15 Integral Transformations

15.1 Notion of Integral Transformation

15.1.1 General Definition of Integral Transformations

An*Integral transformation* is a correspondence between two functions $f(t)$ and $F(p)$ in the form

$$F(p) = \int_{-\infty}^{+\infty} K(p,t) f(t)\, dt. \tag{15.1a}$$

The function $f(t)$ is called the *original function*, its domain is the *original space*. The function $F(p)$ is called the *transform*, its domain is the *image space*.
The function $K(p,t)$ is called the *kernel* of the transformation. In general, t is a real variable, and $p = \sigma + \mathrm{i}\omega$ is a complex variable.
The shorter notation can be used by introducing the symbol $\mathcal{T}$ for the integral transformation with kernel $K(p,t)$:

$$F(p) = \mathcal{T}\{f(t)\}. \tag{15.1b}$$

Then, (15.1b) is called $\mathcal{T}$ transformation.

15.1.2 Special Integral Transformations

Different kernels $K(p,t)$ and different original spaces yield different integral transformations. The most widely known transformations are the Laplace transformation, the Laplace-Carson transformation, and the Fourier transformation. In this book an overview is given about the integral transformations of functions of one variable (see also **Table 15.1**). More recently, some additional transformations have been introduced for use in pattern recognition and in characterizing signals, such as the Wavelet transformation, the Gabor transformation and the Walsh transformation (see 15.6, p. 800ff.).

15.1.3 Inverse Transformations

The inverse transformation of a transform into the original function has special importance in applications. With the symbol $\mathcal{T}^{-1}$ the inverse integral transformation of (15.1a) is

$$f(t) = \mathcal{T}^{-1}\{F(p)\}. \tag{15.2a}$$

The operator $\mathcal{T}^{-1}$ is called the *inverse operator* of $\mathcal{T}$, so

$$\mathcal{T}^{-1}\{\mathcal{T}\{f(t)\}\} = f(t). \tag{15.2b}$$

The determination of the inverse transformation means the solution of the integral equation (15.1a), where the function $F(p)$ is given and function $f(t)$ is to be determined. If there is a solution, then it can be written in the form

$$f(t) = \mathcal{T}^{-1}\{F(p)\}. \tag{15.2c}$$

The explicit determination of *inverse operators* for different integral transformations, i.e., for different kernels $K(p,t)$, belongs to the fundamental problems of the theory of integral transformations. The user can solve practical problems by using the given correspondences between transforms and original functions in the corresponding tables (**Table 21.13**, p. 1109, **Table 21.14**, p. 1114, and **Table 21.15**, p. 1128).

15.1.4 Linearity of Integral Transformations

If $f_1(t)$ and $f_2(t)$ are transformable functions, then

$$\mathcal{T}\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 \mathcal{T}\{f_1(t)\} + k_2 \mathcal{T}\{f_2(t)\}, \tag{15.3}$$

where k_1 and k_2 are arbitrary numbers. That is, an integral transformation represents a linear operation on the set T of the $\mathcal{T}$-transformable functions.

I.N. Bronshtein et al., *Handbook of Mathematics*,
DOI 10.1007/978-3-662-46221-8_15

Table 15.1 Overview of integral transformations of functions of one variable

Transformation	**Kernel $K(p,t)$**	**Symbol**	**Remark**
Laplace transformation	$\begin{cases} 0 & \text{for } t<0 \\ e^{-pt} & \text{for } t>0 \end{cases}$	$\mathcal{L}\{f(t)\} = \int\limits_0^\infty e^{-pt} f(t)dt$	$p=\sigma+\mathrm{i}\omega$
Two-sided Laplace transformation	e^{-pt}	$\mathcal{L}_{\mathrm{II}}\{f(t)\} = \int\limits_{-\infty}^{+\infty} e^{-pt} f(t)dt$	$\mathcal{L}_{\mathrm{II}}\{f(t)\mathit{1}(t)\} = \mathcal{L}\{f(t)\}$ where $\mathit{1}(t) = \begin{cases} 0 \text{ for } t<0 \\ 1 \text{ for } t>0 \end{cases}$
Finite Laplace transformation	$\begin{cases} 0 & \text{for } t<0 \\ e^{-pt} & \text{for } 0<t<a \\ 0 & \text{for } t>a \end{cases}$	$\mathcal{L}_a\{f(t)\} = \int\limits_0^a e^{-pt} f(t)dt$	
Laplace-Carson transformation	$\begin{cases} 0 & \text{for } t<0 \\ pe^{-pt} & \text{for } t>0 \end{cases}$	$\mathcal{C}\{f(t)\} = \int\limits_0^\infty pe^{-pt} f(t)dt$	The Carson transformation can also be a two-sided and finite transformation.
Fourier transformation	$e^{-\mathrm{i}\omega t}$	$\mathcal{F}\{f(t)\} = \int\limits_{-\infty}^{+\infty} e^{-\mathrm{i}\omega t} f(t)dt$	$p=\sigma+\mathrm{i}\omega \quad \sigma=0$
One-sided Fourier transformation	$\begin{cases} 0 & \text{for } t<0 \\ e^{-\mathrm{i}\omega t} & \text{for } t>0 \end{cases}$	$\mathcal{F}_{\mathrm{I}}\{f(t)\} = \int\limits_0^\infty e^{-\mathrm{i}\omega t} f(t)dt$	$p=\sigma+\mathrm{i}\omega \quad \sigma=0$
Finite Fourier transformation	$\begin{cases} 0 & \text{for } t<0 \\ e^{-\mathrm{i}\omega t} & \text{for } 0<t<a \\ 0 & \text{for } t>a \end{cases}$	$\mathcal{F}_a\{f(t)\} = \int\limits_0^a e^{-\mathrm{i}\omega t} f(t)dt$	$p=\sigma+\mathrm{i}\omega \quad \sigma=0$
Fourier cosine transformation	$\begin{cases} 0 & \text{for } t<0 \\ \mathrm{Re}\,[e^{\mathrm{i}\omega t}] & \text{for } t>0 \end{cases}$	$\mathcal{F}_c\{f(t)\} = \int\limits_0^\infty f(t)\cos\omega t\, dt$	$p=\sigma+\mathrm{i}\omega \quad \sigma=0$
Fourier sine transformation	$\begin{cases} 0 & \text{for } t<0 \\ \mathrm{Im}\,[e^{\mathrm{i}\omega t}] & \text{for } t>0 \end{cases}$	$\mathcal{F}_s\{f(t)\} = \int\limits_0^\infty f(t)\sin\omega t\, dt$	$p=\sigma+\mathrm{i}\omega \quad \sigma=0$
Mellin transformation	$\begin{cases} 0 & \text{for } t<0 \\ t^{p-1} & \text{for } t>0 \end{cases}$	$\mathcal{M}\{f(t)\} = \int\limits_0^\infty t^{p-1} f(t)dt$	
Hankel transformation of order ν	$\begin{cases} 0 & \text{for } t<0 \\ tJ_\nu(\sigma t) & \text{for } t>0 \end{cases}$	$\mathcal{H}_\nu\{f(t)\} = \int\limits_0^\infty tJ_\nu(\sigma t) f(t)dt$	$p=\sigma+\mathrm{i}\omega \quad \omega=0$ $J_\nu(\sigma t)$ is the ν-th order Bessel function of the first kind.
Stieltjes transformation	$\begin{cases} 0 & \text{for } t<0 \\ \dfrac{1}{p+t} & \text{for } t>0 \end{cases}$	$\mathcal{S}\{f(t)\} = \int\limits_0^\infty \dfrac{f(t)}{p+t} dt$	

15.1.5 Integral Transformations for Functions of Several Variables

Integral transformations for functions of several variables are also called *multiple integral transformations* (see [15.13]). The best-known ones are the double Laplace transformation, i.e., the Laplace transformation for functions of two variables, the double Laplace-Carson transformation and the double Fourier transformation. The definition of the double Laplace transformation is

$$F(p,q) = \mathcal{L}^2\{f(x,y)\} \equiv \int\limits_{x=0}^{\infty}\int\limits_{y=0}^{\infty} e^{-px-qy} f(x,y)\,dx\,dy. \tag{15.4}$$

The symbol $\mathcal{L}$ denotes the Laplace transformation for functions of one variable (see **Table 15.1**).

15.1.6 Applications of Integral Transformations

1. Fields of Applications

Besides the great theoretical importance that integral transformations have in such basic fields of mathematics as the theory of integral equations and the theory of linear operators, they have a large field of applications in the solutions of practical problems in physics and engineering. Methods with applications of integral transformations are often called *operator methods*. They are suitable to solve ordinary and partial differential equations, integral equations and difference equations.

2. Scheme of the Operator Method

The general scheme to the use of an operator method with an integral transformation is represented in **Fig. 15.1**. One gets the solution of a problem not directly from the original defining equation; first an integral transformation is applied. The inverse transformation of the solution of the transformed equation gives the solution of the original problem.

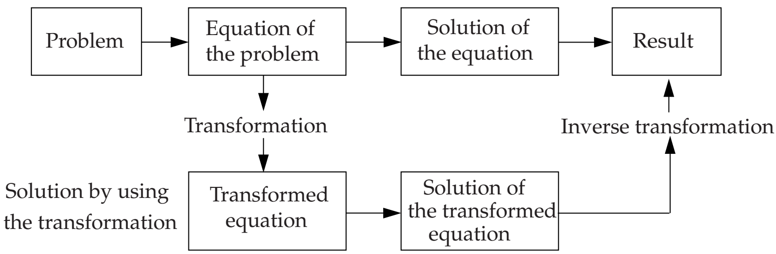

Figure 15.1

The application of the operator method to solve ordinary differential equations consists of the following three steps:

1. Transition from a differential equation of an unknown function to an equation of its transform.

2. Solution of the transformed equation in the image space. The transformed equation is usually no longer a differential equation, but an algebraic equation.

3. Inverse transformation of the transform with help of $\mathcal{T}^{-1}$ into the original space, i.e., determination of the solution of the original problem.

The major difficulty of the operator method is usually not the solution of the transformed equation, but the transformation of the function and the inverse transformation.

15.2 Laplace Transformation

15.2.1 Properties of the Laplace Transformation

15.2.1.1 Laplace Transformation, Original and Image Space

1. Definition of the Laplace Transformation

The Laplace transformation

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t)\,dt = F(p) \tag{15.5}$$

assigns a function $F(p)$ of a complex variable p to a function $f(t)$ of a real variable t, if the given improper integral exists. $f(t)$ is called the *original function*, $F(p)$ is called the *transform* of $f(t)$. In the further discussion it is assumed that the improper integral exists if the original function $f(t)$ is piecewise smooth in its domain $t \geq 0$, in the so called *original space*, and for $t \to \infty$, $|f(t)| \leq Ke^{\alpha t}$ holds with certain constants $K > 0$, $\alpha > 0$. The domain of the transform $F(p)$ is called the *image space*.
In the literature the Laplace transformation is often found also in the Wagner or Laplace-Carson form

$$\mathcal{L}_W\{f(t)\} = p\int_0^\infty e^{-pt} f(t)\,dt = p\,F(p). \tag{15.6}$$

2. Convergence

The Laplace integral $\mathcal{L}\{f(t)\}$ converges in the half-plane $\mathrm{Re}\,p > \alpha$ **(Fig. 15.2)**. The transform $F(p)$ is an analytic function with the properties:

1. $$\lim_{\mathrm{Re}\,p\to\infty} F(p) = 0. \tag{15.7a}$$

This property is a necessary condition for $F(p)$ to be a transform.

