6 Differentiation

6.1 Differentiation of Functions of One Variable

6.1.1 Differential Quotient

1. Differential Quotient or Derivative of a Function

The *differential quotient* of a function $y = f(x)$ at x_0 is equal to $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ if this limit exists and is finite. The *derivative function* of a function $y = f(x)$ with respect to the variable x is another function of x denoted by the symbols y', $\dot{y}$, Dy, $\dfrac{dy}{dx}$, $f'(x)$, $Df(x)$, or $\dfrac{df(x)}{dx}$, and its value for every x is equal to the limit of the quotient of the increment of the function Δy and the corresponding increment Δx for $\Delta x \to 0$, if this limit exists:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{6.1}$$

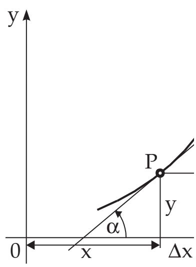

Figure 6.1

2. Geometric Representation of the Derivative

If $y = f(x)$ is represented as a curve in a Cartesian coordinate system as in **Fig. 6.1**, and if the x-axis and the y-axis have the same unit, then

$$f'(x) = \tan\alpha \tag{6.2}$$

is valid. The angle α between the x-axis and the tangent line of the curve at the considered point defines the *angular coefficient* or *slope of the tangent* (see 3.6.1.2, **2.**, p. 245). The angle is measured from the positive x-axis to the tangent in a counterclockwise direction, and it is called the *angle of slope* or *angle of inclination.*

3. Differentiability

From the definition of the derivative it obviously follows that $f(x)$ is differentiable with respect to x for the values of x where the differential quotient (6.1) has a finite value. The domain of the derivative function is a subset (proper or trivial) of the domain of the original function. If the function is continuous at x but the derivative does not exist, then perhaps there is no determined tangent line at that point, or the tangent line is perpendicular to the x-axis. In this last case the limit in (6.1) is infinity. For this case is used the notation $f'(x) = +\infty$ or $-\infty$.

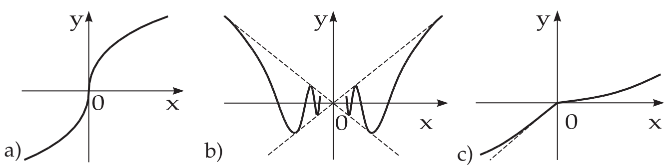

Figure 6.2

■ **A:** $f(x) = \sqrt[3]{x}$: $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, $f'(0) = \infty$. At the point 0 the limit (6.1) tends to infinity, so the derivative does not exist at the point 0 **(Fig. 6.2a)**.

■ **B:** $f(x) = x \sin\dfrac{1}{x}$ for $x \neq 0$. At the point $x = 0$ the function $f(x)$ is not defined, but it has zero limit, so one writes $f(0) = 0$. However the limit (6.1) does not exist at $x = 0$ **(Fig. 6.2b)**.

I.N. Bronshtein et al., *Handbook of Mathematics*,
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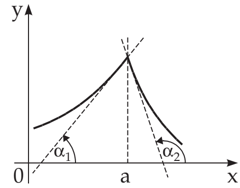

Figure 6.3

4. Left-Hand and Right-Hand Differentiability

If the limit (6.1) does not exist for a value $x = a$, but the left-hand limit or the right-hand limit exists, this limit is called the left-hand derivative or right-hand derivative respectively. If both exist, the curve has two tangents here:

$$f'(a-0) = \tan\alpha_1, \qquad f'(a+0) = \tan\alpha_2. \tag{6.3}$$

Geometrically this means that the curve has a knee **(Fig. 6.2c, Fig. 6.3)**.

■ $f(x) = \dfrac{x}{1+e^{\frac{1}{x}}}$ for $x \neq 0$. For $x = 0$ the function is not defined, but it has zero limit at $x = 0$, so one writes $f(x) = 0$. At the point $x = 0$ there is no limit of type (6.1), but there is a left-hand and a right-hand limit $f'(-0) = 1$ and $f'(+0) = 0$, i.e., the curve has a knee here **(Fig. 6.2c)**.

6.1.2 Rules of Differentiation for Functions of One Variable

6.1.2.1 Derivatives of the Elementary Functions

The elementary functions have a derivative on all their domains except perhaps some points, as represented in **Fig. 6.2**.

A summary of the derivatives of elementary functions can be found in **Table 6.1**. Further derivatives of elementary functions can be found by reversing the results of the indefinite integrals in **Table 8.1**.

Remark: In practice, it is often useful to transform the function into a more convenient form to perform differentiation, e.g., to transform it into a sum where parentheses are removed (see 1.1.6.1, p. 11) or to separate the integral rational part of the expression (see 1.1.7, p. 14) or to take the logarithm of the expression (see 1.1.4.3, p. 9).

■ **A:** $y = \dfrac{2-3\sqrt{x}+4\sqrt[3]{x}+x^2}{x} = \dfrac{2}{x} - 3x^{-\frac{1}{2}} + 4x^{-\frac{2}{3}} + x\,;\quad \dfrac{dy}{dx} = -2x^{-2} + \dfrac{3}{2}x^{-\frac{3}{2}} - \dfrac{8}{3}x^{-\frac{5}{3}} + 1\,.$

■ **B:** $y = \ln\sqrt{\dfrac{x^2+1}{x^2-1}} = \dfrac{1}{2}\ln(x^2+1) - \dfrac{1}{2}\ln(x^2-1)\,;\quad \dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{2x}{x^2+1}\right) - \dfrac{1}{2}\left(\dfrac{2x}{x^2-1}\right) = -\dfrac{2x}{x^4-1}\,.$

6.1.2.2 Basic Rules of Differentiation

Assume u, v, w, and y are functions of the independent variable x, and u', v', w', and y' are the derivatives with respect to x. The differential is denoted by du, dv, dw, and dy (see 6.2.1.3, p. 446). The basic rules of differentiation, which are explained separately, are summarized in **Table 6.2**, p. 439.

1. Derivative of a Constant Function

The derivative of a constant function c is the zero function:

$$c' = 0. \tag{6.4}$$

2. Derivative of a Scalar Multiple

A constant factor c can be factored out from the differential sign:

$$(c\,u)' = c\,u', \quad d(c\,u) = c\,du. \tag{6.5}$$

3. Derivative of a Sum

If the functions u, v, w, etc. are differentiable one by one, their sum and difference is also differentiable, and equal to the sum or difference of the derivatives:

$$(u+v-w)' = u' + v' - w', \tag{6.6a}$$

$$d(u+v-w) = du + dv - dw. \tag{6.6b}$$

It is possible that the summands are not differentiable separately, but their sum or difference is. Then the derivative must be calculated by definition formula (6.1).

Table 6.1 Derivatives of elementary functions in the intervals on which they are defined and the occurring numerators are not equal to zero

Function	**Derivative**	**Function**	**Derivative**
C (constant)	0	$\sec x$	$\frac{\sin x}{\cos^2 x}$
x	1	$\operatorname{cosec} x$	$\frac{-\cos x}{\sin^2 x}$
$x^n \quad (n \in \mathbb{R})$	nx^{n-1}	$\arcsin x \quad (\lvert x\rvert < 1)$	$\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x} \quad (x \neq 0)$	$-\frac{1}{x^2} \quad (x \neq 0)$	$\arccos x \quad (\lvert x\rvert < 1)$	$-\frac{1}{\sqrt{1-x^2}}$
$\frac{1}{x^n} \quad (x \neq 0)$	$-\frac{n}{x^{n+1}}$	$\arctan x$	$\frac{1}{1+x^2}$
$\sqrt{x} \quad (x > 0)$	$\frac{1}{2\sqrt{x}}$	$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$
$\sqrt[n]{x} \quad (n \in \mathbb{R},\ n \neq 0,\ x > 0)$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$	$\operatorname{arcsec} x \quad (x > 1)$	$\frac{1}{x\sqrt{x^2-1}}$
e^x	e^x	$\operatorname{arccosec} x \quad (x > 1)$	$-\frac{1}{x\sqrt{x^2-1}}$
$e^{bx} \quad (b \in \mathbb{R})$	be^{bx}	$\sinh x$	$\cosh x$
$a^x \quad (a > 0)$	$a^x \ln a$	$\cosh x$	$\sinh x$
$a^{bx} \quad (b \in \mathbb{R},\ a > 0)$	$ba^{bx} \ln a$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\ln x \quad (x > 0)$	$\frac{1}{x}$	$\coth x \quad (x \neq 0)$	$-\frac{1}{\sinh^2 x}$
$\log_a x \quad (a > 0,\ a \neq 1,\ x > 0)$	$\frac{1}{x}\log_a e = \frac{1}{x\ln a}$	$\operatorname{Arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\lg x \quad (x > 0)$	$\frac{1}{x}\lg e \approx \frac{0.4343}{x}$	$\operatorname{Arcosh} x \quad (x > 1)$	$\frac{1}{\sqrt{x^2-1}}$
$\sin x$	$\cos x$	$\operatorname{Artanh} x \quad (\lvert x\rvert < 1)$	$\frac{1}{1-x^2}$
$\cos x$	$-\sin x$	$\operatorname{Arcoth} x \quad (\lvert x\rvert > 1)$	$-\frac{1}{x^2-1}$
$\tan x \quad (x \neq (2k+1)\frac{\pi}{2},\ k \in \mathbb{Z})$	$\frac{1}{\cos^2 x} = \sec^2 x$	$[f(x)]^n \quad (n \in \mathbb{R})$	$n[f(x)]^{n-1}f'(x)$
$\cot x \quad (x \neq k\pi,\ k \in \mathbb{Z})$	$\frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$	$\ln f(x) \quad (f(x) > 0)$	$\frac{f'(x)}{f(x)}$

4. Derivative of a Product

If two, three, or n functions are differentiable one by one, then their product is differentiable, and can be calculated as follows:

a) Derivative of the Product of Two Functions:

$$(u\,v)' = u'\,v + u\,v', \quad d(u\,v) = v\,du + u\,dv. \tag{6.7a}$$

It is possible that the terms are not differentiable separately, but their product is. Then the derivative must be calculated by definition formula (6.1).

b) Derivative of the Product of Three Functions:

$$(u\,v\,w)' = u'\,v\,w + u\,v'\,w + u\,v\,w', \quad d(u\,v\,w) = v\,w\,du + u\,w\,dv + u\,v\,dw. \tag{6.7b}$$

c) Derivative of the Product of n Functions:

$$(u_1\,u_2\,\cdots\,u_n)' = \sum_{i=1}^{n} u_1\,u_2\,\cdots\,u_i'\,\cdots\,u_n. \tag{6.7c}$$

■ **A:** $y = x^3 \cos x, \quad y' = 3x^2 \cos x - x^3 \sin x.$

■ **B:** $y = x^3 e^x \cos x, \quad y' = 3x^2 e^x \cos x + x^3 e^x \cos x - x^3 e^x \sin x.$

5. Derivative of a Quotient

If both u and v are differentiable, and $v(x) \neq 0$, their ratio is also differentiable:

$$\left(\frac{u}{v}\right)' = \frac{v\,u' - u\,v'}{v^2}, \quad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}. \tag{6.8}$$

■ $y = \tan x = \dfrac{\sin x}{\cos x}, \quad y' = \dfrac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}.$

6. Chain Rule

The composite function (see 2.1.5.5, **2.**, p. 61) $y = u(v(x))$ has the derivative

$$\frac{dy}{dx} = u'(v)\,v'(x) = \frac{du}{dv}\frac{dv}{dx}, \tag{6.9}$$

where the functions $u = u(v)$ and $v = v(x)$ must be differentiable functions with respect to their own variables. $u(v)$ is called the exterior function, and $v(x)$ is called the interior function. According to this, $\dfrac{du}{dv}$ is the *exterior derivative* and $\dfrac{dv}{dx}$ is the *interior derivative*. It is possible that the functions u and v are not differentiable separately, but the composite function is. Then one gets the derivative by the definition formula (6.1).

