neighborhood of the given initial values, the functions $f_i$ satisfy the following inequalities:

$$\begin{aligned} &|f_i(x, y_1 + \Delta y_1, y_2 + \Delta y_2, \ldots, y_n + \Delta y_n) - f_i(x, y_1, y_2, \ldots, y_n)| \\ &\qquad \le K\left(|\Delta y_1| + |\Delta y_2| + \cdots + |\Delta y_n|\right) \end{aligned} \tag{9.24a}$$

with a common constant $K$ (see also 9.1.1.1, **2.**, p. 541).
This fact implies that if the function $f(x, y, y', \ldots, y^{(n-1)})$ is continuous and satisfies the Lipschitz condition (9.24a), then the equation

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \tag{9.24b}$$

has a unique solution with the initial values $y(x_0) = y_0$, $y'(x_0) = y_0{}'$, ..., $y^{(n-1)}(x_0) = y_0{}^{(n-1)}$, and it is $(n-1)$ times continuously differentiable.

## 2. General Solution

**1.** The general solution of the differential equation (9.24b) contains $n$ independent arbitrary constants:

$$y = y(x, C_1, C_2, \ldots, C_n). \tag{9.25a}$$

**2.** In the geometrical interpretation the equation (9.25a) defines a family of curves depending on $n$ parameters. Every single one of these integral curves, i.e., the graph of the corresponding particular solution, can be obtained by a suitable choice of the constants $C_1, C_2, \ldots, C_n$. If the solution has to satisfy the above initial conditions, then the values $C_1, C_2, \ldots, C_n$ are determined from the following equations:

$$\begin{aligned} y(x_0, C_1, \ldots, C_n) &= y_0, \\ \left[\frac{d}{dx} y(x, C_1, \ldots, C_n)\right]_{x=x_0} &= y_0{}', \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left[\frac{d^{n-1}}{dx^{n-1}} y(x, C_1, \ldots, C_n)\right]_{x=x_0} &= y_0{}^{(n-1)}. \end{aligned} \tag{9.25b}$$

If these equations are inconsistent for any initial values in a certain domain, then the solution is not general in this domain, i.e., the arbitrary constants cannot be chosen independently.

**3.** The general solution of system (9.23a) also contains $n$ arbitrary constants. This general solution can be represented in two different ways: Either it is given in a form which is solved for the unknown functions

$$y_1 = F_1(x, C_1, \ldots, C_n),\ \ y_2 = F_2(x, C_1, \ldots, C_n), \ldots, y_n = F_n(x, C_1, \ldots, C_n) \tag{9.26a}$$

or in the form which is solved for the constants

$$\varphi_1(x, y_1, \ldots, y_n) = C_1,\ \ \varphi_2(x, y_1, \ldots, y_n) = C_2, \ldots, \varphi_n(x, y_1, \ldots, y_n) = C_n. \tag{9.26b}$$

In the case of (9.26b) each relation

$$\varphi_i(x, y_1, \ldots, y_n) = C_i \tag{9.26c}$$

is a *first integral* of the system (9.23a). The first integral can be defined independently of the general solution as a relation (9.26c). That is, (9.26c) will be an identity replacing $y_1, y_2, \ldots, y_n$ by any particular solution of the given system and replacing the constant by the arbitrary constant $C_i$ determined by this particular solution.
If any first integral is known in the form (9.26c), then the function $\varphi_i(x, y_1, \ldots, y_n)$ satisfies the partial different equation

$$\frac{\partial \varphi_i}{\partial x} + f_1(x, y_1, \ldots, y_n)\frac{\partial \varphi_i}{\partial y_1} + \cdots + f_n(x, y_1, \ldots, y_n)\frac{\partial \varphi_i}{\partial y_n} = 0. \tag{9.26d}$$

Conversely, each solution $\varphi_i(x, y_1, \ldots, y_n)$ of the partial differential equation (9.26d) defines a first integral of the system (9.23a) in the form (9.26c). The general solution of the system (9.23a) can be represented as a system of $n$ first integrals of system (9.23a), if the corresponding functions $\varphi_i(x, y_1, \ldots, y_n)$ $(i = 1, 2, \ldots, n)$ are linearly independent (see 9.1.2.3, **2.**, p. 553).

### 9.1.2.2 Lowering the Order

One of the most important solution methods for $n$-th order differential equations

$$f\left(x, y, y', \ldots, y^{(n)}\right) = 0 \tag{9.27}$$

is the substitution of variables in order to obtain a simpler differential equation, especially one of lower order. Different cases can be distinguished.

**1.** $\boldsymbol{f = f(y, y', \ldots, y^{(n)})}$**, i.e.,** $\boldsymbol{x}$ **does not appear explicitly:**

$$f\left(y, y', \ldots, y^{(n)}\right) = 0. \tag{9.28a}$$

By substitution

$$\frac{dy}{dx} = p, \quad \frac{d^2y}{dx^2} = p\frac{dp}{dy}, \ldots \tag{9.28b}$$

the order of the differential equation can be reduced from $n$ to $(n-1)$.

■ Reducing the order of the differential equation $yy'' - y'^2 = 0$ to one, with the substitution $y' = p, p\,dp/dy = y''$ it becomes a first-order differential equation $y\,p\,dp/dy - p^2 = 0$, and $y\,dp/dy - p = 0$ results in $p = C\,y = dy/dx$, $y = C_1 e^{Cx}$. Canceling $p$ does not result in a loss of a solution, since $p = 0$ gives the solution $y = C_1$, which is included in the general solution with $C = 0$.

**2.** $\boldsymbol{f = f(x, y', \ldots, y^{(n)})}$**, i.e.,** $\boldsymbol{y}$ **does not appear explicitly:**

$$f\left(x, y', ..., y^{(n)}\right) = 0. \tag{9.29a}$$

The order of the differential equation can be reduced from $n$ to $(n-1)$ by the substitution

$$y' = p. \tag{9.29b}$$

If the first $k$ derivatives are missing in the initial equation, then a suitable substitution is

$$y^{(k+1)} = p. \tag{9.29c}$$

■ The order of the differential equation $y'' - xy''' + (y''')^3 = 0$ will be reduced by the substitution $y'' = p$, so one gets a Clairaut differential equation $p - x\dfrac{dp}{dx} + \left(\dfrac{dp}{dx}\right)^3 = 0$ whose general solution is $p = C_1\,x + {C_1}^3$. Therefore, $y = \dfrac{C_1 x^3}{6} - \dfrac{{C_1}^3 x^2}{2} + C_2 x + C_3$. From the singular solution of the Clairaut differential equation $p = \dfrac{2\sqrt{3}}{9}x^{3/2}$ one gets the singular solution of the original equation: $y = \dfrac{8\sqrt{3}}{315}x^{7/2} + C_1 x + C_2$.

**3.** $\boldsymbol{f\left(x, y, y', \ldots, y^{(n)}\right)}$ **is a homogeneous function** (see 2.18.2.4, **4.**, p. 122) **in** $\boldsymbol{y, y', y'', \ldots, y^{(n)}}$**:**

$$f\left(x, y, y', \ldots, y^{(n)}\right) = 0. \tag{9.30a}$$

One can reduce the order by the substitution

$$z = \frac{y'}{y}, \quad \text{i.e.,} \quad y = e^{\int z\,dx}. \tag{9.30b}$$

■ Transforming the differential equation $yy'' - y'^2 = 0$ by the substitution $z = y'/y$, results in $\frac{dz}{dx} = \frac{yy'' - y'^2}{y^2} = 0$ so the order is reduced by one. One gets $z = C_1$, therefore, $\ln|y| = C_1x + C_2$, or $y = Ce^{C_1x}$ with $\ln|C| = C_2$.

**4.** $\boldsymbol{f = f(x, y, y', \ldots, y^{(n)})}$ **is a function of only** $\boldsymbol{x}$**:**

$$y^{(n)} = f(x). \tag{9.31a}$$

One gets the general solution by $n$ repeated integrations. It has the form

$$y = C_1 + C_2x + C_3x^2 + \cdots + C_nx^{n-1} + \psi(x) \tag{9.31b}$$

with

$$\psi(x) = \iint \cdots \int f(x)\,(dx)^n = \frac{1}{(n-1)!}\int_{x_0}^{x} f(t)(x-t)^{n-1}\,dt. \tag{9.31c}$$

It has to be mentioned here that $x_0$ is not an additional arbitrary constant, since the change in $x_0$ results in the change of $C_k$ because of the relation

$$C_k = \frac{1}{(k-1)!}y^{(k-1)}(x_0). \tag{9.31d}$$

### 9.1.2.3 Linear $n$-th Order Differential Equations

#### 1. Classification

A differential equation of the form

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_{n-1}y' + a_ny = F \tag{9.32}$$

is called an $n$-th order linear differential equation. Here $F$ and the coefficients $a_i$ are functions of $x$, which are supposed to be continuous in a certain interval. If $a_1, a_2, \ldots, a_n$ are constants, it is called a *differential equation with constant coefficients*. If $F \equiv 0$, then the linear differential equation is *homogeneous*, and if $F \not\equiv 0$, then it is *inhomogeneous*.

#### 2. Fundamental System of Solutions

A system of $n$ solutions $y_1, y_2, \ldots, y_n$ of a homogeneous linear differential equation is called a *fundamental system* if these functions are *linearly independent* on the considered interval, i.e., their linear combination $C_1\,y_1 + C_2\,y_2 + \cdots + C_n\,y_n$ is not identically zero for any system of values $C_1, C_2, \ldots, C_n$, except for the values $C_1 = C_2 = \cdots = C_n = 0$. The solutions $y_1, y_2, \ldots, y_n$ of a linear homogeneous differential equation form a fundamental system on the considered interval if and only if their *Wronskian determinant*

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{9.33}$$

is non-zero. For every solution system of a homogeneous linear differential equation the *formula of Liouville* is valid:

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} a_{n-1}(x)\,dx\right). \tag{9.34}$$

It follows from (9.34) that if the Wronskian determinant is zero somewhere in the solution interval, then it can be only identically zero. This means: The $n$ solutions $y_1, y_2, \ldots, y_n$ of the homogeneous linear differential equation are linearly dependent if even for a single point $x_0$ of the considered interval $W(x_0) = 0$. If the solutions $y_1, y_2, \ldots, y_n$ form a fundamental system of the differential equation, then the general solution of the linear homogeneous differential equation (9.32) is given as

$$y = C_1\,y_1 + C_2\,y_2 + \cdots + C_n\,y_n. \tag{9.35}$$

A linear $n$-th order homogeneous differential equation has exactly $n$ linearly independent solutions on an interval, where the coefficient functions $a_i(x)$ are continuous.

### 3. Lowering the Order

If a particular solution $y_1$ of a homogeneous differential equation is known, by assuming

$$y = y_1(x)u(x) \tag{9.36}$$

one can determine further solutions from a homogeneous linear differential equation of order $n-1$ for $u'(x)$.

### 4. Superposition Principle

If $y_1$ and $y_2$ are two solutions of the differential equation (9.32) for different right-hand sides $F_1$ and $F_2$, then their sum $y = y_1 + y_2$ is a solution of the same differential equation with the right-hand side $F = F_1 + F_2$. From this observation it follows that to get the general solution of an inhomogeneous differential equation it is sufficient to add any particular solution of the inhomogeneous differential equation to the general solution of the corresponding homogeneous differential equation.

### 5. Decomposition Theorem

If an inhomogeneous differential equation (9.32) has real coefficients and its right-hand side is complex in the form $F = F_1 + \mathrm{i}F_2$ with some real functions $F_1$ and $F_2$, then the solution $y = y_1 + \mathrm{i}y_2$ is also complex, where $y_1$ and $y_2$ are the two solutions of the two inhomogeneous differential equations (9.32) with the corresponding right-hand sides $F_1$ and $F_2$.

### 6. Solution of Inhomogeneous Differential Equations (9.32) by Means of Quadratures

If the fundamental system of the corresponding homogeneous differential equation is already known, there are the following two solution methods to continue the calculations:

**1. Method of Variation of Constants** Looking for the solution in the form

$$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n \tag{9.37a}$$

where $C_1, C_2, \ldots, C_n$, here treated as functions of $x$. There are infinitely many such functions, but requiring that they satisfy the equations

$$\begin{aligned} C_1'y_1 + C_2'y_2 + \cdots + C_n'y_n &= 0,\\ C_1'y_1' + C_2'y_2' + \cdots + C_n'y_n' &= 0,\\ &\;\cdots\cdots\cdots\cdots\cdots\\ C_1'y_1^{(n-2)} + C_2'y_2^{(n-2)} + \cdots + C_n'y_n^{(n-2)} &= 0 \end{aligned} \tag{9.37b}$$

and substituting $y$ into (9.32) with these equalities follows

$$C_1'y_1^{(n-1)} + C_2'y_2^{(n-1)} + \cdots + C_n'y_n^{(n-1)} = F. \tag{9.37c}$$

Because the Wronskian determinant of the coefficients in the linear system of equations (9.37b) and (9.37c) is different from zero, one gets a unique solution for the unknown functions $C_1', C_2', \ldots, C_n'$, and their integrals give the functions $C_1, C_2, \ldots, C_n$.

■ $y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = x - 1.$ (9.37d)

In the interval $x > 1$ or $x < 1$ all assumptions on the coefficients are fulfilled. First the homogeneous equation $\overline{y}'' + \frac{x}{1-x}\overline{y}' - \frac{1}{1-x}\overline{y} = 0$ is solved. A particular solution is $\varphi_1 = e^x$. Then one looks for a second one in the form $\varphi_2 = e^x u(x)$, and with the notation $u'(x) = v(x)$ one gets the first-order differential equation $v' + \left(1 + \frac{1}{1-x}\right)v = 0$. A solution of this equation is $v(x) = (1-x)e^{-x}$, and therefore, $u(x) = \int v(x)\,dx = \int (1-x)e^{-x}\,dx = xe^{-x}$. With this result $\varphi_2 = x$ is obtained for the second element of the fundamental system. The general solution of the homogeneous equation is $\overline{y}(x) = C_1 e^x + C_2 x$. The variation of constants with $u_1(x)$ and $u_2(x)$ instead of $C_1(x)$ and $C_2(x)$ is now:

$$\begin{aligned}
y(x) &= u_1(x)e^x + u_2(x)x, \\
y'(x) &= u_1(x)e^x + u_2(x) + u_1{}'(x)e^x + u_2{}'(x)x, \quad && u_1{}'(x)e^x + u_2{}'(x)x = 0, \\
y''(x) &= u_1(x)e^x + u_1{}'(x)e^x + u_2{}'(x), && u_1{}'(x)e^x + u_2{}'(x) = x - 1, \quad \text{so} \\
u_1{}'(x) &= xe^{-x}, \quad u_2{}'(x) = -1, \quad \text{i.e.,} \quad u_1(x) = -(1+x)e^{-x} + C_1, \quad u_2(x) = -x + C_2.
\end{aligned}$$

With this result the general solution of the inhomogeneous differential equation is:

$$y(x) = -(1+x^2) + C_1 e^x + (C_2 - 1)x = -(1+x^2) + C_1{}^* e^x + C_2{}^* x. \tag{9.37e}$$

**2. Method of Cauchy** In the general solution

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n \tag{9.38a}$$

of the homogeneous differential equation associated to (9.32) one determines the constants such that for an arbitrary parameter $\alpha$ the equations $y = 0,\ y' = 0, \ldots,\ y^{(n-2)} = 0,\ y^{(n-1)} = F(\alpha)$ are satisfied. In this way one gets a particular solution of the homogeneous equation, denoted by $\varphi(x, \alpha)$, and then

$$y = \int_{x_0}^{x} \varphi(x, \alpha)\,d\alpha \tag{9.38b}$$

is a particular solution of the inhomogeneous differential equation (9.32). This solution and their derivatives up to order $(n-1)$ are equal to zero at the point $x = x_0$.

■ The general solution of the homogeneous equation associated to the differential equation (9.37d) which has been solved by the method variation of constants is $y = C_1 e^x + C_2 x$. From this result follows $y(\alpha) = C_1 e^\alpha + C_2 \alpha = 0$, $y'(\alpha) = C_1 e^\alpha + C_2 = \alpha - 1$ and $\varphi(x, \alpha) = \alpha e^{-\alpha} e^x - x$, so that the particular solution $y(x)$ of the inhomogeneous differential equation with $y(x_0) = y'(x_0) = 0$ is: $y(x) = \int_{x_0}^{x} (\alpha e^{-\alpha} e^x - x)\,d\alpha = (x_0 + 1)e^{x - x_0} + (x_0 - 1)x - x^2 - 1$. With this result one can get the general solution $y(x) = C_1{}^* e^x + C_2{}^* x - (x^2 + 1)$ of the inhomogeneous differential equation.

### 9.1.2.4 Solution of Linear Differential Equations with Constant Coefficients

#### 1. Operational Notation

The differential equation (9.32) can be written symbolically in the form

$$P_n(D)y \equiv \left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1}D + a_n\right) y = F, \tag{9.39a}$$

where $D$ is a differential operator:

$$Dy = \frac{dy}{dx}, \quad D^k y = \frac{d^k y}{dx^k}. \tag{9.39b}$$

If the coefficients $a_i$ are constants, then $P_n(D)$ is a usual polynomial in the operator $D$ of degree $n$.

### 2. Solution of the Homogeneous Differential Equation with Constant Coefficients

To determine the general solution of the homogeneous differential equation (9.39a) with $F = 0$, i.e.,

$$P_n(D)y = 0 \tag{9.40a}$$

one has to find the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation

$$P_n(r) = r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0. \tag{9.40b}$$

Every root $r_i$ determines a solution $e^{r_i x}$ of the equation $P_n(D)y = 0$. If a root $r_i$ has a higher multiplicity $k$, then $xe^{r_i x}$, $x^2 e^{r_i x}, \ldots,$ $x^{k-1}e^{r_i x}$ are also solutions. The linear combination of all these solutions is the general solution of the homogeneous differential equation:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + e^{r_i x}\left(C_i + C_{i+1}x + \cdots + C_{i+k-1}x^{k-1}\right) + \cdots. \tag{9.40c}$$

If the coefficients $a_i$ are all real, then the complex roots of the characteristic equation are pairwise conjugate with the same multiplicity. In this case, for $r_1 = \alpha + \mathrm{i}\beta$ and $r_2 = \alpha - \mathrm{i}\beta$ one can replace the corresponding complex solution functions $e^{r_1 x}$ and $e^{r_2 x}$ by the real functions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$. The resulting expression $C_1\cos\beta x + C_2\sin\beta x$ can be written in the form $A\cos(\beta x + \varphi)$ with some constants $A$ and $\varphi$.

■ In the case of the differential equation $y^{(6)} + y^{(4)} - y'' - y = 0$, the characteristic equation is $r^6 + r^4 - r^2 - 1 = 0$ with roots $r_1 = 1$, $r_2 = -1$, $r_{3,4} = \mathrm{i}$, $r_{5,6} = -\mathrm{i}$. The general solution can be given in two forms:

$$y = C_1 e^x + C_2 e^{-x} + (C_3 + C_4 x)\cos x + (C_5 + C_6 x)\sin x, \quad \text{or}$$
$$y = C_1 e^x + C_2 e^{-x} + A_1\cos(x + \varphi_1) + xA_2\cos(x + \varphi_2).$$

### 3. Hurwitz Theorem

In different applications, e.g., in vibration theory, it is important to know whether a solution of a given homogeneous differential equation with constant coefficients tend to zero for $x \to +\infty$ or not. It tends to zero, obviously, if the real parts of the roots of the characteristic equation (9.40b) are negative. According to the *Hurwitz theorem* an equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{9.41a}$$

has only roots with negative real part if and only if all the determinants

$$D_1 = a_1, \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \ldots, \quad D_n = \begin{vmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix}$$

$$(\text{with } a_m = 0 \quad \text{for} \quad m > n) \tag{9.41b}$$

are positive. The determinants $D_k$ have on their diagonal the coefficients $a_1, a_2, \ldots, a_k$ $(k = 1, 2, \ldots, n)$, and the coefficient-indices are decreasing from left to right. Coefficients with negative indices and also with indices larger than $n$ are all put to 0.

■ For a cubic polynomial the determinants have in accordance to (9.41b) the following form:

$$D_1 = a_1, \quad D_2 = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_2 \end{vmatrix}, D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix}.$$

### 4. Solution of Inhomogeneous Differential Equations with Constant Coefficients

These differential equations can be solved by the method variation of constants, or by the method of Cauchy, or with the operator method (see 9.2.2.3, **5.**, p. 588). If the right-hand side of the inhomogeneous differential equation (9.32) has a special form, then a particular solution can be determined easily.

**1. Form:** $\boldsymbol{F(x) = Ae^{\alpha x}, \quad P_n(\alpha) \neq 0}$ (9.42a)

A particular solution is

$$y = \frac{Ae^{\alpha x}}{P_n(\alpha)}. \tag{9.42b}$$

If $\alpha$ is a root of the characteristic equation of multiplicity $m$, i.e., if

$$P_n(\alpha) = P_n{}'(\alpha) = \ldots = P_n{}^{(m-1)}(\alpha) = 0, \tag{9.42c}$$

then $y = \dfrac{Ax^m e^{\alpha x}}{P_n{}^{(m)}(\alpha)}$ is a particular solution. These formulas can also be used by applying the decomposition theorem, if the right side is

$$F(x) = Ae^{\alpha x}\cos\omega x \quad \text{or} \quad Ae^{\alpha x}\sin\omega x. \tag{9.42d}$$

The corresponding particular solutions are the real or the imaginary part of the solution of the same differential equation for

$$F(x) = Ae^{\alpha x}(\cos\omega x + \mathrm{i}\sin\omega x) = Ae^{(\alpha+\mathrm{i}\omega)x} \tag{9.42e}$$

on the right-hand side.

■ **A:** For the differential equation $y'' - 6y' + 8y = e^{2x}$, the characteristic polynomial is $P(D) = D^2 - 6D + 8$ with $P(2) = 0$ and $P'(D) = 2D - 6$ with $P'(2) = 2\cdot 2 - 6 = -2$, so the particular solution is $y = -\dfrac{xe^{2x}}{2}$.

■ **B:** The differential equation $y'' + y' + y = e^x \sin x$ results in the equation $(D^2 + D + 1)y = e^{(1+\mathrm{i})x}$. From its solution $y = \dfrac{e^{(1+\mathrm{i})x}}{(1+\mathrm{i})^2 + (1+\mathrm{i}) + 1} = \dfrac{e^x(\cos x + \mathrm{i}\sin x)}{2+3\mathrm{i}}$ one gets a particular solution $y_1 = \dfrac{e^x}{13}(2\sin x - 3\cos x)$. Here $y_1$ is the imaginary part of $y$.

**2. Form: $F(x) = Q_n(x)e^{\alpha x}$, $Q_n(x)$ is a polynomial of degree $n$** (9.43)

A particular solution can always be found in the same form, i.e., as an expression $y = R(x)e^{\alpha x}$. $R(x)$ is a polynomial of degree $n$ multiplied by $x^m$ if $\alpha$ is a root of the characteristic equation with a multiplicity $m$. Considering the coefficients of the polynomial $R(x)$ as unknowns and substituting the expression into the inhomogeneous differential equation a linear system of equations is obtained for the coefficients, and this system of equations always has a unique solution.

This method is very useful especially in the cases of $F(x) = Q_n(x)$ for $\alpha = 0$ and $F(x) = Q_n(x)e^{rx}\cos\omega x$ or $F(x) = Q_n(x)e^{rx}\sin\omega x$ for $\alpha = r \pm \mathrm{i}\omega$. There is a solution in the form $y = x^m e^{rx}[M_n(x)\cos\omega x + N_n(x)\sin\omega x]$.

■ The roots of the characteristic equation associated to the differential equation $y^{(4)} + 2y''' + y'' = 6x + 2x\sin x$ are $k_1 = k_2 = 0$, $k_3 = k_4 = -1$. Because of the superposition principle (see 9.1.2.3, **4.**, p. 554), one can calculate the particular solutions of the inhomogeneous differential equation for the summands of the right-hand side separately. For the first summand the substitution of the given form $y_1 = x^2(ax+b)$ results in a right-hand side $12a + 2b + 6ax = 6x$, and so: $a = 1$ and $b = -6$. For the second summand one substitutes $y_2 = (cx+d)\sin x + (fx+g)\cos x$. One gets the coefficients by coefficient comparison from $(2g+2f-6c+2fx)\sin x - (2c+2d+6f+2cx)\cos x = 2x\sin x$, so $c = 0$, $d = -3$, $f = 1$, $g = -1$. Therefore, the general solution is $y = c_1 + c_2x - 6x^2 + x^3 + (c_3x + c_4)e^{-x} - 3\sin x + (x-1)\cos x$.