2. $$\lim_{\substack{p\to 0\\(p\to\infty)}} p\,F(p) = A, \tag{15.7b}$$

if the original function $f(t)$ has a finite limit $\lim\limits_{\substack{t\to\infty\\(t\to 0)}} f(t) = A$.

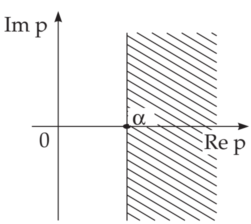

Figure 15.2

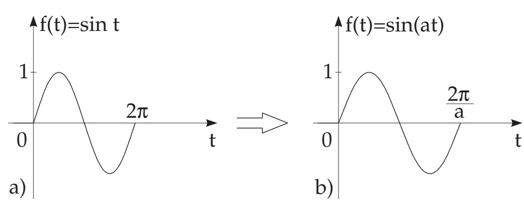

Figure 15.3

3. Inverse Laplace Transformation

One can retrieve the original function from the transform with the formula

$$\mathcal{L}^{-1}\{F(p)\} = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{\,pt} F(p)\,dp = \begin{cases} f(t) & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{15.8}$$

The path of integration of this complex integral is a line $\mathrm{Re}\,p = c$ parallel to the imaginary axis, where $\mathrm{Re}\,p = c > \alpha$. If the function $f(t)$ has a jump at $t = 0$, i.e., $\lim_{t\to+0} f(t) \neq 0$, then the integral has the value $\frac{1}{2}f(+0)$ there.

15.2.1.2 Rules for the Evaluation of the Laplace Transformation

The rules for evaluation are the mappings of operations in the original domain into operations in the transform space.
Hereafter the original functions will be denoted by lowercase letters, the transforms are denoted by the corresponding capital letters.

1. Addition or Linearity Law

The Laplace transform of a linear combination of functions is the same linear combination of the Laplace transforms, if they exist. With constants $\lambda_1, \ldots, \lambda_n$ that is

$$\mathcal{L}\{\lambda_1 f_1(t) + \lambda_2 f_2(t) + \cdots + \lambda_n f_n(t)\} = \lambda_1 F_1(p) + \lambda_2 F_2(p) + \cdots + \lambda_n F_n(p). \tag{15.9}$$

2. Similarity Laws

The Laplace transform of $f(at)$ $(a > 0, a \text{ real})$ is the Laplace transform of the original function divided by a and with the argument p/a:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{p}{a}\right) \quad (a > 0,\ \text{real}). \tag{15.10a}$$

Analogously for the inverse transformation

$$\mathcal{L}^{-1}\{F(ap)\} = \frac{1}{a}f\left(\frac{t}{a}\right) \quad (a > 0). \tag{15.10b}$$

Fig. 15.3 shows the application of the similarity laws for a sine function.

■ Determination of the Laplace transform of $f(t) = \sin(\omega t)$. The transform of the sine function is $\mathcal{L}\{\sin(t)\} = F(p) = 1/(p^2+1)$. Application of the similarity law gives $\mathcal{L}\{\sin(\omega t)\} = \frac{1}{\omega}F(p/\omega) = \frac{1}{\omega}\frac{1}{(p/\omega)^2+1} = \frac{\omega}{p^2+\omega^2}$.

3. Translation Laws

1. Shifting to the Right The Laplace transform of an original function shifted to the right by a $(a > 0)$ is equal to the Laplace transform of the non-shifted original function multiplied by the factor e^{-ap}:

$$\mathcal{L}\{f(t-a)\} = e^{-ap}\,F(p). \tag{15.11a}$$

2. Shifting to the Left The Laplace transform of an original function shifted to the left by a is equal to e^{ap} multiplied by the difference of the transform of the non-shifted function and the integral $\int_0^a f(t)\,e^{-pt}\,dt$:

$$\mathcal{L}\{f(t+a)\} = e^{ap}\left[F(p) - \int_0^a e^{-pt}\,f(t)\,dt\right]. \tag{15.11b}$$

Figs. 15.4 and **15.5** show the cosine function shifted to the right and a line shifted to the left.

4. Frequency Shift Theorem

The Laplace transform of an original function multiplied by e^{-bt} is equal to the Laplace transform with the argument $p + b$ (b is an arbitrary complex number):

$$\mathcal{L}\{e^{-bt}f(t)\} = F(p+b). \tag{15.12}$$

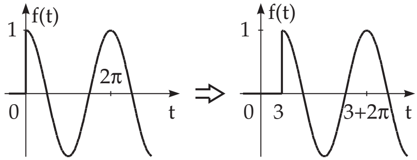

Figure 15.4

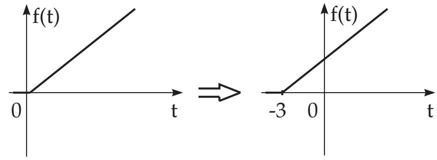

Figure 15.5

5. Differentiation in the Original Space

If the derivatives $f'(t), f''(t), \ldots, f^{(n)}(t)$ exist for $t > 0$ and the highest derivative of $f(t)$ has a transform, then the lower derivatives of $f(t)$ and also $f(t)$ have a transform, and:

$$\left.\begin{array}{l} \mathcal{L}\{f'(t)\} = p\,F(p) - f(+0), \\ \mathcal{L}\{f''(t)\} = p^2\,F(p) - f(+0)\,p - f'(+0), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathcal{L}\{f^{(n)}(t)\} = p^n\,F(p) - f(+0)\,p^{n-1} - f'(+0)\,p^{n-2} - \cdots \\ \qquad - f^{(n-2)}(+0)\,p - f^{(n-1)}(+0) \text{ with} \\ f^{(\nu)}(+0) = \lim\limits_{t\to+0} f^{(\nu)}(t). \end{array}\right\} \tag{15.13}$$

Equation (15.13) gives the following representation of the Laplace integral, which can be used for approximating the Laplace integral:

$$\mathcal{L}\{f(t)\} = \frac{f(+0)}{p} + \frac{f'(+0)}{p^2} + \frac{f''(+0)}{p^3} + \cdots + \frac{f^{(n-1)}(+0)}{p^{n-1}} + \frac{1}{p^n}\mathcal{L}\{f^{(n)}(t)\}. \tag{15.14}$$

6. Differentiation in the Image Space

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(p). \tag{15.15}$$

The n-th derivative of the transform is equal to the Laplace transform of the $(-t)^n$-th multiple of the original function $f(t)$:

$$\mathcal{L}\{(-1)^n\, t^n\, f(t)\} = F^{(n)}(p) \qquad (n = 1, 2, \ldots). \tag{15.16}$$

7. Integration in the Original Space

The transform of an integral of the original function is equal to $1/p^n$ $(n > 0)$ multiplied by the transform of the original function:

$$\mathcal{L}\left\{\int\limits_0^t d\tau_1 \int\limits_0^{\tau_1} d\tau_2 \ldots \int\limits_0^{\tau_{n-1}} f(\tau_n)\, d\tau_n\right\} = \frac{1}{(n-1)\,!}\,\mathcal{L}\left\{\int\limits_0^t (t-\tau)^{(n-1)}\, f(\tau)\, d\tau\right\} = \frac{1}{p^n}F(p). \tag{15.17a}$$

In the special case of the ordinary simple integral

$$\mathcal{L}\left\{\int\limits_0^t f(\tau)\, d\tau\right\} = \frac{1}{p}F(p) \tag{15.17b}$$

holds. In the original space, differentiation and integration act in converse ways if the initial values are zeros.

8. Integration in the Image Space

$$\mathcal{L}\left\{\frac{f(t)}{t^n}\right\} = \int\limits_p^\infty dp_1 \int\limits_{p_1}^\infty dp_2 \ldots \int\limits_{p_{n-1}}^\infty F(p_n)\, dp_n = \frac{1}{(n-1)\,!}\int\limits_p^\infty (z-p)^{n-1} F(z)\, dz. \tag{15.18}$$

This formula is valid only if $f(t)/t^n$ has a Laplace transform. For this purpose, $f(x)$ must tend to zero fast enough as $t \to 0$. The path of integration can be any ray starting at p, which forms an acute angle with the positive half of the real axis.

9. Division Law

In the special case of $n = 1$ of (15.18)

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(z)\,dz \tag{15.19}$$

holds. For the existence of the integral (15.19), the limit $\lim\limits_{t\to 0}\dfrac{f(t)}{t}$ must also exist.

10. Differentiation and Integration with Respect to a Parameter

$$\mathcal{L}\left\{\frac{\partial f(t,\alpha)}{\partial\alpha}\right\} = \frac{\partial F(p,\alpha)}{\partial\alpha}, \quad \text{(15.20a)} \qquad \mathcal{L}\left\{\int_{\alpha_1}^{\alpha_2} f(t,\alpha)\,d\alpha\right\} = \int_{\alpha_1}^{\alpha_2} F(p,\alpha)\,d\alpha. \quad \text{(15.20b)}$$

Sometimes one can calculate Laplace integrals from known integrals with the help of these formulas.

11. Convolution

1. Convolution in the Original Space The convolution of two functions $f_1(t)$ and $f_2(t)$ is the integral

$$f_1 * f_2 = \int_0^t f_1(\tau)\cdot f_2(t-\tau)\,d\tau\,. \tag{15.21}$$

Equation (15.21) is also called the *one-sided convolution* in the interval $(0,t)$. A *two-sided convolution* occurs for the Fourier transformation (convolution in the interval $(-\infty,\infty)$ see 15.3.1.3, **9.**, p. 789). The convolution (15.21) has the properties

a) Commutative law: $f_1 * f_2 = f_2 * f_1$. (15.22a)

b) Associative law: $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$. (15.22b)

c) Distributive law: $(f_1 + f_2) * f_3 = f_1 * f_3 + f_2 * f_3$. (15.22c)

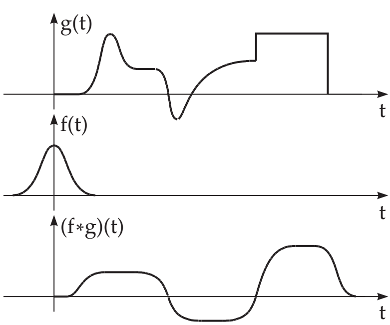

Figure 15.6

In the image domain, the usual multiplication corresponds to the convolution:

$$\mathcal{L}\{f_1 * f_2\} = F_1(p)\cdot F_2(p). \tag{15.23}$$

The convolution of two functions is shown in **Fig. 15.6**. One can apply the convolution theorem to determine the original function:

a) Factoring the transform $F(p) = F_1(p)\cdot F_2(p)$.

b) Determining the original functions $f_1(t)$ and $f_2(t)$ of the transforms $F_1(p)$ and $F_2(p)$ (from a table).

c) Determining the original function associated to $F(p)$ by convolution of $f_1(t)$ and $f_2(t)$ in the original space ($f(t) = f_1(t) * f_2(t)$).