Similarly one has to proceed if there is a longer "chain", i.e., in the case of a composite function of several *intermediate variables*. For example for $y = u(v(w(x)))$:

$$y' = \frac{dy}{dx} = \frac{du}{dv}\,\frac{dv}{dw}\,\frac{dw}{dx}. \tag{6.10}$$

■ **A:** $y = e^{\sin^2 x}, \quad \dfrac{dy}{dx} = \dfrac{d\left(e^{\sin^2 x}\right)}{d\left(\sin^2 x\right)}\,\dfrac{d\left(\sin^2 x\right)}{d\,(\sin x)}\,\dfrac{d\,(\sin x)}{dx} = e^{\sin^2 x}\,2\,\sin x \cos x.$

■ **B:** $y = e^{\tan\sqrt{x}}; \quad \dfrac{dy}{dx} = \dfrac{d\left(e^{\tan\sqrt{x}}\right)}{d\,(\tan\sqrt{x})}\,\dfrac{d\,(\tan\sqrt{x})}{d\,(\sqrt{x})}\,\dfrac{d(\sqrt{x})}{dx} = e^{\tan\sqrt{x}}\,\dfrac{1}{\cos^2\sqrt{x}}\,\dfrac{1}{2\sqrt{x}}.$

7. Logarithmic Differentiation

If $y(x) > 0$ holds, one can calculate the derivative y' starting with the function $\ln y(x)$, whose derivative (considering the chain rule) is

$$\frac{d(\ln y(x))}{dx} = \frac{1}{y(x)}\,y'. \tag{6.11}$$

From this rule

$$y' = y(x)\,\frac{d(\ln y(x))}{dx} \tag{6.12}$$

follows.
Remark 1: With the help of logarithmic differentiation it is possible to simplify some differentiation problems, and there are functions such that this is the only way to calculate the derivative, for instance, when the function has the form

$$y = u(x)^{v(x)} \quad \text{with} \quad u(x) > 0. \tag{6.13}$$

The logarithmic differentiation of this equality follows from the formula (6.12)

$$y' = y\,\frac{d\,(\ln u^v)}{dx} = y\,\frac{d(v \ln u)}{dx} = u^v \left(v' \ln u + \frac{vu'}{u}\right). \tag{6.14}$$

■ $y = (2x+1)^{3x}$, $\ln y = 3x \ln(2x+1)$, $\dfrac{y'}{y} = 3\ln(2x+1) + \dfrac{3x \cdot 2}{2x+1}$;
$y' = 3\,(2x+1)^{3x}\left(\ln(2x+1) + \dfrac{2x}{2x+1}\right)$.

Remark 2: Logarithmic differentiation is often used in the case to differentiate a product of several functions.

■ **A:** $y = \sqrt{x^3 e^{4x} \sin x}$, $\quad \ln y = \dfrac{1}{2}(3\ln x + 4x + \ln \sin x)$,
$\dfrac{y'}{y} = \dfrac{1}{2}\left(\dfrac{3}{x} + 4 + \dfrac{\cos x}{\sin x}\right)$, $\quad y' = \dfrac{1}{2}\sqrt{x^3 e^{4x} \sin x}\left(\dfrac{3}{x} + 4 + \cot x\right)$.

■ **B:** $y = u\,v$, $\quad \ln y = \ln u + \ln v$, $\quad \dfrac{y'}{y} = \dfrac{1}{u}u' + \dfrac{1}{v}v'$. From this identity it follows that $y' = (u\,v)' = v\,u' + u\,v'$, so one gets the formula for the derivative of a product (6.7a) (under the assumption $u, v > 0$).

■ **C:** $y = \dfrac{u}{v}$, $\quad \ln y = \ln u - \ln v$, $\quad \dfrac{y'}{y} = \dfrac{1}{u}u' - \dfrac{1}{v}v'$. From this identity it follows that $y' = \left(\dfrac{u}{v}\right)' = \dfrac{u'}{v} - \dfrac{uv'}{v^2} = \dfrac{v\,u' - u\,v'}{v^2}$, which is the formula for the derivative of a quotient (6.8) (under the assumption $u, v > 0$).

8. Derivative of the Inverse Function

If $y = \varphi(x)$ is the inverse function of the original function $y = f(x)$, then both forms $y = f(x)$ and $x = \varphi(y)$ are equivalent. For every corresponding value of x and y such that f is differentiable with respect to x, and φ is differentiable with respect to y, e.g., none of the derivatives is equal to zero, between the derivatives of f and its inverse function φ is valid the following relation:

$$f'(x) = \frac{1}{\varphi'(y)} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}. \tag{6.15}$$

■ The function $y = f(x) = \arcsin x$ for $-1 < x < 1$ is equivalent to the function $x = \varphi(y) = \sin y$ for $-\pi/2 < y < \pi/2$. From (6.15) it follows that
$(\arcsin x)' = \dfrac{1}{(\sin y)'} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1 - \sin^2 y}} = \dfrac{1}{\sqrt{1 - x^2}}$, because $\cos y \neq 0$ for $-\pi/2 < y < \pi/2$.

9. Derivative of an Implicit Function

Suppose the function $y = f(x)$ is given in implicit form by the equation $F(x, y) = 0$. Considering the rules of differentiation for functions of several variables (see 6.2, p. 445) calculating the derivative with respect to x gives

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' = 0 \quad \text{and so} \quad y' = -\frac{F_x}{F_y}, \tag{6.16}$$

if the partial derivative F_y differs from zero.

■ The equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ of an ellipse with semi-axes a and b can be written in the form $F(x, y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$. For the slope of the tangent line at the point of the ellipse (x, y) one gets according to (6.16)

$$y' = -\frac{2x}{a^2}\Big/\frac{2y}{b^2} = -\frac{b^2}{a^2}\frac{x}{y}.$$

10. Derivative of a Function Given in Parametric Form

If a function $y = f(x)$ is given in parametric form $x = x(t)$, $y = y(t)$, then the derivative y' can be calculated by the formula

$$\frac{dy}{dx} = f'(x) = \frac{\dot{y}}{\dot{x}} \tag{6.17}$$

with the help of the derivatives $\dot{y}(t) = \dfrac{dy}{dt}$ and $\dot{x}(t) = \dfrac{dx}{dt}$ with respect to the variable t, if of course $\dot{x}(t) \neq 0$ holds.

■ **Polar Coordinate Representation:** If a function is given with polar coordinates (see 3.5.2.2, **3.**, p. 192) $\rho = \rho(\varphi)$, then the parametric form is

$$x = \rho(\varphi)\cos\varphi, \quad y = \rho(\varphi)\sin\varphi \tag{6.18}$$

with the angle φ as a parameter. For the slope y' of the tangent of the curve (see 3.6.1.2, **2.**, p. 245 or 6.1.1, **2.**, p. 432) one gets from (6.17)

$$y' = \frac{\dot{\rho}\sin\varphi + \rho\cos\varphi}{\dot{\rho}\cos\varphi - \rho\sin\varphi} \quad \text{where } \dot{\rho} = \frac{d\rho}{d\varphi}. \tag{6.19}$$

Remarks:

1. The derivatives $\dot{x}$, $\dot{y}$ are the components of the tangent vector at the point $(x(t), y(t))$ of the curve.

2. It is often useful to consider the complex relation:

$$x(t) + \mathrm{i}\,y(t) = z(t), \quad \dot{x}(t) + \mathrm{i}\,\dot{y}(t) = \dot{z}(t). \tag{6.20}$$

■ **Circular Movement:** $z(t) = re^{\mathrm{i}\omega t}$ $(r, \omega \text{ const})$, $\dot{z}(t) = r\mathrm{i}\omega e^{\mathrm{i}\omega t} = r\omega e^{\mathrm{i}(\omega t + \frac{\pi}{2})}$. The tangent vector runs ahead by a phase-shift $\pi/2$ with respect to the position vector.

11. Graphical Differentiation

If a differentiable function $y = f(x)$ is represented by its curve Γ in the Cartesian coordinate system in an interval $a < x < b$, then the curve Γ' of its derivative can be constructed approximately. The construction of a tangent estimated by eye is pretty inaccurate. However, if the direction of the tangent MN **(Fig. 6.4)** is given, then one can determine the point of contact A more precisely.

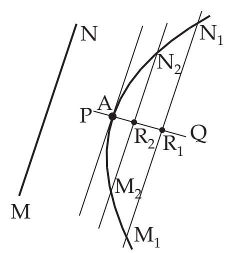

Figure 6.4

1. Construction of the Point of Contact of a Tangent

One draws two secants $\overline{M_1N_1}$ and $\overline{M_2N_2}$ parallel to the direction MN of the tangent so that the curve is intersected in points being not far from each other. Then there are to be determine the midpoints of the secants, and a straight line through them must be drawn. This line PQ intersects the curve at the point A, which is approximately the point, where the tangent has the given direction MN. To check the accuracy, one draws a third line close to and parallel to the first two lines, and the line PQ should intersect it at the midpoint.

2. Construction of the Derivative Curve

a) Choose some directions $l_1, l_2, \ldots, l_n$ which could be the directions of some tangents of the curve

$y = f(x)$ in the considered interval as in **Fig. 6.5**, and determine the corresponding points of contact $A_1, A_2, \ldots, A_n$, where the tangents themselves must not be constructed.

b) Choose a point P, a "pole", on the negative side of the x-axis, where the longer the segment $PO = a$, the flatter the curve is.

c) Draw the lines through the pole P parallel to the directions $l_1, l_2, \ldots l_n$, and denote their intersection points with the y-axis by $B_1, B_2, \ldots B_n$.

d) Construct the horizontal lines $B_1C_1, B_2C_2, \ldots, B_nC_n$ through the points $B_1, B_2, \ldots, B_n$ to the intersection points $C_1, C_2, \ldots, C_n$ with the orthogonal lines from the points $A_1, A_2, \ldots, A_n$.

e) Connect the points $C_1, C_2, \ldots, C_n$ with the help of a curved ruler. The resulting curve satisfies the equation $y = af'(x)$. If the segment a is chosen so that it corresponds to the unit length on the y-axis, then the curve one gets is the curve of the derivative. Otherwise, one has to multiply the ordinates of $C_1, C_2, \ldots, C_n$ by the factor $1/a$. The points $D_1, D_2, \ldots, D_n$ given in **Fig. 6.5** are on the correctly scaled curve Γ' of the derivative.