**3. Euler Differential Equation**

The Euler differential equation

$$\sum_{k=0}^{n} a_k(cx+d)^k y^{(k)} = F(x) \tag{9.44a}$$

can be transformed with the substitution

$$cx + d = e^t \tag{9.44b}$$

into a linear differential equation with constant coefficients.

■ The differential equation $x^2y'' - 5xy' + 8y = x^2$ is a special case of the Euler differential equation for $n = 2$. With the substitution $x = e^t$ it becomes the differential equation discussed earlier in ■ **A**, p. 557: $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = e^{2t}$. The general solution is $y = C_1e^{2t} + C_2e^{4t} - \frac{t}{2}e^{2t} = C_1x^2 + C_2x^4 - \frac{x^2}{2}\ln|x|$.

### 9.1.2.5 Systems of Linear Differential Equations with Constant Coefficients

#### 1. Normal Form

The following simple case of a system of first-order linear differential equations with constant coefficients is called a *normal system* or a *normal form*:

$$\left.\begin{array}{l} y_1{}' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n, \\ y_2{}' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n{}' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n. \end{array}\right\} \tag{9.45a}$$

To find the general solution of such a system, one has to find first the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - r & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - r & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - r \end{vmatrix} = 0. \tag{9.45b}$$

To every single root $r_i$ of this equation there is a system of particular solutions

$$y_1 = A_1e^{r_ix}, \quad y_2 = A_2e^{r_ix}, \ldots, y_n = A_ne^{r_ix}, \tag{9.45c}$$

whose coefficients $A_k$ $(k = 1, 2, \ldots, n)$ are determined from the homogeneous linear equation system

$$\begin{array}{l} (a_{11} - r_i)A_1 + a_{12}A_2 + \cdots + a_{1n}A_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}A_1 + a_{n2}A_2 + \cdots + (a_{nn} - r_i)A_n = 0. \end{array} \tag{9.45d}$$

This system gives the relations between the values of the coefficients $A_k$ (see Trivial Solution and Fundamental System in 4.5.2.1, **2.**, p. 309). For every $r_i$, the particular solutions determined this way will contain an arbitrary constant. If all the roots of the characteristic equation are different, the sum of these particular solutions contains $n$ independent arbitrary constants, so in this way one gets the general solution. If a root $r_i$ has a multiplicity $m$ in the characteristic equation, the system of particular solutions corresponding to this root has the form

$$y_1 = A_1(x)e^{r_ix}, \quad y_2 = A_2(x)e^{r_ix}, \ldots, y_n = A_n(x)e^{r_ix}, \tag{9.45e}$$

where $A_1(x), \ldots, A_n(x)$ are polynomials of degree at most $m - 1$. After substituting these expressions with unknown coefficients of the polynomials $A_k(x)$ into the differential equation system one first can cancel the factor $e^{r_ix}$, then one compares the coefficients of the different powers of $x$ to have linear equations for the unknown coefficients of the polynomials, and among them $m$ can be chosen freely. In this way, one gets a part of the solution with $m$ arbitrary constants. The degree of the polynomials can be less than $m - 1$.

In the special case when the system (9.45a) is symmetric, i.e., when $a_{ik} = a_{ki}$, then it is sufficient to substitute $A_i(x) = \text{const}$. For complex roots of the characteristic equation, the general solution can be transformed into a real form in the same way as has been shown for the case of a differential equation with constant coefficients (see 9.1.2.4, p. 555).

■ For the system $y_1{}' = 2y_1 + 2y_2 - y_3$, $y_2{}' = -2y_1 + 4y_2 + y_3$, $y_3{}' = -3y_1 + 8y_2 + 2y_3$ the characteristic equation has the form

$$\begin{vmatrix} 2-r & 2 & -1 \\ -2 & 4-r & 1 \\ -3 & 8 & 2-r \end{vmatrix} = -(r-6)(r-1)^2 = 0.$$

For the simple root $r_1 = 6$ one gets $-4A_1 + 2A_2 - A_3 = 0$, $-2A_1 - 2A_2 + A_3 = 0$, $-3A_1 + 8A_2 - 4A_3 = 0$. From this system one has $A_1 = 0$, $A_2 = \frac{1}{2}A_3 = C_1$, $y_1 = 0$, $y_2 = C_1 e^{6x}$, $y_3 = 2C_1 e^{6x}$. For the multiple root $r_2 = 1$ one gets $y_1 = (P_1 x + Q_1)e^x$, $y_2 = (P_2 x + Q_2)e^x$, $y_3 = (P_3 x + Q_3)e^x$. Substitution into the differential equations yields

$$P_1 x + (P_1 + Q_1) = (2P_1 + 2P_2 - P_3)x + (2Q_1 + 2Q_2 - Q_3),$$

$$P_2 x + (P_2 + Q_2) = (-2P_1 + 4P_2 + P_3)x + (-2Q_1 + 4Q_2 + Q_3),$$

$$P_3 x + (P_3 + Q_3) = (-3P_1 + 8P_2 + 2P_3)x + (-3Q_1 + 8Q_2 + 2Q_3),$$

which implies that $P_1 = 5C_2$, $P_2 = C_2$, $P_3 = 7C_2$, $Q_1 = 5C_3 - 6C_2$, $Q_2 = C_3$, $Q_3 = 7C_3 - 11C_2$. The general solution is $y_1 = (5C_2 x + 5C_3 - 6C_2)e^x$, $y_2 = C_1 e^{6x} + (C_2 x + C_3)e^x$, $y_3 = 2C_1 e^{6x} + (7C_2 x + 7C_3 - 11C_2)e^x$.

## 2. Homogeneous Systems of First-Order Linear Differential Equations with Constant Coefficients

have the general form

$$\sum_{k=1}^{n} a_{ik} y_k' + \sum_{k=1}^{n} b_{ik} y_k = 0 \quad (i = 1, 2, \ldots, n). \tag{9.46a}$$

If the determinant $\det(a_{ik})$ does not disappear, i.e.,

$$\det(a_{ik}) \neq 0, \tag{9.46b}$$

then the system (9.46a) can be transformed into the normal form (9.45a).
In the case of $\det(a_{ik}) = 0$ further investigations are necessary (see [9.15]).
The solution can be determined from the general form in the same way as shown for the normal form. The characteristic equation has the form

$$\det(a_{ik} r + b_{ik}) = 0. \tag{9.46c}$$

The coefficients $A_i$ in the solution (9.45c) corresponding to a single root $r_j$ are determined from the equation system

$$\sum_{k=1}^{n} (a_{ik} r_j + b_{ik}) A_k = 0 \quad (i = 1, 2, \ldots, n). \tag{9.46d}$$

Otherwise the solution method follows the same ideas as in the case of the normal form.

■ The characteristic equation of the two differential equations $5y_1' + 4y_1 - 2y_2' - y_2 = 0$, $y_1' + 8y_1 - 3y_2 = 0$ is:

$$\begin{vmatrix} 5r+4 & -2r-1 \\ r+8 & -3 \end{vmatrix} = 2r^2 + 2r - 4 = 0, \quad r_1 = 1, \quad r_2 = -2.$$

The coefficients $A_1$ and $A_2$ for $r_1 = 1$ can be got from the equations $9A_1 - 3A_2 = 0$, $9A_1 - 3A_2 = 0$ so $A_2 = 3A_1 = 3C_1$. For $r_2 = -2$ one gets analogously $\overline{A}_2 = 2\overline{A}_1 = 2C_2$. The general solution is $y_1 = C_1 e^x + C_2 e^{-2x}$, $y_2 = 3C_1 e^x + 2C_2 e^{-2x}$.

## 3. Inhomogeneous Systems of First-Order Linear Differential Equations

have the general form

$$\sum_{k=1}^{n} a_{ik} y_k' + \sum_{k=1}^{n} b_{ik} y_k = F_i(x) \quad (i = 1, 2, \ldots, n). \tag{9.47}$$

**1. Superposition Principle** If ${y_j}^{(1)}$ and ${y_j}^{(2)}$ $(j = 1, 2, \ldots, n)$ are solutions of inhomogeneous systems which differ from each other only in their right-hand sides ${F_i}^{(1)}$ and ${F_i}^{(2)}$, then the sum $y_j = {y_j}^{(1)} + {y_j}^{(2)}$ $(j = 1, 2, \ldots, n)$ is a solution of this system with the right-hand side $F_i(x) = {F_i}^{(1)}(x) + {F_i}^{(2)}(x)$. Because of this, to get the general solution of an inhomogeneous system it is enough to add a particular solution to the general solution of the corresponding homogeneous system.

**2. The Variation of Constants** can be used to get a particular solution of the inhomogeneous differential equation system. To do this one uses the general solution of the homogeneous system, and considers the constants $C_1, C_2, \ldots, C_n$ as unknown functions $C_1(x), C_2(x), \ldots, C_n(x)$. Then it is to be substituted into the inhomogeneous system. In the expressions of the derivatives of ${y_k}'$ there is the derivative of the new unknown functions $C_k(x)$. Because $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous system, the terms containing the new unknown functions will be canceled; only their derivatives remain in the equations. This gives for the functions ${C_k}'(x)$ an inhomogeneous linear algebraic equation system which always has a unique solution. After $n$ integrations one gets the functions $C_1(x), C_2(x), \ldots, C_n(x)$. Substituting them into the solution of the homogeneous system instead of the constants results in the particular solution of the inhomogeneous system.

■ For the system of two inhomogeneous differential equations $5{y_1}' + 4y_1 - 2{y_2}' - y_2 = e^{-x}$, ${y_1}' + 8y_1 - 3y_2 = 5e^{-x}$ the general solution of the homogeneous system is (see p. 559) $y_1 = C_1e^x + C_2e^{-2x}$, $y_2 = 3C_1e^x + 2C_2e^{-2x}$. Considering the constants $C_1$ and $C_2$ as functions of $x$ and substituting into the original equations gives $5{C_1}'e^x + 5{C_2}'e^{-2x} - 6{C_1}'e^x - 4{C_2}'e^{-2x} = e^{-x}$, ${C_1}'e^x + {C_2}'e^{-2x} = 5e^{-x}$ or ${C_2}'e^{-2x} - {C_1}'e^{-x} = e^{-x}$, ${C_1}'e^x + {C_2}'e^{-2x} = 5e^{-x}$. Therefore, $2{C_1}'e^x = 4e^{-x}$, $C_1 = -e^{-2x} + \text{const}$, $2{C_2}'e^{-2x} = 6e^{-x}$, $C_2 = 3e^x + \text{const}$. Since a particular solution is searched for, one can replace every constant by zero and the result is $y_1 = 2e^{-x}$, $y_2 = 3e^{-x}$. The general solution is finally $y_1 = 2e^{-x} + C_1e^x + C_2e^{-2x}$, $y_2 = 3e^{-x} + 3C_1e^x + 2C_2e^{-2x}$.

**3. The Method of Unknown Coefficients** is especially useful if on the right-hand side there are special functions in the form $Q_n(x)e^{\alpha x}$. The application is similar to the one, used for differential equations of $n$-th order (see 9.1.2.5, p. 558).

### 4. Second-Order Systems

The methods introduced above can also be used for differential equations of higher order. For the system

$$\sum_{k=1}^{n} a_{ik}{y_k}'' + \sum_{k=1}^{n} b_{ik}{y_k}' + \sum_{k=1}^{n} c_{ik}y_k = 0 \quad (i = 1, 2, \ldots, n) \tag{9.48}$$

one can determine particular solutions in the form $y_i = A_ie^{r_ix}$. To do this, one gets $r_i$ from the characteristic equation $\det(a_{ik}r^2 + b_{ik}r + c_{ik}) = 0$, and $A_i$ from the corresponding linear homogeneous algebraic equations.

## 9.1.2.6 Linear Second-Order Differential Equations

Many special differential equations belong to this class, which often occur in practical applications. Several of them are discussed in this paragraph. For more details of representation, properties and solution methods see [9.15].

### 1. General Methods

**1. Solving the Inhomogeneous Differential Equation by the Help of the Superposition Principle**

$$y'' + p(x)y' + q(x)y = F(x). \tag{9.49a}$$

To get the general solution of an inhomogeneous differential equation it is enough to add a particular solution of the inhomogeneous equation to the general solution of the corresponding homogeneous equation.

**a)** The general solution of the corresponding homogeneous differential equation, i.e., with $F(x) \equiv 0$, is

$$y = C_1y_1 + C_2y_2. \tag{9.49b}$$

Here $y_1$ and $y_2$ are two linearly independent particular solutions of (9.49a) (see 9.1.2.3, **2.**, p. 553). If a particular solution $y_1$ is already known, then the second one $y_2$ can be determined by the equation (9.34) of Liouville. From (9.34) follows:

$$\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1y_2{}' - y_1{}'y_2 = y_1^2\,\frac{y_1y_2{}' - y_1{}'y_2}{y_1^2} = y_1^2\left(\frac{y_2}{y_1}\right)' = A\exp\left(-\int p(x)\,dx\right) \tag{9.49c}$$

giving

$$y_2 = Ay_1\int\frac{\exp\left(-\int p\,dx\right)}{y_1{}^2}\,dx \tag{9.49d}$$

where $A$ can be chosen arbitrarily.

**b)** A particular solution of the inhomogeneous equation can be determined by the formula

$$y = \frac{1}{A}\int\limits_{x_0}^{x} F(\xi)\exp\left(\int p(\xi)\,d\xi\right)[y_2(x)y_1(\xi) - y_1(x)y_2(\xi)]\,d\xi, \tag{9.49e}$$

where $y_1$ and $y_2$ are two particular solutions of the corresponding homogeneous differential equation.

**c)** A particular solution of the inhomogeneous differential equation can be determined also by variation of constants (see 9.1.2.3, **6.**, p. 554).

**2. Solving the Inhomogeneous Differential Equation by the Method of Undetermined Coefficients**

$$s(x)y'' + p(x)y' + q(x)y = F(x) \tag{9.50a}$$

If the functions $s(x)$, $p(x)$, $q(x)$ and $F(x)$ are polynomials or functions which can be expanded into a convergent power series around $x_0$ in a certain domain, where $s(x_0) \neq 0$, then the solutions of this differential equation can also be expanded into a similar series, and these series are convergent in the same domain. Here they should be determined by the method of undetermined coefficients: The solution to be looking for as a series has the form

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots, \tag{9.50b}$$

and it has to be substituted into the differential equation (9.50a). Equating corresponding coefficients (of the same powers of $(x - x_0)$) results in equations to determine the coefficients $a_0, a_1, a_2, \ldots$.

■ To solve the differential equation $y'' + xy = 0$ one substitutes $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, $y' = a_1 + 2a_2x + 3a_3x^2 + \cdots$, and $y'' = 2a_2 + 6a_3x + \cdots$ getting $2a_2 = 0$, $6a_3 + a_0 = 0, \ldots$. The solution of these equations is $a_2 = 0$, $a_3 = -\dfrac{a_0}{2\cdot 3}$, $a_4 = -\dfrac{a_1}{3\cdot 4}$, $a_5 = 0, \ldots$, so the solution is $y = a_0\left(1 - \dfrac{x^3}{2\cdot 3} + \dfrac{x^6}{2\cdot 3\cdot 5\cdot 6} - \cdots\right) + a_1\left(x - \dfrac{x^4}{3\cdot 4} + \dfrac{x^7}{3\cdot 4\cdot 6\cdot 7} - \cdots\right)$.

**3. The Homogeneous Differential Equation**

$$x^2y'' + xp(x)y' + q(x)y = 0 \tag{9.51a}$$

can be solved by the method of undetermined coefficients if the functions $p(x)$ and $q(x)$ can be expanded as a convergent power series of $x$. The solutions have the form

$$y = x^r(a_0 + a_1x + a_2x^2 + \cdots), \tag{9.51b}$$

whose exponent $r$ can be determined from the *defining equation*

$$r(r-1) + p(0)r + q(0) = 0. \tag{9.51c}$$

If the roots of this equation are different and their difference is not an integer number, then one gets two linearly independent solutions of (9.51a). Otherwise the method of undetermined coefficients results only one solution. Then with the help of (9.49b) one can get a second solution or at least one can find a form which gives a second solution with the method of undetermined coefficients.

■ For the Bessel differential equation (9.52a) one gets only one solution with the method of the undetermined coefficients in the form $y_1 = \sum\limits_{k=0}^{\infty} a_k x^{n+2k}$ $(a_0 \neq 0)$, which coincides with $J_n(x)$ up to a constant factor. Since $\exp\left(-\int p\,dx\right) = \frac{1}{x}$ one finds a second solution by using formula (9.49d)

$$y_2 = Ay_1 \int \frac{dx}{x \cdot x^{2n}\left(\sum a_k x^{2k}\right)^2} = Ay_1 \int \frac{\sum\limits_{k=0}^{\infty} c_k x^{2k}}{x^{2n+1}} dx = By_1 \ln x + x^{-n} \sum_{k=0}^{\infty} d_k x^{2k}.$$

The determination of the unknown coefficients $c_k$ and $d_k$ is difficult from the $a_k$'s. But this last expression can be used to get the solution with the method of undetermined coefficients. Obviously this form is a series expansion of the function $Y_n(x)$ (9.53c).

## 2. Bessel Differential Equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{9.52a}$$

**1. The Defining Equation** is in this case

$$r(r-1) + r - n^2 \equiv r^2 - n^2 = 0, \tag{9.52b}$$

so, $r = \pm n$. Substituting

$$y = x^n(a_0 + a_1 x + \cdots) \tag{9.52c}$$

into this equation and equating the coefficients of $x^{n+k}$ to zero gives

$$k(2n+k)a_k + a_{k-2} = 0. \tag{9.52d}$$

For $k = 1$ follows $(2n+1)a_1 = 0$. For the values $k = 2, 3, \ldots$ one obtains

$$a_{2m+1} = 0 \quad (m = 1, 2, \ldots), \quad a_2 = -\frac{a_0}{2(2n+2)},$$

$$a_4 = \frac{a_0}{2 \cdot 4 \cdot (2n+2)(2n+4)}, \ldots, \quad a_0 \text{ is arbitrary.} \tag{9.52e}$$

**2. Bessel or Cylindrical Functions** The series obtained above for $a_0 = \frac{1}{2^n \Gamma(n+1)}$, where $\Gamma$ is the gamma function (see 8.2.5, **6.**, p. 514), is a particular solution of the Bessel differential equation (9.52a) for integer values of $n$. It defines the *Bessel* or *cylindrical function* of the first kind of index $n$

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4 \cdot (2n+2)(2n+4)} - \cdots\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\,\Gamma(n+k+1)}. \tag{9.53a}$$

The graphs of functions $J_0$ and $J_1$ are shown in **Fig. 9.12**.
The general solution of the Bessel differential equation for non-integer $n$ has the form

$$y = C_1 J_n(x) + C_2 J_{-n}(x), \tag{9.53b}$$

where $J_{-n}(x)$ is defined by the infinite series obtained from the series representation of $J_n(x)$ by replacing $n$ with $-n$. For integer $n$, holds $J_{-n}(x) = (-1)^n J_n(x)$. In this case, the term $J_{-n}(x)$ in the general solution should be replaced with the Bessel function of the second kind

$$Y_n(x) = \lim_{m \to n} \frac{J_m(x)\cos m\pi - J_{-m}(x)}{\sin m\pi}, \tag{9.53c}$$

which is also called the *Weber function*. For the series expansion of $Y_n(x)$ see, e.g., [9.15]. The graphs of the functions $Y_0$ and $Y_1$ are shown in **Fig. 9.13**.

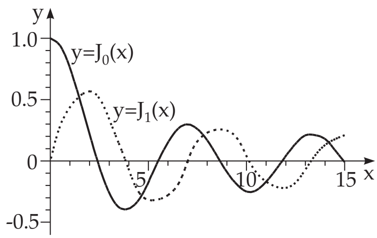


Figure 9.12

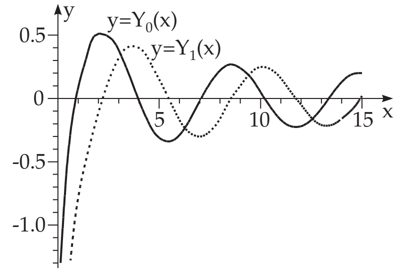


Figure 9.13

**3. Bessel Functions with Imaginary Variables** In some applications one uses Bessel functions with pure imaginary variables. In this case it is to be considered the product $\mathrm{i}^{-n}J_n(\mathrm{i}x)$ which will be denoted by $I_n(x)$:

$$I_n(x) = \mathrm{i}^{-n}J_n(\mathrm{i}x) = \frac{\left(\frac{x}{2}\right)^n}{\Gamma(n+1)} + \frac{\left(\frac{x}{2}\right)^{n+2}}{1!\Gamma(n+2)} + \frac{\left(\frac{x}{2}\right)^{n+4}}{2!\Gamma(n+3)} + \cdots. \tag{9.54a}$$

The functions $I_n(x)$ are solutions of the differential equation

$$x^2y'' + xy' - (x^2 + n^2)y = 0. \tag{9.54b}$$

A second solution of this differential equation is the *MacDonald function*

$$K_n(x) = \frac{\pi}{2}\frac{I_{-n}(x) - I_n(x)}{\sin n\pi}. \tag{9.54c}$$

If $n$ converges to an integer number, this expression also converges.

The functions $I_n(x)$ and $K_n(x)$ are called *modified Bessel functions*. The graphs of functions $I_0$ and $I_1$ are shown in **Fig. 9.14**; the graphs of functions $K_0$ and $K_1$ are illustrated in **Fig. 9.15**. The values of functions $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$, $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$ are given in **Table 21.11**, p. 1106.

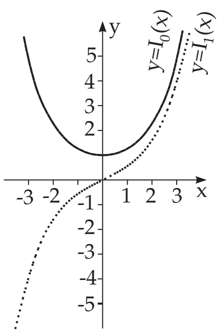


Figure 9.14

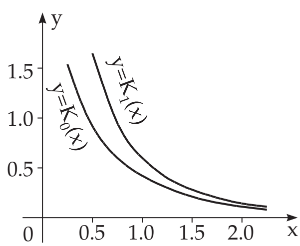


Figure 9.15

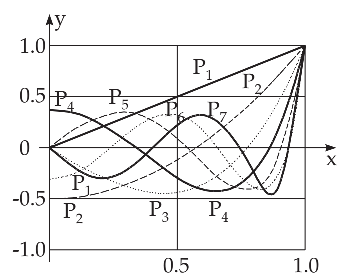


Figure 9.16

**4. Important Formulas for the Bessel Functions $J_n(x)$**

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x), \quad \frac{dJ_n(x)}{dx} = -\frac{n}{x}J_n(x) + J_{n-1}(x). \tag{9.55a}$$

The formulas (9.55a) are also valid for the Weber functions $Y_n(x)$.

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2nI_n(x)}{x}, \quad \frac{dI_n(x)}{dx} = I_{n-1}(x) - \frac{n}{x}I_n(x), \tag{9.55b}$$

$$K_{n+1}(x) - K_{n-1}(x) = \frac{2nK_n(x)}{x}, \quad \frac{dK_n(x)}{dx} = -K_{n-1}(x) - \frac{n}{x}K_n(x). \tag{9.55c}$$

For integer numbers $n$ the following formulas are valid:

$$J_{2n}(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\varphi)\cos 2n\varphi\, d\varphi, \tag{9.55d}$$

$$J_{2n+1}(x) = \frac{2}{\pi}\int_0^{\pi/2} \sin(x\sin\varphi)\sin(2n+1)\varphi\, d\varphi \tag{9.55e}$$

or, in complex form,

$$J_n(x) = \frac{-(\mathrm{i})^n}{\pi}\int_0^{\pi} e^{\mathrm{i}x\cos\varphi}\cos n\varphi\, d\varphi. \tag{9.55f}$$

The functions $J_{n+1/2}(x)$ can be expressed by using elementary functions. In particular,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad (9.56a) \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{9.56b}$$

By applying the recursion formulas (9.55a)–(9.55f) the expression for $J_{n+1/2}(x)$ for arbitrary integer $n$ can be given. For large values of $x$ the following asymptotic formulas are valid:

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right], \tag{9.57a}$$

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 + O\left(\frac{1}{x}\right)\right], \tag{9.57b}$$

$$Y_n(x) = \sqrt{\frac{2}{\pi x}}\left[\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right], \tag{9.57c}$$

$$K_n(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left[1 + O\left(\frac{1}{x}\right)\right]. \tag{9.57d}$$

The expression $O\left(\frac{1}{x}\right)$ means an infinitesimal quantity of the same order as $\frac{1}{x}$ (see the Landau symbol, 2.1.4.9, p. 57).
For further properties of the Bessel functions see [21.1].