2. Convolution in the Image Space (Complex Convolution)

$$\mathcal{L}\{f_1(t)\cdot f_2(t)\} = \begin{cases} \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_{x_1-\mathrm{i}\infty}^{x_1+\mathrm{i}\infty} F_1(z)\cdot F_2(p-z)\,dz, \\ \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_{x_2-\mathrm{i}\infty}^{x_2+\mathrm{i}\infty} F_1(p-z)\cdot F_2(z)\,dz. \end{cases} \tag{15.24}$$

The integration is performed along a line parallel to the imaginary axis. In the first integral, x_1 and p must be chosen so that z is in the half plane of convergence of $\mathcal{L}\{f_1\}$ and $p-z$ is in the half plane of convergence of $\mathcal{L}\{f_2\}$. The corresponding requirements must be valid for the second integral.

15.2.1.3 Transforms of Special Functions

1. Step Function

The unit jump at $t = t_0$ is called a step function **(Fig. 15.7)** (see also 14.4.3.2, **3.**, p. 757); it is also called the *Heaviside unit step function*:

$$u(t-t_0) = \begin{cases} 1 \text{ for } t > t_0, \\ 0 \text{ for } t < t_0 \end{cases} \quad (t_0 > 0). \tag{15.25}$$

■ **A:** $f(t) = u(t-t_0)\sin\omega t,$ $\qquad F(p) = e^{-t_0 p}\dfrac{\omega\,\cos\omega\,t_0 + p\,\sin\omega\,t_0}{p^2+\omega^2}$ **(Fig. 15.8)**.

■ **B:** $f(t) = u(t-t_0)\sin\omega\,(t-t_0),$ $\qquad F(p) = e^{-t_0 p}\dfrac{\omega}{p^2+\omega^2}$ **(Fig. 15.9)**.

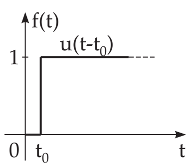

Figure 15.7

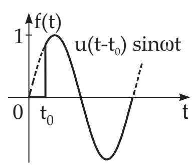

Figure 15.8

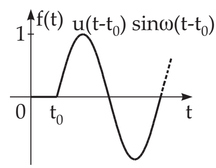

Figure 15.9

2. Rectangular Impulse

A rectangular impulse of height 1 and width T **(Fig. 15.10)** is composed by the superposition of two step functions in the form

$$u_T(t-t_0) = u(t-t_0) - u(t-t_0-T) = \begin{cases} 0 & \text{for } t < t_0, \\ 1 & \text{for } t_0 < t < t_0+T, \\ 0 & \text{for } t > t_0+T; \end{cases} \tag{15.26}$$

$$\mathcal{L}\{u_T(t-t_0)\} = \frac{e^{-t_0 p}(1-e^{-Tp})}{p}. \tag{15.27}$$

3. Impulse Function (Dirac δ Function)

(See also 12.9.5.4, p. 700.) The impulse function $\delta(t-t_0)$ can obviously be interpreted as a limit of the rectangular impulse of width T and height $1/T$ at the point $t = t_0$ **(Fig. 15.11)**:

$$\delta(t-t_0) = \lim_{T\to 0}\frac{1}{T}[\,u(t-t_0) - u(t-t_0-T)\,]. \tag{15.28}$$

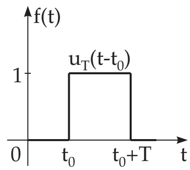

Figure 15.10

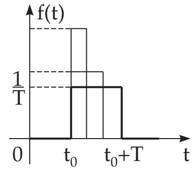

Figure 15.11

For a continuous function $h(t)$,

$$\int_a^b h(t)\,\delta(t-t_0)\,dt = \begin{cases} h(t_0), & \text{if } t_0 \text{ is inside } (a,b), \\ 0, & \text{if } t_0 \text{ is outside } (a,b). \end{cases} \tag{15.29}$$

Relations such as

$$\delta(t-t_0) = \frac{du(t-t_0)}{dt}, \qquad \mathcal{L}\{\,\delta(t-t_0)\} = e^{-t_0 p} \qquad (t_0 \geq 0) \tag{15.30}$$

are investigated generally in *distribution theory* (see 12.9.5.3, p. 699).

4. Piecewise Differentiable Functions

The transform of a piecewise differentiable function can be determined easily with the help of the δ function: If $f(t)$ is piecewise differentiable and at the points t_ν $(\nu = 1, 2, \ldots, n)$ it has jumps a_ν, then its first derivative can be represented in the form

$$\frac{df(t)}{dt} = f_s'(t) + a_1\delta(t-t_1) + a_2\delta(t-t_2) + \cdots + a_n\delta(t-t_n) \tag{15.31}$$

where $f_s'(t)$ is the usual derivative of $f(t)$, where it is differentiable.
If jumps occur first in the derivative, then similar formulas are valid. In this way, one can easily determine the transform of functions which correspond to curves composed of parabolic arcs of arbitrarily high degree, e.g., curves found empirically. In formal application of (15.13), the values $f(+0), f'(+0), \ldots$ should be replaced by zero in the case of a jump.

■ **A:**
$f(t) = \begin{cases} at+b \text{ for } 0<t<t_0, \\ 0 \quad\quad \text{otherwise,} \end{cases}$ **(Fig. 15.12)**; $f'(t) = a\,u_{t_0}(t) + b\,\delta(t) - (at_0+b)\,\delta(t-t_0)$; $\mathcal{L}\{f'(t)\} =$
$\dfrac{a}{p}(1-e^{-t_0 p}) + b - (at_0+b)\,e^{-t_0 p}$; $\mathcal{L}\{f(t)\} = \dfrac{1}{p}\left[\dfrac{a}{p} + b - e^{-t_0 p}\left(\dfrac{a}{p} + at_0 + b\right)\right]$.

■ **B:**
$f(t) = \begin{cases} t & \text{for } 0<t<t_0, \\ 2t_0 - t & \text{for } t_0<t<2t_0, \\ 0 & \text{for } t>2t_0, \end{cases}$ **(Fig. 15.13)**; $f'(t) = \begin{cases} 1 & \text{for } 0<t<t_0, \\ -1 & \text{for } t_0<t<2t_0, \\ 0 & \text{for } t>2t_0, \end{cases}$ **(Fig. 15.14)**;

$f''(t) = \delta(t) - \delta(t-t_0) - \delta(t-t_0) + \delta(t-2t_0)$; $\mathcal{L}\{f''(t)\} = 1 - 2e^{-t_0 p} + e^{-2t_0 p}$; $\mathcal{L}\{f(t)\} = \dfrac{(1-e^{-t_0 p})^2}{p^2}$.

■ **C:** $f(t) = \begin{cases} E\,t/t_0 & \text{for} & 0<t<t_0, \\ E & \text{for} & t_0<t<T-t_0, \\ -E(t-T)/t_0 & \text{for} & T-t_0<t<T, \\ 0 & \text{otherwise,} \end{cases}$ **(Fig. 15.15)**;

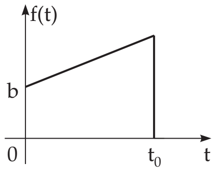

Figure 15.12

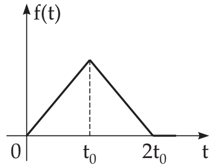

Figure 15.13

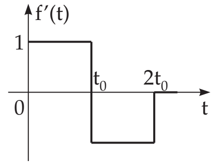

Figure 15.14

$$f'(t) = \begin{cases} E/t_0 & \text{for} \quad 0 < t < t_0, \\ 0 & \text{for} \quad t_0 < t < T - t_0,\ (t > T), \\ -E/t_0 & \text{for} \quad T - t_0 < t < T, \\ 0 & \text{otherwise,} \end{cases} \quad \textbf{(Fig. 15.16)};$$

$$f''(t) = \frac{E}{t_0}\delta(t) - \frac{E}{t_0}\delta(t-t_0) - \frac{E}{t_0}\delta(t-T+t_0) + \frac{E}{t_0}\delta(t-T);\ \mathcal{L}\{f''(t)\} = \frac{E}{t_0}\left[1 - e^{-t_0 p} - e^{-(T-t_0)p} + e^{-Tp}\right];$$

$$\mathcal{L}\{f(t)\} = \frac{E}{t_0}\,\frac{(1 - e^{-t_0 p})(1 - e^{-(T-t_0)p})}{p^2}.$$

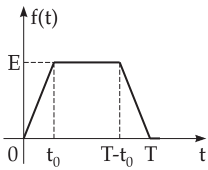

Figure 15.15

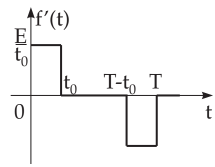

Figure 15.16

■ **D:**

$$f(t) = \begin{cases} t - t^2 & \text{for } 0 < t < 1, \\ 0 & \text{otherwise,} \end{cases} \quad \textbf{(Fig. 15.17)}; \quad f'(t) = \begin{cases} 1 - 2t & \text{for } 0 < t < 1, \\ 0 & \text{otherwise,} \end{cases} \quad \textbf{(Fig. 15.18)};$$

$$f''(t) = -2u_1(t) + \delta(t) + \delta(t-1);$$

$$\mathcal{L}\{f''(t)\} = -\frac{2}{p}(1 - e^{-p}) + 1 + e^{-p}; \quad \mathcal{L}\{f(t)\} = \frac{1 + e^{-p}}{p^2} - \frac{2\,(1 - e^{-p})}{p^3}.$$

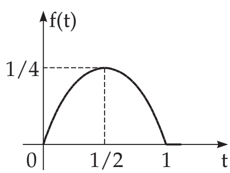

Figure 15.17

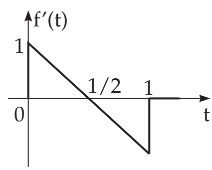

Figure 15.18

5. Periodic Functions

The transform of a periodic function $f^*(t)$ with period T, which is a periodic continuation of a function $f(t)$, can be obtained from the Laplace transform of $f(t)$ multiplied by the *periodization factor*

$$(1 - e^{-Tp})^{-1}. \tag{15.32}$$

■ **A:** The periodic continuation of $f(t)$ from example **B** (see above) with period $T = 2t_0$ is $f^*(t)$ with $\mathcal{L}\{f^*(t)\} = \dfrac{(1-e^{-t_0 p})^2}{p^2} \cdot \dfrac{1}{1-e^{-2t_0 p}} = \dfrac{1-e^{-t_0 p}}{p^2(1+e^{-t_0 p})}$.