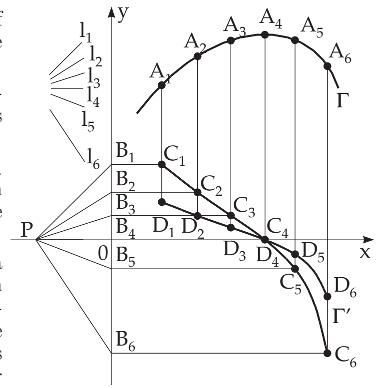

Figure 6.5

6.1.3 Derivatives of Higher Order

6.1.3.1 Definition of Derivatives of Higher Order

The derivative of $y' = f'(x)$, which means $(y')'$ or $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, is called the second derivative of the function $y = f(x)$ and it is denoted by $y'', \ddot{y}, \dfrac{d^2y}{dx^2}, \; f''(x)$ or $\dfrac{d^2f(x)}{dx^2}$. Higher derivatives can be defined analogously. The notation for the n-th *derivative* of the function $y = f(x)$ is:

$$y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x) = \frac{d^nf(x)}{dx^n} \qquad \left(n = 0, 1, \ldots\, ; \; y^{(0)}(x) = f^{(0)}(x) = f(x)\right). \tag{6.21}$$

6.1.3.2 Derivatives of Higher Order of some Elementary Functions

The n-th derivatives of the simplest functions are collected in **Table 6.3**.

6.1.3.3 Leibniz's Formula

To calculate the n-th-order derivative of a product of two functions, the Leibniz formula can be used:

$$\begin{aligned} D^n(uv) &= u\,D^nv + \frac{n}{1!}Du\,D^{n-1}v + \frac{n(n-1)}{2!}D^2u\,D^{n-2}v + \cdots \\ &\quad + \frac{n(n-1)\ldots(n-m+1)}{m!}D^mu\,D^{n-m}v + \cdots + D^nu\,v \end{aligned} \tag{6.22}$$

Here the notation $D^n = \dfrac{d^n}{dx^n}$ is used. If D^0u is replaced by u and D^0v by v, then one gets the formula (6.23) whose structure corresponds to the binomial formula (see 1.1.6.4, p. 12):

$$D^n(uv) = \sum_{m=0}^{n} \binom{n}{m} D^muD^{n-m}v\,. \tag{6.23}$$

Table 6.2 Differentiation rules

Expression	**Formula for the derivative**
Constant function	$c' = 0 \quad (c \text{ const})$
Constant multiple	$(cu)' = cu' \quad (c \text{ const})$
Sum	$(u \pm v)' = u' \pm v'$
Product of two functions	$(uv)' = u'v + uv'$
Product of n functions	$(u_1 u_2 \cdots u_n)' = \sum\limits_{i=1}^{n} u_1 \cdots u_i' \cdots u_n$
Quotient	$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \quad (v \neq 0)$
Chain rule for two functions	$y = u(v(x)): \quad y' = \frac{du}{dv}\frac{dv}{dx}$
Chain rule for three functions	$y = u(v(w(x))): \quad y' = \frac{du}{dv}\frac{dv}{dw}\frac{dw}{dx}$
Power	$(u^\alpha)' = \alpha u^{\alpha-1}u' \quad (\alpha \in \mathbb{R},\ \alpha \neq 0)$ specially : $\left(\frac{1}{u}\right)' = -\frac{u'}{u^2} \quad (u \neq 0)$
Logarithmic differentiation	$\frac{d(\ln y(x))}{dx} = \frac{1}{y}y' \Longrightarrow y' = y\frac{d(\ln y)}{dx}$ special : $(u^v)' = u^v\left(v' \ln u + \frac{vu'}{u}\right) \quad (u > 0)$
Differentiation of the inverse function	φ inverse function of f, i.e. $y = f(x) \Longleftrightarrow x = \varphi(y)$: $f'(x) = \frac{1}{\varphi'(y)}$ or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$
Implicit differentiation	$F(x, y) = 0: \quad F_x + F_y y' = 0$ or $y' = -\frac{F_x}{F_y} \quad \left(F_x = \frac{\partial F}{\partial x},\ F_y = \frac{\partial F}{\partial y};\ F_y \neq 0\right)$
Derivative in parameter form	$x = x(t),\ y = y(t)$ (t parameter): $y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \quad \left(\dot{x} = \frac{dx}{dt},\ \dot{y} = \frac{dy}{dt}\right)$
Derivative in polar coordinates	$r = r(\varphi): \quad \begin{matrix} x = \rho(\varphi)\cos\varphi \\ y = \rho(\varphi)\sin\varphi \end{matrix}$ (angle φ as parameter) $y' = \frac{dy}{dx} = \frac{\dot{\rho}\sin\varphi + \rho\cos\varphi}{\dot{\rho}\cos\varphi - r\sin\varphi} \quad \left(\dot{\rho} = \frac{dx}{d\varphi}\right)$

■ **A:** $(x^2 \cos ax)^{(50)}$: If $v = x^2,\ u = \cos ax$ are substituted, then follows $u^{(k)} = a^k \cos\left(ax + k\frac{\pi}{2}\right),\ v' = 2x,\ v'' = 2,\ v''' = v^{(4)} = \cdots = 0$. Except the first three cases, all the summands are equal to zero, so $(uv)^{(50)} = x^2 a^{50} \cos\left(ax + 50\frac{\pi}{2}\right) + \frac{50}{1} \cdot 2xa^{49} \cos\left(ax + 49\frac{\pi}{2}\right) + \frac{50 \cdot 49}{1 \cdot 2} \cdot 2a^{48} \cos\left(ax + 48\frac{\pi}{2}\right)$

$= a^{48}[(2450 - a^2x^2)\cos ax - 100ax \sin ax]\,.$

■ **B:** $(x^3e^x)^{(6)} = \binom{6}{0} \cdot x^3e^x + \binom{6}{1} \cdot 3x^2e^x + \binom{6}{2} \cdot 6xe^x + \binom{6}{3} \cdot 6e^x = e^x(x^3 + 18x^2 + 90y + 120).$

Table 6.3 Derivatives of higher order of some elementary functions

Function	***n*-th-order derivative**
x^m	$m(m-1)(m-2)\ldots(m-n+1)x^{m-n}$ (for integer m and $n > m$ the n-th derivative is 0)
$\ln x \;\; (x > 0)$	$(-1)^{n-1}(n-1)!\,\dfrac{1}{x^n}$
$\log_a x \;\; (x > 0)$	$(-1)^{n-1}\dfrac{(n-1)!}{\ln a}\dfrac{1}{x^n}$
e^{kx}	$k^n e^{kx}$
a^x	$(\ln a)^n a^x$
a^{kx}	$(k \ln a)^n a^{kx}$
$\sin x$	$\sin(x + \dfrac{n\pi}{2})$
$\cos x$	$\cos(x + \dfrac{n\pi}{2})$
$\sin kx$	$k^n \sin(kx + \dfrac{n\pi}{2})$
$\cos kx$	$k^n \cos(kx + \dfrac{n\pi}{2})$
$\sinh x$	$\sinh x$ for even n, $\cosh x$ for odd n
$\cosh x$	$\cosh x$ for even n, $\sinh x$ for odd n

6.1.3.4 Higher Derivatives of Functions Given in Parametric Form

If a function $y = f(x)$ is given in the parametric form $x = x(t)$, $y = y(t)$, then its higher derivatives (y'', y''', etc.) can be calculated by the following formulas, where $\dot{y}(t) = \dfrac{dy}{dt}$, $\dot{x}(t) = \dfrac{dx}{dt}$, $\ddot{y}(t) = \dfrac{d^2y}{dt^2}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, etc., denote the derivatives with respect to the parameter t:

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}, \quad \frac{d^3y}{dx^3} = \frac{\dot{x}^2\dddot{y} - 3\dot{x}\ddot{x}\ddot{y} + 3\dot{y}\ddot{x}^2 - \dot{x}\dot{y}\dddot{x}}{\dot{x}^5}, \ldots \quad (\dot{x}(t) \neq 0). \tag{6.24}$$

6.1.3.5 Derivatives of Higher Order of the Inverse Function

If $y = \varphi(x)$ is the inverse function of the original function $y = f(x)$, then both forms $y = f(x)$ and $x = \varphi(y)$ are equivalent. Supposing $\varphi'(y) \neq 0$ holds, the relation (6.15) is valid for the derivatives of the function f and its inverse function φ. For higher derivatives (y'', y''', etc.) one gets

$$\frac{d^2y}{dx^2} = -\frac{\varphi''(y)}{[\varphi'(y)]^3}, \quad \frac{d^3y}{dx^3} = \frac{3[\varphi''(y)]^2 - \varphi'(y)\varphi'''(y)}{[\varphi'(y)]^5}, \ldots\,. \tag{6.25}$$

6.1.4 Fundamental Theorems of Differential Calculus

6.1.4.1 Monotonicity

If a function $f(x)$ is defined and continuous in a connected interval, and if it is differentiable at every interior point of this interval, then the relations

$$f'(x) \geq 0 \quad \text{for a monotone increasing function,} \tag{6.26a}$$

$$f'(x) \leq 0 \quad \text{for a monotone decreasing function} \tag{6.26b}$$

are necessary and sufficient. If the function is strictly monotone increasing or decreasing, then the derivative function $f'(x)$ must not be identically zero on any subinterval of the given interval. In **Fig. 6.6b** this condition is not fulfilled on the segment $\overline{BC}$.

The geometrical meaning of monotonicity is that the curve of an increasing function never falls for increasing values of the argument, i.e., it either rises or runs horizontally **(Fig. 6.6a)**. Therefore the tangent line at any point of the curve forms an acute angle with the positive x-axis or it is parallel to it. For monotonically decreasing functions **(Fig. 6.6b)** analogous statements are valid. If the function is strictly monotone, then the tangent can be parallel to the x-axis only at some single points, e.g., at the point A in **Fig. 6.6a**, i.e., not on a subinterval such as $\overline{BC}$ in **Fig. 6.6b**.