**5. Important Formulas for the Spherical Bessel Functions** Spherical Bessel functions of the first and second kind $j_l(z)$ and $n_l(z)$ follow from the Bessel functions of the first and second kind $J_n(z)$ (9.53a)and $Y_n(z)$ (9.53c) for half odd order index $n = \frac{1}{2}, \frac{3}{2}, \ldots$ as in $j_l(z) = \sqrt{\frac{\pi}{2z}}J_{l+\frac{1}{2}}(z)$ and $n_l(z) = \sqrt{\frac{\pi}{2z}}Y_{l+\frac{1}{2}}(z)$ with $l = 0, 1, 2, \ldots$. They occur as regular or singular solutions of the potential free radial Schroedinger equation (see 9.2.4.6,**3.**, (9.137b), p. 599) with $V(r) = 0$, $E = \dfrac{\hbar^2 k^2}{2m}$, $z = kr$ and $s_l(z) = R_l(r)$:

$$z\frac{d^2}{dz^2}[z\, s_l(z)] + [z^2 - l(l+1)]s_l(z) = 0\,, \quad s_l(z) = j_l(z) \text{ or } n_l(z). \tag{9.58a}$$

They also occur in the quantum mechanical scattering theory, where the $n_l(z)$ are called the spherical von Neumann functions. By the help of the Rayleigh formulas

$$j_l(z) = (-z)^l \left(\frac{d}{zdz}\right)^l \frac{\sin z}{z}, \quad n_l(z) = (-z)^l \left(\frac{d}{zdz}\right)^l (-1)\frac{\cos z}{z} \tag{9.58b}$$

follows, so

$$j_0(z) = \frac{\sin z}{z}, \quad j_1(z) = \frac{\sin z - z\cos z}{z^2}, \cdots, \tag{9.58c}$$

$$n_0(z) = -\frac{\cos z}{z}, \quad n_1(z) = -\frac{\cos z + z\sin z}{z^2}, \cdots. \tag{9.58d}$$

Complex spherical functions are used with $\Phi_m(\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}$ in the form $Y_L(\vec{\mathbf{e}}) = \Theta_l^m(\vartheta)\Phi_m(\varphi)$, e.g. in 9.2.4.6, (9.136e), p. 599. In the combined index $L = (l, m)$ $l = 0, 1, 2, \ldots$ denotes the quantum number of the orbital angular momentum. The magnetic quantum number $m$ is restricted to the $2l + 1$ values $m = -l, -l + 1, \cdots, +l$. With the abbreviations

$$j_L(k\vec{\mathbf{r}}) = j_l(kr)Y_L(\vec{\mathbf{e}}_r)\,, \quad n_L(k\vec{\mathbf{r}}) = n_l(kr)Y_L(\vec{\mathbf{e}}_r), \quad \vec{\mathbf{e}}_r = \frac{\vec{\mathbf{r}}}{r} \tag{9.59a}$$

one gets the Kasterinian formulas

$$\mathrm{i}^l j_L(k\vec{\mathbf{r}}) = Y_L\left(\frac{\nabla}{\mathrm{i}k}\right)\frac{\sin kr}{kr}, \quad \mathrm{i}^l n_L(k\vec{\mathbf{r}}) = Y_L\left(\frac{\nabla}{\mathrm{i}k}\right)(-1)\frac{\cos kr}{kr}, \tag{9.59b}$$

where $\nabla$ denotes the nabla operator (s. 13.2.6.1, S. 715). The expansion of a plane wave in terms of spherical or Bessel functions gives

$$\mathrm{e}^{\mathrm{i}\vec{\mathbf{k}}\vec{\mathbf{r}}} = 4\pi\sum_L \mathrm{i}^l j_L(k\vec{\mathbf{r}})Y_L^*(\vec{\mathbf{e}}_k), \quad \vec{\mathbf{e}}_k = \frac{\vec{\mathbf{k}}}{k}, \quad \sum_L \cdots = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} \cdots. \tag{9.59c}$$

There are the following addition theorems

$$\mathrm{i}^l j_L(k(\vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_2)) = 4\pi\sum_{L_1,L_2} C_{LL_1L_2}\mathrm{i}^{l_1+l_2} j_{L_1}(k\vec{\mathbf{r}}_1)j_{L_2}^*(k\vec{\mathbf{r}}_2)\,, \quad r_{1,2} = \text{arbitrarily}, \tag{9.59d}$$

$$\mathrm{i}^l n_L(k(\vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_2)) = 4\pi\sum_{L_1,L_2} C_{LL_1L_2}\mathrm{i}^{l_1+l_2} n_{L_1}(k\vec{\mathbf{r}}_1)j_{L_2}^*(k\vec{\mathbf{r}}_2)\,, \quad r_1 > r_2 \tag{9.59e}$$

with the Clebsch-Gordan coefficients (see 5.3.4.7, p. 345)

$$C_{LL_1L_2} = \int d^2e\, Y_L(\vec{\mathbf{e}})Y_{L_1}^*(\vec{\mathbf{e}})Y_{L_2}^*(\vec{\mathbf{e}})\,. \tag{9.59f}$$

For further details see [21.1], [9.28] until [9.31].

### 3. Legendre Differential Equation

Restricting the investigations in this book to the case of real variables and integer parameters $n = 0, 1, 2, \ldots$ the Legendre differential equation has the form

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{or} \quad ((1 - x^2)y')' + n(n+1)y = 0. \tag{9.60a}$$

**1. Legendre Polynomials or Spherical Harmonics of the First Kind** are the particular solutions of the Legendre differential equation for integer $n$, which can be expanded into the power series $y = \sum_{\nu=0}^{\infty} a_\nu x^\nu$. The method of undetermined coefficients yields the polynomials

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - + \cdots\right],$$

$$(|x| < \infty; n = 0, 1, 2, \ldots). \quad (9.60b)$$

$$P_n(x) = F\left(n+1, -n, 1; \frac{1-x}{2}\right) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}, \quad (9.60c)$$

where $F$ denotes the hypergeometric series (see **4.**, p. 567). The first eight polynomials have the following simple form (see 21.12, p. 1108):

$$P_0(x) = 1, \quad (9.60d) \qquad P_1(x) = x, \quad (9.60e)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \quad (9.60f) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad (9.60g)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad (9.60h) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \quad (9.60i)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5), \quad (9.60j) \qquad P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x). \quad (9.60k)$$

The graphs of $P_n(x)$ for the values from $n = 1$ to $n = 7$ are represented in **Fig. 9.16**. The numerical values can be calculated easily by pocket calculators or from function tables.

**2. Properties of the Legendre Polynomials of the First Kind**

**a) Integral Representation:**

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x \pm \cos\varphi\sqrt{x^2-1})^n\, d\varphi = \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{(x \pm \cos\varphi\sqrt{x^2-1})^{n+1}}. \quad (9.61a)$$

The signs can be chosen arbitrarily in both equations.

**b) Recursion Formulas:**

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (n \ge 1; P_0(x) = 1,\ P_1(x) = x), \quad (9.61b)$$

$$(x^2-1)\frac{dP_n(x)}{dx} = n[xP_n(x) - P_{n-1}(x)] \quad (n \ge 1). \quad (9.61c)$$

**c) Orthogonality Relation:**

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & \text{for } m \ne n, \\ \dfrac{2}{2n+1} & \text{for } m = n. \end{cases} \quad (9.61d)$$

**d) Root Theorem:** All the $n$ roots of $P_n(x)$ are real and single and are in the interval $(-1, 1)$.

**e) Generating Function:** The Legendre polynomial of the first kind can be represented as the power series expansion of the function

$$\frac{1}{\sqrt{1-2rx+r^2}} = \sum_{n=0}^{\infty} P_n(x)r^n. \quad (9.61e)$$

For further properties of the Legendre polynomials of the first kind see [21.1].

**3. Legendre Functions or Spherical Harmonics of the Second Kind** A second particular solution $Q_n(x)$ can be got, which is valid for $|x| > 1$ and linearly independent of $P_n(x)$, see (9.61a), by the power series expansion $\sum\limits_{\nu=-\infty}^{-(n+1)} b_\nu x^\nu$:

$$\begin{aligned} Q_n(x) &= \frac{2^n(n!)^2}{(2n+1)!} x^{-(n+1)} F\left(\frac{n+1}{2}, \frac{n+2}{2}, \frac{2n+3}{2}; \frac{1}{x^2}\right) \\ &= \frac{2^n(n!)^2}{(2n+1)!} \left[ x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-(n+3)} \right. \\ &+ \left. \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot (2n+3)(2n+5)} x^{-(n+5)} + \cdots \right]. \end{aligned} \tag{9.62a}$$

The representation of $Q_n(x)$ valid for $|x| < 1$ is:

$$Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(x) P_{n-k}(x). \tag{9.62b}$$

The spherical harmonics of the first and second kind are also called the *associated Legendre functions* (see also 9.2.4.6, **4.**, (9.138c), p. 600).

## 4. Hypergeometric Differential Equation

The *hypergeometric differential equation* is the equation

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta y = 0, \tag{9.63a}$$

where $\alpha$, $\beta$, $\gamma$ are parameters. It contains several important special cases.

**a)** For $\alpha = n+1$, $\beta = -n$, $\gamma = 1$, and $x = \dfrac{1-z}{2}$ it is the Legendre differential equation.

**b)** If $\gamma \neq 0$ or $\gamma$ is not a negative integer, it has the hypergeometric series or hypergeometric function as a particular solution :

$$\begin{aligned} F(\alpha, \beta, \gamma; x) = 1 &+ \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots \\ &+ \frac{\alpha(\alpha+1)\ldots(\alpha+n)\beta(\beta+1)\ldots(\beta+n)}{1 \cdot 2 \ldots (n+1) \cdot \gamma(\gamma+1)\ldots(\gamma+n)} x^{n+1} + \cdots, \end{aligned} \tag{9.63b}$$

which is absolutely convergent for $|x| < 1$. The convergence for $x = \pm 1$ depends on the value of $\delta = \gamma - \alpha - \beta$. For $x = 1$ it is convergent if $\delta > 0$, it is divergent if $\delta \leq 0$. For $x = -1$ it is absolutely convergent if $\delta < 0$, it is conditionally convergent for $-1 < \delta \leq 0$, and it is divergent for $\delta \leq -1$.

**c)** For $2 - \gamma \neq 0$ or not equal to a negative integer it has a particular solution

$$y = x^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, x). \tag{9.63c}$$

**d)** In some special cases the hypergeometric series can be reduced to elementary functions, e.g.,

$$F(1, \beta, \beta; x) = F(\alpha, 1, \alpha; x) = \frac{1}{1-x}, \tag{9.64a}$$

$$F(-n, \beta, \beta; -x) = (1+x)^n, \tag{9.64b}$$

$$F(1, 1, 2; -x) = \frac{\ln(1+x)}{x}, \tag{9.64c}$$

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x^2\right) = \frac{\arcsin x}{x}, \tag{9.64d}$$

$$\lim_{\beta \to \infty} F\left(1, \beta, 1; \frac{x}{\beta}\right) = e^x. \tag{9.64e}$$

## 5. Laguerre Differential Equation

Restricting the investigation to integer parameters ($n = 0, 1, 2, \ldots$) and real variables, the *Laguerre differential equation* has the form

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{9.65a}$$

Particular solutions are the *Laguerre polynomials*

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}. \tag{9.65b}$$

The recursion formula for $n \geq 1$ is:

$$(n+1)L_{n+1}^{(\alpha)}(x) = (-x + 2n + \alpha + 1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \tag{9.65c}$$

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)} = 1 + \alpha - x. \tag{9.65d}$$

An orthogonality relation for $\alpha > -1$ holds:

$$\int_0^\infty e^{-x}x^\alpha L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,dx = \begin{cases} 0 & \text{for } m \neq n, \\ \binom{n+\alpha}{n}\Gamma(1+\alpha) & \text{for } m = n. \end{cases} \tag{9.65e}$$

$\Gamma$ denotes the gamma function (see 8.2.5, **6.**, p. 514).

## 6. Hermite Differential Equation

Two defining equations are often used in the literature:

**a) Defining Equation of Type 1:**

$$y'' - xy' + ny = 0 \quad (n = 0, 1, 2, \ldots). \tag{9.66a}$$

**b) Defining Equation of Type 2:**

$$y'' - 2xy' + ny = 0 \quad (n = 0, 1, 2, \ldots). \tag{9.66b}$$

Particular solutions are the *Hermite polynomials*, $He_n(x)$ for the defining equation of type 1, and $H_n(x)$ for the defining equation of type 2.

**a) Hermite Polynomials for Defining Equation of Type 1:**

$$\begin{aligned} He_n(x) &= (-1)^n \exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2}\right) \\ &= x^n - \binom{n}{2}x^{n-2} + 1\cdot 3\binom{n}{4}x^{n-4} - 1\cdot 3\cdot 5\binom{n}{6}x^{n-6} + \cdots \quad (n \in \mathbb{N}). \end{aligned} \tag{9.66c}$$

For $n \geq 1$ the following recursion formulas are valid:

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x), \tag{9.66d}$$ $$He_0(x) = 1, \quad He_1(x) = x. \tag{9.66e}$$

The orthogonality relation is:

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2}\right) He_m(x)He_n(x)\,dx = \begin{cases} 0 & \text{for } m \neq n, \\ n!\sqrt{2\pi} & \text{for } m = n. \end{cases} \tag{9.66f}$$

**b) Hermite Polynomials for Defining Equation of Type 2:**

$$H_n(x) = (-1)^n \exp\left(x^2\right)\frac{d^n}{dx^n}\exp\left(-x^2\right) \quad (n \in \mathbb{N}). \tag{9.66g}$$

The relation with the Hermite polynomials for defining equation of type 1 is the following:

$$He_n(x) = 2^{-n/2}H_n\left(\frac{x}{\sqrt{2}}\right) \quad (n \in \mathbb{N}). \tag{9.66h}$$

## 9.1.3 Boundary Value Problems

### 9.1.3.1 Problem Formulation

#### 1. Notion of the Boundary Value Problem

In different applications, e.g., in mathematical physics, differential equations must be solved as so-called *boundary value problems* (see 9.2.3, p. 589), where the required solution must satisfy previously given relations at the endpoints of an interval of the independent variable. A special case is the linear boundary value problem, where a solution of a linear differential equation should satisfy linear boundary value conditions. In the following section the discussion is restricted to second-order linear differential equations with linear boundary values.

#### 2. Self-Adjoint Differential Equation

*Self-adjoint differential equations* are important special second-order differential equations of the form

$$[py']' - qy + \lambda \varrho y = f. \tag{9.67a}$$

The linear boundary values are the homogeneous conditions

$$A_0 y(a) + B_0 y'(a) = 0, \quad A_1 y(b) + B_1 y'(b) = 0. \tag{9.67b}$$

The functions $p(x)$, $p'(x)$, $q(x)$, $\varrho(x)$, and $f(x)$ are supposed to be continuous in the finite interval $a \le x \le b$. In the case of an infinite interval the results change considerably (see [9.5]). Furthermore, it is supposed that $p(x) > p_0 > 0$, $\varrho(x) > \varrho_0 > 0$. The quantity $\lambda$, a parameter of the differential equation, is a constant. For $f = 0$, it is called the *homogeneous boundary value problem* associated to the *inhomogeneous boundary value problem.*

Every second-order differential equation of the form

$$Ay'' + By' + Cy + \lambda R y = F \tag{9.67c}$$

can be reduced to the self-adjoint equation (9.67a) by multiplying it by $p/A$ if in $[a, b]$, $A \ne 0$, and performing the following substitutions

$$p = \exp\left(\int \frac{R}{A}\, dx\right), \quad q = -\frac{pC}{A}, \quad \varrho = \frac{pR}{A}. \tag{9.67d}$$

To find a solution satisfying the inhomogeneous conditions

$$A_0 y(a) + B_0 y'(a) = C_0, \quad A_1 y(b) + B_1 y'(b) = C_1 \tag{9.67e}$$

one returns to the problem with homogeneous boundary conditions, but the right-hand side $f(x)$ changes and $y = z + u$ is substituted where $u$ is an arbitrary twice differentiable function satisfying the inhomogeneous boundary conditions and $z$ is a new unknown function satisfying the corresponding homogeneous conditions.

#### 3. Sturm-Liouville Problem

For a given value of the parameter $\lambda$ there are two cases:

**1.** Either the inhomogeneous boundary value problem has a unique solution for arbitrary $f(x)$, while the corresponding homogeneous problem has only the trivial, identically zero solution, or,

**2.** The corresponding homogeneous problem also has non-trivial, i.e., not identically zero solutions, but in this case the inhomogeneous problem does not have a solution for arbitrary right-hand side; and if a solution exists, it is not unique.

The values of the parameter $\lambda$, for which the second case occurs, i.e., the homogeneous problem has a non-trivial solution, are called the *eigenvalues of the boundary value problem*, the corresponding non-trivial solutions are called the *eigenfunctions.* The problem of determining the eigenvalues and eigenfunctions of a differential equation (9.67a) is called the *Sturm-Liouville problem.*

### 9.1.3.2 Fundamental Properties of Eigenfunctions and Eigenvalues

**1.** The eigenvalues of a boundary value problem form a monotone increasing sequence of real numbers

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \tag{9.68a}$$

tending to infinity.

**2.** The eigenfunction associated to the eigenvalue $\lambda_n$ has exactly $n$ roots in the interval $a < x < b$.
**3.** If $y(x)$ and $z(x)$ are two eigenfunctions belonging to the same eigenvalue $\lambda$, they differ only in a constant multiplier $c$, i.e.,

$$z(x) = cy(x). \tag{9.68b}$$

**4.** Two eigenfunctions $y_1(x)$ and $y_2(x)$, associated to different eigenvalues $\lambda_1$ and $\lambda_2$, are *orthogonal* to each other with the *weight function* $\varrho(x)$

$$\int_a^b y_1(x)\, y_2(x)\, \varrho(x)\, dx = 0. \tag{9.68c}$$

**5.** If in (9.67a) the coefficients $p(x)$ and $q(x)$ are replaced by $\tilde{p}(x) \geq p(x)$ and $\tilde{q}(x) \geq q(x)$, then the eigenvalues will not decrease, i.e., $\tilde{\lambda}_n \geq \lambda_n$, where $\tilde{\lambda}_n$ and $\lambda_n$ are the $n$-th eigenvalues of the modified and the original equations respectively. But if the coefficient $\varrho(x)$ is replaced by $\tilde{\varrho}(x) \geq \varrho(x)$, then the eigenvalues will not increase, i.e., $\tilde{\lambda}_n \leq \lambda_n$. The $n$-th eigenvalue depends continuously on the coefficients of the equation, i.e., small changes in the coefficients will result in small variations of the $n$-th eigenvalue.
**6.** Reduction of the interval $[a, b]$ into a smaller one does not result in smaller eigenvalues.

## 9.1.3.3 Expansion in Eigenfunctions

### 1. Normalization of the Eigenfunction

For every $\lambda_n$ an eigenfunction $\varphi_n(x)$ is chosen such that

$$\int_a^b [\varphi_n(x)]^2 \varrho(x)\, dx = 1. \tag{9.69a}$$

It is called a *normalized eigenfunction*.

### 2. Fourier Expansion

To every function $g(x)$ defined in the interval $[a, b]$, one can assign its *Fourier series*

$$g(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x), \quad c_n = \int_a^b g(x)\, \varphi_n(x)\, \varrho(x)\, dx \tag{9.69b}$$

with the eigenfunctions of the corresponding boundary value problem, if the integrals in (9.69b) exist.

### 3. Expansion Theorem

If the function $g(x)$ has a continuous derivative and satisfies the boundary conditions of the given problem, then the Fourier series of $g(x)$ (in the eigenfunctions of this boundary value problem) is absolutely and uniformly convergent to $g(x)$.

### 4. Parseval Equation

If the integral on the left-hand side exists, then

$$\int_a^b [g(x)]^2 \varrho(x)\, dx = \sum_{n=0}^{\infty} {c_n}^2 \tag{9.69c}$$

is always valid. The Fourier series of the function $g(x)$ converges in this case to $g(x)$ in mean, that is

$$\lim_{N\to\infty} \int_a^b \left[ g(x) - \sum_{n=0}^{N} c_n \varphi_n(x) \right]^2 \varrho(x)\, dx = 0. \tag{9.69d}$$

## 9.1.3.4 Singular Cases

Boundary value problems of the above type occur very often in solving problems of theoretical physics by the Fourier method, however at the endpoints of the interval $[a, b]$ some singularities of the differential equation may occur, e.g., $p(x)$ vanishes. At such singular points some restrictions are imposed on the solutions, e.g., continuity or being finite or unlimited growth with a bounded order. These conditions play the role of homogeneous boundary conditions (see 9.2.3.3, p. 591). In addition, often occurs the case where in certain boundary value problems homogeneous boundary conditions should be considered, such that they connect the values of the function or its derivative at different endpoints of the interval. Often occur the relations

$$y(a) = y(b), \quad p(a)y'(a) = p(b)y'(b), \tag{9.70}$$

which represent periodicity in the case of $p(a) = p(b)$. For such boundary value problems everything being introduced above remains valid, except statement (9.68b). For further discussion of this topic see [9.5].

# 9.2 Partial Differential Equations

## 9.2.1 First-Order Partial Differential Equations

### 9.2.1.1 Linear First-Order Partial Differential Equations

#### 1. Linear and Quasilinear Partial Differential Equations

The equation

$$X_1 \frac{\partial z}{\partial x_1} + X_2 \frac{\partial z}{\partial x_2} + \cdots + X_n \frac{\partial z}{\partial x_n} = Y \tag{9.71a}$$

is called a *linear first-order partial differential equation.* Here $z$ is an unknown function of the independent variables $x_1, \ldots, x_n$, and $X_1, \ldots, X_n, Y$ are given functions of these variables. If functions $X_1, \ldots, X_n, Y$ depend also on $z$, the equation is called a *quasilinear partial differential equation.* In the case of

$$Y \equiv 0, \tag{9.71b}$$

the equation is called homogeneous.

#### 2. Solution of a Homogeneous Partial Linear Differential Equation

The solution of a homogeneous partial linear differential equation and the solution of the so-called *characteristic system*

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \tag{9.72a}$$

are equivalent. This system can be solved in two different ways:

**1.** Any $x_k$, for which $X_k \neq 0$, can be chosen as an independent variable, so the system is transformed into the form

$$\frac{dx_j}{dx_k} = \frac{X_j}{X_k} \quad (j = 1, \ldots, n). \tag{9.72b}$$

**2.** A more convenient way is to keep symmetry and to introduce a new variable $t$ getting

$$\frac{dx_j}{dt} = X_j \quad (j = 1, 2, \ldots, n). \tag{9.72c}$$

Every first integral of the system (9.72a) is a solution of the homogeneous linear partial differential equation (9.72a,b), and conversely, every solution of (9.72a,b) is a first integral of (9.72a) (see 9.1.2.1, **2.**, p. 551). If the $n-1$ first integrals

$$\varphi_i(x_1, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, n-1) \tag{9.72d}$$

are independent (see 9.1.2.3, **2.**, p. 553), then the general solution is

$$z = \Phi(\varphi_1, \ldots, \varphi_{n-1}). \tag{9.72e}$$

Here $\Phi$ is an arbitrary function of the $n-1$ arguments $\varphi_i$ and a general solution of the homogeneous linear differential equation.

### 3. Solution of Inhomogeneous Linear and Quasilinear Partial Differential Equations

To solve an inhomogeneous linear and quasilinear partial differential equation (9.71a) one can try to find the solution $z$ in the implicit form $V(x_1, \ldots, x_n, z) = C$. The function $V$ is a solution of the homogeneous linear differential equation with $n+1$ independent variables

$$X_1 \frac{\partial V}{\partial x_1} + X_2 \frac{\partial V}{\partial x_2} + \cdots + X_n \frac{\partial V}{\partial x_n} + Y \frac{\partial V}{\partial z} = 0, \tag{9.73a}$$

whose characteristic system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} = \frac{dz}{Y} \tag{9.73b}$$

is called the *characteristic system of the original equation* (9.71a).

### 4. Geometrical Representation and Characteristics of the System

In the case of the equation

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z) \tag{9.74a}$$

with two independent variables $x_1 = x$ and $x_2 = y$, a solution $z = f(x,y)$ is a surface in $x, y, z$ space, and it is called the *integral surface* of the differential equation. Equation (9.74a) means that at every point of the integral surface $z = f(x,y)$ the normal vector $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ is orthogonal to the vector $(P, Q, R)$ given at that point. Here the system (9.73b) has the form

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}. \tag{9.74b}$$

It follows (see 13.1.3.5, p. 708) that the *integral curves of this system*, the so-called *characteristics*, are tangent to the vector $(P, Q, R)$. Therefore, a characteristic having a common point with the integral surface $z = f(x,y)$ lies completely on this surface. Since the conditions for the existence theorem 13.1.3.5, **1.**, p. 551 hold, there is an integral curve of the characteristic system passing through every point of space, so the integral surface consists of characteristics.