■ **B:** The periodic continuation of $f(t)$ from example **C** (see above) with period T is $f^*(t)$ with $\mathcal{L}\{f^*(t)\} = \dfrac{E\,(1-e^{-t_0 p})\,(1-e^{-(T-t_0)p})}{t_0\,p^2\,(1-e^{-Tp})}$.

15.2.1.4 Dirac δ Function and Distributions

In describing certain technical systems by linear differential equations, functions $u(t)$ and $\delta(t)$ often occur as perturbation or input functions, although the conditions required in 15.2.1.1, **1.** p. 770, are not satisfied: $u(t)$ is discontinuous, and $\delta(t)$ cannot be defined in the sense of classical analysis.
Distribution theory offers a solution by introducing so-called *generalized functions* (*distributions*), so that with the known continuous real functions $\delta(t)$ can also be examined, where the necessary differentiability is also guaranteed. Distributions can be represented in different ways. One of the best known representations is the continuous real linear form, introduced by L. Schwartz (see 12.9.5, p. 698).
Fourier coefficients and Fourier series can be associated uniquely to periodic distributions, analogously to real functions (see 7.4, p. 474).

1. Approximations of the δ Function

Analogously to (15.28), the impulse function $\delta(t)$ can be approximated by a rectangular impulse of width ε and height $1/\varepsilon$ $(\varepsilon > 0)$:

$$f(t,\varepsilon) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2, \\ 0 & \text{for } |t| \geq \varepsilon/2. \end{cases} \tag{15.33a}$$

Further examples of the approximation of $\delta(t)$ are the error curve (see 2.6.3, p. 73) and Lorentz function (see 2.11.2, p. 95):

$$f(t,\varepsilon) = \frac{1}{\varepsilon\sqrt{2\pi}} e^{-\frac{t^2}{2\varepsilon^2}} \quad (\varepsilon > 0), \tag{15.33b}$$

$$f(t,\varepsilon) = \frac{\varepsilon/\pi}{t^2+\varepsilon^2} \qquad (\varepsilon > 0). \tag{15.33c}$$

These functions have the common properties:

1. $$\int_{-\infty}^{\infty} f(t,\varepsilon)\,dt = 1. \tag{15.34a}$$

2. $$f(-t,\varepsilon) = f(t,\varepsilon), \text{ i.e., they are even functions.} \tag{15.34b}$$

3. $$\lim_{\varepsilon\to 0} f(t,\varepsilon) = \begin{cases} \infty & \text{for } t = 0, \\ 0 & \text{for } t \neq 0. \end{cases} \tag{15.34c}$$

2. Properties of the δ Function

Important properties of the δ function are:

1. $$\int_{x-a}^{x+a} f(t)\delta(x-t)\,dt = f(x) \quad (f \text{ is continuous, } a > 0). \tag{15.35}$$

2. $$\delta(\alpha x) = \frac{1}{\alpha}\delta(x) \quad (\alpha > 0). \tag{15.36}$$

3. $$\delta\,(g(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|}\delta(x-x_i) \quad \text{with } g(x_i) = 0 \text{ and } g'(x_i) \neq 0 \;\; (i = 1, 2, \ldots, n). \tag{15.37}$$

Here all roots of $g(x)$ are considered and they must be simple.

4. n-th Derivative of the δ Function: After n repeated partial integrations of

$$f^{(n)}(x) = \int_{x-a}^{x+a} f^{(n)}(t)\,\delta(x-t)\,dt, \tag{15.38a}$$

a rule is obtained for the n-th derivative of the δ function:

$$(-1)^n f^{(n)}(x) = \int_{x-a}^{x+a} f(t)\,\delta^{(n)}(x-t)\,dt. \tag{15.38b}$$

15.2.2 Inverse Transformation into the Original Space

To perform an inverse transformation, there are the following possibilities:
1. Using a table of correspondences, i.e., a table with the corresponding original functions and transforms (see **Table 21.13**, p. 1109).
2. Reducing to known correspondences by using some properties of the transformation (see 15.2.2.2, p. 778, and 15.2.2.3, p. 779).
3. Evaluating the inverse formula (see 15.2.2.4, p. 780).

15.2.2.1 Inverse Transformation with the Help of Tables

The use of a table is shown here by an example with **Table 21.13**, p. 1109.
Further tables can be found, e.g., in [15.3].

■ $F(p) = \dfrac{1}{(p^2+\omega^2)(p+c)} = F_1(p)\cdot F_2(p)$, $\mathcal{L}^{-1}\{F_1(p)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{p^2+\omega^2}\right\} = \dfrac{1}{\omega}\sin\omega t = f_1(t)$,
$\mathcal{L}^{-1}\{F_2(p)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{p+c}\right\} = e^{-ct} = f_2(t)$. Applying the convolution theorem (15.23) yields:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F_1(p)\cdot F_2(p)\} \\ &= \int_0^t f_1(\tau)\cdot f_2(t-\tau)\,d\tau = \int_0^t e^{-c(t-\tau)}\frac{\sin\omega\tau}{\omega}\,d\tau = \frac{1}{c^2+\omega^2}\left(\frac{c\sin\omega t - \omega\cos\omega t}{\omega} + e^{-ct}\right). \end{aligned}$$

15.2.2.2 Partial Fraction Decomposition

1. Principle

In many applications, there are transforms in the form $F(p) = H(p)/G(p)$, where $G(p)$ is a polynomial of p. If the original functions for $H(p)$ and $1/G(p)$ are already known, then the required original function of $F(p)$ can be got by applying the convolution theorem.

2. Simple Real Roots of $G(p)$

If the transform $1/G(p)$ has only simple poles p_ν $(\nu = 1, 2, \ldots, n)$, then it has the following partial fraction decomposition:

$$\frac{1}{G(p)} = \sum_{\nu=1}^{n} \frac{1}{G'(p_\nu)(p-p_\nu)}. \tag{15.39}$$

The corresponding original function is

$$q(t) = \mathcal{L}^{-1}\left\{\frac{1}{G(p)}\right\} = \sum_{\nu=1}^{n} \frac{1}{G'(p_\nu)} e^{p_\nu t}. \tag{15.40}$$

3. The Heaviside Expansion Theorem

If the numerator $H(p)$ is also a polynomial of p with a lower degree than $G(p)$, then we can obtain the original function of $F(p)$ with the help of the Heaviside formula

$$f(t) = \sum_{\nu=1}^{n} \frac{H(p_\nu)}{G'(p_\nu)} e^{p_\nu t}. \tag{15.41}$$

4. Complex Roots

Even in cases when the denominator has simple complex roots, the Heaviside expansion theorem can be used in the same way. The terms belonging to complex conjugate roots can be collected into one quadratic expression, whose inverse transformation can be found in tables also in the case of roots of higher multiplicity.

■ $F(p) = \dfrac{1}{(p+c)(p^2+\omega^2)}$, i.e., $H(p) = 1$, $G(p) = (p+c)(p^2+\omega^2)$, $G'(p) = 3p^2 + 2pc + \omega^2$. The zeroes of $G(p)$ $p_1 = -c$, $p_2 = \mathrm{i}\omega$, $p_3 = -\mathrm{i}\omega$ are all simple. According to the Heaviside theorem one gets $f(t) = \dfrac{1}{\omega^2+c^2} e^{-ct} - \dfrac{1}{2\omega(\omega - \mathrm{i}c)} e^{\mathrm{i}\omega t} - \dfrac{1}{2\omega(\omega+\mathrm{i}c)} e^{-\mathrm{i}\omega t}$ or by using partial fraction decomposition and the table $F(p) = \dfrac{1}{\omega^2+c^2}\left[\dfrac{1}{p+c} + \dfrac{c-p}{p^2+\omega^2}\right]$, $f(t) = \dfrac{1}{\omega^2+c^2}\left[e^{-ct} + \dfrac{c}{\omega}\sin\omega t - \cos\omega t\right]$. These expressions for $f(t)$ are identical.

15.2.2.3 Series Expansion

In order to obtain $f(t)$ from $F(p)$ one can try to expand $F(p)$ into a series $F(p) = \sum\limits_{n=0}^{\infty} F_n(p)$, whose terms $F_n(p)$ are transforms of known functions, i.e., $F_n(p) = \mathcal{L}\{f_n(t)\}$.

1. $F(p)$ is an Absolutely Convergent Series

If $F(p)$ has an absolutely convergent series

$$F(p) = \sum_{n=0}^{\infty} \frac{a_n}{p^{\lambda_n}}, \tag{15.42}$$

for $|p| > R$, where the values λ_n form an arbitrary increasing sequences of numbers $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots < \cdots \to \infty$, then a termwise inverse transformation is possible:

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\lambda_n - 1}}{\Gamma(\lambda_n)}. \tag{15.43}$$

Γ denotes the gamma function (see 8.2.5, **6.**, p. 514). In particular, for $\lambda_n = n+1$, i.e., for $F(p) = \sum\limits_{n=0}^{\infty} \dfrac{a_{n+1}}{p^{n+1}}$ the series $f(t) = \sum\limits_{n=0}^{\infty} \dfrac{a_{n+1}}{n!} t^n$ is obtained, which is convergent for every real and complex t. Furthermore, one can have an estimation in the form $|f(t)| < C\,e^{c|t|}$ (C, c real constants).

■ $F(p) = \dfrac{1}{\sqrt{1+p^2}} = \dfrac{1}{p}\left(1 + \dfrac{1}{p^2}\right)^{-1/2} = \sum\limits_{n=0}^{\infty} \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} \dfrac{1}{p^{2n+1}}$. After a termwise transformation into the original space the result is $f(t) = \sum\limits_{n=0}^{\infty} \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} \dfrac{t^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(n!)^2} \left(\dfrac{t}{2}\right)^{2n} = J_0(t)$ (Bessel function of 0 order).

2. $F(p)$ is a Meromorphic Function

If $F(p)$ is a *meromorphic function*, which can be represented as the quotient of two integer functions (of two functions having everywhere convergent power series expansions) which do not have common roots, and so can be rewritten as the sum of an integer function and infinitely many partial fractions, then the equality

$$\frac{1}{2\pi \mathrm{i}}\int\limits_{c-\mathrm{i}y_n}^{c+\mathrm{i}y_n} e^{tp}\,F(p)\,dp = \sum_{\nu=1}^{n} b_\nu e^{p_\nu t} - \frac{1}{2\pi \mathrm{i}}\int\limits_{(K_n)} e^{tp}F(p)\,dp \tag{15.44}$$

is obtained. Here p_ν ($\nu = 1, 2, \ldots, n$) are the first-order poles of the function $F(p)$, b_ν are the corresponding residues (see 14.3.5.4, p. 753), y_ν are certain values and K_ν are certain curves, for example, half circles in the sense represented in **Fig. 15.19**. The solution $f(t)$ has the form

$$f(t) = \sum_{\nu=1}^{\infty} b_\nu e^{p_\nu t}, \qquad \text{if} \qquad \frac{1}{2\pi \mathrm{i}}\int\limits_{(K_n)} e^{tp}F(p)\,dp \;\to\; 0 \tag{15.45}$$

as $y \to \infty$, what is often not easy to verify.