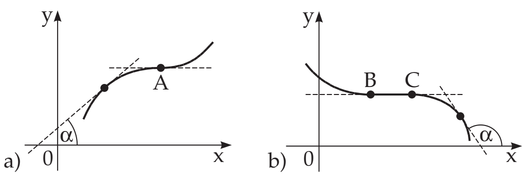

Figure 6.6

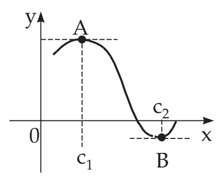

Figure 6.7

6.1.4.2 Fermat's Theorem

If a function $y = f(x)$ is defined on a connected interval, and it has a maximum or a minimum value at an interior point $x = c$ of this interval **(Fig. 6.7)**, i.e., if for every x in this interval

$$f(c) > f(x) \quad (6.27a) \qquad \text{or} \qquad f(c) < f(x), \quad (6.27b)$$

holds, and if the derivative exists at the point c, then the derivative must be equal to zero there:

$$f'(c) = 0. \tag{6.27c}$$

The geometrical meaning of the Fermat theorem is that if a function satisfies the assumptions of the theorem, then its curve has tangents parallel to the x-axis at A and B **(Fig. 6.7)**.

The Fermat theorem gives only a necessary condition for the existence of a maximum or minimum value at a point. From **Fig. 6.6a** it is obvious that having a zero derivative is not sufficient to give an extreme value: At the point A, $f'(x) = 0$ holds, but there is no maximum or minimum here.

To have an extreme value differentiability is not a necessary condition. The function in **Fig. 6.8d** has a maximum at e, but the derivative does not exist here.

6.1.4.3 Rolle's Theorem

If a function $y = f(x)$ is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), and

$$f(a) = 0, \quad f(b) = 0 \quad (a < b) \tag{6.28a}$$

hold, then there exists at least one point c between a and b such that

$$f'(c) = 0 \quad (a < c < b) \tag{6.28b}$$

holds. The geometrical meaning of Rolle's theorem is that if the graph of a function $y = f(x)$ which is continuous on the interval (a, b) intersects the x-axis at two points A and B, and it has a non-vertical tangent at every point, then there is at least one point C between A and B such that the tangent is parallel to the x-axis here **(Fig. 6.8a)**. It is possible, that there are several such points in this inter-

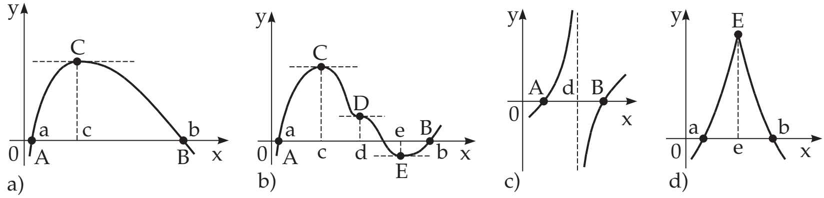

Figure 6.8

val, e.g., the points C, D, and E in **Fig. 6.8b**. The properties of continuity and differentiability are important in the theorem: in **Fig. 6.8c** the function is not continuous at $x = d$, and in **Fig. 6.8d** the function is not differentiable at $x = e$. In both cases $f'(x) \neq 0$ holds everywhere where the derivative exists.

6.1.4.4 Mean Value Theorem of Differential Calculus

If a function $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one point c between a and b which satisfies the following relation:

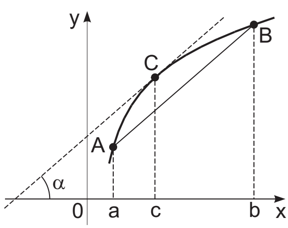

Figure 6.9

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad (a < c < b) \tag{6.29a}$$

holds. Substituting $b = a + h$, and Θ means a number between 0 and 1, then the theorem can be written in the form

$$f(a + h) = f(a) + h\, f'(a + \Theta\, h) \quad (0 < \Theta < 1). \tag{6.29b}$$

1. Geometrical Meaning The geometrical meaning of the theorem is that if a function $y = f(x)$ satisfies the conditions of the theorem, then its graph has at least one point C between A and B such that the tangent line at this point is parallel to the line segment between A and B **(Fig. 6.9)**. There can be several such points **(Fig. 6.8b)**.

That the properties of continuity and differentiability are important can be shown in examples and also as can be observed in **Fig. 6.8c,d**.

2. Applications The mean value theorem has several useful applications.

■ **A:** This theorem can be used to prove some inequalities in the form

$$|f(b) - f(a)| < K|b - a|, \tag{6.30}$$

where K is an upper bound of $|f'(x)|$ for every x in the interval $[a, b]$.

■ **B:** How accurate is the value of $f(\pi) = \dfrac{1}{1 + \pi^2}$ if π is replaced by the approximate value $\overline{\pi} = 3.14$?

We have: $|f(\pi) - f(\overline{\pi})| = \left|\dfrac{2c}{(1 + c^2)^2}\right| |\pi - \overline{\pi}| \leq 0.053 \cdot 0.0016 = 0.000\,085$, which means $\dfrac{1}{1 + \pi^2}$ is between $0.092\,084 \pm 0.000\,085$.

6.1.4.5 Taylor's Theorem of Functions of One Variable

If a function $y = f(x)$ is continuously differentiable (it has continuous derivatives) $n - 1$ times on the interval $[a, a + h]$, and if also the n-th derivative exists in the interior of the interval, then the Taylor formula or Taylor expansion is

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\Theta h) \tag{6.31}$$

with $0 < \Theta < 1$. The quantity h can be positive or negative. The mean value theorem (6.29b) is a special case of the Taylor formula for $n = 1$.

6.1.4.6 Generalized Mean Value Theorem of Differential Calculus (Cauchy's Theorem)

If two functions $y = f(x)$ and $y = \varphi(x)$ are continuous on the closed interval $[a, b]$ and they are differentiable at least in the interior of the interval, and $\varphi'(x)$ is never equal to zero in this interval, then there exists at least one value c between a and b such that

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(c)}{\varphi'(c)} \quad (a < c < b). \tag{6.32}$$

The geometrical meaning of the generalized mean value theorem corresponds to that of the first mean value theorem. Supposing, e.g., that the curve in **Fig.6.9** is given in parametric form $x = \varphi(t),\ y = f(t)$, where the points A and B belong to the parameter values $t = a$ and $t = b$ respectively. Then for the point C

$$\tan\alpha = \frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(c)}{\varphi'(c)} \tag{6.33}$$

is valid. For $\varphi(x) = x$ the generalized mean value theorem is simplified into the first mean value theorem.

6.1.5 Determination of the Extreme Values and Inflection Points

6.1.5.1 Maxima and Minima

The substitution value $f(x_0)$ of a function $f(x)$ is called the *relative maximum* (M) or *relative minimum* (m) if one of the inequalities

$$f(x_0 + h) < f(x_0) \quad \text{(for maximum)}, \tag{6.34a}$$

$$f(x_0 + h) > f(x_0) \quad \text{(for minimum)} \tag{6.34b}$$

holds for arbitrary positive or negative values of h small enough. At a relative maximum the value $f(x_0)$ is greater than the values in the neighborhood, and similarly, at a minimum it is smaller. The relative maxima and minima are called *relative* or *local extrema*. The greatest or the smallest value of a function in an interval is called the *global* or *absolute maximum* or *global* or *absolute minimum* in this interval.

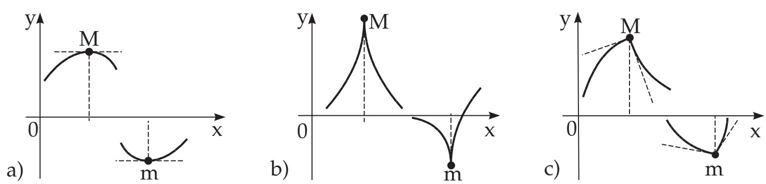

Figure 6.10

6.1.5.2 Necessary Conditions for the Existence of a Relative Extreme Value

A function can have a relative maximum or minimum only at the points where its derivative is equal to zero or does not exist. That is: At the points of the graph of the function corresponding to the relative extrema the tangent line is whether parallel to the x-axis **(Fig. 6.10a)** or parallel to the y-axis

(Fig. 6.10b) or does not exist **(Fig. 6.10c)**. Anyway, these are not sufficient conditions, e.g., at the points A, B, C in **Fig. 6.11** these conditions are obviously fulfilled, but there are no extreme values of the function.
If a continuous function has relative extreme values, then maxima and minima follow alternately, that means, between two neighboring maxima there is a minimum, and conversely.

6.1.5.3 Determination of the Relative Extreme Values and the Inflection Points of a Differentiable Explicit Function $y = f(x)$

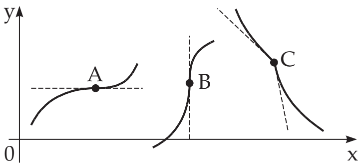

Figure 6.11

Since $f'(x) = 0$ is a necessary condition where the derivative exists, after determining the derivative $f'(x)$, first one calculates all the real roots $x_1, x_2, \ldots, x_i, \ldots, x_n$ of the equation $f'(x) = 0$. Then each of them has to be checked , e.g., x_i with one of the following methods.

1. Method of Sign Change
For values x_- and x_+, which are slightly smaller and greater than x_i, and for which between x_i and x_- and x_+ no more roots or points of discontinuity of $f'(x)$ exist, one checks the sign of $f'(x)$. When during the transition from $f'(x_-)$ to $f'(x_+)$ the sign of $f'(x)$ changes from "+" to "−", then there is a relative maximum of the function $f(x)$ at $x = x_i$ **(Fig. 6.12a)**; if it changes from "−" to "+", then there is a relative minimum there **(Fig. 6.12b)**. If the derivative does not change its sign **(Fig. 6.12c,d)**, then there is no extremum at $x = x_i$, but it has an inflection point with a tangent parallel to the x-axis.

2. Method of Higher Derivatives
If a function has higher derivatives at $x = x_i$, then one can substitute, e.g., the root x_i into the second derivative $f''(x)$. If $f''(x_i) < 0$ holds, then there is a relative maximum at x_i, and if $f''(x_i) > 0$ holds, a relative minimum. If $f''(x_i) = 0$ holds, then x_i must be substituted into the third derivative $f'''(x)$. If $f'''(x_i) \neq 0$ holds, then there is no extremum at $x = x_i$ but an inflection point. If still $f'''(x_i) = 0$ holds, then one substitutes it into the forth derivative, etc. If the first non-zero derivative at $x = x_i$ is an even one, then $f(x)$ has an extremum here: If the derivative is positive, then there is minimum, if it is negative, then there is a maximum. If the first non-zero derivative is an odd one, then there is no extremum there (actually, there is an inflection point).