### 5. Cauchy Problem

There are given $n$ functions of $n-1$ independent variables $t_1, t_2, \ldots, t_{n-1}$:

$$x_1 = x_1(t_1, t_2, \ldots, t_{n-1}),\; x_2 = x_2(t_1, t_2, \ldots, t_{n-1}), \ldots,\; x_n = x_n(t_1, t_2, \ldots, t_{n-1}). \tag{9.75a}$$

The Cauchy problem for the differential equation (9.71a) is to find a solution

$$z = \varphi(x_1, x_2, \ldots, x_n) \tag{9.75b}$$

such that if one substitutes (9.75a), the result is a previously given function $\psi(t_1, t_2, \ldots, t_{n-1})$:

$$\varphi[x_1(t_1, t_2, \ldots, t_{n-1}),\; x_2(t_1, t_2, \ldots, t_{n-1}), \ldots,\; x_n(t_1, t_2, \ldots, t_{n-1})] = \psi(t_1, t_2, \ldots, t_{n-1}). \tag{9.75c}$$

In the case of two independent variables, the problem reduces to find an integral surface passing through the given curve. If this curve has a tangent depending continuously on a point and it is not tangent to the characteristics at any point, then the Cauchy problem has a unique solution in a certain neighborhood of this curve. Here the integral surface consists of the set of all characteristics intersecting the given curve. For more mathematical discussion on theorems about the existence of the solution of the Cauchy problem see [9.15].

■ **A:** For the linear first-order inhomogeneous partial differential equation $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$ ($l, m, n$ are constants), the equations of the characteristics are $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} =$

$\dfrac{dz}{ly - mx}$. The integrals of this system are $lx + my + nz = C_1$, $x^2 + y^2 + z^2 = C_2$. One gets circles as characteristics, whose centers are on a line passing through the origin, and this line has direction cosines proportional to $l, m, n$. The integral surfaces are rotation surfaces with this line as an axis.

■ **B:** Determine the integral surface of the first-order linear inhomogeneous differential equation $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = z$, which passes through the curve $x = 0$, $z = \varphi(y)$. The equations of characteristics are $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{z}$. The characteristics passing through the point $(x_0, y_0, z_0)$ are $y = x - x_0 + y_0$, $z = z_0 e^{x-x_0}$. A parametric representation of the required integral surface is $y = x + y_0$, $z = e^x\varphi(y_0)$, if we substitute $x_0 = 0$, $z_0 = \varphi(y_0)$. The elimination of $y_0$ results in $z = e^x\varphi(y - x)$.

### 9.2.1.2 Non-Linear First-Order Partial Differential Equations

#### 1. General Form of First-Order Partial Differential Equation

is the implicit equation

$$F\left(x_1, \ldots, x_n, z, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right) = 0. \tag{9.76a}$$

**1. Complete Integral** is the solution

$$z = \varphi(x_1, \ldots, x_n; a_1, \ldots, a_n), \tag{9.76b}$$

depending on $n$ parameters $a_1, \ldots, a_n$ if at the considered values of $x_1, \ldots, x_n, z$ the functional determinant (or Jacobian determinant, see 2.18.2.6, **3.**, p. 123) is non-zero:

$$\frac{\partial\left(\varphi_{x_1}, \ldots, \varphi_{x_n}\right)}{\partial(a_1, \ldots, a_n)} \neq 0. \tag{9.76c}$$

**2. Characteristic Strip** The solution of (9.76a) is reduced to the solution of the characteristic system

$$\frac{dx_1}{P_1} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{p_1P_1 + \cdots + p_nP_n} = \frac{-dp_1}{X_1 + p_1Z} = \cdots = \frac{-dp_n}{X_n + p_nZ} \tag{9.76d}$$

with

$$Z = \frac{\partial F}{\partial z}, \quad X_i = \frac{\partial F}{\partial x_i}, \quad p_i = \frac{\partial z}{\partial x_i}, \quad P_i = \frac{\partial F}{\partial p_i} \quad (i = 1, \ldots, n). \tag{9.76e}$$

The solutions of the characteristic system satisfying the additional condition

$$F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0 \tag{9.76f}$$

are called the *characteristic strips*.

#### 2. Canonical Systems of Differential Equations

Sometimes it is more convenient to consider an equation not involving explicitly the unknown function $z$. Such an equation can be obtained by introducing an additional independent variable $x_{n+1} = z$ and an unknown function $V(x_1, \ldots, x_n, x_{n+1})$, which defines the function $z(x_1, x_2, \ldots, x_n)$ with the equation

$$V(x_1, \ldots, x_n, z) = C. \tag{9.77a}$$

At the same time, one substitutes the functions $-\dfrac{\partial V}{\partial x_i} \Big/ \dfrac{\partial V}{\partial x_{n+1}}$ $(i = 1, \ldots, n)$ for $\dfrac{\partial z}{\partial x_i}$ in (9.76a). Then one solves the differential equation (9.76a) for an arbitrary partial derivative of the function $V$.

The corresponding independent variable will be denoted by $x$ after a suitable renumbering of the other variables. Finally, one gets the equation (9.76a) in the form

$$p + H(x_1, \dots, x_n, x, p_1, \dots, p_n) = 0, \quad p = \frac{\partial V}{\partial x}, \quad p_i = \frac{\partial V}{\partial x_i} \quad (i = 1, \dots, n). \tag{9.77b}$$

The system of characteristic differential equations is transformed into the system

$$\frac{dx_i}{dx} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dx} = -\frac{\partial H}{\partial x_i} \quad (i = 1, \dots, n) \quad \text{and} \tag{9.77c}$$

$$\frac{dV}{dx} = p_1 \frac{\partial H}{\partial p_1} + \dots + p_n \frac{\partial H}{\partial p_n} - H, \quad \frac{dp}{dx} = -\frac{\partial H}{\partial x}. \tag{9.77d}$$

Equations (9.77c) represent a system of $2n$ ordinary differential equations, which corresponds to an arbitrary function $H(x_1, \dots, x_n, x, p_1, \dots, p_n)$ with $2n + 1$ variables. It is called a *canonical system* or a *normal system* of differential equations.

Many problems of mechanics and theoretical physics lead to equations of this form. Knowing a complete integral

$$V = \varphi(x_1, \dots, x_n, x, a_1, \dots, a_n) + a \tag{9.77e}$$

of the equation (9.77b) one can find the general solution of the canonical system (9.77c), since the equations $\dfrac{\partial \varphi}{\partial a_i} = b_i$, $\dfrac{\partial \varphi}{\partial x_i} = p_i$ $(i = 1, 2, \dots, n)$ with $2n$ arbitrary parameters $a_i$ and $b_i$ determine a $2n$-parameter solution of the canonical system (9.77c).

### 3. Clairaut Differential Equation

If the given differential equation can be transformed into the form

$$z = x_1 p_1 + x_2 p_2 + \dots + x_n p_n + f(p_1, p_2, \dots, p_n), \quad p_i = \frac{\partial z}{\partial x_i} \quad (i = 1, \dots, n), \tag{9.78a}$$

it is called a Clairaut differential equation. The determination of the complete integral is particularly simple, because a complete integral with the arbitrary parameters $a_1, a_2, \dots, a_n$ is

$$z = a_1 x_1 + a_2 x_2 + \dots + a_n x_n + f(a_1, a_2, \dots, a_n). \tag{9.78b}$$

■ **Two-Body Problem with Hamilton Function:** Consider two particles moving in a plane under their mutual gravitational attraction according to the Newton field (see also 13.4.3.2, p. 728). Choosing the origin as the initial position of one of the particles, the equations of motion have the form

$$\frac{d^2x}{dt^2} = \frac{\partial V}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial V}{\partial y}; \quad V = \frac{k^2}{\sqrt{x^2 + y^2}}. \tag{9.79a}$$

Introducing the Hamiltonian function

$$H = \frac{1}{2}(p^2 + q^2) - \frac{k^2}{\sqrt{x^2 + y^2}}, \tag{9.79b}$$

the system (9.79a) is transformed into the normal system (into the system of canonical differential equations)

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial q}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dq}{dt} = -\frac{\partial H}{\partial y} \tag{9.79c}$$

with variables

$$x, y, p = \frac{dx}{dt}, \quad q = \frac{dy}{dt}. \tag{9.79d}$$

Now, the partial differential equation has the form

$$\frac{\partial z}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] - \frac{k^2}{\sqrt{x^2+y^2}} = 0. \tag{9.79e}$$

Introducing the polar coordinates $\rho$, $\varphi$ in (9.79e) one obtains a new differential equation having the solution

$$z = -at - b\varphi + c - \int_{\rho_0}^{\rho} \sqrt{2a + \frac{2k^2}{r} - \frac{b^2}{r^2}}\, dr \tag{9.79f}$$

with the parameters $a, b, c$. The general solution of the system (9.79c) follows from the equations

$$\frac{\partial z}{\partial a} = -t_0, \quad \frac{\partial z}{\partial b} = -\varphi_0. \tag{9.79g}$$

## 4. First-Order Differential Equation in Two Independent Variables

For $x_1 = x$, $x_2 = y$, $p_1 = p$, $p_2 = q$ the characteristic strip (see 9.2.1.2, **1.**, p. 573) can be geometrically interpreted as a curve at every point $(x, y, z)$ of which a plane $p(\xi - x) + q(\eta - y) = \zeta - z$ being tangent to the curve is prescribed. So, the problem of finding an integral surface of the equation

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \tag{9.80}$$

passing through a given curve, i.e., to solve the Cauchy problem (see 9.2.1.1, **5.**, p. 572), is transformed into another problem: To find the characteristic strips passing through the points of the initial curve such that the corresponding tangent plane to each strip is tangent to that curve. One gets the values $p$ and $q$ at the points of the initial curve from the equations $F(x, y, z, p, q) = 0$ and $p\,dx + q\,dy = dz$. There can be several solutions in the case of non-linear differential equations.

Therefore, under the formulation of the Cauchy problem, in order to obtain a unique solution one can assume two continuous functions $p$ and $q$ satisfying the above relations along the initial curve.
For the existence of solutions of the Cauchy problem see [9.15].

■ For the partial differential equation $p\,q = 1$ and the initial curve $y = x^3$, $z = 2x^2$, one can choose $p = x$ and $q = 1/x$ along the curve. The characteristic system has the form

$$\frac{dx}{dt} = q, \quad \frac{dy}{dt} = p, \quad \frac{dz}{dt} = 2p\,q, \quad \frac{dp}{dt} = 0, \quad \frac{dq}{dt} = 0.$$

The characteristic strip with initial values $x_0, y_0, z_0, p_0$ and $q_0$ for $t = 0$ satisfies the equations $x = x_0 + q_0 t$, $y = y_0 + p_0 t$, $z = 2p_0 q_0 t + z_0$, $p = p_0, q = q_0$. For the case of $p_0 = x_0$, $q_0 = 1/x_0$ the equation of the curve belonging to the characteristic strip that passes through the point $(x_0, y_0, z_0)$ of the initial curve is

$$x = x_0 + \frac{t}{x_0}, \quad y = {x_0}^3 + t x_0, \quad z = 2t + 2{x_0}^2.$$

Eliminating the parameters $x_0$ and $t$ gives $z^2 = 4xy$. For other chosen values of $p$ and $q$ along the initial curve one can get different solutions.

**Remark:** The envelope of a one-parameter family of integral surfaces is also an integral surface. Considering this fact one can solve the Cauchy problem with a complete integral. One finds a one-parameter family of solutions tangent to the planes given at the points of the initial curve. Then one determines the envelope of this family.

■ Determine the integral surface for the Clairaut differential equation $z - px - qy + pq = 0$ passing through the curve $y = x$, $z = x^2$. The complete integral of the differential equation is $z = ax + by - ab$. Since along the initial curve $p = q = x$, one determines the one-parameter family of integral surfaces by the condition $a = b$. When the envelope of this family is found then one gets $z = \frac{1}{4}(x + y)^2$.

### 5. Linear First-Order Partial Differential Equations in Total Differentials

Equations of this kind have the form

$$dz = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n, \tag{9.81a}$$

where $f_1, f_2, \ldots, f_n$ are given functions of the variables $x_1, x_2, \ldots, x_n, z$. The equation is called a *completely integrable* or *exact differential equation* when there exists a unique relation between $x_1, x_2, \ldots, x_n$, $z$ with one arbitrary constant, which leads to equation (9.81a). Then there exists a unique solution $z = z(x_1, x_2, \ldots, x_n)$ of (9.81a), which has a given value $z_0$ for the initial values ${x_1}^0, \ldots, {x_n}^0$ of the independent variables. Therefore, for $n = 2$, $x_1 = x$, $x_2 = y$ a unique integral surface passes through every point of space.

The differential equation (9.81a) is *completely integrable* if and only if the $\dfrac{n(n-1)}{2}$ equalities

$$\frac{\partial f_i}{\partial x_k} + f_k \frac{\partial f_i}{\partial z} = \frac{\partial f_k}{\partial x_i} + f_i \frac{\partial f_k}{\partial z} \quad (i, k = 1, \ldots, n) \tag{9.81b}$$

in all variables $x_1, x_2, \ldots, x_n, z$ are identically satisfied.
If the differential equation is given in symmetric form

$$f_1 dx_1 + \cdots + f_n dx_n = 0, \tag{9.81c}$$

then the condition for complete integrability is

$$f_i \left( \frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \right) + f_j \left( \frac{\partial f_i}{\partial x_k} - \frac{\partial f_k}{\partial x_i} \right) + f_k \left( \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right) = 0 \tag{9.81d}$$

for all possible combinations of the indices $i, j, k$. If the equation is completely integrable, then the solution of the differential equation (9.81a) can be reduced to the solution of an ordinary differential equation with $n - 1$ parameters.

## 9.2.2 Linear Second-Order Partial Differential Equations

### 9.2.2.1 Classification and Properties of Second-Order Differential Equations with Two Independent Variables

#### 1. General Form

of a linear second-order partial differential equation with two independent variables $x, y$ and an unknown function $u$ is an equation in the form

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = f, \tag{9.82a}$$

where the coefficients $A, B, C, a, b, c$ and $f$ on the right-hand side are known functions of $x$ and $y$.
The form of the solution of this differential equation depends on the sign of the *discriminant*

$$\delta = AC - B^2 \tag{9.82b}$$

in a considered domain. The following cases should be distinguished.

**1. $\delta < 0$: Hyperbolic type.**

**2. $\delta = 0$: Parabolic type.**

**3. $\delta > 0$: Elliptic type.**

**4.** $\delta$ changes its sign: **Mixed type.**

An important property of the discriminant $\delta$ is that its sign is invariant with respect to arbitrary transformation of the independent variables, e.g., to introduction new coordinates in the $x, y$ plane. Therefore, the type of the differential equation is invariant with respect to the choice of the independent variables.

## 2. Characteristics

of linear second-order partial differential equations are the integral curves of the differential equation

$$A dy^2 - 2B dx dy + C dx^2 = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{B \pm \sqrt{-\delta}}{A}. \tag{9.83}$$

For the characteristics of the above three types of differential equations the following statements are valid:

**1. Hyperbolic type:** There exist two families of real characteristics.

**2. Parabolic type:** There exists only one family of real characteristics.

**3. Elliptic type:** There exists no real characteristic.

**4.** A differential equation obtained by coordinate transformation from (9.82a) has the same characteristics as (9.82a).

**5.** If a family of characteristics coincides with a family of coordinate lines, then the term with the second derivative of the unknown function with respect to the corresponding independent variable is missing in (9.82a). In the case of a parabolic differential equation, the mixed derivative term is also missing.

## 3. Normal Form or Canonical Form

One has the following possibilities to transform (9.82a) into the normal form of linear second-order partial differential equations.

**1. Transformation into Normal Form:** The differential equation (9.82a) can be transformed into normal form by introducing the new independent variables

$$\xi = \varphi(x, y) \quad \text{and} \quad \eta = \psi(x, y), \tag{9.84a}$$

which according to the sign of the discriminant (9.82b) belongs to one of the three considered types:

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + \cdots = 0, \quad \delta < 0, \quad \text{hyperbolic type;} \tag{9.84b}$$

$$\frac{\partial^2 u}{\partial \eta^2} + \cdots = 0, \quad \delta = 0, \quad \text{parabolic type;} \tag{9.84c}$$

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \cdots = 0, \quad \delta > 0, \quad \text{elliptic type.} \tag{9.84d}$$

The terms not containing second-order partial derivatives of the unknown function are denoted by dots.

**2. Reduction of a Hyperbolic Type Equation to Canonical Form** (9.84b)**:** If, in the hyperbolic case, one chooses two families of characteristics as the coordinate lines of the new coordinate system (9.84a), i.e., if substituting $\xi_1 = \varphi(x, y)$, $\eta_1 = \psi(x, y)$, where $\varphi(x, y) =$ constant, $\psi(x, y) =$ constant are the equations of the characteristics, then (9.82a) becomes the form

$$\frac{\partial^2 u}{\partial \xi_1 \partial \eta_1} + \cdots = 0. \tag{9.84e}$$

This form is also called the *canonical form of a hyperbolic type differential equation*. From here one gets the canonical form (9.84b) by the substitution

$$\xi = \xi_1 + \eta_1, \quad \eta = \xi_1 - \eta_1. \tag{9.84f}$$

**3. Reduction of a Parabolic Type Equation to Canonical Form** (9.84c)**:** The only family of characteristics given in this case is selected for the family $\xi =$ const, where an arbitrary function of $x$ and $y$ can be chosen for $\eta$, which must not be dependent on $\xi$.

**4. Reduction of an Elliptic Type Equation to Canonical Form** (9.84d)**:** If the coefficients $A(x, y)$, $B(x, y)$, $C(x, y)$ are analytic functions (see 14.1.2.1, p. 732) in the elliptic case, then the characteristics define two complex conjugate families of curves $\varphi(x, y) =$ constant, $\psi(x, y) =$ constant. By substituting $\xi = \varphi + \psi$ and $\eta = \mathrm{i}(\varphi - \psi)$, the equation becomes of the form (9.84d).

### 4. Generalized Form

Every statement for the classification and reduction to canonical forms remains valid for equations given in a more general form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0, \tag{9.85}$$

where $F$ is a non-linear function of the unknown function $u$ and its first-order partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$, in contrast to (9.82a).

## 9.2.2.2 Classification and Properties of Linear Second-Order Differential Equations with More than Two Independent Variables

### 1. General Form

A differential equation of this kind for $u = u(x_1, x_2, \ldots, x_n)$ has the form

$$\sum_{i,k} a_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + \cdots = 0, \tag{9.86}$$

where $a_{ik}$ are given functions of the independent variables and the dots in (9.86) mean terms not containing second-order derivatives of the unknown function.
In general, the differential equation (9.86) cannot be reduced to a simple canonical form by transforming the independent variables. However, there is an important classification, similar to the one introduced above in 9.2.2.1, p. 576 (see [9.5]).

### 2. Linear Second-Order Partial Differential Equations with Constant Coefficients

If all coefficients $a_{ik}$ in (9.86) are constants, then the equation can be reduced by a linear homogeneous transformation of the independent variables into a simpler canonical form

$$\sum_{i} \kappa_i \frac{\partial^2 u}{\partial {x_i}^2} + \cdots = 0, \tag{9.87}$$

where the coefficients $\kappa_i$ are $\pm 1$ or 0. Several characteristic cases have to be distinguished.

**1. Elliptic Differential Equation** If all coefficients $\kappa_i$ are different from zero, and they have the same sign, then it is the case of an *elliptic differential equation*.

**2. Hyperbolic and Ultra-Hyperbolic Differential Equation** If all coefficients $\kappa_i$ are different from zero, but one has a sign different from the other's, then it is the case of a *hyperbolic differential equation*. If both types of signs occur at least twice, then it is an *ultra-hyperbolic differential equation*.

**3. Parabolic Differential Equation** If one of the coefficients $\kappa_i$ is equal to zero, the others are different from zero and they have the same sign, then it is the case of a *parabolic differential equation*.

**4. Simple Case for Elliptic and Hyperbolic Differential Equations** If not only the coefficients of the second order derivatives of the unknown function are constants, but also those of the first order derivatives, then it is possible to eliminate the terms of the first order derivatives, for which $\kappa_i \neq 0$, by substitution. For this purpose is

$$u = v \exp\left(-\frac{1}{2}\sum \frac{b_k}{\kappa_k} x_k\right), \tag{9.88}$$

substituted where $b_k$ is the coefficient of $\dfrac{\partial u}{\partial x_k}$ in (9.87) and the summation is performed for all $\kappa_i \neq 0$.
In this way, every elliptic and hyperbolic differential equation with constant coefficients can be reduced to a simple form:

**a) Elliptic Case:** $\Delta v + kv = g.$ (9.89) **b) Hyperbolic Case:** $\dfrac{\partial^2 v}{\partial t^2} - \Delta v + kv = g.$ (9.90)

Here $\Delta$ denotes the Laplace operator (see 13.2.6.5, p. 716).

#### 9.2.2.3 Integration Methods for Linear Second-Order Partial Differential Equations

##### 1. Method of Separation of Variables

Certain solutions of several differential equations of physics can be determined by special substitutions, and although these are not general solutions, one gets a family of solutions depending on arbitrary parameters. Linear differential equations, especially those of second order, can often be solved if looking for a solution in the *form of a product*

$$u(x_1, \dots, x_n) = \varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n). \tag{9.91}$$

Next, one tries to separate the functions $\varphi_k(x_k)$, i.e., for each of them one wants to determine an ordinary differential equation containing only one variable $x_k$. This *separation of variables* is successful in many cases when the trial solution in the form of a product (9.91) is substituted into the given differential equation. In order to guarantee that the solution of the original equation satisfies the required homogeneous boundary conditions, it may appear to be sufficient that some of functions $\varphi_1(x_1)$, $\varphi_2(x_2), \dots, \varphi_n(x_n)$ satisfy certain boundary conditions.
By means of summation, differentiation and integration, new solutions can be acquired from the obtained ones; the parameters should be chosen so that the remaining boundary and initial conditions are satisfied (see examples).
Finally, don't forget that the solutions obtained in this way, often infinite series and improper integrals, are only *formal solutions*. That is, one has to check whether the solution makes a physical sense, e.g., whether it is convergent, satisfies the original differential equation and the boundary conditions, whether it is differentiable termwise and whether the limit at the boundary exists.

The infinite series and improper integrals in the examples of this paragraph are convergent if the functions defining the boundary conditions satisfy the required conditions, e.g., the continuity assumption for the second derivatives in the first and the second examples.