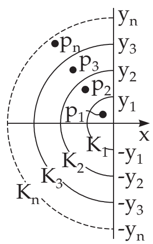

Figure 15.19

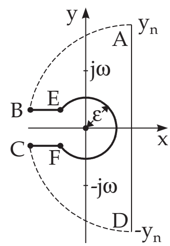

Figure 15.20

In certain cases, e.g., when the rational part of the meromorphic function $F(p)$ is identically zero, the above result is a formal application of the Heaviside expansion theorem to meromorphic functions.

15.2.2.4 Inverse Integral

The inverse formula

$$f(t) = \lim_{y_n\to\infty} \frac{1}{2\pi \mathrm{i}}\int\limits_{c-\mathrm{i}y_n}^{c+\mathrm{i}y_n} e^{tp}\,F(p)\,dp \tag{15.46}$$

represents a complex integral of a function analytic in a certain domain. The usual methods of integration for complex functions can be used, e.g., the residue calculation or certain changes of the path of integration according to the Cauchy integral theorem.

■ $F(p) = \dfrac{p}{p^2+\omega^2}e^{-\sqrt{p}\alpha}$ is double valued because of $\sqrt{p}$. Therefore, we chose the following path of integration **(Fig. 15.20)**: $\displaystyle\frac{1}{2\pi \mathrm{i}}\oint\limits_{(K)} e^{tp}\frac{p}{p^2+\omega^2}e^{-\sqrt{p}\alpha}\,dp = \int\limits_{\overset{\frown}{AB}}\cdots + \int\limits_{\overset{\frown}{CD}}\cdots + \int\limits_{\overset{\frown}{EF}}\cdots + \int\limits_{\overline{DA}}\cdots + \int\limits_{\overline{BE}}\cdots + \int\limits_{\overline{FC}}\cdots =$

$\sum \mathrm{Res}\, e^{tp}F(p) = e^{-\alpha\sqrt{\omega/2}}\cos(\omega t - \alpha\sqrt{\omega/2})$. According to the Jordan lemma (see 14.4.3, p. 755), the integral part over $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ vanishes as $y_n \to \infty$. The integrand remains bounded on the circular arc $\overset{\frown}{EF}$ (radius ε), and the length of the path of integration tends to zero for $\varepsilon \to 0$; so this term of the integral also vanishes. There are to investigate the integrals on the two horizontal segments $\overline{BE}$ and $\overline{FC}$, where it is to consider the upper side ($p = re^{\mathrm{i}\pi}$) and the lower side ($p = re^{-\mathrm{i}\pi}$) of the negative real axis:

$$\int_{-\infty}^{0} F(p)e^{tp}\,dp = -\int_0^\infty e^{-tr}\frac{r}{r^2+\omega^2}e^{-\mathrm{i}\alpha\sqrt{r}}\,dr, \qquad \int_0^{-\infty} F(p)e^{tp}\,dp = \int_0^\infty e^{-tr}\frac{r}{r^2+\omega^2}e^{\,\mathrm{i}\alpha\sqrt{r}}\,dr.$$

Finally one gets:

$$f(t) = e^{-\alpha\sqrt{\omega/2}}\cos\left(\omega\,t - \alpha\sqrt{\frac{\omega}{2}}\right) - \tfrac{1}{\pi}\int_0^\infty e^{-tr}\frac{r\sin\alpha\sqrt{r}}{r^2+\omega^2}\,dr.$$

15.2.3 Solution of Differential Equations using Laplace Transformation

It has been noticed already from the rules of calculation of the Laplace transformation (see 15.2.1.2, p. 771), that complicated operations, such as differentiation or integration in the original space, can be replaced by simple algebraic operations in the image space using the Laplace transform. Here, some additional conditions are considered, such as initial conditions in using the differentiation rule. These conditions are necessary for the solution of differential equations.

15.2.3.1 Ordinary Linear Differential Equations with Constant Coefficients

1. Principle

The n-th order differential equation of the form

$$y^{(n)}(t) + c_{n-1}\,y^{(n-1)}(t) + \cdots + c_1\,y'(t) + c_0\,y(t) = f(t) \tag{15.47a}$$

with the initial values $y(+0) = y_0,\, y'(+0) = y_0',\ldots, y^{(n-1)}(+0) = y_0^{(n-1)}$ can be transformed by Laplace transformation into the equation

$$\sum_{k=0}^{n} c_k p^k Y(p) - \sum_{k=1}^{n} c_k \sum_{\nu=0}^{k-1} p^{k-\nu-1} y_0^{(\nu)} = F(p) \qquad (c_n = 1). \tag{15.47b}$$

Here $G(p) = \sum\limits_{k=0}^{n} c_k p^k = 0$ is the characteristic equation of the differential equation (see 4.6.2.1, p. 315).

2. First-Order Differential Equations

The original and the transformed equations are:

$$y'(t) + c_0 y(t) = f(t), \quad y(+0) = y_0, \qquad (15.48a) \qquad (p+c_0)\,Y(p) - y_0 = F(p), \qquad (15.48b)$$

where $c_0 = \text{const}$. The solution for $Y(p)$ results in

$$Y(p) = \frac{F(p)+y_0}{p+c_0}. \tag{15.48c}$$

Special case: For $f(t) = \lambda\, e^{\mu t}$ with $F(p) = \dfrac{\lambda}{p-\mu}$, $(\lambda, \mu \text{ const})$: (15.49a)

$$Y(p) = \frac{\lambda}{(p-\mu)(p+c_0)} + \frac{y_0}{p+c_0}, \tag{15.49b}$$

$$y(t) = \frac{\lambda}{\mu+c_0}e^{\mu t} + \left(y_0 - \frac{\lambda}{\mu+c_0}\right)e^{-c_0 t}. \tag{15.49c}$$

3. Second-Order Differential Equations

The original and transformed equations are:

$$y''(t) + 2ay'(t) + by(t) = f(t), \qquad y(+0) = y_0, \qquad y'(+0) = y_0'. \tag{15.50a}$$

$$(p^2 + 2ap + b)\,Y(p) - 2ay_0 - (py_0 + y_0') = F(p). \tag{15.50b}$$

The solution for $Y(p)$ results in

$$Y(p) = \frac{F(p) + (2a+p)\,y_0 + y_0'}{p^2 + 2ap + b}. \tag{15.50c}$$

Distinction of Cases:

a) $b < a^2:\quad G(p) = (p-\alpha_1)(p-\alpha_2) \quad (\alpha_1,\ \alpha_2 \text{ real; } \alpha_1 \neq \alpha_2),$ (15.51a)

$$q(t) = \mathcal{L}^{-1}\left\{\frac{1}{G(p)}\right\} = \frac{1}{\alpha_1 - \alpha_2}(e^{\alpha_1 t} - e^{\alpha_2 t}). \tag{15.51b}$$

b) $b = a^2:\quad G(p) = (p-\alpha)^2,$ (15.52a) $\qquad q(t) = t\,e^{\alpha t}.$ (15.52b)

c) $b > a^2:\quad G(p)$ has complex roots, (15.53a)

$$q(t) = \mathcal{L}^{-1}\left\{\frac{1}{G(p)}\right\} = \frac{1}{\sqrt{b-a^2}}\,e^{-at}\sin\sqrt{b-a^2}\,t. \tag{15.53b}$$

The solution $y(t)$ can be obtained as the convolution of the original function of the numerator of $Y(p)$ and $q(t)$. The application of the convolution can be avoided if a direct transformation of the right-hand side can be found.

■ The transformed equation for the differential equation $y''(t) + 2y'(t) + 10y(t) = 37\cos 3t + 9e^{-t}$ with $y_0 = 1$ and $y_0' = 0$ is $Y(p) = \dfrac{p+2}{p^2+2p+10} + \dfrac{37p}{(p^2+9)(p^2+2p+10)} + \dfrac{9}{(p+1)(p^2+2p+10)}$. The representation $Y(p) = \dfrac{-p}{p^2+2p+10} - \dfrac{19}{(p^2+2p+10)} + \dfrac{p}{(p^2+9)} + \dfrac{18}{(p^2+9)} + \dfrac{1}{(p+1)}$ follows from partial fraction decomposition of the second and third terms of the right-hand side but not separating the second-order terms into linear ones. The solution after termwise transformation is (see **Table 21.13**, p. 1109) $y(t) = (-\cos 3t - 6\sin 3t)e^{-t} + \cos 3t + 6\sin 3t + e^{-t}$.

4. n-th Order Differential Equations

The characteristic equation $G(p) = 0$ of this differential equation (see (15.47a)) has only simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, and none of them is equal to zero. Two cases are distinguished for the perturbation function $f(t)$.

1. If the perturbation function $f(t)$ is the jump function $u(t)$ which often occurs in practical problems, then the solution is:

$$u(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0, \end{cases} \tag{15.54a}$$

$$y(t) = \frac{1}{G(0)} + \sum_{\nu=1}^{n} \frac{1}{\alpha_\nu G'(\alpha_\nu)} e^{\alpha_\nu t}. \tag{15.54b}$$

2. For a general perturbation function $f(t)$, one gets the solution $\tilde{y}(t)$ from (15.54b) in the form of the Duhamel formula which uses the convolution (see 15.2.1.2, **11.**, p. 773):

$$\tilde{y}(t) = \frac{d}{dt}\int_0^t y(t-\tau)f(\tau)\,d\tau = \frac{d}{dt}[\,y * f\,]. \tag{15.55}$$

15.2.3.2 Ordinary Linear Differential Equations with Coefficients Depending on the Variable

Differential equations whose coefficients are polynomials in t can also be solved by Laplace transformation. Applying (15.16), in the image space yields a differential equation, whose order can be lower than the original one.

If the coefficients are first-order polynomials, then the differential equation in the image space is a first-order differential equation and may be it can be solved more easily.

■ Bessel differential equation of 0 order: $t\dfrac{d^2f}{dt^2} + \dfrac{df}{dt} + tf = 0$ (see (9.52a, p. 562) for $n = 0$). The transformation into the image space results in

$$-\frac{d}{dp}[\,p^2F(p) - pf(0) - f'(0)\,] + pF(p) - f(0) - \frac{dF(p)}{dp} = 0 \quad \text{or} \quad \frac{dF}{dp} = -\frac{p}{p^2+1}F(p)\,.$$

Separation of the variables and integration yields $\log F(p) = -\displaystyle\int \frac{p\,dp}{p^2+1} = -\log\sqrt{p^2+1} + \log C$,

$F(p) = \dfrac{C}{\sqrt{p^2+1}}$ (C is the integration constant), $f(t) = CJ_0(t)$ (see ■ in 15.2.2.3,**1.**, p. 779 with the Bessel function of 0 order).