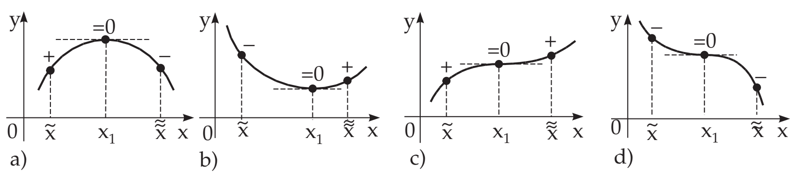

Figure 6.12

3. Further Conditions for Extreme Points and Determination of Inflection Points
If a continuous function is increasing below x_0 and decreasing after, then it has a maximum there; if it is decreasing below and increasing after, then it has a minimum there. Checking the sign change of the derivative is a useful method even if the derivative does not exist at certain points as in **Fig. 6.10b,c** and **Fig. 6.11**. If the first derivative exists at a point where the function has an inflection point, then the first derivative has an extremum there. So, to find the inflection points with the help of derivatives, one has to do the same investigation for the derivative function as one has done for the original function to find its extrema.

Remark: For non-continuous functions, and sometimes also for certain differentiable functions the determination of extrema needs individual ideas. It is possible that a function has an extremum so that the first derivative exists and it is equal to zero, but the second derivative does not exist, and the first one has infinitely many roots in an arbitrary neighborhood of the considered point, so it is meaningless to say it changes its sign there. For instance $f(x) = x^2(2 + \sin(1/x))$ for $x \neq 0$ and $f(0) = 0$.

6.1.5.4 Determination of Absolute Extrema

The considered interval of the independent variable is divided into subintervals such that in these intervals the function has a continuous derivative. The absolute extreme values are among the relative extreme values, or at the endpoints of the subintervals, if their endpoints belong to them. For non-continuous functions or for non-closed intervals it is possible that no maximum or minimum exists on the considered interval.

Examples of the Determination of Extrema:

■ **A:** $y = e^{-x^2}$, interval $[-1, +1]$. Greatest value at $x = 0$, smallest at the endpoints **(Fig. 6.13a).**

■ **B:** $y = x^3 - x^2$, interval $[-1, +2]$. Greatest value at $x = +2$, smallest at $x = -1$, at the ends of the interval **(Fig. 6.13b)**.

■ **C:** $y = \dfrac{1}{1 + e^{\frac{1}{x}}}$, interval $[-3, +3]$, $x \neq 0$. There is no maximum or minimum. Relative minimum at $x = -3$, relative maximum at $x = 3$. If one defines $y = 1$ for $x = 0$, then there will be an absolute maximum at $x = 0$ (**Fig. 6.13c).**

■ **D:** $y = 2 - x^{\frac{2}{3}}$, interval $[-1, +1]$. Greatest value at $x = 0$ (**Fig. 6.13d**, the derivative is not finite).

6.1.5.5 Determination of the Extrema of Implicit Functions

If the function is given in the implicit form $F(x, y) = 0$, and the function F itself and also its partial derivatives F_x, F_y are continuous, then its maxima and minima can be determined in the following way:

1. Solution of the Equation System $\boldsymbol{F(x, y) = 0, F_x(x, y) = 0}$ and substitution of the resulting values $(x_1, y_1), (x_2, y_2), \ldots, (x_i, y_i), \ldots$ in F_y and F_{xx}.

2. Sign Comparison for $\boldsymbol{F_y}$ and $\boldsymbol{F_{xx}}$ at the Point $\boldsymbol{(x_i, y_i)}$: When they have different signs, the function $y = f(x)$ has a minimum at x_i; when F_y and F_{xx} have the same sign, then it has a maximum at x_i. If either F_y or F_{xx} vanishes at (x_i, y_i), then one needs further and rather complicated investigation.

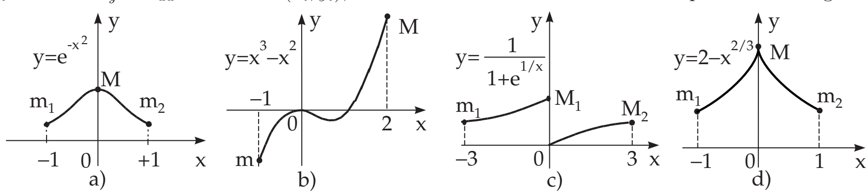

Figure 6.13

6.2 Differentiation of Functions of Several Variables

6.2.1 Partial Derivatives

6.2.1.1 Partial Derivative of a Function

The partial derivative of a function $u = f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ with respect to one of its n variables, e.g., with respect to x_1 is defined by

$$\frac{\partial u}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)}{\Delta x_1}, \tag{6.35}$$

so only one of the n variables is changing, the other $n-1$ are considered as constants. The symbols for the partial derivatives are $\frac{\partial u}{\partial x}$, u'_x, $\frac{\partial f}{\partial x}$, f'_x. A function of n variables can have n first-order partial derivatives: $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \ldots, \frac{\partial u}{\partial x_n}$. The calculation of the partial derivatives can be done following the same rules as there are for the functions of one variable.

■ $u = \frac{x^2 y}{z}, \quad \frac{\partial u}{\partial x} = \frac{2xy}{z}, \quad \frac{\partial u}{\partial y} = \frac{x^2}{z}, \quad \frac{\partial u}{\partial z} = -\frac{x^2 y}{z^2}.$

6.2.1.2 Geometrical Meaning for Functions of Two Variables

If a function $u = f(x, y)$ is represented as a surface in a Cartesian coordinate system, and this surface is intersected through its point P by a plane parallel to the x, u plane **(Fig. 6.14)**, then holds

$$\frac{\partial u}{\partial x} = \tan\alpha, \tag{6.36a}$$

where α is the angle between the positive x-axis and the tangent line of the intersection curve at P, which is the same as the angle between the positive x-axis and the perpendicular projection of the tangent line into the x, u plane. Here, α is measured starting at the x-axis, and the positive direction is counterclockwise if looking toward the positive half of the y-axis. Analogously to α, β is defined with a plane parallel to the y, u plane:

$$\frac{\partial u}{\partial y} = \tan\beta. \tag{6.36b}$$

The derivative with respect to a given direction, the so-called *directional derivative*, and *derivative with respect to volume*, will be discussed in vector analysis (see 13.2.1, p. 708 and p. 709).

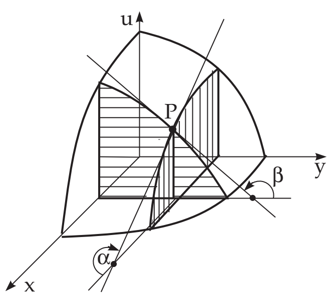

Figure 6.14

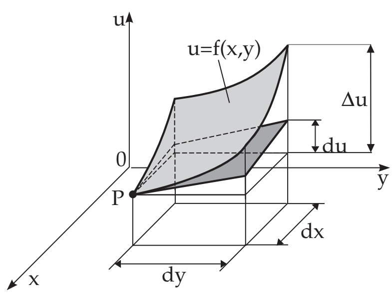

Figure 6.15

6.2.1.3 Differentials of x and $f(x)$

1. The Differential dx of an Independent Variable x
is equal to the increment Δx, i.e.,

$$dx = \Delta x \tag{6.37a}$$

for an arbitrary value of Δx.

2. The Differential dy of a Function $y = f(x)$ of One Variable x
is defined for a given value of x and for a given value of the differential dx as the product

$$dy = f'(x)\, dx. \tag{6.37b}$$

3. The Increment of a Function $y = f(x)$ for $x + \Delta x$
is the difference

$$\Delta y = f(x + \Delta x) - f(x). \tag{6.37c}$$

4. Geometrical Meaning of the Differential
If the function is represented by a curve in a Cartesian coordinate system, then dy is the increment of the ordinate of the tangent line for the change of x by a given increment dx **(Fig. 6.1)**. In an analogous way Δy is the increment of the ordinate of the curve.

6.2.1.4 Basic Properties of the Differential

1. Invariance
Independently of whether x is an independent variable or a function of a further variable t

$$dy = f'(x)\,dx \tag{6.38}$$

is valid.

2. Order of Magnitude
If dx is an arbitrarily small value, then dy and $\Delta y = y(x + \Delta x) - y(x)$ are also arbitrarily small, but of equivalent amounts, i.e., $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{dy} = 1$. Consequently, the difference between them is also arbitrarily small, but of higher order than dx, dy and Δx (except if $dy = 0$ holds). Therefore, one gets the relation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{dy} = 1, \quad \Delta y \approx dy = f'(x)\,dx, \tag{6.39}$$

which allows to reduce the calculation of a small *increment* to the calculation of its differential. This formula is frequently used for approximate calculations (see 6.1.4.4, p. 442 and 16.4.2.1, **2.**, p. 855).

6.2.1.5 Partial Differential

For a function of several variables $u = f(x, y, \ldots)$ one can form the partial differential with respect to one of its variables, e.g., with respect to x, which is defined by the equality

$$d_x u = d_x f = \frac{\partial u}{\partial x}\,dx. \tag{6.40}$$

6.2.2 Total Differential and Differentials of Higher Order

6.2.2.1 Notion of Total Differential of a Function of Several Variables (Complete Differential)

1. Differentiability
The function of several variable $u = f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ is called *differentiable* at the point $P_0(x_{10}, x_{20}, \ldots, x_{i0}, \ldots, x_{n0})$ if at a transition to an arbitrarily close point $P(x_{10}+\Delta x_1, x_{20}+\Delta x_2, \ldots, x_{i0} +\Delta x_i, \ldots, x_{n0} + \Delta x_n)$ with the arbitrarily small quantities $\Delta x_1, \Delta x_2, \ldots, \Delta x_i, \ldots, \Delta x_n$ the complete increment

$$\begin{aligned}\Delta u = {} & f(x_{10} + \Delta x_1, x_{20} + \Delta x_2, \ldots, x_{i0} + \Delta x_i, \ldots, x_{n0} + \Delta x_n) \\ & - f(x_{10}, x_{20}, \ldots, x_{i0}, \ldots, x_{n0})\end{aligned} \tag{6.41a}$$

of the function differs from the sum of the partial differentials of all variables

$$\left(\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_n} dx_n\right)_{x_{10}, x_{20}, \ldots, x_{n0}} \tag{6.41b}$$

by an arbitrarily small amount in higher order than the distance

$$\overline{P_0 P} = \sqrt{\Delta x_1^2 + \Delta x_2^2 + \ldots + \Delta x_n^2} = \sqrt{dx_1^2 + dx_2^2 + \ldots + dx_n^2}\,. \tag{6.41c}$$

A continuous function of several variables is differentiable at a point if its partial derivatives, as functions of several variables, are continuous in a neighborhood of this point. This is a sufficient but not a necessary condition, while the simple existence of the partial derivatives at the considered point is not sufficient even for the continuity of the function.

2. Total Differential

If u is a differentiable function, then the sum (6.41b)

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \ldots + \frac{\partial u}{\partial x_n}dx_n \tag{6.42a}$$

is called the *total differential* of the function. With the n-dimensional vectors

$$\underline{\mathbf{grad\,u}} = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)^{\mathrm{T}}, \quad (6.42b) \qquad \underline{\mathbf{dr}} = (dx_1, dx_2, \ldots, dx_n)^{\mathrm{T}} \tag{6.42c}$$

the total differential can be expressed as the scalar product

$$du = (\underline{\mathbf{grad\,u}})^{\mathrm{T}} \cdot \underline{\mathbf{dr}}. \tag{6.42d}$$

In (6.42b), there is the gradient, defined in 13.2.2, p. 710, for n independent variables.