■ **A: Equation of the Vibrating String** is a linear second-order partial differential equation of hyperbolic type

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}\,. \tag{9.92a}$$

It describes the vibration of a spanned string. The boundary and the initial conditions are:

$$u\Big|_{t=0} = f(x), \ \ \frac{\partial u}{\partial t}\Big|_{t=0} = \varphi(x), \quad u|_{x=0} = 0, \ \ u|_{x=l} = 0. \tag{9.92b}$$

Seeking a solution in the form

$$u = X(x)T(t), \tag{9.92c}$$

and after substituting it into the given equation (9.92a) follows

$$\frac{T''}{a^2 T} = \frac{X''}{X}\,. \tag{9.92d}$$

The variables are separated, the right side depends on only $x$ and the left side depends on only $t$, so each of them is a constant quantity. The constant must be negative, otherwise the boundary conditions cannot be satisfied, i.e., non-negative values give the trivial solution $u(x,t) = 0$. This negative constant is denoted by $-\lambda^2$. The result is an ordinary linear second-order differential equation with constant coefficients for both variables. For the general solution see 9.1.2.4, p. 555. The results are the linear differential equations

$$X'' + \lambda^2 X = 0, \tag{9.92e}$$ and $$T'' + a^2\lambda^2 T = 0. \tag{9.92f}$$

From the boundary conditions follows $X(0) = X(l) = 0$. Hence $X(x)$ is an eigenfunction of the

Sturm-Liouville boundary value problem and $\lambda^2$ is the corresponding eigenvalue (see 9.1.3.1, **3.**, p. 569). Solving the differential equation (9.92e) for $X$ with the corresponding boundary conditions one gets

$$X(x) = C\sin\lambda x \quad \text{with} \quad \sin\lambda l = 0, \quad \text{i.e., with} \quad \lambda = \frac{n\pi}{l} = \lambda_n \quad (n = 1, 2, \ldots). \tag{9.92g}$$

Solving equation (9.92f) for $T$ yields a particular solution of the original differential equation (9.92a) for every eigenvalue $\lambda_n$:

$$u_n(x,t) = \left(a_n \cos\frac{na\pi}{l}t + b_n \sin\frac{na\pi}{l}t\right)\sin\frac{n\pi}{l}x. \tag{9.92h}$$

Requiring that for $t = 0$,

$$u\bigg|_{t=0} = \sum_{n=1}^{\infty} u_n(x,0) \ \text{ is equal to f(x)}, \quad \text{and} \tag{9.92i}$$

$$\frac{\partial u}{\partial t}\bigg|_{t=0} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}(x,0) \ \text{ is equal to } \varphi(x), \tag{9.92j}$$

one gets with a Fourier series expansion in sines (see 7.4.1.1, **1.**, p. 474)

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad b_n = \frac{2}{na\pi}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx. \tag{9.92k}$$

■ **B: Equation of Longitudinal Vibration of a Bar** is a linear second-order partial differential equation of hyperbolic type, which describes the longitudinal vibration of a bar with one end free and a constant force $p$ affecting the fixed end. Here is to solve the same differential equation as in ■ **A** (p. 579), i.e.,

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \tag{9.93a}$$

with the same initial but different boundary conditions:

$$u\bigg|_{t=0} = f(x), \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = \varphi(x), \quad \text{(9.93b)} \qquad \frac{\partial u}{\partial x}\bigg|_{x=0} = 0 \quad \text{(free end)}, \tag{9.93c}$$

$$\frac{\partial u}{\partial x}\bigg|_{x=l} = kp\,. \tag{9.93d}$$

The conditions (9.93c,d) can be replaced by the homogeneous conditions

$$\frac{\partial z}{\partial x}\bigg|_{x=0} = \frac{\partial z}{\partial x}\bigg|_{x=l} = 0 \tag{9.93e}$$

where instead of $u$ is introduced a new unknown function

$$z = u - \frac{kpx^2}{2l}. \tag{9.93f}$$

The differential equation becomes inhomogeneous:

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2} + \frac{a^2kp}{l}. \tag{9.93g}$$

Looking for the solution in the form $z = v + w$, where $v$ satisfies the homogeneous differential equation with the initial and boundary conditions for $z$, i.e.,

$$z\bigg|_{t=0} = f(x) - \frac{kpx^2}{2}, \quad \frac{\partial z}{\partial t}\bigg|_{t=0} = \varphi(x), \tag{9.93h}$$

and $w$ satisfies the inhomogeneous differential equation with zero initial and boundary conditions. This gives $w = \dfrac{ka^2pt^2}{2l}$. Substituting the product form of the unknown function $v(x,t)$ into the differential equation (9.93a)

$$v = X(x)T(t) \tag{9.93i}$$

gives the separated ordinary differential equations as in ■ **A** (p. 579)

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda^2. \tag{9.93j}$$

Integrating the differential equation for $X$ with the boundary conditions $X'(0) = X'(l) = 0$ one finds the eigenfunctions

$$X_n = \cos\frac{n\pi x}{l} \tag{9.93k}$$

and the corresponding eigenvalues

$$\lambda_n{}^2 = \frac{n^2\pi^2}{l^2} \quad (n = 0, 1, 2, \ldots). \tag{9.93l}$$

Proceeding as in ■ **A** (p. 579) one finally obtains

$$u = \frac{ka^2pt^2}{2l} + \frac{kpx^2}{2l} + a_0 + \frac{a\pi}{l}b_0t + \sum_{n=1}^{\infty}\left(a_n\cos\frac{an\pi t}{l} + \frac{b_n}{n}\sin\frac{an\pi t}{l}\right)\cos\frac{n\pi x}{l}, \tag{9.93m}$$

where $a_n$ and $b_n$ $(n = 0, 1, 2, \ldots)$ are the coefficients of the Fourier series expansion in cosines of the functions $f(x) - \dfrac{kpx^2}{2}$ and $\dfrac{l}{a\pi}\varphi(x)$ in the interval $(0, l)$ (see 7.4.1.1, **1.**, p. 474).

■ **C: Equation of a Vibrating Round Membrane** fixed along the boundary:
The differential equation is linear, partial and it is of hyperbolic type. It has the form in Cartesian and in polar coordinates (see 3.5.3.2,**3.**, p. 211)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \tag{9.94a}$$

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}. \tag{9.94b}$$

The initial and boundary conditions are

$$u|_{t=0} = f(\rho, \varphi), \tag{9.94c}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = F(\rho, \varphi), \tag{9.94d}$$

$$u|_{\rho=R} = 0. \tag{9.94e}$$

The substitution of the product form

$$u = U(\rho)\Phi(\varphi)T(t) \tag{9.94f}$$

with three variables into the differential equation in polar coordinates yields

$$\frac{U''}{U} + \frac{U'}{\rho U} + \frac{\Phi''}{\rho^2\Phi} = \frac{1}{a^2}\frac{T''}{T} = -\lambda^2. \tag{9.94g}$$

Three ordinary differential equations are obtained for the separated variables analogously to examples **A** (p. 579) and **B** (p. 580):

$$T'' + a^2\lambda^2T = 0, \tag{9.94h}$$

$$\frac{\rho^2U'' + \rho U'}{U} + \lambda^2\rho^2 = -\frac{\Phi''}{\Phi} = \nu^2, \tag{9.94i}$$

$$\Phi'' + \nu^2\Phi = 0. \tag{9.94j}$$

From the conditions $\Phi(0) = \Phi(2\pi)$, $\Phi'(0) = \Phi'(2\pi)$ it follows that:

$$\Phi(\varphi) = a_n\cos n\varphi + b_n\sin n\varphi, \quad \nu^2 = n^2 \quad (n = 0, 1, 2, \ldots). \tag{9.94k}$$

$U$ and $\lambda$ will be determined from the equations $[\rho U']' - \frac{n^2}{\rho}U = -\lambda^2 \rho U$ and $U(R) = 0$. Considering the obvious condition of boundedness of $U(\rho)$ at $\rho = 0$ and substituting $\lambda\rho = z$ gives

$$z^2U'' + zU' + (z^2 - n^2)U = 0, \quad \text{i.e.,} \quad U(\rho) = J_n(z) = J_n\left(\mu\frac{\rho}{R}\right), \tag{9.94l}$$

where $J_n$ are the Bessel functions (see 9.1.2.6, **2.**, p. 562) with $\lambda = \frac{\mu}{R}$ and $J_n(\mu) = 0$. The system of functions

$$U_{nk}(\rho) = J_n\left(\mu_{nk}\frac{\rho}{R}\right) \quad (k = 1, 2, \dots) \tag{9.94m}$$

with $\mu_{nk}$ as the $k$-th positive root of the function $J_n(z)$ is a complete system of eigenfunctions of the self-adjoint Sturm-Liouville problem which are orthogonal with the weight function $\rho$.
The solution of the problem can have the form of a double series:

$$U = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\left[(a_{nk}\cos n\varphi + b_{nk}\sin n\varphi)\cos\frac{a\mu_{nk}t}{R} + (c_{nk}\cos n\varphi + d_{nk}\sin n\varphi)\sin\frac{a\mu_{nk}t}{R}\right] J_n\left(\mu_{nk}\frac{\rho}{R}\right). \tag{9.94n}$$

From the initial conditions at $t = 0$ one obtains

$$f(\rho,\varphi) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(a_{nk}\cos n\varphi + b_{nk}\sin n\varphi)J_n\left(\mu_{nk}\frac{\rho}{R}\right), \tag{9.94o}$$

$$F(\rho,\varphi) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{a\mu_{nk}}{R}(c_{nk}\cos n\varphi + d_{nk}\sin n\varphi)J_n\left(\mu_{nk}\frac{\rho}{R}\right), \tag{9.94p}$$

where

$$a_{nk} = \frac{2}{\pi R^2 J_{n-1}^2(\mu_{nk})}\int_0^{2\pi}d\varphi\int_0^R f(\rho,\varphi)\cos n\varphi J_n\left(\mu_{nk}\frac{\rho}{R}\right)\rho\,d\rho, \tag{9.94q}$$

$$b_{nk} = \frac{2}{\pi R^2 J_{n-1}^2(\mu_{nk})}\int_0^{2\pi}d\varphi\int_0^R f(\rho,\varphi)\sin n\varphi J_n\left(\mu_{nk}\frac{\rho}{R}\right)\rho\,d\rho. \tag{9.94r}$$

In the case of $n = 0$, the numerator 2 should be changed to 1. To determine the coefficients $c_{nk}$ and $d_{nk}$ the function $f(\rho,\varphi)$ is replaced by $F(\rho,\varphi)$ in the formulas for $a_{nk}$ and $b_{nk}$ and finely it is multiplied by $\frac{R}{a\mu_{nk}}$.

■ **D: Dirichlet Problem** (see 13.5.1, p. 729) for the rectangle $0 \le x \le a$, $0 \le y \le b$ **(Fig. 9.17)**:

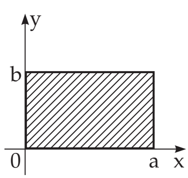

Figure 9.17

Find a function $u(x,y)$ satisfying the elliptic type Laplace differential equation

$$\Delta u = 0 \tag{9.95a}$$

and the boundary conditions

$$u(0,y) = \varphi_1(y), \quad u(a,y) = \varphi_2(y),$$
$$u(x,0) = \psi_1(x), \quad u(x,b) = \psi_2(x). \tag{9.95b}$$

First there is to determine a particular solution for the boundary conditions $\varphi_1(y) = \varphi_2(y) = 0$. Substituting the product form

$$u = X(x)Y(y) \tag{9.95c}$$

into (9.95a) gives the separated differential equations

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \tag{9.95d}$$

with the eigenvalue $\lambda$ analogously to examples **A** (p. 579) through **C** (p. 581). Since $X(0) = X(a) = 0$,

$$X = C \sin \lambda x, \quad \lambda = \frac{n\pi}{a} = \lambda_n \quad (n = 1, 2, \ldots). \tag{9.95e}$$

In the second step the general solution of the differential equation is obtained:

$$Y'' - \frac{n^2\pi^2}{a^2} Y = 0 \quad (9.95\text{f}) \qquad \text{in the form} \quad Y = a_n \sinh \frac{n\pi}{a}(b - y) + b_n \sinh \frac{n\pi}{a} y\,. \tag{9.95g}$$

From these equations one gets a particular solution of (9.95a) satisfying the boundary conditions $u(0, y) = u(a, y) = 0$, which has the form

$$u_n = \left[a_n \sinh \frac{n\pi}{a}(b - y) + b_n \sinh \frac{n\pi}{a} y\right] \sin \frac{n\pi}{a} x\,. \tag{9.95h}$$

In the third step one considers the general solution as a series

$$u = \sum_{n=1}^{\infty} u_n, \tag{9.95i}$$

so from the boundary conditions for $y = 0$ and $y = b$

$$u = \sum_{n=1}^{\infty} \left(a_n \sinh \frac{n\pi}{a}(b - y) + b_n \sinh \frac{n\pi}{a} y\right) \sin \frac{n\pi}{a} x \tag{9.95j}$$

follows with the coefficients

$$a_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a \psi_1(x) \sin \frac{n\pi}{a} x\, dx, \quad b_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a \psi_2(x) \sin \frac{n\pi}{a} x\, dx. \tag{9.95k}$$

The problem with the boundary conditions $\psi_1(x) = \psi_2(x) = 0$ can be solved in a similar manner, and taking the series (9.95j) one gets the general solution of (9.95a) and (9.95b).

■ **E: Heat Conduction Equation** Heat conduction in a homogeneous bar with one end at infinity and the other end kept at a constant temperature is described by the linear second-order partial differential equation of parabolic type

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{9.96a}$$

which satisfies the initial and boundary conditions

$$u|_{t=0} = f(x), \quad u|_{x=0} = 0 \tag{9.96b}$$

in the domain $0 \le x < +\infty$, $t \ge 0$. It is also to be supposed that the temperature tends to zero at infinity. Substituting

$$u = X(x)T(t) \tag{9.96c}$$

into (9.96a) one obtains the ordinary differential equations

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda^2, \tag{9.96d}$$

whose parameter $\lambda$ is introduced analogously to the previous examples **A** (p. 579) through **D** (p. 582). One gets

$$T(t) = C_\lambda e^{-\lambda^2 a^2 t} \tag{9.96e}$$

as a solution for $T(t)$. Using the boundary condition $X(0) = 0$, gives

$$X(x) = C \sin \lambda x \qquad (9.96f) \qquad \text{and so} \quad u_\lambda = C_\lambda e^{-\lambda^2 a^2 t} \sin \lambda x, \qquad (9.96g)$$

where $\lambda$ is an arbitrary real number. The solution can be obtained in the form

$$u(x,t) = \int_0^\infty C(\lambda) e^{-\lambda^2 a^2 t} \sin \lambda x \, d\lambda. \qquad (9.96h)$$

From the initial condition $u|_{t=0} = f(x)$ follows (9.96i) with (9.96j) for the constant (see 7.4.1.1, **1.**, p. 474).

$$f(x) = \int_0^\infty C(\lambda) \sin \lambda x \, d\lambda\,, \qquad (9.96i) \qquad C(\lambda) = \frac{2}{\pi} \int_0^\infty f(s) \sin \lambda s \, ds\,. \qquad (9.96j)$$

Combining (9.96j) and (9.96h) gives

$$u(x,t) = \frac{2}{\pi} \int_0^\infty f(s) \left( \int_0^\infty e^{-\lambda^2 a^2 t} \sin \lambda s \sin \lambda x \, d\lambda \right) ds \qquad (9.96k)$$

or after replacing the product of the two sines with one half of the difference of two cosines ((2.122), p. 83) and using formula (21.27), in **Table 21.8.2**, p. 1100, it follows that

$$u(x,t) = \int_0^\infty f(s) \frac{1}{2a\sqrt{\pi t}} \left[ \exp\left( -\frac{(x-s)^2}{4a^2 t} \right) - \exp\left( -\frac{(x+s)^2}{4a^2 t} \right) \right] ds. \qquad (9.96l)$$

## 2. Riemann Method for Solving Cauchy's Problem for the Hyperbolic Differential Equation

$$\frac{\partial^2 u}{\partial x \partial y} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = F \qquad (9.97a)$$

**1. Riemann Function** is a function $v(x, y; \xi, \eta)$, where $\xi$ and $\eta$ are considered as parameters, satisfying the homogeneous equation

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial (av)}{\partial x} - \frac{\partial (bv)}{\partial y} + cv = 0 \qquad (9.97b)$$

which is the adjoint of (9.97a) and the conditions

$$v(x, \eta; \xi, \eta) = \exp\left( \int_\xi^x b(s, \eta) \, ds \right), \quad v(\xi, y; \xi, \eta) = \exp\left( \int_\eta^y a(\xi, s) \, ds \right). \qquad (9.97c)$$

In general, linear second-order differential equations and their adjoint differential equations have the form

$$\sum_{i,k} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i b_i \frac{\partial u}{\partial x_i} + cu = f \quad (9.97d) \quad \text{and} \quad \sum_{i,k} \frac{\partial^2 (a_{ik} v)}{\partial x_i \partial x_k} - \sum_i \frac{\partial (b_i v)}{\partial x_i} + cv = 0. \quad (9.97e)$$

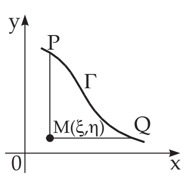

Figure 9.18

**2. Riemann Formula** is the integral formula which is used to determine function $u(\xi, \eta)$ satisfying the given differential equation (9.97a) and taking the previously given values along the previously given curve $\Gamma$ **(Fig. 9.18)** together with its derivative in the direction of the curve normal (see 3.6.1.2, **2.**, p. 244):

$$u(\xi, \eta) = \frac{1}{2}(uv)_P + \frac{1}{2}(uv)_Q - \int_{\overset{\frown}{QP}} \left[ buv + \frac{1}{2} \left( v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) \right] dx$$

$$-\left[auv + \frac{1}{2}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right)\right] dy + \iint\limits_{PMQ} Fv\,dx\,dy. \tag{9.97f}$$

The smooth curve $\Gamma$ **(Fig. 9.18)** must not have tangents parallel to the coordinate axes, i.e., the curve must not be tangent to the characteristics. The line integral in this formula can be calculated, since the values of both partial derivatives can be determined from the function values and from its derivatives in a non-tangential direction along the curve arc.

In the Cauchy problem, the values of the partial derivatives of the unknown function, e.g., $\frac{\partial u}{\partial y}$ are often given instead of the normal derivative along the curve. Then another form of the Riemann formula is used:

$$u(\xi,\eta) = (uv)_P - \int\limits_{\widehat{QP}} \left(buv - u\frac{\partial v}{\partial x}\right) dx - \left(auv + v\frac{\partial u}{\partial y}\right) dy + \iint\limits_{PMQ} Fv\,dx\,dy. \tag{9.97g}$$

■ **Telegraph Equation** (Telegrapher's Equation) is a linear second-order partial differential equation of hyperbolic type

$$a\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial u}{\partial t} + cu = \frac{\partial^2 u}{\partial x^2} \tag{9.98a}$$

where $a > 0$, $b$, and $c$ are constants. The equation describes the current flow in wires. It is a generalization of the differential equation of a vibrating string.
Replacing the unknown function $u(x,t)$ by $u = z\exp(-(b/a)t)$, so (9.98a) is reduced to the form

$$\frac{\partial^2 z}{\partial t^2} = m^2\frac{\partial^2 z}{\partial x^2} + n^2 z \quad \left(m^2 = \frac{1}{a},\ n^2 = \frac{b^2 - ac}{a^2}\right). \tag{9.98b}$$

Replacing the independent variables by

$$\xi = \frac{n}{m}(mt + x), \quad \eta = \frac{n}{m}(mt - x) \tag{9.98c}$$

finally one gets the canonical form

$$\frac{\partial^2 z}{\partial\xi\partial\eta} - \frac{z}{4} = 0 \tag{9.98d}$$

of a hyperbolic type linear partial differential equation (see 9.2.2.1, **1.**, p. 577). The Riemann function $v(\xi,\eta;\xi_0,\eta_0)$ should satisfy this equation with unit value at $\xi = \xi_0$ and $\eta = \eta_0$. Choosing the form

$$w = (\xi - \xi_0)(\eta - \eta_0) \tag{9.98e}$$

for $w$ in $v = f(w)$, then $f(w)$ is a solution of the differential equation

$$w\frac{d^2 f}{dw^2} + \frac{df}{dw} - \frac{1}{4}f = 0 \tag{9.98f}$$

with initial condition $f(0) = 1$. The substitution $w = \alpha^2$ reduces this differential equation to Bessel's differential equation of order zero (see 9.1.2.6, **2.**, p. 562)

$$\frac{d^2 f}{d\alpha^2} + \frac{1}{\alpha}\frac{df}{d\alpha} - f = 0, \tag{9.98g}$$

hence the solution is

$$v = I_0\left[\sqrt{(\xi - \xi_0)(\eta - \eta_0)}\right]. \tag{9.98h}$$

A solution of the original differential equation (9.98a) satisfying the boundary conditions

$$z\Big|_{t=0} = f(x), \quad \frac{\partial z}{\partial t}\Big|_{t=0} = g(x) \tag{9.98i}$$

can be obtained substituting the found value of $v$ into the Riemann formula and then returning to the original variables:

$$z(x,t) = \frac{1}{2}[f(x-mt) + f(x+mt)]$$
$$+\frac{1}{2}\int\limits_{x-mt}^{x+mt}\left[g(s)\frac{I_0\left(\frac{n}{m}\sqrt{m^2t^2-(s-x)^2}\right)}{m} - f(s)\frac{ntI_1\left(\frac{n}{m}\sqrt{m^2t^2-(s-x)^2}\right)}{\sqrt{m^2t^2-(s-x)^2}}\right]ds. \tag{9.98j}$$

## 3. Green's Method of Solving the Boundary Value Problem for Elliptic Differential Equations with Two Independent Variables

This method is very similar to the Riemann method of solving the Cauchy problem for hyperbolic differential equations.
If one wants to find a function $u(x,y)$ satisfying the elliptic type of linear second-order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\,u = f \tag{9.99a}$$

in a given domain and taking the prescribed values on its boundary, first the *Green function* $G(x,y,\xi,\eta)$ has to be determined for this domain, where $\xi$ and $\eta$ are regarded as parameters. The Green function must satisfy the following conditions:

**1.** The function $G(x,y;\xi,\eta)$ satisfies the homogeneous adjoint differential equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} - \frac{\partial(a\,G)}{\partial x} - \frac{\partial(b\,G)}{\partial y} + c\,G = 0 \tag{9.99b}$$

everywhere in the given domain except at the point $x=\xi$, $y=\eta$.

**2.** The function $G(x,y;\xi,\eta)$ has the form

$$U\ln\frac{1}{r} + V \tag{9.99c}$$ with $$r = \sqrt{(x-\xi)^2+(y-\eta)^2}\,, \tag{9.99d}$$

where $U$ has unit value at the point $x=\xi$, $y=\eta$ and $U$ and $V$ are continuous functions in the entire domain together with their second derivatives.

**3.** The function $G(x,y;\xi,\eta)$ is equal to zero on the boundary of the given domain.
The second step is to give the solution of the boundary value problem with the Green function by the formula

$$u(\xi,\eta) = \frac{1}{2\pi}\int\limits_S u(x,y)\frac{\partial}{\partial n}G(x,y;\xi,\eta)\,ds - \frac{1}{2\pi}\iint\limits_D f(x,y)G(x,y;\xi,\eta)\,dx\,dy, \tag{9.99e}$$

where $D$ is the considered domain, $S$ is its boundary on which the function is assumed to be known and $\frac{\partial}{\partial n}$ denotes the normal derivative directed toward the interior of $D$.
Condition **3** depends on the formulation of the problem. For instance, if instead of the function values the values of the derivative of the unknown function are given in the direction normal to the boundary of the domain, then in **3** the condition

$$\frac{\partial G}{\partial n} - (a\cos\alpha + b\cos\beta)G = 0 \tag{9.99f}$$

holds on the boundary. $\alpha$ and $\beta$ denote here the angles between the interior normal to the boundary of the domain and the coordinate axes. In this case, the solution is given by the formula

$$u(\xi,\eta) = -\frac{1}{2\pi}\int\limits_S \frac{\partial u}{\partial n} G\, ds - \frac{1}{2\pi}\iint\limits_D fG\, dx\, dy. \tag{9.99g}$$

## 4. Green's Method for the Solution of Boundary Value Problems with Three Independent Variables

The solution of the differential equation

$$\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial z} + e\,u = f \tag{9.100a}$$

should take the given values on the boundary of the considered domain. As the first step, one constructs again the Green function, but now it depends on three parameters $\xi$, $\eta$, and $\zeta$. The adjoint differential equation satisfied by the Green function has the form

$$\Delta G - \frac{\partial(a\,G)}{\partial x} - \frac{\partial(b\,G)}{\partial y} - \frac{\partial(c\,G)}{\partial z} + e\,G = 0. \tag{9.100b}$$

As in condition **2**, the function $G(x,y,z;\xi,\eta,\zeta)$ has the form

$$U\frac{1}{r} + V \tag{9.100c}$$

$$\text{with} \quad r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}. \tag{9.100d}$$

The solution of the problem is:

$$u(\xi,\eta,\zeta) = \frac{1}{4\pi}\iint\limits_S u\frac{\partial G}{\partial n}\, ds - \frac{1}{4\pi}\iiint\limits_D fG\, dx\, dy\, dz. \tag{9.100e}$$

Both methods, Riemann's and Green's, have the common idea first to determine a special solution of the differential equation, which can then be used to obtain a solution with arbitrary boundary conditions. An essential difference between the Riemann and the Green function is that the first one depends only on the form of the left-hand side of the differential equation, while the second one depends also on the considered domain. Finding the Green function is, in practice, an extremely difficult problem, even if it is known to exist; therefore, Green's method is used mostly in theoretical research.

■ **A:** Construction of the Green function for the Dirichlet problem of the Laplace differential equation (see 13.5.1, p. 729)

$$\Delta u = 0 \tag{9.101a}$$

for the case, when the considered domain is a circle **(Fig. 9.19)**. The Green function is

$$G(x,y;\xi,\eta) = \ln\frac{1}{r} + \ln\frac{r_1\rho}{R}, \tag{9.101b}$$

where $r = \overline{MP}$, $\rho = \overline{OM}$, $r_1 = \overline{M_1P}$ and $R$ is the radius of the considered circle **(Fig. 9.19)**. The points $M$ and $M_1$ are symmetric with respect to the circle, i.e., both points are on the same ray starting from the center and

$$\overline{OM}\cdot\overline{OM_1} = R^2. \tag{9.101c}$$

The formula (9.99e) for a solution of Dirichlet's problem, after substituting the normal derivative of the Green function and after certain calculations, yields the so-called Poisson integral

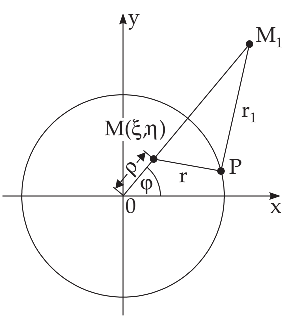


Figure 9.19

$$u(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\psi-\varphi)}u(\varphi)\, d\varphi. \tag{9.101d}$$

The notation is the same as above. The known values of $u$ are given on the boundary of the circle by $u(\varphi)$. For the coordinates of the point $M(\xi, \eta)$ follows $\xi = \rho\cos\psi$, $\eta = \rho\sin\psi$.