15.2.3.3 Partial Differential Equations

1. General Introduction

The solution of a partial differential equation is a function of at least two variables: $u = u(x,t)$. Since the Laplace transformation represents an integration with respect to only one variable, the other variable should be considered as a constant in the transformation:

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-pt}u(x,t)\,dt = U(x,p). \tag{15.56}$$

x also remains fixed in the transformation of derivatives:

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\} &= p\,\mathcal{L}\{u(x,t)\} - u(x,+0),\\ \mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} &= p^2\mathcal{L}\{u(x,t)\} - u(x,+0)p - u_t(x,+0). \end{aligned} \tag{15.57}$$

The differentiation with respect to x is supposed to be interchangeable with the Laplace integral:

$$\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\} = \frac{\partial}{\partial x}U(x,p). \tag{15.58}$$

In this way, an ordinary differential equation is obtained in the image space. Furthermore, the boundary and initial conditions are to be transformed into the image space.

2. Solution of the One-Dimensional Heat Conduction Equation for a Homogeneous Medium

1. Formulation of the Problem Suppose the one-dimensional heat conduction equation with vanishing perturbation and for a homogeneous medium is given in the form

$$u_{xx} - a^{-2}u_t = u_{xx} - u_y = 0 \tag{15.59a}$$

in the original space $0 < t < \infty$, $0 < x < l$ and with the initial and boundary conditions

$$u(x,+0) = u_0(x), \quad u(+0,t) = a_0(t), \quad u(l-0,t) = a_1(t). \tag{15.59b}$$

The time coordinate is replaced by $y = at$. (15.59a) is also a parabolic type equation, just as the three-dimensional heat conduction equation (see 9.2.3.3, p. 591).

2. Laplace Transformation The transformed equation is

$$\frac{d^2U}{dx^2} = pU - u_0(x), \tag{15.60a}$$

and the boundary conditions are

$$U(+0,p) = A_0(p), \quad U(l-0,p) = A_1(p). \tag{15.60b}$$

The solution of the transformed equation for zero starting temperature $u_0(x) = 0$ is

$$U(x,p) = c_1e^{x\sqrt{p}} + c_2e^{-x\sqrt{p}}. \tag{15.60c}$$

It is a good idea to produce two particular solutions U_1 and U_2 with the properties

$$U_1(0,p) = 1, \quad U_1(l,p) = 0, \qquad (15.61a) \qquad U_2(0,p) = 0, \quad U_2(l,p) = 1, \text{ i.e.,} \qquad (15.61b)$$

$$U_1(x,p) = \frac{e^{(l-x)\sqrt{p}} - e^{-(l-x)\sqrt{p}}}{e^{l\sqrt{p}} - e^{-l\sqrt{p}}}, \qquad (15.61c) \qquad U_2(x,p) = \frac{e^{x\sqrt{p}} - e^{-x\sqrt{p}}}{e^{l\sqrt{p}} - e^{-l\sqrt{p}}}. \qquad (15.61d)$$

The required solution of the transformed equation has the form

$$U(x,p) = A_0(p)\,U_1(x,p) + A_1(p)\,U_2(x,p). \qquad (15.62)$$

3. Inverse Transformation The inverse transformation is especially easy in the case of $l \to \infty$:

$$U(x,p) = a_0(p)e^{-x\sqrt{p}}, \qquad (15.63a) \qquad u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{a_0(t-\tau)}{\tau^{3/2}} \exp\left(-\frac{x^2}{4\tau}\right) d\tau. \qquad (15.63b)$$

15.3 Fourier Transformation

15.3.1 Properties of the Fourier Transformation

15.3.1.1 Fourier Integral

1. Fourier Integral in Complex Representation

The basis of the Fourier transformation is the Fourier integral, also called the *integral formula of Fourier*. If a non-periodic function $f(t)$ satisfies the Dirichlet conditions (see 7.4.1.2, **3.**, p. 475) in an arbitrary finite interval, and furthermore the integral

$$\int_{-\infty}^{+\infty} |f(t)|\,dt \qquad (15.64a)$$

is convergent, then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega(t-\tau)} f(\tau)\,d\omega\,d\tau \qquad (15.64b)$$

at every point where the function $f(t)$ is continuous, and

$$\frac{f(t+0) + f(t-0)}{2} = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{+\infty} f(\tau)\cos\omega\,(t-\tau)\,d\tau \qquad (15.64c)$$

at the points of discontinuity.

2. Equivalent Representations

Other equivalent forms for the Fourier integral (15.64b) are:

1. $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau)\cos[\,\omega\,(t-\tau)\,]\,d\omega\,d\tau. \qquad (15.65a)$$

2. $$f(t) = \int_0^{\infty} [\,a(\omega)\cos\omega t + b(\omega)\sin\omega t\,]\,d\omega \qquad (15.65b)$$ with the coefficients

$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos\omega t\,dt \qquad (15.65c) \qquad b(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin\omega t\,dt. \qquad (15.65d)$$

3. $$f(t) = \int_0^{\infty} A(\omega)\cos[\,\omega t + \psi(\omega)\,]\,d\omega. \qquad (15.66)$$

4. $f(t) = \int\limits_0^{\infty} A(\omega) \sin[\omega t + \varphi(\omega)]\, d\omega.$ (15.67)

The following relations are valid here:

$$A(\omega) = \sqrt{a^2(\omega) + b^2(\omega)}, \quad (15.68a) \qquad \varphi(\omega) = \psi(\omega) + \frac{\pi}{2}, \quad (15.68b)$$

$$\cos\psi(\omega) = \frac{a(\omega)}{A(\omega)}, \quad (15.68c) \qquad \sin\psi(\omega) = \frac{b(\omega)}{A(\omega)}, \quad (15.68d)$$

$$\cos\varphi(\omega) = \frac{b(\omega)}{A(\omega)}, \quad (15.68e) \qquad \sin\varphi(\omega) = \frac{a(\omega)}{A(\omega)}. \quad (15.68f)$$

15.3.1.2 Fourier Transformation and Inverse Transformation

1. Definition of the Fourier Transformation

The Fourier transformation is an integral transformation of the form (15.1a), which comes from the Fourier integral (15.64b) by substituting

$$F(\omega) = \int\limits_{-\infty}^{+\infty} e^{-i\omega\tau} f(\tau)\, d\tau. \quad (15.69)$$

The following relation is valid between the real original function $f(t)$ and the usually complex transform $F(\omega)$:

$$f(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} e^{i\omega t} F(\omega)\, d\omega. \quad (15.70)$$

In the brief notation one uses $\mathcal{F}$:

$$F(\omega) = \mathcal{F}\{f(t)\} = \int\limits_{-\infty}^{+\infty} e^{-i\omega t} f(t)\, dt. \quad (15.71)$$

The original function $f(t)$ is Fourier transformable if the integral (15.69), i.e., an improper integral with the parameter ω, exists. If the Fourier integral does not exist as an ordinary improper integral, then it is considered as the Cauchy principal value (see 8.2.3.3, **1.**, p. 510). The transform $F(\omega)$ is also called the *Fourier transform*; it is bounded, continuous, and it tends to zero for $|\omega| \to \infty$:

$$\lim_{|\omega| \to \infty} F(\omega) = 0. \quad (15.72)$$

The existence and boundedness of $F(\omega)$ follow directly from the obvious inequality

$$|F(\omega)| \le \int\limits_{-\infty}^{+\infty} |e^{-i\omega t} f(t)|\, dt \le \int\limits_{-\infty}^{+\infty} |f(t)|\, dt. \quad (15.73)$$

The existence of the Fourier transform is a sufficient condition for the continuity of $F(\omega)$ and for the properties $F(\omega) \to 0$ for $|\omega| \to \infty$. This statement is often used in the following form: If the function $f(t)$ in $(-\infty, \infty)$ is absolutely integrable, then its Fourier transform is a continuous function of ω, and (15.72) holds.

The following functions are not Fourier transformable: Constant functions, arbitrary periodic functions (e.g., $\sin\omega t, \cos\omega t$), power functions, polynomials, exponential functions (e.g., $e^{\alpha t}$, hyperbolic functions).

2. Fourier Cosine and Fourier Sine Transformation

In the Fourier transformation (15.71), the integrand can be decomposed into a sine and a cosine part. So, one gets the sine and the cosine Fourier transformation.

1. Fourier Sine Transformation

$$F_s(\omega) = \mathcal{F}_s\{\, f(t)\,\} = \int_0^{\infty} f(t)\sin(\omega t)\,dt. \tag{15.74a}$$

2. Fourier Cosine Transformation

$$F_c(\omega) = \mathcal{F}_c\{\, f(t)\,\} = \int_0^{\infty} f(t)\cos(\omega\, t)\,dt. \tag{15.74b}$$

3. Conversion Formulas Between the Fourier sine (15.74a) and the Fourier cosine transformation (15.74b) on one hand, and the Fourier transformation (15.71) on the other hand, the following relations are valid:

$$F(\omega) = \mathcal{F}\{\, f(t)\,\} = \mathcal{F}_c\{\, f(t) + f(-t)\,\} - \mathrm{i}\mathcal{F}_s\{\, f(t) - f(-t)\,\}, \tag{15.75a}$$

$$F_s(\omega) = \frac{\mathrm{i}}{2}\mathcal{F}\{\, f(|t|)\operatorname{sign} t\,\}, \quad (15.75\text{b}) \qquad F_c(\omega) = \frac{1}{2}\mathcal{F}\{\, f(|t|)\,\}. \quad (15.75\text{c})$$

For an even or for an odd function $f(t)$ the following representations hold:

$$\begin{aligned} &f(t)\ \text{even:}\ \mathcal{F}\{\, f(t)\,\} = 2\mathcal{F}_c\{\, f(t)\,\},\\ &f(t)\ \text{odd:}\ \ \mathcal{F}\{\, f(t)\,\} = -2\mathrm{i}\mathcal{F}_s\{\, f(t)\,\}. \end{aligned} \tag{15.75d}$$

3. Exponential Fourier Transformation

Differently from the definition of $F(\omega)$ in (15.71), the transform

$$F_e(\omega) = \mathcal{F}_e\{f(t)\} = \frac{1}{2}\int_{-\infty}^{+\infty} e^{\mathrm{i}\omega t} f(t)\,dt \tag{15.76}$$

is called the *exponential Fourier transformation*, so that

$$F(\omega) = 2F_e(-\omega). \tag{15.77}$$

4. Tables of the Fourier Transformation

Based on formulas (15.75a,b,c) one either does not need special tables for the corresponding Fourier sine and Fourier cosine transformations, or one uses tables for Fourier sine and Fourier cosine transformations and calculates $\mathcal{F}(\omega)$ with the help of (15.75a,b,c). In **Table 21.14.1** (see p. 1114) and **Table 21.14.2** (see p. 1120) the Fourier sine transforms $\mathcal{F}_s(\omega)$, the Fourier cosine transforms $\mathcal{F}_c(\omega)$ respectively, in **Table 21.14.3** (see p. 1125) for some functions the Fourier transform $\mathcal{F}(\omega)$ and in **Table 21.14.4** (see p. 1127) the exponential transform $\mathcal{F}_e(\omega)$ are given.