3. Geometrical Representation

The geometrical meaning of the total differential of a function of two variables $u = f(x, y)$, represented in a Cartesian coordinate system as a surface **(Fig. 6.15)**, is that du is the same as the increment of the applicate (see 3.5.3.1, **3.**, p. 210) of the tangent plane (at the same point) if dx and dy are the increments of x and y.

From the Taylor formula (see 6.2.2.3, **1.**, p. 449) it follows for functions of two variables that

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + R_1. \tag{6.43a}$$

Ignoring the remainder R_1, holds that

$$u = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \tag{6.43b}$$

gives the equation of the tangent plane of the surface $u = f(x, y)$ at the point $P_0(x_0, y_0, u_0)$.

4. The Fundamental Property of the Total Differential

is the invariance with respect to the variables as formulated in (6.38) for the one-variable case.

5. Application in Error Calculations

In error calculations one uses the total differential du for an estimation of the error Δu (see (6.41a)) (see, e.g., 16.4.1.3, **5.**, p. 852). From the Taylor formula (see 6.2.2.3, **1.**, p. 449) follows

$$|\Delta u| = |du + R_1| \le |du| + |R_1| \approx |du|, \tag{6.44}$$

i.e., the absolute error $|\Delta u|$ can be replaced by $|du|$ as a first approximation. It follows that du is a linear approximation for Δu.

6.2.2.2 Derivatives and Differentials of Higher Order

1. Partial Derivatives of Second Order, Schwarz's Exchange Theorem

The second-order partial derivative of a function $u = f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ can be calculated with respect to the same variable as the first one was, i.e., $\frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \ldots$, or with respect to another variable, i.e. $\frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^2 u}{\partial x_2 \partial x_3}, \frac{\partial^2 u}{\partial x_3 \partial x_1}, \ldots$. In this second case one talks about mixed derivatives.

If at the considered point the mixed derivatives are continuous, then

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1} \tag{6.45}$$

holds for given x_1 and x_2 independently of the order of sequence of the differentiation (*Schwarz's exchange theorem*).

Partial derivatives of higher order such as, e.g., $\frac{\partial^3 u}{\partial x^3}$, $\frac{\partial^3 u}{\partial x \partial y^2}$, ... are defined analogously.

2. Second-Order Differential of a Function of One Variable $y = f(x)$

The second-order differential of a function $y = f(x)$ of one variable, denoted by the symbols d^2y, $d^2f(x)$, is the differential of the first differential: $d^2y = d(dy) = f''(x)dx^2$. These symbols are appropriate only if x is an independent variable, and they are not appropriate if x is given, e.g., in the form $x = z(v)$. Differentials of higher order are defined analogously. If the variables $x_1, x_2, \ldots, x_i, \ldots, x_n$ are themselves functions of other variables, then one gets more complicated formulas (see 6.2.4, p. 452).

3. Total Differential of Second Order of a Function of Two Variables $u = f(x, y)$

$$d^2u = d(du) = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}\,dx\,dy + \frac{\partial^2 u}{\partial y^2}\,dy^2 \tag{6.46a}$$

or symbolically

$$d^2u = \left(\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}dy\right)^2 u. \tag{6.46b}$$

4. Total Differential of n-th Order of a Function of Two Variables

$$d^n u = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n u. \tag{6.47}$$

5. Total Differential of n-th Order of a Function $u = f(x_1, x_2, \ldots, x_m)$ of m Variables

$$d^n u = \left(\frac{\partial}{\partial x_1}\,dx_1 + \frac{\partial}{\partial x_2}\,dx_2 + \ldots + \frac{\partial}{\partial x_m}\,dx_m\right)^n u. \tag{6.48}$$

6.2.2.3 Taylor's Theorem for Functions of Several Variables

1. Taylor's Formula for Functions of Two Variables

a) First Form of Representation:

$$\begin{aligned} f(x,y) &= f(a,b) + \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(a,b)}(x-a) + \left.\frac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(a,b)}(y-b) \\ &+\frac{1}{2!}\left\{\left.\frac{\partial^2 f(x,y)}{\partial x^2}\right|_{(x,y)=(a,b)}(x-a)^2 + 2\left.\frac{\partial^2 f(x,y)}{\partial x \partial y}\right|_{(x,y)=(a,b)}(x-a)(y-b)\right. \\ &\left.+\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{(x,y)=(a,b)}(y-b)^2\right\} + \frac{1}{3!}\{\ldots\} + \cdots + \frac{1}{n!}\{\ldots\} + R_n. \end{aligned} \tag{6.49a}$$

Here (a, b) is the center of expansion and R_n is the remainder. Sometimes one writes, e.g., instead of $\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(x_0,y_0)}$ the shorter expression $\frac{\partial f}{\partial x}(x_0, y_0)$.

The terms of higher order in (6.49a) can be represented in a clear way with the help of operators:

$$\begin{aligned} f(x,y) &= f(a,b) + \frac{1}{1!}\left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}f(x,y)\Big|_{(x,y)=(a,b)} \\ &+\frac{1}{2!}\left\{(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right\}^2 f(x,y)\Big|_{(x,y)=(a,b)} \end{aligned}$$

$$+\frac{1}{3!}\{\dots\}^3 f(x,y)\Big|_{(x,y)=(a,b)} + \dots + \frac{1}{n!}\{\dots\}^n f(x,y)\Big|_{(x,y)=(a,b)} + R_n \,. \quad (6.49b)$$

This symbolic form means that after using the binomial theorem the powers of the differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ represent the higher-order derivatives of the function $f(x,y)$. Then the derivatives must be taken at the point (a,b).

b) Second Form of the Representation:

$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left(\frac{\partial}{\partial x}h + \frac{\partial}{\partial y}k\right) f(x,y) + \frac{1}{2!}\left(\frac{\partial}{\partial x}h + \frac{\partial}{\partial y}k\right)^2 f(x,y)$$
$$+\frac{1}{3!}\left(\frac{\partial}{\partial x}h + \frac{\partial}{\partial y}k\right)^3 f(x,y) + \dots + \frac{1}{n!}\left(\frac{\partial}{\partial x}h + \frac{\partial}{\partial y}k\right)^n f(x,y) + R_n. \quad (6.49c)$$

c) Remainder: The expression for the remainder is

$$R_n = \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}h + \frac{\partial}{\partial y}k\right)^{n+1} f(x+\Theta h, y+\Theta k) \qquad (0<\Theta<1). \quad (6.49d)$$

2. Taylor Formula for Functions of m Variables

The analogous representation with differential operators is

$$f(x+h,\ y\ +k, \dots, t+l)$$
$$= f(x,y,\dots,t) + \sum_{i=1}^{n}\frac{1}{i!}\left(\frac{\partial}{\partial x}h + \frac{\partial}{\partial y}k + \dots + \frac{\partial}{\partial t}l\right)^i f(x,y,\dots,t) + R_n, \quad (6.50a)$$

where the remainder can be calculated by the expression

$$R_n = \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}\,h + \frac{\partial}{\partial y}\,k + \dots + \frac{\partial}{\partial t}\,l\right)^{n+1} f(x+\Theta\,h\,, y+\Theta\,k\,,\dots,t+\Theta\,l)$$
$$(0<\Theta<1). \quad (6.50b)$$

6.2.3 Rules of Differentiation for Functions of Several Variables

6.2.3.1 Differentiation of Composite Functions

1. Composite Function of One Independent Variable

$$u = f(x_1, x_2, \dots, x_n)\,, \quad x_1 = x_1(\xi), \quad x_2 = x_2(\xi)\,,\dots, \quad x_n = x_n(\xi) \quad (6.51a)$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x_1}\frac{dx_1}{d\xi} + \frac{\partial u}{\partial x_2}\frac{dx_2}{d\xi} + \ldots + \frac{\partial u}{\partial x_n}\frac{dx_n}{d\xi}\,. \quad (6.51b)$$

2. Composite Function of Several Independent Variables

$$u = f(x_1, x_2, \dots, x_n),$$
$$x_1 = x_1(\xi,\eta,\dots,\tau), \quad x_2 = x_2(\xi,\eta,\dots,\tau),\dots, \quad x_n = x_n(\xi,\eta,\dots,\tau) \quad (6.52a)$$

$$\left.\begin{array}{l}
\dfrac{\partial u}{\partial \xi} = \dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial \xi} + \dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial \xi} + \dots + \dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial \xi}, \\
\dfrac{\partial u}{\partial \eta} = \dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial \eta} + \dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial \eta} + \dots + \dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial \eta}, \\
\ \vdots\ = \quad \vdots \quad + \quad \vdots \quad + \vdots + \quad \vdots \\
\dfrac{\partial u}{\partial \tau} = \dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial \tau} + \dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial \tau} + \dots + \dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial \tau}.
\end{array}\right\} \quad (6.52b)$$

6.2.3.2 Differentiation of Implicit Functions

1. A Function $y = f(x)$ of One Variable is given by the equation

$$F(x, y) = 0. \tag{6.53a}$$

Differentiating (6.53a) with respect to x with the help of (6.51b) one gets

$$F_x + F_y y' = 0 \tag{6.53b}$$ and $$y' = -\frac{F_x}{F_y} \quad (F_y \neq 0). \tag{6.53c}$$

Differentiation of (6.53b) yields in the same way

$$F_{xx} + 2F_{xy}y' + F_{yy}(y')^2 + F_y y'' = 0, \tag{6.53d}$$

so considering (6.53b) one has

$$y'' = \frac{2F_xF_yF_{xy} - (F_y)^2F_{xx} - (F_x)^2F_{yy}}{(F_y)^3}. \tag{6.53e}$$

In an analogous way one can calculate the third derivative

$$F_{xxx} + 3F_{xxy}y' + 3F_{xyy}(y')^2 + F_{yyy}(y')^3 + 3F_{xy}y'' + 3F_{yy}y'y'' + F_y y''' = 0, \tag{6.53f}$$

from which y''' can be expressed.

2. A Function $u = f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ of Several Variables is given by the equation

$$F(x_1, x_2, \ldots, x_i, \ldots, x_n, u) = 0. \tag{6.54a}$$

The partial derivatives

$$\frac{\partial u}{\partial x_1} = -\frac{F_{x_1}}{F_u}, \frac{\partial u}{\partial x_2} = -\frac{F_{x_2}}{F_u}, \ldots, \frac{\partial u}{\partial x_n} = -\frac{F_{x_n}}{F_u} \tag{6.54b}$$

can be calculated similarly as it has been shown above but here the formulas (6.52b) are to be used. The higher-order derivatives can be calculated in the same way.