■ **B:** Construction of the Green function for the Dirichlet problem of the Laplace differential equation (see 13.5.1, p. 729)

$$\Delta u = 0, \tag{9.102a}$$

for the case when the considered domain is a sphere with radius $R$. The Green function now has the form

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{r} - \frac{R}{r_1 \rho}, \tag{9.102b}$$

with $\rho = \sqrt{\xi^2 + \eta^2 + \zeta^2}$ as the distance of the point $(\xi, \eta, \zeta)$ from the center, $r$ as the distance between the points $(x, y, z)$ and $(\xi, \eta, \zeta)$, and $r_1$ as the distance of the point $(x, y, z)$ from the symmetric point of $(\xi, \eta, \zeta)$ according to (9.101c), i.e., from the point $\left(\frac{R\xi}{\rho}, \frac{R\eta}{\rho}, \frac{R\zeta}{\rho}\right)$. In this case, the Poisson integral has the form (with the same notation as in ■ **A** (p. 587))

$$u(\xi, \eta, \zeta) = \frac{1}{4\pi} \iint_S \frac{R^2 - \rho^2}{Rr^3} u \, ds. \tag{9.102c}$$

## 5. Operational Method

Operational methods can be used not only to solve ordinary differential equations but also for partial differential equations (see 15.1.6, p. 769). They are based on transition from the unknown function to its transform (see 15.1, p. 767). In this process the unknown function is regarded as a function of only one variable and the transformation is performed with respect to this variable. The remaining variables are considered as parameters. The differential equation to determine the transform of the unknown function contains one less independent variable than the original equation. In particular, if the original equation is a partial differential equation of two independent variables, then one obtains an ordinary differential equation for the transform. If the transform of the unknown function can be found from the obtained equation, then the original function is obtained either from the formula for the inverse function or from the table of transforms.

## 6. Approximation Methods

In order to solve practical problems with partial differential equations, different approximation methods are used. They can be divided into analytical and numerical methods.

**1. Analytical Methods** make possible the determination of approximate analytical expressions for the unknown function.

**2. Numerical Methods** result in approximate values of the unknown function for certain values of the independent variables. Here the following methods (see 19.5, p. 976) are used:

**a) Finite Difference Method**, or *Lattice-Point Method*: The derivatives are replaced by divided differences, so the differential equation including the initial and boundary conditions becomes an algebraic equation system. A linear differential equation with linear initial and boundary conditions becomes a linear equation system.

**b) Finite Element Method**, or briefly **FEM** (see 19.5.3, p. 978), for boundary value problems: Here a variational problem is assigned to the boundary value problem. The approximation of the unknown function is performed by a spline approach, whose coefficients should be chosen to get the best possible solution. The domain of the boundary value problem is decomposed into regular sub-domains. The coefficients are determined by solving an extreme value problem.

**c) Integral Equation Method (along a Closed Curve)** for special boundary problems: The boundary value problem is formulated as an equivalent integral equation problem along the boundary of the domain of the boundary value problem. To do this, one applies the theorems of vector analysis (see 13.3.3, p. 724, and followings), e.g., Green formulas. The remaining integrals along the closed curve are to be determined numerically by a suitable quadrature formula.

**3. Physical Solutions** of differential equations can be given by experimental methods. This is based on the fact that various physical phenomena can be described by the same differential equation. To solve a given equation, first a model is constructed by which the given problem is simulated and the values of the unknown function are obtained directly from this model. Since such models are often known and can be constructed by varying the parameters in a wide range, the differential equation can also be applied in a wide domain of the variables.

## 9.2.3 Some further Partial Differential Equations from Natural Sciences and Engineering

### 9.2.3.1 Formulation of the Problem and the Boundary Conditions

#### 1. Problem Formulation

The modeling and the mathematical treatment of different physical phenomena in classical theoretical physics, especially in modeling media considered structureless or continuously changing, such as gases, fluids, solids, the fields of classical physics, leads to the introduction of partial differential equations. Examples are the wave (see 9.2.3.2, p. 590) and the heat equations (see 9.2.3.3, p. 591). Many problems in non-classical theoretical physics are also governed by partial differential equations. An important area is quantum mechanics, which is based on the recognition that media and fields are discontinuous. The most famous relation is the Schroedinger equation. Linear second-order partial differential equations occur most frequently and they have special importance in today's natural sciences.

#### 2. Initial and Boundary Conditions

The solution of the problems of physics, engineering, and the natural sciences must usually fulfill two basic requirements:

**1.** The solution must satisfy not only the differential equation, but also certain initial and/or boundary conditions. There are problems with only initial condition or only with boundary conditions or with both. All the conditions together must determine the unique solution of the differential equation.

**2.** The solution must be stable with respect to small changes in the initial and boundary conditions, i.e., its change should be arbitrarily small if the *perturbations* of these conditions are small enough. Then a *correct problem formulation* is given.
One can assume that the mathematical model of the given problem to describe the real situation is adequate only in cases when these conditions are fulfilled.
For instance, the Cauchy problem (see 9.2.1.1, **5.**, p. 572) is correctly defined with a differential equation of hyperbolic type for investigating vibration processes in continuous media. This means that the values of the required function, and the values of its derivatives in a non-tangential (mostly in a normal) direction are given on an initial manifold, i.e., on a curve or on a surface.

In the case of differential equations of elliptic type, which occur in investigations of steady state and equilibrium problems in continuous media, the formulation of the boundary value problem is correct. If the considered domain is unbounded, then the unknown function must satisfy certain given properties with unlimited increase of the independent variables.

#### 3. Inhomogeneous Conditions and Inhomogeneous Differential Equations

The solution of homogeneous or inhomogeneous linear partial differential equations with inhomogeneous initial or boundary conditions can be reduced to the solution of an equation which differs from the original one only by a free term not containing the unknown function, and which has homogeneous conditions. It is sufficient to replace the original function by its difference from an arbitrary twice differentiable function satisfying the given inhomogeneous conditions.

In general, one uses the fact that the solution of a linear inhomogeneous partial differential equation with given inhomogeneous initial or boundary conditions is the sum of the solutions of the same differential equation with zero conditions and the solution of the corresponding homogeneous differential equation with the given conditions.

To reduce the solution of the linear inhomogeneous partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - L[u] = g(x,t) \tag{9.103a}$$

with homogeneous initial conditions

$$u\Big|_{t=0} = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \tag{9.103b}$$

to the solution of the Cauchy problem for the corresponding homogeneous differential equation, one substitutes

$$u = \int_0^t \varphi(x,t;\tau)\, d\tau. \tag{9.103c}$$

Here $\varphi(x,t;\tau)$ is the solution of the differential equation

$$\frac{\partial^2 u}{\partial t^2} - L[u] = 0, \tag{9.103d}$$

which satisfies the boundary conditions

$$u\Big|_{t=\tau} = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=\tau} = g(x,\tau). \tag{9.103e}$$

In this equation, $x$ represents symbolically all the $n$ variables $x_1, x_2, \ldots, x_n$ of the $n$-dimensional problem. $L[u]$ denotes a linear differential expression, which may contain the derivative $\frac{\partial u}{\partial t}$, but not higher-order derivatives with respect to $t$.

### 9.2.3.2 Wave Equation

The extension of oscillations in a homogeneous media is described by the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} - a^2 \Delta u = Q(x,t), \tag{9.104a}$$

whose right-hand side $Q(x,t)$ vanishes when there is no perturbation. The symbol $x$ represents the $n$ variables $x_1, \ldots, x_n$ of the $n$-dimensional problem. The Laplace operator $\Delta$ (see also 13.2.6.5, 716,) is defined in the following way:

$$\Delta u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2}. \tag{9.104b}$$

The solution of the wave equation is the *wave function* $u$. The differential equation (9.104a) is of hyperbolic type.

#### 1. Homogeneous Problem

The solution of the homogeneous problem with $Q(x,t) = 0$ and with the initial conditions

$$u\Big|_{t=0} = \varphi(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \tag{9.105}$$

is given for the cases $n = 1, 2, 3$ by the following integrals.

**Case $n = 3$ (Kirchhoff Formula):**

$$u(x_1,x_2,x_3,t) = \frac{1}{4\pi a^2}\left[\iint\limits_{(S_{\rm at})} \frac{\psi(\alpha_1,\alpha_2,\alpha_3)}{t}\, d\sigma + \frac{\partial}{\partial t}\iint\limits_{(S_{\rm at})} \frac{\varphi(\alpha_1,\alpha_2,\alpha_3)}{t}\, d\sigma\right], \tag{9.106a}$$

where the integration is performed over the spherical surface $S_{at}$ given by the equation $(\alpha_1 - x_1)^2 + (\alpha_2 - x_2)^2 + (\alpha_3 - x_3)^2 = a^2 t^2$.

**Case $n = 2$ (Poisson Formula):**

$$u(x_1, x_2, t) = \frac{1}{2\pi a}\Big[\iint_{(C_{\mathrm{at}})} \frac{\psi(\alpha_1, \alpha_2)\, d\alpha_1 d\alpha_2}{\sqrt{a^2t^2 - (\alpha_1 - x_1)^2 - (\alpha_2 - x_2)^2}}$$
$$+\frac{\partial}{\partial t}\iint_{(C_{\mathrm{at}})} \frac{\varphi(\alpha_1, \alpha_2)\, d\alpha_1 d\alpha_2}{\sqrt{a^2t^2 - (\alpha_1 - x_1)^2 - (\alpha_2 - x_2)^2}}\Big], \tag{9.106b}$$

where the integration is performed along the circle $C_{\mathrm{at}}$ given by the equation $(\alpha_1 - x_1)^2 + (\alpha_2 - x_2)^2 \le a^2t^2$.

**Case $n = 1$ (d'Alembert formula):**

$$u(x_1, t) = \frac{\varphi(x_1 + at) + \varphi(x_1 - at)}{2} + \frac{1}{2a}\int_{x_1-at}^{x_1+at} \psi(\alpha)\, d\alpha. \tag{9.106c}$$

## 2. Inhomogeneous Problem

In the case, when $Q(x, t) \ne 0$, one has to add to the right-hand sides of (9.106a,b,c) the correcting terms:

**Case $n = 3$ (Retarded Potential):** For a domain $K$ given by $r \le at$ with $r = \sqrt{(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2}$, the correction term is

$$\frac{1}{4\pi a^2}\iiint_{(K)} \frac{Q\left(\xi_1, \xi_2, \xi_3, t - \dfrac{r}{a}\right)}{r}\, d\xi_1 d\xi_2 d\xi_3\,. \tag{9.107a}$$

**Case $n = 2$:** $$\frac{1}{2\pi a}\iiint_{(K)} \frac{Q(\xi_1, \xi_2, \tau)\, d\xi_1 d\xi_2 d\tau}{\sqrt{a^2(t-\tau)^2 - (\xi_1 - x_1)^2 - (\xi_2 - x_2)^2}}\,, \tag{9.107b}$$

where $K$ is a domain of $\xi_1, \xi_2, \tau$ space defined by the inequalities $0 \le \tau \le t$, $(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 \le a^2(t - \tau)^2$.

**Case $n = 1$:** $$\frac{1}{2a}\iint_{(T)} Q(\xi, \tau)\, d\xi d\tau, \tag{9.107c}$$

where $T$ is the triangle $0 \le \tau \le t$, $|\xi - x_1| \le a|t - \tau|$. $a$ denotes the wave velocity of the perturbation.

### 9.2.3.3 Heat Conduction and Diffusion Equation for Homogeneous Media

## 1. Three-Dimensional Heat Conduction Equation

The propagation of heat in a homogeneous medium is described by a linear second-order partial differential equation of parabolic type

$$\frac{\partial u}{\partial t} - a^2 \Delta u = Q(x, t), \tag{9.108a}$$

where $\Delta$ is the three-dimensional Laplace operator defined in three directions of propagation $x_1$, $x_2$, $x_3$, determined by the position vector $\vec{\mathbf{r}}$. If the heat flow has neither source nor sink, the right-hand side vanishes since $Q(x, t) = 0$.

The Cauchy problem can be posed in the following way: It is to determine a bounded solution $u(x, t)$ for

$t > 0$, where $u|_{t=0} = f(x)$. The requirement of boundedness guarantees the uniqueness of the solution. For the homogeneous differential equation with $Q(x,t) = 0$, one gets the *wave function*

$$u(x_1, x_2, x_3, t) = \frac{1}{(2a\sqrt{\pi t})^n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha_1, \alpha_2, \alpha_3) \cdot \exp\left(-\frac{(x_1-\alpha_1)^2 + (x_2-\alpha_2)^2 + (x_3-\alpha_3)^2}{4a^2 t}\right) d\alpha_1 d\alpha_2 d\alpha_3. \tag{9.108b}$$

In the case of an inhomogeneous differential equation with $Q(x,t) \neq 0$, one has to add to the right-hand side of (9.108b) the following expression:

$$\int_0^t \left[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{Q(\alpha_1, \alpha_2, \alpha_3)}{[2a\sqrt{\pi(t-\tau)}]^n} \cdot \exp\left(-\frac{(x_1-\alpha_1)^2 + (x_2-\alpha_2)^2 + (x_3-\alpha_3)^2}{4a^2(t-\tau)}\right) d\alpha_1 d\alpha_2 d\alpha_3 \right] d\tau. \tag{9.108c}$$

The problem of determining $u(x,t)$ for $t < 0$, if the values $u(x,0)$ are given, cannot be solved in this way, since the Cauchy problem is not correctly formulated in this case.
Since the temperature difference is proportional to the heat, one often introduces $u = T(\vec{\mathbf{r}}, t)$ (temperature field) and $a^2 = D_W$ (heat diffusion constant or thermal conductivity) to get

$$\frac{\partial T}{\partial t} - D_W \Delta T = Q_W(\vec{\mathbf{r}}, t). \tag{9.108d}$$

**2. Three-Dimensional Diffusion Equation**

In analogy to the heat equation, the propagation of a concentration $C$ in a homogeneous medium is described by the same linear partial differential equation (9.108a) and (9.108d), where $D_W$ is replaced by the three-dimensional *diffusion coefficient* $D_C$. The *diffusion equation* is:

$$\frac{\partial C}{\partial t} - D_C \Delta C = Q_C(\vec{\mathbf{r}}, t). \tag{9.109}$$

One gets the solutions by changing the symbols in the wave equations (9.108b) and (9.108c).

### 9.2.3.4 Potential Equation

The linear second-order partial differential equation

$$\Delta u = -4\pi\varrho \tag{9.110a}$$

is called the *potential equation* or *Poisson differential equation* (see 13.5.2, p. 729), which makes the determination of the potential $u(x)$ of a scalar field determined by a scalar point function $\varrho(x)$ possible, where $x$ has the coordinates $x_1$, $x_2$, $x_3$ and $\Delta$ is the Laplace operator. The solution, the potential $u_M(x_1, x_2, x_3)$ at the point $M$, is discussed in 13.5.2, p. 729.
One gets the *Laplace differential equation* (see 13.5.1, p. 729) for the homogeneous differential equation with $\varrho \equiv 0$:

$$\Delta u = 0. \tag{9.110b}$$

The differential equations (9.110a) and (9.110b) are of elliptic type.

## 9.2.4 Schroedinger's Equation

### 9.2.4.1 Notion of the Schroedinger Equation

**1. Determination and Dependencies**

The solutions of the Schroedinger equation, the *wave functions* $\psi$, describe the properties of a quantum mechanical system, i.e., the properties of the states of a particle. The Schroedinger equation is a second-order partial differential equation with the second-order derivatives of the wave function with respect to the space coordinates and first-order with respect to the time coordinate:

$$\mathrm{i}\,\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + U(x_1, x_2, x_3, t)\,\psi = \hat{H}\,\psi \tag{9.111a}$$

$$\hat{H} \equiv \frac{\hat{p}^2}{2m} + U(\vec{\mathbf{r}}, t)\,, \quad \hat{p} \equiv \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\vec{\mathbf{r}}} \equiv \frac{\hbar}{\mathrm{i}}\nabla. \tag{9.111b}$$

Here, $\Delta$ is the Laplace operator, $\hbar = \dfrac{h}{2\pi}$ is the reduced Planck's constant, i is the imaginary unit and $\nabla$ is the nabla operator. The relation between the impulse $p$ of a free particle with mass $m$ and wave length $\lambda$ is $\lambda = h/p$.

**2. Remarks:**

**a)** In quantum mechanics, an operator is assigned to every measurable quantity. The operator occurring in (9.111a) and (9.111b) is called the *Hamilton operator* $\hat{H}$ ("*Hamiltonian*") . It has the same role as the Hamilton function of classical mechanical systems (see, e.g., the example on Two-Body Problem on p. 574). It represents the total energy of the system which is divided into kinetic and potential energy. The first term in $\hat{H}$ is the operator for the kinetic energy, the second one for the potential energy.

**b)** The imaginary unit appears explicitly in the Schroedinger equation. Consequently, the wave functions are complex functions. Both real functions occurring in $\psi^{(1)} + \mathrm{i}\psi^{(2)}$ are needed to calculate the observable quantities. The square $|\Psi|^2$ of the wave function, describing the probability $dw$ of the particle being in an arbitrary volume element $dV$ of the observed domain, must satisfy special further conditions.

**c)** Besides the potential of the *interaction*, every special solution depends also on the initial and boundary conditions of the given problem. In general, there is a linear second-order boundary value problem, whose solutions have physical meaning only for the eigenvalues. The squares of the absolute value of meaningful solutions are everywhere unique and regular, and tend to zero at infinity.

**d)** The micro-particles also have wave and particle properties based on the *wave–particle duality*, so the Schroedinger equation is a wave equation (see 9.2.3.2, p. 590) for the De Broglie matter waves.

**e)** The restriction to the non-relativistic case means that the velocity $v$ of the particle is very small with respect to the velocity of light $c$ ($v \ll c$).

The application of the Schroedinger equations is discussed in detail in the literature of theoretical physics (see, e.g., [9.15], [9.7],[9.10], [22.15]). In this chapter only some most important examples are demonstrated.

### 9.2.4.2 Time-Dependent Schroedinger Equation

The time-dependent Schroedinger equation (9.111a) describes the general non-relativistic case of a spin-less particle with mass $m$ in a position-dependent and time-dependent potential field $U(x_1, x_2, x_3, t)$. The special conditions, which must be satisfied by the wave function, are:

**a)** The function $\psi$ must be bounded and continuous.
**b)** The partial derivatives $\partial\psi/\partial x_1$, $\partial\psi/\partial x_2$, and $\partial\psi/\partial x_3$ must be continuous.
**c)** The function $|\psi|^2$ must be integrable, i.e.,

$$\iiint\limits_V |\psi(x_1, x_2, x_3, t)|^2\,dV < \infty. \tag{9.112a}$$

According to the normalization condition, the probability that the particle is in the considered domain must be equal to one. (9.112a) is sufficient to guarantee the condition, since multiplying $\psi$ by an appropriate constant the value of the integral becomes one.
A solution of the time-dependent Schroedinger equation has the form

$$\psi(x_1, x_2, x_3, t) = \Psi(x_1, x_2, x_3)\exp\left(-\mathrm{i}\tfrac{E}{\hbar}t\right)\,. \tag{9.112b}$$

The state of the particle is described by a periodic function of time with angular frequency $\omega = E/\hbar$. If the energy of the particle has the fixed value $E = \text{const}$, then the probability $dw$ of finding the particle in a space element $dV$ is independent of time:

$$d\omega = |\psi|^2\, dV = \psi\psi^*\, dV. \tag{9.112c}$$

Then one speaks about a *stationary state* of the particle.

### 9.2.4.3 Time-Independent Schroedinger Equation

If the potential $U$ does not depend on time, i.e., $U = U(x_1, x_2, x_3)$, then it is the time-independent Schroedinger equation and the wave function $\Psi(x_1, x_2, x_3)$ is sufficient to describe the state. Reducing it from the time-dependent Schroedinger equation (9.111a) with the solution (9.112b) gives

$$\Delta\Psi + \frac{2m}{\hbar^2}(E - U)\Psi = 0. \tag{9.113a}$$

In this non-relativistic case, the energy of the particle is

$$E = \frac{p^2}{2m} \quad \left(p = \frac{h}{\lambda}, \quad h = 2\pi\hbar\right). \tag{9.113b}$$

The wave functions $\Psi$ satisfying this differential equation are the *eigenfunctions*; they exist only for certain *energy values* $E$, which are given for the considered problem of the special boundary conditions. The union of the eigenvalues forms the *energy spectrum* of the particle. If $U$ is a potential of finite depth and it tends to zero at infinity, then the negative eigenvalues form a *discrete spectrum*.

If the considered domain is the entire space, then it can be required as a boundary condition that $\Psi$ is quadratically integrable in the entire space in the Lebesgue sense (see 12.9.3.2, p. 696 and [8.5]). If the domain is finite, e.g., a sphere or a cylinder, then one can require, e.g., $\Psi = 0$ for the boundary as the first boundary condition problem.
This gives the *Helmholtz differential equation* in the special case of $U(x) = 0$:

$$\Delta\Psi + \lambda\Psi = 0 \tag{9.114a}$$

$$\text{with the eigenvalue} \quad \lambda = \frac{2mE}{\hbar^2}. \tag{9.114b}$$

$\Psi = 0$ is often required here as a boundary condition. (9.114a) represents the initial mathematical equation for acoustic oscillation in a finite domain.

### 9.2.4.4 Statistical Interpretation of the Wave Function

The quantum mechanics postulates that the complete description of a regarded single-particle system in the time $t$ is to be performed by the complex *wave function* $\psi(\vec{\mathbf{r}}, t)$ as a *state function* and normalized solution of the Schroedinger equation. So, the wave function contains all possible experimental information, which can be got by measurements on this system. There exist no hidden sub-structures of the theory and no hidden parameters which could eliminate the *principal statistical character* of quantum mechanics, as it contains the connection of state function and $\psi$ and measurement results.

#### 1. Observable and Probability Amplitude

A physical expression (position, momentum, angular momentum, energy), which can be determined by a suitable measuring instrument, is called an *observable*. In quantum mechanics every observable $A$ is represented by a linear, hermitian operator $\hat{A}$ with $\hat{A}^+ = \hat{A}$, which interacts on the wave function. At the same time the operator of the quantum mechanics takes over the structure of the classical expression.

■ For the operator $\hat{\vec{\mathbf{l}}}$ of the angular momentum, where $\hat{\vec{\mathbf{r}}}$ is the position operator $\hat{\vec{\mathbf{p}}}$ the momentum operator:

$$\hat{\vec{\mathbf{l}}} = (\hat{l}_x, \hat{l}_y, \hat{l}_z) = \hat{\vec{\mathbf{r}}} \times \hat{\vec{\mathbf{p}}} \quad \text{i.e.,} \qquad \hat{l}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \tag{9.115a}$$

$$\hat{l}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad \hat{l}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{9.115b}$$

In general, it is not possible to assign a certain numerical value to an observable by determining the wave function, but first only as the result of a measurement. The only possible measurement values $A$ are the real eigenvalues $a_i$ of $\hat{A}$; the associated eigenfunctions $\varphi_i$ form a complete orthogonal system:

$$\hat{A}\varphi_i = a_i\,\varphi_i \quad (i,k = 1,2,\dots), \qquad \iiint\limits_V \varphi_i^*\varphi_k\,dV = \delta_{i,k}\,. \tag{9.116}$$

If the system is in an arbitrarily general state $\psi$, the result of a single experiment, i.e. the occurrence of a certain measure value $a_i$ in a single measurement can not be predicted. If imaging to perform the measurement on $N \to \infty$ identical systems being in the same state $\psi$, then among the measurement results every possible result $a_i$ can be found with a frequency $N_i$. The probability $W_i$ to find the value $a_i$ in a single measurement can be determined:

$$W_i = \lim_{N\to\infty}\frac{N_i}{N}, \qquad \sum_i N_i = N\,. \tag{9.117}$$

To determine this probability from the wave function $\psi$, one performs an expansion of $\psi$ as a series of eigenfunctions $\varphi_i$:

$$\psi = \sum_i c_i\,\varphi_i\,, \qquad c_i = \iiint\limits_V \varphi_i^*\psi\,dV\,. \tag{9.118}$$

The coefficient of expansion $c_i$ is the probability to find the system $\psi$ in its characteristic state $\varphi_i$, i.e., to obtain the measuring value $a_i$. From the absolute square of $c_i$ one gets the probability $W_i$ for the measuring value $a_i$:

$$W_i = |c_i|^2\,, \quad \sum_i W_i = \iiint\limits_V \psi^*\psi\,dV = 1\,. \tag{9.119}$$

Because in every measurement it is sure to find one of the possible measuring values $a_i$, the sum of the probabilities $W_i$ fulfills the condition of normalization for the wave function $\psi$.