■ The function of the unipolar rectangular impulse $f(t) = 1$ for $|t| < t_0,\ f(t) = 0$ for $|t| > t_0$ (A.1) **(Fig. 15.21)** satisfies the assumptions of the existence of the Fourier integral (15.64a). According to (15.65c,d) the coefficients are $a(\omega) = \frac{1}{\pi}\int_{-t_0}^{+t_0}\cos\omega\, t\,dt = \frac{2}{\pi\omega}\sin\omega\, t_0$ and $b(\omega) = \frac{1}{\pi}\int_{-t_0}^{+t_0}\sin\omega\, t\,dt = 0$ (A.2) and so from (15.65b) follows $f(t) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\, t_0\cos\omega\, t}{\omega}\,d\omega$ (A.3).

5. Spectral Interpretation of the Fourier Transformation

Analogously to the Fourier series of a periodic function, the Fourier integral for a non-periodic function has a simple physical interpretation. A function $f(t)$, for which the Fourier integral exists, can be represented according to (15.66) and (15.67) as a sum of sinusoidal vibrations with continuously changing frequency ω in the form

$$A(\omega)\,d\omega\,\sin[\,\omega\,t+\varphi(\omega)\,], \quad (15.78a) \qquad A(\omega)\,d\omega\,\cos[\,\omega\,t+\psi(\omega)\,]. \quad (15.78b)$$

The expression $A(\omega)\,d\omega$ gives the amplitude of the wave components and $\varphi(\omega)$ and $\psi(\omega)$ are the phases. The same interpretation holds for the complex formulation: The function $f(t)$ is a sum (or integral) of summands depending on ω of the form

$$\frac{1}{2\pi}F(\omega)\,d\omega\,e^{\mathrm{i}\omega t}, \qquad (15.79)$$

where the quantity $\frac{1}{2\pi}F(\omega)$ also determines the amplitude and the phase of all the parts.
This *spectral interpretation* of the Fourier integral and the Fourier transformation has a big advantage in applications in physics and engineering. The transform

$$F(\omega)=|F(\omega)|\,e^{\mathrm{i}\psi(\omega)} \quad \text{or} \quad F(\omega)=|F(\omega)|\,e^{\mathrm{i}\varphi(\omega)} \qquad (15.80a)$$

is called the *spectrum* or *frequency spectrum* of the function $f(t)$, the quantity

$$|F(\omega)|=\pi\,A(\omega) \qquad (15.80b)$$

is the *amplitude spectrum* and $\varphi(\omega)$ and $\psi(\omega)$ are the *phase spectra* of the function $f(t)$. The relation between the spectrum $F(\omega)$ and the coefficients (15.65c,d) is

$$F(\omega)=\pi[\,a(\omega)-\mathrm{i}b(\omega)\,], \qquad (15.81)$$

from which one gets the following statements:

1. If $f(t)$ is a real function, then the amplitude spectrum $|F(\omega)|$ is an even function of ω, and the phase spectrum is an odd function of ω.

2. If $f(t)$ is a real and even function, then its spectrum $F(\omega)$ is real, and if $f(t)$ is real and odd, then the spectrum $F(\omega)$ is imaginary.

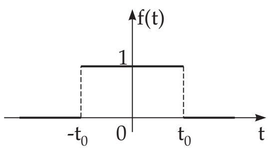

Figure 15.21

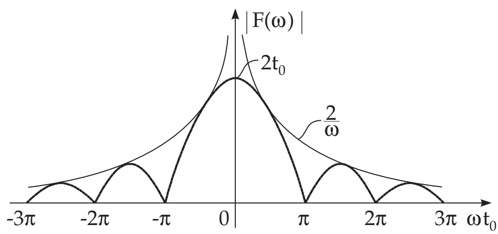

Figure 15.22

■ Substituting the result (A.2) for the unipolar rectangular impulse function on p. 786 into (15.81), then one gets for the transform $F(\omega)$ and for the amplitude spectrum $|F(\omega)|$ **(Fig. 15.22)**
$F(\omega)=\mathcal{F}\{\,f(t)\,\}=\pi a(\omega)=2\frac{\sin\omega\,t_0}{\omega}$ (A.3), $|F(\omega)|=2\left|\frac{\sin\omega\,t_0}{\omega}\right|$ (A.4). The points of contact of the amplitude spectrum $|F(\omega)|$ with the hyperbola $\frac{2}{\omega}$ are at $\omega t_0=\pm(2n+1)\frac{\pi}{2}$ $(n=0,1,2,\ldots)$.

15.3.1.3 Rules of Calculation with the Fourier Transformation

As it has been already pointed out for the Laplace transformation, the rules of calculation with integral transformations mean the mappings of certain operations in the original space into operations in the image space. Supposing that both functions $f(t)$ and $g(t)$ are absolutely integrable in the interval $(-\infty,\ \infty)$ and their Fourier transforms are

$$F(\omega)=\mathcal{F}\{\,f(t)\,\} \quad \text{and} \quad G(\omega)=\mathcal{F}\{\,g(t)\,\} \qquad (15.82)$$

then the following rules are valid.

1. Addition or Linearity Laws
If α and β are two coefficients from $(-\infty, \infty)$, then:

$$\mathcal{F}\{\,\alpha f(t) + \beta g(t)\,\} = \alpha F(\omega) + \beta G(\omega). \tag{15.83}$$

2. Similarity Law
For real $\alpha \neq 0$,

$$\mathcal{F}\{\,f(t/\alpha)\,\} = |\alpha|\,F(\alpha\omega). \tag{15.84}$$

3. Shifting Theorem
For real $\alpha \neq 0$ and real β,

$$\mathcal{F}\{\,f(\alpha t + \beta)\,\} = (1/|\alpha|)\,e^{\mathrm{i}\beta\omega/\alpha}F(\omega/\alpha) \quad \text{or} \tag{15.85a}$$

$$\mathcal{F}\{\,f(t - t_0)\,\} = e^{-\mathrm{i}\omega t_0}F(\omega). \tag{15.85b}$$

If t_0 is replaced by $-t_0$ in (15.85b), then

$$\mathcal{F}\{\,f(t + t_0)\,\} = e^{\mathrm{i}\omega t_0}F(\omega). \tag{15.85c}$$

4. Frequency-Shift Theorem
For real $\alpha > 0$ and $\beta \in (-\infty, \infty)$,

$$\mathcal{F}\{\,e^{\mathrm{i}\beta t}f(\alpha t)\,\} = (1/\alpha)F((\omega - \beta)/\alpha) \quad \text{or} \tag{15.86a}$$

$$\mathcal{F}\{\,e^{\mathrm{i}\omega_0 t}f(t)\,\} = F(\omega - \omega_0). \tag{15.86b}$$

5. Differentiation in the Image Space
If the function $t^n f(t)$ is absolutely integrable in $(-\infty, \infty)$, then the Fourier transform of the function $f(t)$ has n continuous derivatives, which can be determined for $k = 1, 2, \ldots, n$ as

$$\frac{d^k F(\omega)}{d\omega^k} = \int_{-\infty}^{+\infty} \frac{\partial^k}{\partial\omega^k}\left[e^{-\mathrm{i}\omega t}f(t)\right] dt = (-1)^k \int_{-\infty}^{+\infty} e^{-\mathrm{i}\omega t}t^k f(t)\,dt, \tag{15.87a}$$

where

$$\lim_{\omega\to\pm\infty} \frac{d^k F(\omega)}{d\omega^k} = 0. \tag{15.87b}$$

With the above assumptions these relations imply that

$$\mathcal{F}\{\,t^n f(t)\,\} = \mathrm{i}^n \frac{d^n F(\omega)}{d\omega^n}. \tag{15.87c}$$

6. Differentiation in the Original Space
1. First Derivative If a function $f(t)$ is continuous and absolutely integrable in $(-\infty, \infty)$ and it tends to zero for $t \to \pm\infty$, and the derivative $f'(t)$ exists everywhere except, maybe, at certain points, and this derivative is absolutely integrable in $(-\infty, \infty)$, then

$$\mathcal{F}\{\,f'(t)\,\} = \mathrm{i}\omega\,\mathcal{F}\{\,f(t)\,\}. \tag{15.88a}$$

2. *n*-th Derivative If the requirements of the theorem for the first derivative are valid for all derivatives up to $f^{(n-1)}$, then

$$\mathcal{F}\{\,f^{(n)}(t)\,\} = (\mathrm{i}\omega)^n\mathcal{F}\{\,f(t)\,\}. \tag{15.88b}$$

These rules of differentiation will be used in the solution of differential equations (see 15.3.2, p. 791).

7. Integration in the Image Space

$$\int_{\alpha_1}^{\alpha_2} F(\omega)\,d\omega = \mathrm{i}[G(\alpha_2) - G(\alpha_1)] \ \text{ with } \ G(\omega) = \mathcal{F}\{g(t)\} \ \text{ and } \ g(t) = \frac{f(t)}{t}. \tag{15.89}$$

8. Integration in the Original Space and the Parseval Formula

1. Integration Theorem If the assumption

$$\int_{-\infty}^{+\infty} f(t)\,dt = 0 \quad (15.90a) \qquad \text{is fulfilled, then} \qquad \mathcal{F}\left\{\int_{-\infty}^{t} f(t)\,dt\right\} = \frac{1}{\mathrm{i}\omega}F(\omega). \quad (15.90b)$$

2. Parseval Formula If the function $f(t)$ and its square are integrable in the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{+\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2\,d\omega. \quad (15.91)$$

9. Convolution

The *two-sided convolution*

$$f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_1(\tau)f_2(t-\tau)\,d\tau \quad (15.92)$$

is considered in the interval $(-\infty, \infty)$ and it exists under the assumptions that the functions $f_1(t)$ and $f_2(t)$ are absolutely integrable in the interval $(-\infty, \infty)$. If $f_1(t)$ and $f_2(t)$ both vanish for $t < 0$, then one gets the *one-sided convolution* from (15.92)

$$f_1(t) * f_2(t) = \begin{cases} \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} \quad (15.93)$$

So, it is a special case of the two-sided convolution. While the Fourier transformation uses the two-sided convolution, the Laplace transformation uses the one-sided convolution.
For the Fourier transformation of a two-sided convolution

$$\mathcal{F}\{ f_1(t) * f_2(t) \} = \mathcal{F}\{ f_1(t) \} \cdot \mathcal{F}\{ f_2(t) \} \quad (15.94)$$

holds, if both integrals

$$\int_{-\infty}^{+\infty} |f_1(t)|^2\,dt \qquad \text{and} \qquad \int_{-\infty}^{+\infty} |f_2(t)|^2\,dt \quad (15.95)$$

exist, i.e., the functions and their squares are integrable in the interval $(-\infty, \infty)$.