3. Two Functions $y = f(x)$ and $z = \varphi(x)$ of One Variable are given by the system of equations

$$F(x, y, z) = 0 \quad \text{and} \quad \Phi(x, y, z) = 0. \tag{6.55a}$$

Then differentiation of (6.55a) according to (6.51b) results in

$$F_x + F_y y' + F_z z' = 0, \quad \Phi_x + \Phi_y y' + \Phi_z z' = 0, \tag{6.55b}$$

$$y' = \frac{F_z\Phi_x - \Phi_z F_x}{F_y\Phi_z - F_z\Phi_y}, \quad z' = \frac{F_x\Phi_y - F_y\Phi_x}{F_y\Phi_z - F_z\Phi_y}. \tag{6.55c}$$

The second derivatives y'' and z'' are calculated in the same way by differentiation of (6.55b) considering y' and z'.

4. n Functions of One Variable Let the functions $y_1 = f(x), y_2 = \varphi(x), \ldots, y_n = \psi(x)$ be given by a system

$$F(x, y_1, y_2, \ldots, y_n) = 0, \quad \Phi(x, y_1, y_2, \ldots, y_n) = 0, \quad \ldots, \Psi(x, y_1, y_2, \ldots, y_n) = 0 \tag{6.56a}$$

of n equations. Differentiation of (6.56a) using (6.51b) results in

$$\left.\begin{array}{l} F_x + F_{y_1}y_1' + F_{y_2}y_2' + \cdots + F_{y_n}y_n' = 0, \\ \Phi_x + \Phi_{y_1}y_1' + \Phi_{y_2}y_2' + \cdots + \Phi_{y_n}y_n' = 0, \\ \vdots + \quad \vdots \quad + \quad \vdots \quad + \vdots + \quad \vdots \quad = 0, \\ \Psi_x + \Psi_{y_1}y_1' + \Psi_{y_2}y_2' + \cdots + \Psi_{y_n}y_n' = 0. \end{array}\right\} \tag{6.56b}$$

Solving (6.56b) yields the derivatives $y_1', y_2', \ldots, y_n'$, which are to be looking for. In the same way one can calculate the higher-order derivatives.

5. Two Functions $u = f(x,y), v = \varphi(x,y)$ of Two Variables are given by the system of equations

$$F(x,y,u,v) = 0 \quad \text{and} \quad \Phi(x,y,u,v) = 0. \tag{6.57a}$$

Then differentiation of (6.57a) with respect to x and y with the help of (6.52b) results in

$$\left.\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} &= 0,\\ \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \Phi}{\partial v}\frac{\partial v}{\partial x} &= 0, \end{aligned}\right\} \quad (6.57\text{b}) \qquad \left.\begin{aligned} \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} &= 0,\\ \frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \Phi}{\partial v}\frac{\partial v}{\partial y} &= 0. \end{aligned}\right\} \quad (6.57\text{c})$$

Solving the system (6.57b) for $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ and the system (6.57c) for $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$ give the first-order partial derivatives. The higher-order derivatives should be calculated in the same way.

6. n Functions of m Variables Given by a System of n Equations The first-order and higher-order partial derivatives can be calculated in the same way as in the previous cases.

6.2.4 Substitution of Variables in Differential Expressions and Coordinate Transformations

6.2.4.1 Function of One Variable

Suppose, given a function $y(x)$ and a differential expression F containing the independent variable, the function, and its derivatives:

$$y = f(x), \quad (6.58\text{a}) \qquad F = F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\right). \quad (6.58\text{b})$$

If the variables are substituted, then the derivatives can be calculated in the following way:

Case 1a: The variable x is replaced by the variable t, and they have the relation

$$x = \varphi(t). \tag{6.59a}$$

Then holds

$$\frac{dy}{dx} = \frac{1}{\varphi'(t)}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{1}{[\varphi'(t)]^3}\left\{\varphi'(t)\frac{d^2y}{dt^2} - \varphi''(t)\frac{dy}{dt}\right\}, \tag{6.59b}$$

$$\frac{d^3y}{dx^3} = \frac{1}{[\varphi'(t)]^5}\left\{[\varphi'(t)]^2\frac{d^3y}{dt^3} - 3\,\varphi'(t)\,\varphi''(t)\frac{d^2y}{dt^2} + [3[\varphi''(t)]^2 - \varphi'(t)\,\varphi'''(t)]\frac{dy}{dx}\right\}, \ldots . \tag{6.59c}$$

Case 1b: If the relation between the variables is not explicit but it is given in implicit form

$$\Phi(x,t) = 0, \tag{6.60}$$

then the derivatives $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}$ are calculated by the same formulas, but the derivatives $\varphi'(t), \varphi''(t), \varphi'''(t)$ must be calculated according to the rules for implicit functions. In this case it can happen that the relation (6.58b) contains the variable x. To eliminate x, the relation (6.60) is used.

Case 2: If the function y is replaced by a function $u(x)$, and the relation between them is

$$y = \varphi(u), \tag{6.61a}$$

then the calculation of the derivatives can be performed using the following formulas:

$$\frac{dy}{dx} = \varphi'(u)\frac{du}{dx}, \quad \frac{d^2y}{dx^2} = \varphi'(u)\frac{d^2u}{dx^2} + \varphi''(u)\left(\frac{du}{dx}\right)^2, \tag{6.61b}$$

$$\frac{d^3y}{dx^3} = \varphi'(u)\frac{d^3u}{dx^3} + 3\varphi''(u)\frac{du}{dx}\frac{d^2u}{dx^2} + \varphi'''(u)\left(\frac{du}{dx}\right)^3, \ldots . \tag{6.61c}$$

Case 3: The variables x and y are replaced by the new variables t and u, and the relations between them are given by

$$x = \varphi(t,u), \quad y = \psi(t,u). \tag{6.62a}$$

For the calculation of the derivatives the following formulas are used:

$$\frac{dy}{dx} = \frac{\dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial u}\dfrac{du}{dt}}{\dfrac{\partial\varphi}{\partial t} + \dfrac{\partial\varphi}{\partial u}\dfrac{du}{dt}}, \tag{6.62b}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{\dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial u}\dfrac{du}{dt}}{\dfrac{\partial\varphi}{\partial t} + \dfrac{\partial\varphi}{\partial u}\dfrac{du}{dt}}\right] = \frac{1}{\dfrac{\partial\varphi}{\partial t} + \dfrac{\partial\varphi}{\partial u}\dfrac{du}{dt}}\,\frac{d}{dt}\left[\frac{\dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial u}\dfrac{du}{dt}}{\dfrac{\partial\varphi}{\partial t} + \dfrac{\partial\varphi}{\partial u}\dfrac{du}{dt}}\right], \tag{6.62c}$$

$$\frac{1}{B}\frac{d}{dt}\left(\frac{A}{B}\right) = \frac{1}{B^3}\left(B\frac{dA}{dt} - A\frac{dB}{dt}\right), \tag{6.62d}$$

with $A = \dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial u}\dfrac{du}{dt}$ (6.62e) and $B = \dfrac{\partial\varphi}{\partial t} + \dfrac{\partial\varphi}{\partial u}\dfrac{du}{dt}$. (6.62f)

The determination of the third derivative $\dfrac{d^3y}{dx^3}$ can be done in an analogous way.

■ For the transformation from Cartesian coordinates into polar coordinates according to

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi \tag{6.63a}$$

the first and second derivatives should be calculated as follows:

$$\frac{dy}{dx} = \frac{\rho'\sin\varphi + \rho\cos\varphi}{\rho'\cos\varphi - \rho\sin\varphi}, \tag{6.63b}$$

$$\frac{d^2y}{dx^2} = \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{(\rho'\cos\varphi - \rho\sin\varphi)^3}. \tag{6.63c}$$

6.2.4.2 Function of Two Variables

Suppose given a function $\omega(x,y)$ and a differential expression F containing the independent variables, the function and its partial derivatives:

$$\omega = f(x,y), \tag{6.64a}$$

$$F = F\left(x, y, \omega, \frac{\partial\omega}{\partial x}, \frac{\partial\omega}{\partial y}, \frac{\partial^2\omega}{\partial x^2}, \frac{\partial^2\omega}{\partial x\partial y}, \frac{\partial^2\omega}{\partial y^2}, \ldots\right). \tag{6.64b}$$

If x and y are replaced by the new variables u and v given by the relations

$$x = \varphi(u,v), \quad y = \psi(u,v), \tag{6.65a}$$

then the first-order partial derivatives can be expressed from the system of equations

$$\frac{\partial\omega}{\partial u} = \frac{\partial\omega}{\partial x}\frac{\partial\varphi}{\partial u} + \frac{\partial\omega}{\partial y}\frac{\partial\psi}{\partial u}, \quad \frac{\partial\omega}{\partial v} = \frac{\partial\omega}{\partial x}\frac{\partial\varphi}{\partial v} + \frac{\partial\omega}{\partial y}\frac{\partial\psi}{\partial v} \tag{6.65b}$$

with the new functions A, B, C, and D of the new variables u and v

$$\frac{\partial\omega}{\partial x} = A\frac{\partial\omega}{\partial u} + B\frac{\partial\omega}{\partial v}, \quad \frac{\partial\omega}{\partial y} = C\frac{\partial\omega}{\partial u} + D\frac{\partial\omega}{\partial v}. \tag{6.65c}$$

The second-order partial derivatives are calculated with the same formulas, only without using ω in them but its partial derivatives $\dfrac{\partial\omega}{\partial x}$ and $\dfrac{\partial\omega}{\partial y}$, e.g.,

$$\frac{\partial^2\omega}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\omega}{\partial x}\right) = \frac{\partial}{\partial x}\left(A\frac{\partial\omega}{\partial u} + B\frac{\partial\omega}{\partial v}\right) = A\left(A\frac{\partial^2\omega}{\partial u^2} + B\frac{\partial^2\omega}{\partial u\partial v} + \frac{\partial A}{\partial u}\frac{\partial\omega}{\partial u} + \frac{\partial B}{\partial u}\frac{\partial\omega}{\partial v}\right)$$

$$+B\left(A\frac{\partial^2\omega}{\partial u\partial v}+B\frac{\partial^2\omega}{\partial v^2}+\frac{\partial A}{\partial v}\frac{\partial\omega}{\partial u}+\frac{\partial B}{\partial v}\frac{\partial\omega}{\partial v}\right). \quad (6.66)$$

The higher partial derivatives can be calculated in the same way.