If two states $\psi_1, \psi_2$ of a physical system are known, then from the linearity of the Schroedinger equation follows, that the superposition

$$\psi = \psi_1 + \psi_2 \tag{9.120}$$

also represents a possible physical state. This fundamental *superposition principle* of quantum mechanics is the reason why at the determination of probabilities with the state function $\psi$, e.g.,

$$|\psi|^2 = |\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\mathrm{Re}(\psi_1\,\psi_2^*) \tag{9.121}$$

besides the single probabilities $|\psi_1|^2\,, |\psi_2|^2$ occurs an additional term with sign. This explains the surprising interference effects of quantum mechanics, e.g. (*wave-particle duality*).

## 2. Expectation Value and Uncertainty

The *quantum mechanical expectation value* $\overline{A}$ is defined as the mean value of the measurement results obtained from measurements with $N \to \infty$ identical systems:

$$\overline{A} = \lim_{N\to\infty}\frac{1}{N}\sum_i a_i\,N_i = \sum_i a_i\,W_i = \iiint\limits_V \psi^*\,\hat{A}\,\psi\,dV\,. \tag{9.122}$$

The expectation value usually is not identical to a possible measurement result.

■ The calculation of the expectation value $\overline{\vec{\mathbf{r}}} = (\overline{x}, \overline{y}, \overline{z})$ of a position measurement for a particle in the state $\psi(\vec{\mathbf{r}}, t)$, e.g.

$$\overline{x} = \iiint\limits_V x|\psi(\vec{\mathbf{r}}, t)|^2\,dV$$

shows, that the wave function $\psi(\vec{\mathbf{r}}, t)$ is to be interpreted as a probability amplitude. The absolute square $|\psi|^2$ then is a probability density. The expression

$$dW = |\psi(\vec{\mathbf{r}}, t)|^2 \, dV\,, \qquad \int dW = \iiint\limits_V |\psi(\vec{\mathbf{r}}, t)|^2 \, dV = 1$$

is to be understood as a probability, to find the particle at the time $t$ in the volume element $dV$ in the position $\vec{\mathbf{r}}$ (*probability of the position*).

As a measure of the distribution of the measured results for an observable $A$, given for a general state in some measurements, can be defined by the help of the so-called *uncertainty* $\Delta A$ near the expectation value $\overline{A}$, which is to be introduced via the standard error:

$$(\Delta A)^2 = \lim_{N\to\infty} \frac{1}{N} \sum_i N_i \, (a_i - \overline{A})^2 = \sum_i W_i \, (a_i - \overline{A})^2 \,. \tag{9.123}$$

By the help of the wave function $\psi$ one can determine the uncertainty $\Delta A$ of a observable as expectation value of the quadratic deviation from the mean value $\overline{A}$:

$$(\Delta A)^2 = \overline{(A - \overline{A})^2} = \overline{A^2} - \overline{A}^2 = \iiint\limits_V \psi^* (\hat{A} - \overline{A})^2 \psi \, dV \,. \tag{9.124}$$

If the system is in an eigenstate $\varphi_i$ of $\hat{A}$, then all measurements give the same measuring value $a_i$:

$$\overline{A} = a_i, \qquad \Delta A = 0 \,. \tag{9.125}$$

A distribution near the expectation value $\overline{A}$ does not appear.

### 3. Uncertainty Relation

Considering two observable $A\,, B$, whose operators commutate (see Lie brackets in 5.3.6.4, **2.**, p. 356),

$$\hat{C} = [\hat{A}\,, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0, \tag{9.126}$$

then (and only then) exists a simultaneous system of eigenfunctions $\varphi_{i,\nu}$ $(i, \nu = 1, 2, \ldots)$

$$\hat{A}\varphi_{i,\nu} = a_i \, \varphi_{i,\nu} \qquad \hat{B}\varphi_{i,\nu} = b_\nu \, \varphi_{i,\nu} \,. \tag{9.127}$$

In this case exist physical states, in which the expected values of both operators are eigenvalues, so that the uncertainties $\Delta A, \Delta B$ simultaneously disappear:

$$\overline{A} = a_i\,, \qquad \overline{B} = b_\nu\,, \qquad \Delta A = \Delta B = 0. \tag{9.128}$$

Performing in this system a measurement of the observable $A$, which leads with the measured value $a_i$ to the state $\varphi_{i,\nu}$, then a following measurement of $B$ gives the measured value $b_\nu$, without interfering the state generated in the first measurement (compatible observable, tolerance measurement).

For two observable $A$ and $B$, which are represented by non-commutative operators there does not exist a simultaneous system of eigenfunctions. In this case it is impossible to find a physical state, for which the uncertainties $\Delta A, \Delta B$ can be simultaneously arbitrarily small. For the product of the uncertainties exists a lower bound, defined by the expectation value of the *commutator* $\hat{C}$:

$$\Delta A \, \Delta B \ge \left| \frac{1}{2i} \, \overline{[\hat{A}, \hat{B}]} \right| \,. \tag{9.129}$$

This relation is called the *uncertainty relation.* The commutation relation (9.130) (see also 5.3.6.4, **2.**, p. 356) between the components, e.g., of the position and the momentum operator into the same direction

$$[\hat{p}_x\,, \hat{x}] = \frac{\hbar}{i} \tag{9.130}$$

$$\Delta p_x \, \Delta x \ge \frac{\hbar}{2} \,. \tag{9.131}$$

leads to the Heisenberg uncertainty relation (9.131). In other words: there is a fundamental limitation on how precisely both the position and the momentum of a particle can be simultaneously known.

### 9.2.4.5 Force-Free Motion of a Particle in a Block

#### 1. Formulation of the Problem

A particle with a mass $m$ is moving freely in a block with impenetrable walls of edge lengths $a$, $b$, $c$, therefore, it is in a potential box which is infinitely high in all three directions because of the impenetrability of the walls. That is, the probability of the presence of the particle, and also the wave function $\Psi$, vanishes outside the box. The Schroedinger equation and the boundary conditions for this problem are

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}+\frac{2m}{\hbar^2}E\Psi=0, \quad (9.132a) \qquad \Psi=0 \quad \text{for} \quad \begin{cases} x=0, & x=a,\\ y=0, & y=b,\\ z=0, & z=c. \end{cases} \quad (9.132b)$$

#### 2. Solution Approach

Separating the variables

$$\Psi(x,y,z)=\Psi_x(x)\,\Psi_y(y)\,\Psi_z(z) \tag{9.133a}$$

and substituting into (9.132a) gives

$$\frac{1}{\Psi_x}\frac{d^2\Psi_x}{dx^2}+\frac{1}{\Psi_y}\frac{d^2\Psi_y}{dy^2}+\frac{1}{\Psi_z}\frac{d^2\Psi_z}{dz^2}=-\frac{2m}{\hbar^2}E=-B. \tag{9.133b}$$

Every term on the left-hand side depends only on one independent variable. Their sum can be a constant $-B$ for arbitrary $x$, $y$, $z$ only if every single term is a constant. In this case the partial differential equation is reduced to three ordinary differential equations:

$$\frac{d^2\Psi_x}{dx^2}=-{k_x}^2\Psi_x,\quad \frac{d^2\Psi_y}{dy^2}=-{k_y}^2\Psi_y,\quad \frac{d^2\Psi_z}{dz^2}=-{k_z}^2\Psi_z. \tag{9.133c}$$

The relation for the *separation constants* $-{k_x}^2$, $-{k_y}^2$, $-{k_z}^2$ is

$${k_x}^2+{k_y}^2+{k_z}^2=B, \quad (9.133d) \qquad \text{consequently} \quad E=\frac{\hbar^2}{2m}({k_x}^2+{k_y}^2+{k_z}^2). \quad (9.133e)$$

#### 3. Solutions

of the three equations (9.133c) are the functions

$$\Psi_x=A_x\sin k_x\,x,\quad \Psi_y=A_y\sin k_y\,y,\quad \Psi_z=A_z\sin k_z\,z \tag{9.134a}$$

with the constants $A_x$, $A_y$, $A_z$. With these functions $\Psi$ satisfies the boundary conditions $\Psi=0$ for $x=0$, $y=0$ and $z=0$.

$$\sin k_x\,a=\sin k_y\,b=\sin k_z\,c=0 \tag{9.134b}$$

must be valid to satisfy also the relation $\Psi=0$ for $x=a$, $y=b$ and $z=c$, i.e., the relations

$$k_x=\frac{\pi n_x}{a},\quad k_y=\frac{\pi n_y}{b},\quad k_z=\frac{\pi n_z}{c} \tag{9.134c}$$

must be satisfied, where $n_x$, $n_y$, and $n_z$ are integers.
One gets for the total energy

$$E_{n_x,n_y,n_z}=\frac{\hbar^2}{2m}\left[\left(\frac{n_x}{a}\right)^2+\left(\frac{n_y}{b}\right)^2+\left(\frac{n_z}{c}\right)^2\right] \quad (n_x,n_y,n_z=\pm1,\pm2,\ldots). \tag{9.134d}$$

It follows from this formula that the changes of energy of a particle by interchange with the neighborhood is not continuous, which is possible only in quantum systems. The numbers $n_x$, $n_y$, and $n_z$, belonging to the *eigenvalues* of the energy, are called the *quantum numbers*.

After calculating the product of constants $A_xA_yA_z$ from the *normalization condition*

$$(A_xA_yA_z)^2 \int_0^a\int_0^b\int_0^c \sin^2\frac{\pi n_x x}{a}\,\sin^2\frac{\pi n_y y}{b}\,\sin^2\frac{\pi n_z z}{c}\,dx\,dy\,dz = 1 \tag{9.134e}$$

one gets the complete *eigenfunctions* of the states characterized by the three quantum numbers

$$\Psi_{n_x,n_y,n_z} = \sqrt{\frac{8}{abc}}\,\sin\frac{\pi n_x x}{a}\,\sin\frac{\pi n_y y}{b}\,\sin\frac{\pi n_z z}{c}\,. \tag{9.134f}$$

The eigenfunctions vanish at the walls since one of the three sine functions is equal to zero. This is always the case outside the walls if the following relations are valid

$$x = \frac{a}{n_x}, \frac{2a}{n_x}, \ldots, \frac{(n_x-1)a}{n_x},\quad y = \frac{b}{n_y}, \frac{2b}{n_y}, \ldots, \frac{(n_y-1)b}{n_y},\quad z = \frac{c}{n_z}, \frac{2c}{n_z}, \ldots, \frac{(n_z-1)c}{n_z}. \tag{9.134g}$$

So, there are $n_x-1$ and $n_y-1$ and $n_z-1$ planes perpendicular to the $x$- or $y$- or $z$-axis, in which $\Psi$ vanishes. These planes are called the *nodal planes*.

### 4. Special Case of a Cube, Degeneracy

In the special case of a cube with $a=b=c$, a particle can be in different states which are described by different linearly independent eigenfunctions and they have the same energy. This is the case when the sum ${n_x}^2+{n_y}^2+{n_z}^2$ has the same value in different states. They are called *degenerate states*, and if there are $i$ states with the same energy, they are called *i-fold degeneracy*.

The quantum numbers $n_x$, $n_y$ and $n_z$ can run through all real numbers, except zero. This last case would mean that the wave function is identically zero, i.e., the particle does not exist at any place in the box. The particle energy must remain finite, even if the temperature reaches absolute zero. This *zero-point translational energy* for a block is

$$E_0 = \frac{\hbar^2}{2m}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right). \tag{9.134h}$$

## 9.2.4.6 Particle Movement in a Symmetric Central Field (see 13.1.2.2, p. 702)

### 1. Formulation of the Problem

The considered particle moves in a central symmetric potential $V(r)$. This model reproduces the movement of an electron in the electrostatic field of a positively charged nucleus. Since this is a spherically symmetric problem, it is reasonable to use spherical coordinates **(Fig. 9.20)**. The following relations hold:

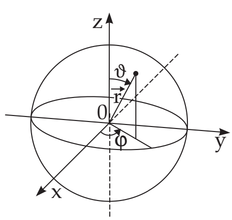


Figure 9.20

$$r = \sqrt{x^2+y^2+z^2}\,,\qquad x = r\sin\vartheta\cos\varphi,$$
$$\vartheta = \arccos\frac{z}{r}\,,\qquad y = r\sin\vartheta\sin\varphi, \tag{9.135a}$$
$$\varphi = \arctan\frac{y}{x}\,,\qquad z = r\cos\vartheta,$$

where $r$ is the absolute value of the radius vector, $\vartheta$ is the angle between the radius vector and the $z$-axis (polar angle) and $\varphi$ is the angle between the projection of the radius vector onto the $x,y$ plane and the $x$-axis (azimuthal angle). For the Laplace operator

$$\Delta\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\vartheta^2} + \frac{\cos\vartheta}{r^2\sin\vartheta}\frac{\partial\Psi}{\partial\vartheta} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Psi}{\partial\varphi^2}\,, \tag{9.135b}$$

holds, so the time-independent Schroedinger equation is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Psi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{2m}{\hbar^2}[E-V(r)]\Psi = 0. \tag{9.135c}$$

## 2. Solution

Looking for a solution in the form

$$\Psi(r,\vartheta,\varphi) = R_l(r)Y_l^m(\vartheta,\varphi), \tag{9.136a}$$

where $R_l$ is the radial wave function depending only on $r$, and $Y_l^m(\vartheta,\varphi)$ is the wave function depending on both angles. Substituting (9.136a) in (9.135c) gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_l}{\partial r}\right)Y_l^m + \frac{2m}{\hbar^2}[E - V(r)]R_lY_l^m$$
$$= -\left\{\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y_l^m}{\partial\vartheta}\right)R_l + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 Y_L^m}{\partial\varphi^2}R_l\right\}. \tag{9.136b}$$

Dividing by $R_lY_l^m$ and multiplying by $r^2$ gives

$$\frac{1}{R_l}\frac{d}{dr}\left(r^2\frac{dR_l}{dr}\right) + \frac{2mr^2}{\hbar^2}[E - V(r)] = -\frac{1}{Y_l^m}\left\{\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y_l^m}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_l^m}{\partial\varphi^2}\right\}. \tag{9.136c}$$

Equation (9.136c) can be satisfied if the expression on the left-hand side depending only on $r$ and expression on the right-hand side depending only on $\vartheta$ and $\varphi$ are equal to a constant, i.e., both sides being independent of each other are equal to the same constant. From the partial differential equation two ordinary differential equations follow. If the constant is chosen equal to $l(l+1)$, then the so-called *radial equation* results depending only on $r$ and the potential $V(r)$:

$$\frac{1}{R_lr^2}\frac{d}{dr}\left(r^2\frac{dR_l}{dr}\right) + \frac{2m}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2mr^2}\right] = 0. \tag{9.136d}$$

To find a solution for the angle-dependent part also in the separated form

$$Y_l^m(\vartheta,\varphi) = \Theta(\vartheta)\Phi(\varphi) \tag{9.136e}$$

one substitutes (9.136e) into (9.136c) giving

$$\sin^2\vartheta\left\{\frac{1}{\Theta\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\Theta}{d\vartheta}\right) + l(l+1)\right\} = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}. \tag{9.136f}$$

If the separation constant is chosen as $m^2$ in a reasonable way, then the so-called *polar equation* is

$$\frac{1}{\Theta\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\Theta}{d\vartheta}\right) + l(l+1) - \frac{m^2}{\sin^2\vartheta} = 0 \tag{9.136g}$$

and the *azimuthal equation* is

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0. \tag{9.136h}$$

Both equations are potential-independent, so they are valid for every central symmetric potential. There are three requirements for (9.136a): It should tend to zero for $r \to \infty$, it should be one-valued and quadratically integrable on the surface of the sphere.

## 3. Solution of the Radial Equation

Beside the potential $V(r)$ the radial equation (9.136d) also contains the separation constant $l(l+1)$. Substituting

$$u_l(r) = r \cdot R_l(r), \tag{9.137a}$$

since the square of the function $u_l(r)$ gives the last required probability $|u_l(r)|^2dr = |R_l(r)|^2r^2dr$ of the presence of the particle in a spherical shell between $r$ and $r + dr$. The substitution leads to the one-dimensional Schroedinger equation

$$\frac{d^2u_l(r)}{dr^2} + \frac{2m}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2mr^2}\right]u_l(r) = 0. \tag{9.137b}$$

This one contains the effective potential

$$V_{\text{eff}} = V(r) + V_l(l), \tag{9.137c}$$

which has two parts. The rotation energy

$$V_l(l) = V_{\text{rot}}(l) = \frac{l(l+1)\hbar^2}{2mr^2} \tag{9.137d}$$

is called the *centrifugal potential*.
The physical meaning of $l$ as the *orbital angular momentum* follows from analogy with the classical rotation energy

$$E_{rot} = \frac{1}{2}\Theta\vec{\omega}^2 = \frac{(\Theta\vec{\omega})^2}{2\Theta} = \frac{\vec{l}^{\,2}}{2\Theta} = \frac{\vec{l}^{\,2}}{2mr^2} \tag{9.137e}$$

a rotating particle with moment of inertia $\Theta = \mu r^2$ and orbital angular momentum $\vec{l} = \Theta\vec{\omega}$:

$$\vec{l}^{\,2} = l(l+1)\hbar^2, \quad \left|\vec{l}^{\,2}\right| = \hbar\sqrt{l(l+1)}\,. \tag{9.137f}$$

## 4. Solution of the polar equation

The polar equation (9.136g), containing both separation constants $l(l+1)$ and $m^2$, is a Legendre differential equation (9.60a), p. 565. Its solution is denoted by $\Theta_l^m(\vartheta)$, and it can be determined by a power series expansion. Finite, single-valued and continuous solutions exist only for $l(l+1) = 0, 2, 6, 12, \ldots$. One gets for $l$ and $m$:

$$l = 0, 1, 2, \ldots, \quad |m| \le l. \tag{9.138a}$$

So, $m$ can take the $(2l+1)$ values

$$-l,\ (-l+1),\ (-l+2), \ldots,\ (l-2),\ (l-1),\ l. \tag{9.138b}$$

For $m \ne 0$ one gets the *corresponding Legendre polynomials*, which are defined in the following way:

$$P_l^m(\cos\vartheta) = \frac{(-1)^m}{2^l l!}(1-\cos^2\vartheta)^{m/2}\frac{d^{l+m}(\cos^2\vartheta - 1)^l}{(d\cos\vartheta)^{l+m}}. \tag{9.138c}$$

As a special case ($l = 0$, $m = n$, $\cos\vartheta = x$) follow the Legendre polynomials of the first kind (9.60c), p. 566. Their normalization results in the equation

$$\Theta_l^m(\vartheta) = \sqrt{\frac{2l+1}{2}\cdot\frac{(l-m)!}{(l+m)!}}\cdot P_l^m(\cos\vartheta) = N_l^m P_l^m(\cos\vartheta). \tag{9.138d}$$

## 5. Solution of the Azimuthal Equation

Since the motion of the particle in the potential field $V(r)$ is independent of the azimuthal angle even in the case of the physical assignment of a space direction, e.g., by a magnetic field, the general solution $\Phi = \alpha e^{\mathrm{i}m\varphi} + \beta e^{-\mathrm{i}m\varphi}$ can be specified by fixing

$$\Phi_m(\varphi) = Ae^{\pm\mathrm{i}m\varphi}, \tag{9.139a}$$

because in this case $|\Phi_m|^2$ is independent of $\varphi$. The requirement for uniqueness is

$$\Phi_m(\varphi + 2\pi) = \Phi_m(\varphi), \tag{9.139b}$$

so $m$ can take on only the values $0, \pm 1, \pm 2, \ldots$.
It follows from the normalization

$$\int_0^{2\pi} |\Phi|^2\,d\varphi = 1 = |A|^2\int_0^{2\pi} d\varphi = 2\pi|A|^2 \tag{9.139c}$$

that

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{\mathrm{i}m\varphi} \quad (m = 0, \pm 1, \pm 2, \ldots). \tag{9.139d}$$

The quantum number $m$ is called the *magnetic quantum number*.

## 6. Complete Solution for the Dependency of the Angles

In accordance with (9.136e), the solutions for the polar and the azimuthal equations should be multiplied by each other:

$$Y_l^m(\vartheta, \varphi) = \Theta(\vartheta)\,\Phi(\varphi) = \frac{1}{\sqrt{2\pi}} N_l^m P_l^m(\cos\vartheta) e^{im\varphi}. \tag{9.140a}$$

The functions $Y_l^m(\vartheta, \varphi)$ are the so-called *surface spherical harmonics*.

When the radius vector $\vec{\mathbf{r}}$ is reflected with respect to the origin ($\vec{\mathbf{r}} \to -\vec{\mathbf{r}}$), the angle $\vartheta$ becomes $\pi - \vartheta$ and $\varphi$ becomes $\varphi + \pi$, so the sign of $Y_l^m$ may change:

$$Y_l^m(\pi - \vartheta, \varphi + \pi) = (-1)^l Y_l^m(\vartheta, \varphi). \tag{9.140b}$$

Then for the *parity* of the considered wave function holds

$$P = (-1)^l. \tag{9.141a}$$

## 7. Parity

The *parity* property serves the characterization of the behavior of the wave function under *space inversion* $\vec{\mathbf{r}} \to -\vec{\mathbf{r}}$ (see 4.3.5.1, **1.**, p. 287). It is performed by the inversion or parity operator $\mathbf{P}$: $\mathbf{P}\Psi(\vec{\mathbf{r}}, t) = \Psi(-\vec{\mathbf{r}}, t)$. Denoting the eigenvalue of the operator by $P$, then applying $\mathbf{P}$ twice it must yield $\mathbf{P}\,\mathbf{P}\,\Psi(\vec{\mathbf{r}}, t) = P\,P\,\Psi(\vec{\mathbf{r}}, t) = \Psi(\vec{\mathbf{r}}, t)$, the original wave function. So:

$$P^2 = 1, \quad P = \pm 1. \tag{9.141b}$$

It is called an *even wave function* if its sign does not change under space inversion, and it is called an *odd wave function* if its sign changes.

# 9.2.4.7 Linear Harmonic Oscillator

## 1. Formulation of the Problem

*Harmonic oscillation* occurs when the drag forces in the oscillator satisfy Hooke's law $F = -kx$. For the frequency of the oscillation, for the frequency of the oscillation circuit and for the potential energy the following formulas are valid:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \quad (9.142\text{a}) \qquad \omega = \sqrt{\frac{k}{m}}, \quad (9.142\text{b}) \qquad E_{\text{pot}} = \frac{1}{2}kx^2 = \frac{\omega^2}{2}x^2. \quad (9.142\text{c})$$

Substituting into (9.114a), the Schroedinger equation becomes

$$\frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - \frac{\omega^2}{2}mx^2\right]\Psi = 0. \tag{9.143a}$$

With the substitutions

$$y = x\sqrt{\frac{m\omega}{\hbar}}, \quad (9.143\text{b}) \qquad \lambda = \frac{2E}{\hbar\omega}, \quad (9.143\text{c})$$

where $\lambda$ is a parameter and not the wavelength, (9.143a) can be transformed into the simpler form of the Weber differential equation

$$\frac{d^2\Psi}{dy^2} + (\lambda - y^2)\Psi = 0. \tag{9.143d}$$

## 2. Solution

A solution of the Weber differential equation can be got in the form

$$\Psi(y) = e^{-y^2/2}H(y). \tag{9.144a}$$

Differentiation shows that

$$\frac{d^2\Psi}{dy^2} = e^{-y^2/2}\left[\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} + (y^2-1)H\right]. \tag{9.144b}$$

Substitution into (9.143d) yields

$$\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} + (\lambda-1)H = 0. \tag{9.144c}$$

The determination of a solution is convenient in the form of a series:

$$H = \sum_{i=0}^{\infty} a_i y^i \quad \text{with} \quad \frac{dH}{dy} = \sum_{i=1}^{\infty} i a_i y^{i-1}, \quad \frac{d^2H}{dy^2} = \sum_{i=2}^{\infty} i(i-1)a_i y^{i-2}. \tag{9.145a}$$

Substitution of (9.145a) into (9.144c) results in

$$\sum_{i=2}^{\infty} i(i-1)a_i y^{i-2} - \sum_{i=1}^{\infty} 2i a_i y^i + \sum_{i=0}^{\infty} i(\lambda-1)a_i y^i = 0. \tag{9.145b}$$

Comparing the coefficients of $y^j$ leads to the recursion formula

$$(j+2)(j+1)a_{j+2} = [2j-(\lambda-1)]a_j \quad (j=0,1,2,\ldots). \tag{9.145c}$$

The coefficients $a_j$ for even powers of $y$ begin from $a_0$, the coefficients for odd powers begin from $a_1$. So, $a_0$ and $a_1$ can be chosen arbitrarily.