■ The Laplace operator (see 13.2.6.5, p. 716) is to be expressed in polar coordinates (see 3.5.2.1, **2.**, p. 191):

$$\Delta\omega=\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}, \quad (6.67a) \qquad x=\rho\cos\varphi, \quad y=\rho\sin\varphi. \quad (6.67b)$$

The calculations are:

$$\frac{\partial\omega}{\partial\rho}=\frac{\partial\omega}{\partial x}\cos\varphi+\frac{\partial\omega}{\partial y}\sin\varphi, \qquad \frac{\partial\omega}{\partial\varphi}=-\frac{\partial\omega}{\partial x}\rho\sin\varphi+\frac{\partial\omega}{\partial y}\rho\cos\varphi,$$

$$\frac{\partial\omega}{\partial x}=\cos\varphi\frac{\partial\omega}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial\omega}{\partial\varphi}, \qquad \frac{\partial\omega}{\partial y}=\sin\varphi\frac{\partial\omega}{\partial\rho}+\frac{\cos\varphi}{\rho}\frac{\partial\omega}{\partial\varphi},$$

$$\frac{\partial^2\omega}{\partial x^2}=\cos\varphi\frac{\partial}{\partial\rho}\left(\cos\varphi\frac{\partial\omega}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial\omega}{\partial\varphi}\right)-\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\omega}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial\omega}{\partial\varphi}\right).$$

Similarly, $\dfrac{\partial^2\omega}{\partial y^2}$ is calculated, so finally:

$$\Delta\omega=\frac{\partial^2\omega}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2\omega}{\partial\varphi^2}+\frac{1}{\rho}\frac{\partial\omega}{\partial\rho}. \quad (6.67c)$$

Remark: If functions of more than two variables should be substituted, then similar substitution formulas can be derived.

6.2.5 Extreme Values of Functions of Several Variables

6.2.5.1 Definition of a Relative Extreme Value

A function $u=f(x_1,x_2,\dots,x_i,\dots,x_n)$ has a relative extreme value at a point $P_0(x_{10},x_{20},\dots,x_{i0},\dots,x_{n0})$, if there is a number ϵ such that for every point $P(x_1,x_2,\dots,x_n)$ belonging to the domain $x_{10}-\epsilon<x_1<x_{10}+\epsilon, x_{20}-\epsilon<x_2<x_{20}+\epsilon,\dots,x_{n0}-\epsilon<x_n<x_{n0}+\epsilon$ and to the domain of the function but different from P_0, then for a maximum the inequality

$$f(x_1,x_2,\dots,x_n)<f(x_{10},x_{20},\dots,x_{n0}) \quad (6.68a)$$

holds, and for a minimum the inequality

$$f(x_1,x_2,\dots,x_n)>f(x_{10},x_{20},\dots,x_{n0}) \quad (6.68b)$$

holds. Using the terminology of several dimensional spaces (see 2.18.1, p. 118) a function has a relative maximum or a relative minimum at a point if it is greater or smaller there than at the neighboring points.

6.2.5.2 Geometric Representation

In the case of a function of two variables, represented in a Cartesian coordinate system as a surface (see 2.18.1.2, p. 119), the relative extreme value geometrically means that the applicate (see 3.5.3.1, **3.**, p. 210) of the surface in the point A is greater or smaller than the applicate of the surface in any other point in a sufficiently small neighborhood of A **(Fig. 6.16)**.

If the surface has a relative extremum at the point P_0 which is an interior point of its domain, and if the surface has a tangent plane at this point, then the tangent plane is parallel to the x,y plane **(Fig. 6.16a,b)**. This property is necessary but not sufficient for a maximum or minimum at a point

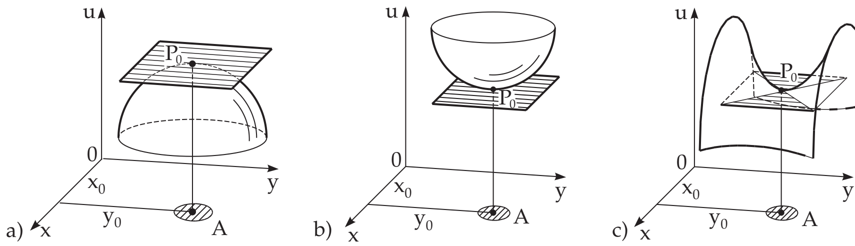

Figure 6.16

P_0. For example **Fig. 6.16c** shows a surface having a horizontal tangent plane at P_0, but there is a saddle point here and not an extremum.

6.2.5.3 Determination of Extreme Values of Differentiable Functions of Two Variables

If $u = f(x, y)$ is given, then one solves the system of equations $f_x = 0$, $f_y = 0$. The resulting pairs of values $(x_1, y_1), (x_2, y_2), \dots$ can be substituted into the second derivatives

$$A = \frac{\partial^2 f}{\partial x^2}\,, \quad B = \frac{\partial^2 f}{\partial x \partial y}\,, \quad C = \frac{\partial^2 f}{\partial y^2}\,. \tag{6.69}$$

Depending on the expression

$$\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = [f_{xx} f_{yy} - (f_{xy})^2]_{x=x_i, y=y_i} \quad (i = 1, 2, \dots) \tag{6.70}$$

it can be decided whether an extreme value exists and of what kind it is:

1. In the case $\Delta > 0$ the function $f(x, y)$ has an extreme value at (x_i, y_i), and for $f_{xx} < 0$ it is a maximum, for $f_{xx} > 0$ it is a minimum (sufficient condition).

2. In the case $\Delta < 0$ the function $f(x, y)$ does not have an extremum.

3. In the case $\Delta = 0$, one needs further investigation.

6.2.5.4 Determination of the Extreme Values of a Function of n Variables

If $u = f(x_1, x_2, \dots, x_n)$ is given, then first it is to find a solution $(x_{10}, x_{20}, \dots, x_{n0})$ of the system of the n equations

$$f_{x_1} = 0,\ f_{x_2} = 0,\ \dots, f_{x_n} = 0, \tag{6.71}$$

because it is a necessary condition for an extreme value. (6.71) is not a sufficient condition. Therefore one prepares a matrix of the second-order partial derivatives such that $a_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$. Then it is to substitute a solution of the system of equations (6.71) into the terms, and to prepare the sequence of left upper subdeterminants $(a_{11},\ a_{11}a_{22} - a_{12}a_{21},\ \dots)$. Then there are the following cases:

1. The signs of the subdeterminants follow the rule $-, +, -, +, \dots$, then there is a maximum there.

2. The signs of the subdeterminants follow the rule $+, +, +, +, \dots$, then there is a minimum there.

3. There are some zero values among the subdeterminants, but the signs of the non-zero subdeterminants coincide with the signs of the corresponding positions of one of the first two cases. Then further investigation is required: Usually one checks the values of the function in a close neighborhood of $x_{10}, x_{20}, \dots, x_{n0}$.

4. The signs of the subdeterminants does not follow the rules given in cases 1. and 2.: There is no extremum at that point.

The case of two variables is of course a special case of the case of n variables, (see [6.4]).

6.2.5.5 Solution of Approximation Problems

Several different approximation problems can be solved with the help of the determination of the extreme values of functions of several variables, e.g., *fitting problems* or *mean squares problems.*

Problems to solve:

• Determination of Fourier coefficients (see 7.4.1.2, p. 475, 19.6.4.1, p. 992).

• Determination of the coefficients and parameters of approximation functions (see 19.6.2, p. 984 ff).

• Determination of an approximate solution of an overdetermined linear system of equations (see 19.2.1.3, p. 958).

Methods: For these problems the following methods are used:

• Gaussian least squares method (see, e.g., 19.6.2, p. 984).

• Least squares method (see 19.6.2.2, p. 985).

• Approximation in mean square (continuous and discrete) (see, e.g., 19.6.2, p.984).

• Calculus of observations (or fitting) (see 19.6.2, p. 984) and regression (see 16.3.4.2, **1.**, p. 841).

6.2.5.6 Extreme Value Problem with Side Conditions

To determine are the extreme values of a function $u = f(x_1, x_2, \ldots, x_n)$ of n variables with the side conditions

$$\varphi(x_1, x_2, \ldots, x_n) = 0,\ \psi(x_1, x_2, \ldots, x_n) = 0, \ldots,\ \chi(x_1, x_2, \ldots, x_n) = 0. \tag{6.72a}$$

Because of the conditions, the variables are not independent, and if the number of conditions is k, obviously $k < n$ must hold. One possibility to determine the extreme values of u is to express k variables with the others from the system of equations of the conditions, to substitute them into the original function. Then the result is an extreme value problem without conditions for $n-k$ variables. The other way is the *Lagrange multiplier method.* Introducing k undefined multipliers $\lambda, \mu, \ldots, \kappa$, and composing the *Lagrange function (Lagrangian)* of $n + k$ variables $x_1, x_2, \ldots, x_n, \lambda, \mu, \ldots, \kappa$ gives:

$$\begin{aligned}\varPhi\ (x_1, x_2, \ldots, x_n, \lambda, \mu, \ldots, \kappa) \\ = f(x_1, x_2, \ldots, x_n) + \lambda\, \varphi(x_1, x_2, \ldots, x_n) + \mu\, \psi(x_1, x_2, \ldots, x_n) + \cdots \\ + \kappa\, \chi(x_1, x_2, \ldots, x_n).\end{aligned} \tag{6.72b}$$

The necessary condition for an extremum of the function $\varPhi$ is a system of $n + k$ equations (6.71) with the unknowns $x_1, x_2, \ldots, x_n, \lambda, \mu, \ldots, \kappa$:

$$\varphi = 0, \psi = 0,\ \ldots,\ \chi = 0,\ \varPhi_{x_1} = 0,\ \varPhi_{x_2} = 0, \ldots, \varPhi_{x_n} = 0 \tag{6.72c}$$

The necessary condition for an extremum of the function f at the point $P_0(x_{10}, x_{20}, \ldots, x_{n0})$ with the side conditions (6.72a) is that the system of values $x_{10}, x_{20}, \ldots, x_{n0}$ must fulfill the equations (6.72c). So it is to look for the extremum points of f among the solutions $x_{10}, x_{20}, \ldots, x_{n0}$ of the system of equations (6.72c). To determine whether there are really extreme values at these points fulfilling the necessary conditions requires further investigations, for which the general rules are fairly complicated. Usually one uses some appropriate and individual calculations depending on the function f to prove if there is an extremum, or not. Often approximation calculations are helpful, i.e., the comparison with values of the function in the neighborhood of P_0.

■ The extreme value of the function $u = f(x, y)$ with the side condition $\varphi(x, y) = 0$ will be determined from the three equations

$$\varphi(x, y) = 0,\ \frac{\partial}{\partial x}[f(x, y) + \lambda\varphi(x, y)] = 0,\ \frac{\partial}{\partial y}[f(x, y) + \lambda\varphi(x, y)] = 0. \tag{6.73}$$

There are three unknowns, x, y, λ. Since the three equations (6.73) are only necessary but not sufficient conditions for the existence of an extremum, further investigation is needed whether there is an extremum at the solution of this system. A mathematical criterion is rather complicated (see [6.3], [6.8]); comparisons of the values of the function at points in the close neighborhood are often useful.