### 3. Physical Solutions

The determination of the probability of the presence of a particle in the different states can be performed by a quadratically integrable wave function $\Psi(x)$ and by an eigenfunction which has physical meaning, i.e., normalizable and for large values of $y$ it tends to zero.
The exponential function $\exp(-y^2/2)$ in (9.144a) guarantees that the solution $\Psi(y)$ tends to zero for $y \to \infty$ if the function $H(y)$ is a polynomial. To get a polynomial, the coefficients $a_j$ in (9.145a), starting from a certain $n$, must vanish for every $j > n$: $a_n \neq 0$, $a_{n+1} = a_{n+2} = a_{n+3} = \ldots = 0$. The recursion formula (9.145c) with $j = n$ is

$$a_{n+2} = \frac{2n-(\lambda-1)}{(n+2)(n+1)}a_n. \tag{9.146a}$$

$a_{n+2} = 0$ can be satisfied for $a_n \neq 0$ only if

$$2n-(\lambda-1) = 0, \quad \lambda = \frac{2E}{\hbar\omega} = 2n+1. \tag{9.146b}$$

The coefficients $a_{n+2}, a_{n+4}, \ldots$ vanish for this choice of $\lambda$. Also $a_{n-1} = 0$ must hold to make the coefficients $a_{n+1}, a_{n+3}, \ldots$ equal to zero.
One gets the *Hermite polynomials* from the second defining equation (see 9.1.2.6, **6.**, p. 568) for the special choice of $a_n = 2^n$, $a_{n-1} = 0$. The first six of them are:

$$\begin{aligned} &H_0(y) = 1, && H_3(y) = -12y + 8y^3,\\ &H_1(y) = 2y, && H_4(y) = 12 - 48y^2 + 16y^4,\\ &H_2(y) = -2 + 4y^2, && H_5(y) = 120y - 160y^3 + 32y^5. \end{aligned} \tag{9.146c}$$

The solution $\Psi(y)$ for the *vibration quantum number* $n$ is

$$\Psi_n = N_n e^{-y^2/2} H_n(y), \tag{9.147a}$$

where $N_n$ is the normalizing factor. One gets it from the normalization condition $\int \Psi_n{}^2\,dy = 1$ as

$$N_n{}^2 = \frac{1}{2^n n!}\sqrt{\frac{\alpha}{\pi}} \quad \text{with} \quad \sqrt{\alpha} = \frac{y}{x} = \sqrt{\frac{m\omega}{\hbar}} \quad \text{(see (9.143b), p. 601).} \tag{9.147b}$$

From the terminating condition of the series (9.143c)

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad (n = 0, 1, 2, \ldots) \tag{9.147c}$$

follows for the eigenvalues of the vibration energy. The spectrum of the energy levels is equidistant. The summand $+1/2$ in the parentheses means that in contrast to the classical case the quantum mechanical oscillator has energy even in the deepest energetic level with $n = 0$, which is known as the *zero-point vibration energy.*

**Fig. 9.21** shows a graphical representation of the equidistant spectra of the energy states, the corresponding wave functions from $\Psi_0$ to $\Psi_5$ and also the function of the potential energy (9.142c). The points of the parabola of the potential energy represent the reversal points of the classical oscillator, which are calculated from the energy $E = \frac{1}{2}m\omega^2 a^2$ as the amplitude $a = \frac{1}{\omega}\sqrt{\frac{2E}{m}}$. The quantum mechanical probability of finding a particle in the interval $(x, x+dx)$ is given by $dw_{qu} = |\Psi(x)|^2\,dx$. It is different from zero also outside of these points. So for, e.g., $n = 1$, hence for $E = (3/2)\hbar\omega$, according to $dw_{qu} = 2\sqrt{\frac{\lambda}{\pi}}e^{-\lambda x^2}\,dx$, the maximum of the probability of presence is at

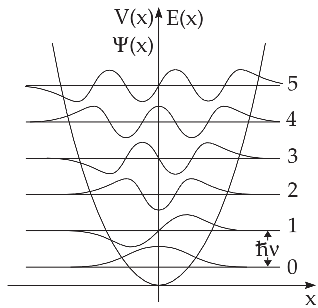


Figure 9.21

$$x_{max,qu} = \frac{\pm 1}{\sqrt{\lambda}} = \pm\sqrt{\frac{\hbar}{m\omega}}. \tag{9.147d}$$

For a corresponding classical oscillator, this is

$$x_{max,kl} = \pm a = \pm\sqrt{\frac{2E}{m\omega^2}} = \pm\sqrt{\frac{3\hbar}{m\omega}}. \tag{9.147e}$$

For large quantum numbers $n$ the quantum mechanical probability density function approaches the classical one in its mean value.

## 9.2.5 Non-Linear Partial Differential Equations: Solitons, Periodic Patterns and Chaos

### 9.2.5.1 Formulation of the Physical-Mathematical Problem

#### 1. Notion of Solitons

*Solitons*, also called solitary waves, from the viewpoint of physics, are pulses, or also localized disturbances of a non-linear medium or field; the energy related to such propagating pulses or disturbances is concentrated in a narrow spatial region. They occur:

- in solids, e.g., in inharmonic lattices, in Josephson contacts, in glass fibres and in quasi-one-dimensional conductors,
- in fluids as surface waves or spin waves,
- in plasmas as Langmuir solitons,
- in linear molecules,
- in classical and quantum field theory.

Solitons have both particle and wave properties; they are localized during their evolution, and the domain of the localization, or the point around which the wave is localized, travels as a free particle; in

particular it can also be at rest. A soliton has a permanent wave structure: based on a balance between nonlinearity and dispersion, the form of this structure does not change.
Mathematically, solitons are special solutions of certain non-linear partial differential equations occurring in physics, engineering and applied mathematics. Their special features are the absence of any dissipation and also that the non-linear terms cannot be handled by perturbation theory. Dissipative solitons are in non-conservative systems.

### 2. Important Examples of Equations with Soliton Solutions

**a) Korteweg de Vries (KdV) Equation** $$u_t + 6uu_x + u_{xxx} = 0, \tag{9.148}$$

**b) Non-Linear Schroedinger (NLS) Equation** $$\mathrm{i}\, u_t + u_{xx} \pm 2|u|^2 u = 0, \tag{9.149}$$

**c) Sine-Gordon (SG) Equation** $$u_{tt} - u_{xx} + \sin u = 0. \tag{9.150}$$

The subscripts $x$ and $t$ denote partial derivatives, e.g., $u_{xx} = \partial^2 u/\partial x^2$.
In these equations the one-dimensional case is considered, i.e., $u$ has the form $u = u(x,t)$, where $x$ is the spatial coordinate and $t$ is the time. The equations are given in a scaled form, i.e., the two independent variables $x$ and $t$ are here dimensionless quantities. In practical applications, they must be multiplied by quantities having the corresponding dimensions and being characteristic of the given problem. The same holds for the velocity.

### 3. Interaction between Solitons

If two solitons, moving with different velocities, collide, they appear again after the interaction as if they had not collided. Every soliton asymptotically keeps its form and velocity; there is only a phase shift. Two solitons can interact without disturbing each other asymptotically. This is called an elastic interaction which is equivalent to the existence of an $N$-soliton solution, where $N$ $(N = 1, 2, 3, \ldots)$ is the number of solitons. Solving an initial value problem with a given initial pulse $u(x,0)$ that disaggregate into solitons, the number of solitons does not depend on the shape of the pulse but on its total amount $\int_{-\infty}^{+\infty} u(x,0)\,dx$.

### 4. Periodic Patterns and Non-Linear Waves

Such non-linear phenomena occur in several classic dissipative systems (i.e. friction or damping systems), when an external impact or force is sufficiently large. E.g., if there is a layer of fluid in the gravitational field, and it is heated from below, the difference of temperature between the upper and lower surface corresponds to an external force. The higher temperature of the lower layer reduces its density and makes it lighter than the upper part, so the layering becomes unstable. At a sufficiently large temperature difference this unstable layering turns spontaneously into periodically arranged convection cells. It is called the *bifurcation* from the state of thermal conductivity (without convection) into the well ordered Rayleigh-Bénard convection. Taking away the external force results because of dissipation into damping of the waves (here the cellular convection). Strengthening of the external force drives the ordered convection into a turbulent convection and into chaos (see 17.3, p. 892). Also in chemical reactions similar phenomena can occur. Important examples for equations describing such phenomena are:

**a) Ginsburg-Landau (GL) Equation** $$u_t - u - (1 + \mathrm{i}b)u_{xx} + (1 + \mathrm{i}c)|u|^2 u = 0, \tag{9.151}$$

**b) Kuramoto-Sivashinsky (KS) Equation** $$u_t + u_{xx} + u_{xxxx} + u_x^2 = 0. \tag{9.152}$$

In contrast to the dissipation-less KdV, NLS, SG, equations, the equations (9.151) and (9.152) are non-linear dissipative equations, which have, besides spatiotemporal periodic solutions, also spatiotemporal disordered (chaotic) solutions. Appearance of spatiotemporal patterns or structures is characteristic which turn into chaos.

### 5. Dissipative Solitons

Solitary (isolated) wave phenomena in non-conservative systems often are called *dissipative solitons.* Contrary to the conservative systems, in which the solitons usually form families of solutions with at least one continuously changing parameter, dissipative solitons can be found at single points of the parameter space, at which a balance is formed between dispersion and nonlinearity from one side and energy or particle flow and dissipation from the other side. This property leads to a special kind of stability of dissipative solitons, although they are not solutions of integrable wave equations. Dissipative solitons are described among others by the complex Ginsburg-Landau equation. They occur ,e.g., in non-linear optic cavitations, in optic semiconductor amplifiers and in reaction-diffusion systems (see also [9.16]).

### 6. Non-Linear Evolution Equations

*Evolution equations* describe the evolution of a physical quantity in time. Examples are the wave equation (see 9.2.3.2, p. 590), the heat equation (see 9.2.3.3, p. 591) and the Schroedinger equation (see 9.2.4.1, **1.**, p. 592). The solutions of the evolution equations are called *evolution functions*.
In contrast to linear evolution equations, the non-linear evolution equations (9.148), (9.149), and (9.150) contain non-linear terms like $u\partial u/\partial x$, $|u|^2 u$, $\sin u$ and $u_x^2$. These equations are (with the exception of (9.151)) parameter-free. From the viewpoint of physics non-linear evolution equations describe structure formations like solitons (dispersive structures) as well as periodic patterns and non-linear waves (dissipative structures).

## 9.2.5.2 Korteweg de Vries Equation (KdV)

### 1. Occurrence

The KdV equation is used in the discussion of
- surface waves in shallow water,
- inharmonic vibrations in non-linear lattices,
- problems of plasma physics and
- non-linear electric networks.

### 2. Equation and Solutions

The KdV equation for the evolution function $u$ is

$$u_t + 6uu_x + u_{xxx} = 0. \tag{9.153}$$

It has the soliton solution

$$u(x,t) = \frac{v}{2\cosh^2\left[\frac{1}{2}\sqrt{v}(x - vt - \varphi)\right]}. \tag{9.154}$$

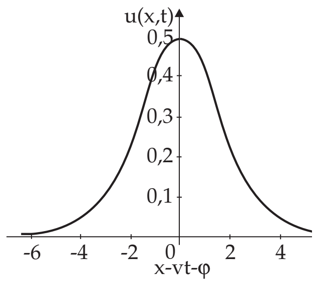


Figure 9.22

This KdV soliton is uniquely defined by the two dimensionless parameters $v$ ($v > 0$) and $\varphi$. In **Fig. 9.22** $v = 1$ is chosen. A typical non-linear effect is that the velocity of the soliton $v$ determines the amplitude and the width of the soliton: KdV solitons with larger amplitude and smaller width move faster than those with smaller amplitude and larger width (taller waves travel faster than shorter ones). The soliton phase $\varphi$ describes the position of the maximum of the soliton at time $t = 0$.

Equation (9.153) also has $N$-solitons solutions. Such an $N$-soliton solution can be represented asymptotically for $t \to \pm\infty$ by the linear superposition of one-soliton solutions:

$$u(x,t) \sim \sum_{n=1}^{N} u_n(x,t). \tag{9.155}$$

Here every evolution function $u_n(x,t)$ is characterized by a velocity $v_n$ and a phase $\varphi_n^{\pm}$. The initial phases $\varphi_n^-$ before the interaction or collision differ from the final phases after the collision $\varphi_n^+$, while the velocities $v_1, v_2, \ldots, v_N$ have no changes, i.e., it is an elastic interaction.

For $N = 2$, (9.153) has a two-soliton solution. It cannot be represented for a finite time by a linear superposition, and with $k_n = \dfrac{1}{2}\sqrt{v_n}$ and $\alpha_n = \dfrac{1}{2}\sqrt{v_n}(x - v_n t - \varphi_n)$ $(n = 1, 2)$ it has the form:

$$u(x,t) = 8\frac{k_1^2 e^{\alpha_1} + k_2^2 e^{\alpha_2} + (k_1-k_2)^2 e^{(\alpha_1+\alpha_2)}\left[2 + \frac{1}{(k_1+k_2)^2}\left(k_1^2 e^{\alpha_1} + k_2^2 e^{\alpha_2}\right)\right]}{\left[1 + e^{\alpha_1} + e^{\alpha_2} + \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 e^{(\alpha_1+\alpha_2)}\right]^2}. \tag{9.156}$$

Equation (9.156) describes two non-interacting solitons for $t \to -\infty$ asymptotically with velocities $v_1 = 4k_1^2$ and $v_2 = 4k_2^2$, which transform after their mutual interaction again into two non-interacting solitons with the same velocities for $t \to +\infty$ asymptotically.

The non-linear evolution equation

$$w_t + 6(w_x)^2 + w_{xxx} = 0 \tag{9.157a}$$

where $w = \frac{F_x}{F}$ has the following properties:

**a)** For $F(x,t) = 1 + e^{\alpha}, \quad \alpha = \frac{1}{2}\sqrt{v}(x - vt - \varphi)$ (9.157b)

it has a soliton solution and

**b)** for $F(x,t) = 1 + e^{\alpha_1} + e^{\alpha_2} + \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 e^{(\alpha_1+\alpha_2)}$ (9.157c)

it has a two-soliton solution. With $2w_x = u$ the KdV equation (9.153) follows from (9.157a). Equation (9.156) and the expression $w$ following from (9.157c) are examples of a non-linear superposition.

If the term $+6uu_x$ is replaced by $-6uu_x$ in (9.153), then the right-hand side of (9.154) has to be multiplied by $(-1)$. In this case the notation *antisoliton* is used.

### 9.2.5.3 Non-Linear Schroedinger Equation (NLS)

#### 1. Occurrence

The NLS equation occurs

• in non-linear optics, where the refractive index $n$ depends on the electric field strength $\vec{\mathbf{E}}$, as, e.g., for the Kerr effect, where $n(\vec{\mathbf{E}}) = n_0 + n_2|\vec{\mathbf{E}}|^2$ with $n_0, n_2 =$ constant holds, and

• in the hydrodynamics of self-gravitating discs which allow us to describe galactic spiral arms.

#### 2. Equation and Solution

The NLS equation for the evolution function $u$ and its solution are:

$$\mathrm{i}\,u_t + u_{xx} \pm 2|u|^2 u = 0, \tag{9.158}$$

$$u(x,t) = 2\eta\frac{\exp\left(-\mathrm{i}[2\xi x + 4(\xi^2-\eta^2)t - \chi]\right)}{\cosh[2\eta(x + 4\xi t - \varphi)]}. \tag{9.159}$$

Here $u(x,t)$ is complex. The NLS soliton is characterized by the four dimensionless parameters $\eta, \xi, \varphi$, and $\chi$. The envelope of the wave packet moves with the velocity $v = -4\xi$; the phase velocity of the wave packet is $2(\eta^2 - \xi^2)/\xi$.

In contrast to the KdV soliton (9.154), the amplitude and the velocity can be chosen independently of each other. **Fig. 9.23** displays the real part of (9.159) with $\eta = 1/2$ and $\xi = 4$.

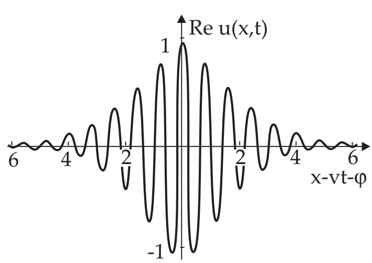


Figure 9.23

The solutions of the form (9.159) often are called *light solitons*; they solve the focusing NLS equation (9.158) for the case "+". The defocusing NLS equation (case "−") gives solitons, for which $|u|^2$ at the position of the solitons is reduced in comparison to a constant background $|u(x \to \pm\infty)| = \eta$. Such *dark solitons* have the form

$$u(x,t) = \left(\mathrm{i}\frac{v}{2} + \sqrt{\eta^2 - \frac{v^2}{4}}\tanh\left[\sqrt{\eta^2 - \frac{v^2}{4}}(x - vt)\right]\right) \cdot \exp\left[-\mathrm{i}\left(2\eta^2 t + \chi\right)\right]. \qquad (9.160)$$

They depend on the three parameters $\eta$, $v$ and $\chi$ and propagate with the velocity $v < 2\eta$ on a background with trivial (flat) phase (see [9.26], [9.23]).

A general solution has in addition a phase gradient, which can be interpreted as velocity $c$ of the background, relative to which the soliton is moving. Then the solution has the following form:

$$u(x,t) = \left(\mathrm{i}\frac{v}{2} + \sqrt{\eta^2 - \frac{v^2}{4}}\right)\tanh\left[\sqrt{\eta^2 - \frac{v^2}{4}}(x - vt - ct)\right]\exp\left[-\mathrm{i}\left(2\eta^2 t + \chi - \frac{c}{2}x + \frac{c^2}{4}t\right)\right]. \qquad (9.161)$$

Beside of these exponential positioned soliton waves also periodic solutions of the NLS equation exist, which can be interpreted as wave packets of solitons. Such solutions can be find demanding stationarity and by integration of the remaining ordinary differential equation. Generally such solutions are elliptic Jacobian-functions (see 14.6.2, p. 763). Some relevant solutions see [9.17].

In the case of $N$ interacting solitons, one can characterize them by $4N$ arbitrary chosen parameters: $\eta_n, \xi_n, \varphi_n, \chi_n \quad (n = 1, 2, \ldots, N)$.
If the solitons have different velocities, the $N$-soliton solution splits asymptotically for $t \to \pm\infty$ into a sum of $N$ individual solitons of the form (9.159).

### 9.2.5.4 Sine-Gordon Equation (SG)

**1. Occurrence**

The SG equation is obtained from the Bloch equation for spatially inhomogeneous quantum mechanical two-level systems. It describes the propagation of

- ultra-short pulses in resonant laser media (self-induced transparency),
- the magnetic flux in large surface Josephson contacts, i.e., in tunnel contacts between two superconductors and
- spin waves in superfluid helium 3 ($^3$He).

The soliton solution of the SG equation can be illustrated by a mechanical model of pendula and springs. The evolution function goes continuously from 0 to a constant value $c$. The *SG solitons* are often called *kink solitons*. If the evolution function changes from the constant value $c$ to 0, it describes a so-called *antikink soliton*. Walls of domain structures can be described with this type of solutions.

**2. Equation and Solution**

The SG equation for the evolution function $u$ is

$$u_{tt} - u_{xx} + \sin u = 0. \qquad (9.162)$$

It has the following soliton solutions:

**1. Kink Soliton**

$$u(x,t) = 4\arctan e^{\gamma(x - x_0 - vt)}, \qquad (9.163)$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2}}$ and $-1 < v < +1$.

The kink soliton (9.163) for $v = 1/2$ is given in **Fig. 9.24**. The kink soliton is determined by two dimensionless parameters $v$ and $x_0$. The velocity is independent of the amplitude. The time and the position derivatives are ordi-

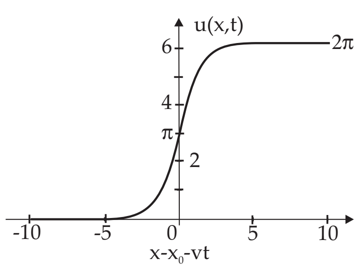


Figure 9.24

nary localized solitons:

$$-\frac{u_t}{v} = u_x = \frac{2\gamma}{\cosh\gamma(x - x_0 - vt)}. \tag{9.164}$$

**2. Antikink Soliton**

$$u(x,t) = 4\arctan e^{-\gamma(x-x_0-vt)}. \tag{9.165}$$

**3. Kink-Antikink Soliton** One gets a static kink-antikink soliton from (9.163, 9.165) with $v = 0$:

$$u(x,t) = 4\arctan e^{\pm(x-x_0)}. \tag{9.166}$$

Further solutions of (9.162) are:

**4. Kink-Kink Collision**

$$u(x,t) = 4\arctan\left[v\frac{\sinh\gamma x}{\cosh\gamma vt}\right]. \tag{9.167}$$

**5. Kink-Antikink Collision**

$$u(x,t) = 4\arctan\left[\frac{1}{v}\frac{\sinh\gamma vt}{\cosh\gamma x}\right]. \tag{9.168}$$

**6. Double or Breather Soliton, also called Kink-Antikink Doublet**

$$u(x,t) = 4\arctan\left[\frac{\sqrt{1-\omega^2}}{\omega}\frac{\sin\omega t}{\cosh\sqrt{1-\omega^2}x}\right]. \tag{9.169}$$

Equation (9.169) represents a stationary wave, whose envelope is modulated by the frequency $\omega$.

**7. Local Periodic Kink Lattice**

$$u(x,t) = 2\arcsin\left[\pm\mathrm{sn}\left(\frac{x - vt}{k\sqrt{1-v^2}}, k\right)\right] + \pi. \tag{9.170a}$$

The relation between the wavelength $\lambda$ and the lattice constant $k$ is

$$\lambda = 4K(k)k\sqrt{1-v^2}. \tag{9.170b}$$

For $k = 1$, i.e., for $\lambda \to \infty$, one gets

$$u(x,t) = 4\arctan e^{\pm\gamma(x-vt)}, \tag{9.170c}$$

which is the kink soliton (9.163) and the antikink soliton (9.165) again, with $x_0 = 0$.

**Remark:** $\mathrm{sn}\, x$ is a Jacobian elliptic function with parameter $k$ and quarter-period $K$ (see 14.6.2, p. 763):

$$\mathrm{sn}x = \sin\varphi(x,k), \tag{9.171a}$$

$$x = \int\limits_0^{\sin\varphi(x,k)} \frac{dq}{\sqrt{(1-q^2)(1-k^2q^2)}}, \tag{9.171b}$$

$$K(k) = \int\limits_0^{\pi/2} \frac{d\Theta}{\sqrt{1-k^2\sin^2\Theta}}. \tag{9.171c}$$

Equation (9.171b) comes from (14.104a), p. 763, by the substitution of $\sin\psi = q$. The series expansion of the complete elliptic integral is given as equation (8.104), in 8.2.5, **7.**, p. 515.

## 9.2.5.5 Further Non-linear Evolution Equations with Soliton Solutions

### 1. Modified KdV Equation

$$u_t \pm 6u^2u_x + u_{xxx} = 0. \tag{9.172}$$

The even more general equation (9.173) has the soliton (9.174) as solution.

$$u_t + u^p u_x + u_{xxx} = 0\,, \quad (9.173) \qquad u(x,t) = \left[\frac{\frac{1}{2}|v|(p+1)(p+2)}{\cosh^2\left(\frac{1}{2}p\sqrt{|v|}(x - vt - \varphi)\right)}\right]^{\frac{1}{p}}. \quad (9.174)$$

**2. Sinh-Gordon Equation**

$$u_{tt} - u_{xx} + \sinh u = 0. \quad (9.175)$$

**3. Boussinesq Equation**

$$u_{xx} - u_{tt} + (u^2)_{xx} + u_{xxxx} = 0. \quad (9.176)$$

This equation occurs in the description of non-linear electric networks as a continuous approximation of the charge-voltage relation.

**4. Hirota Equation**

$$u_t + \mathrm{i}3\alpha|u|^2 u_x + \beta u_{xx} + \mathrm{i}\sigma u_{xxx} + \delta|u|^2 u = 0, \qquad \alpha\beta = \sigma\delta. \quad (9.177)$$

**5. Burgers Equation**

$$u_t - u_{xx} + u u_x = 0. \quad (9.178)$$

This equation occurs when modeling turbulence. With the Hopf-Cole transformation it is transformed into the diffusion equation, i.e., into a linear differential equation.

**6. Kadomzev-Pedviashwili Equation**

The equation

$$(u_t + 6uu_x + u_{xxx})_x = u_{yy} \quad (9.179a)$$

has the soliton

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[\frac{1}{k^2} + \left|x + \mathrm{i}ky - 3k^2 t\right|^2\right] \quad (9.179b)$$

as its solution. The equation (9.179a) is an example of a soliton equation with a higher number of independent variables, e.g., with two spatial variables.

**Remark:** The CD-ROM to the 7th, 8th and 9th German editions of this handbook (see [22.8]) contains more non-linear evolution equations. Furthermore there is shown the application of the Fourier transformation and of the inverse scattering theory to solve linear partial differential equations.