■ Calculation of the two-sided convolution $\psi(t) = f(t) * f(t) = \int_{-\infty}^{+\infty} f(\tau)f(t-\tau)\,d\tau$ (A.1) for the function of the unipolar rectangular impulse function (A.1) in 15.3.1.2, **4.**, p. 786.
Since $\psi(t) = \int_{-t_0}^{t_0} f(t-\tau)\,d\tau = \int_{t-t_0}^{t+t_0} f(\tau)\,d\tau$ (A.2) one gets for $t < -2t_0$ and $t > 2t_0$, $\psi(t) = 0$ and for $-2t_0 \leq t \leq 0$, $\psi(t) = \int_{-t_0}^{t+t_0} d\tau = t + 2t_0$. (A.3)
Analogously, for $0 < t \leq 2t_0$: $\psi(t) = \int_{t-t_0}^{t_0} d\tau = -t + 2t_0$ (A.4) holds.
Altogether, for this convolution **(Fig. 15.23)**

$$\psi(t) = f(t) * f(t) = \begin{cases} t + 2t_0 & \text{for } -2t_0 \leq t \leq 0, \\ -t + 2t_0 & \text{for } 0 < t \leq 2t_0, \\ 0 & \text{for } |t| > 2t_0 \end{cases} \quad (A.5)$$

follows. For the Fourier transform $F(\omega)$ of the unipolar rectangular impulse (A.1) (see p. 786 and

Fig. 15.21) $\Psi(\omega) = \mathcal{F}\{\,\psi(t)\,\} = \mathcal{F}\{\,f(t)\,*\,f(t)\} = [\,F(\omega)\,]^2 = 4\dfrac{\sin^2\omega\,t_0}{\omega^2}$ (A.6) follows and for the amplitude spectrum of the function $f(t)$ $|F(\omega)| = 2\left|\dfrac{\sin\,\omega\,t_0}{\omega}\right|$ (A.7) holds.

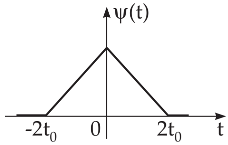

Figure 15.23

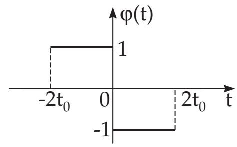

Figure 15.24

10. Comparing the Fourier and Laplace Transformations

There is a strong relation between the Fourier and Laplace transformation, since the Fourier transformation is a special case of the Laplace transformation with $p = \mathrm{i}\omega$. Consequently, every Fourier transformable function is also Laplace transformable, while the reverse statement is not valid for every $f(t)$. **Table 15.2** contains comparisons of several properties of both integral transformations.

Table 15.2 Comparison of the properties of the Fourier and the Laplace transformation

Fourier transformation	**Laplace transformation**
$F(\omega) = \mathcal{F}\{\,f(t)\,\} = \int\limits_{-\infty}^{+\infty} e^{-\mathrm{i}\omega\,t} f(t)\,dt$ ω is real, it has a physical meaning, e.g., frequency.	$F(p) = \mathcal{L}\{\,f(t), p\,\} = \int\limits_{0}^{\infty} e^{-pt} f(t)\,dt$ p is complex, $p = r + \mathrm{i}x$.
One shifting theorem.	Two shifting theorems.
interval: $(-\infty, +\infty)$ Solution of differential equations, problems described by two-sided domain, e.g., the wave equation.	interval: $[\,0, \infty)$ Solution of differential equations, problems described by one-sided domain, e.g., the heat conduction equation.
Differentiation law contains no initial values.	Differentiation law contains initial values.
Convergence of the Fourier integral depends only on $f(t)$.	Convergence of the Laplace integral can be improved by the factor e^{-pt}.
It satisfies the two-sided convolution law.	It satisfies the one-sided convolution law.

15.3.1.4 Transforms of Special Functions

■ **A:** Which image function belongs to the original function $f(t) = e^{-a|t|}$, $\mathrm{Re}\,a > 0$ (A.1)? Considering that $|t| = -t$ for $t < 0$ and $|t| = t$ for $t > 0$ with (15.71) one gets: $\int_{-A}^{+A} e^{-\mathrm{i}\omega\,t - a|t|}dt =$
$\int_{-A}^{0} e^{-(\mathrm{i}\omega-a)t}\,dt + \int_{0}^{+A} e^{-(\mathrm{i}\omega+a)t}\,dt = -\left.\dfrac{e^{-(\mathrm{i}\omega-a)t}}{\mathrm{i}\omega-a}\right|_{-A}^{0} - \left.\dfrac{e^{-(\mathrm{i}\omega+a)t}}{\mathrm{i}\omega+a}\right|_{0}^{+A} = \dfrac{-1+e^{(\mathrm{i}\omega-a)A}}{\mathrm{i}\omega-a} + \dfrac{1-e^{-(\mathrm{i}\omega+a)A}}{\mathrm{i}\omega+a}$ (A.2).
Since $|e^{-aA}| = e^{-A\,\mathrm{Re}\,a}$ and $\mathrm{Re}\,a > 0$, the limit of (A2) exists for $A \to \infty$, so that $F(\omega) = \mathcal{F}\{\,e^{-a|t|}\,\} = \dfrac{2a}{a^2+\omega^2}$ (A.3).

■ **B:** Which image function belongs to the original function $f(t) = e^{-at}$, $\mathrm{Re}\,a > 0$? The function is not Fourier transformable, since the limit of $\int_{-A}^{A} e^{-i\omega t - at} dt$ does not exist for $A \to \infty$.

■ **C:** Determination of the Fourier transform of the bipolar rectangular impulse function **(Fig. 15.24)**

$$\varphi(t) = \begin{cases} 1 & \text{for } -2t_0 < t < 0, \\ -1 & \text{for } 0 < t < 2t_0, \\ 0 & \text{for } |t| > 2t_0, \end{cases} \quad \text{(C.1)}$$

where $\varphi(t)$ can be expressed by using equation (A.1) given for the unipolar rectangular impulse on p. 786. There is $\varphi(t) = f(t + t_0) - f(t - t_0)$ (C.2). With the Fourier transformation according to (15.85b, 15.85c) one gets $\Phi(\omega) = \mathcal{F}\{\,\varphi(t)\,\} = e^{i\omega t_0} F(\omega) - e^{-i\omega t_0} F(\omega)$, (C.3) from which, using (A.1), $\phi(\omega) = (e^{i\omega t_0} - e^{-i\omega t_0}) \dfrac{2 \sin \omega t_0}{\omega} = 4i \dfrac{\sin^2 \omega t_0}{\omega}$ (C.4) follows.

■ **D:** Image function of a damped oscillation: The damped oscillation represented in **Fig. 15.25a** is given by the function $f(t) = \begin{cases} 0 & \text{for } t < 0, \\ e^{-\alpha t} \cos \omega_0 t & \text{for } t \geq 0. \end{cases}$

To simplify the calculations, the Fourier transformation is calculated with the complex function $f^*(t) = e^{(-\alpha + i\omega_0)t}$, with $f(t) = \mathrm{Re}\,(f^*(t))$. The Fourier transformation gives

$$\mathcal{F}\{\,f^*(t)\,\} = \int_0^{\infty} e^{-i\omega t} e^{(-\alpha + i\omega_0)t}\, dt = \int_0^{\infty} e^{(-\alpha + (\omega - \omega_0)i\, t}\, dt = \left. \frac{e^{-\alpha t} e^{i(\omega - \omega_0)t}}{-\alpha + i(\omega_0 - \omega)} \right|_0^{\infty} = \frac{1}{\alpha - i\omega_0 - \omega)} =$$

$\dfrac{\alpha + i(\omega_0 - \omega)}{\alpha^2 + (\omega - \omega_0)^2}$. The result is the Lorentz or Breit-Wigner curve (see also 2.11.2, p. 95)

$\mathcal{F}\{\,f(t)\,\} = \dfrac{\alpha}{\alpha^2 + (\omega - \omega_0)^2}$ **(Fig. 15.25b)**. A damped oscillation in the time domain corresponds to a unique peak in the frequency domain.

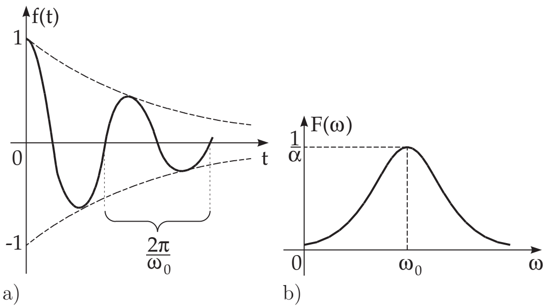

Figure 15.25

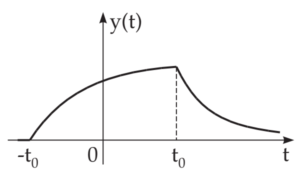

Figure 15.26

15.3.2 Solution of Differential Equations using the Fourier Transformation

Analogously to Laplace transformation, an important field of application of the Fourier transformation is the solution of differential equations, since these equations can be transformed by the integral transformation into a simple form. In the case of ordinary differential equations one gets algebraic equations, in the case of partial differential equations one gets ordinary differential equations.

15.3.2.1 Ordinary Linear Differential Equations

The differential equation

$$y'(t) + a\,y(t) = f(t) \qquad \text{with} \quad f(t) = \begin{cases} 1 & \text{for } |t| < t_0, \\ 0 & \text{for } |t| \geq t_0, \end{cases} \tag{15.96a}$$

i.e., with the function $f(t)$ of **Fig. 15.21**, is transformed by the Fourier transformation

$$\mathcal{F}\{\, y(t)\,\} = Y(\omega) \tag{15.96b}$$

into the algebraic equation

$$\mathrm{i}\omega\, Y + aY = \frac{2\sin\,\omega\, t_0}{\omega}, \quad (15.96\text{c}) \quad \text{giving} \qquad Y(\omega) = 2\,\frac{\sin\,\omega t_0}{\omega(a+\mathrm{i}\omega)}\,. \tag{15.96d}$$

The inverse transformation gives

$$y(t) = \mathcal{F}^{-1}\{\, Y(\omega)\,\} = \mathcal{F}^{-1}\left\{ 2\,\frac{\sin\,\omega t_0}{\omega(a+\mathrm{i}\omega)} \right\} = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \frac{e^{\mathrm{i}\omega t}\sin\,\omega t_0}{\omega(a+\mathrm{i}\omega)}\,d\omega \tag{15.96e}$$

and

$$y(t) = \begin{cases} 0 & \text{for } -\infty < t < -t_0, \\ \dfrac{1}{a}\left[1 - e^{-a(t+t_0)}\right] & \text{for } -t_0 \leq t \leq +t_0, \\ \dfrac{1}{a}\left[e^{-a(t-t_0)} - e^{-a(t-t_0)}\right] & \text{for } \quad t_0 < t < \infty\,. \end{cases} \tag{15.96f}$$

Function (15.96f) is represented graphically in **Fig. 15.26**.

15.3.2.2 Partial Differential Equations

1. General Remarks

The solution of a partial differential equation is a function of at least two variables: $u = u(x,t)$. As the Fourier transformation is an integration with respect to only one variable, the other variable is considered a constant during the transformation. Here the variable x is kept constant and the transformation is to be performed with respect to t:

$$\mathcal{F}\{\, u(x,t)\,\} = \int\limits_{-\infty}^{+\infty} e^{-\mathrm{i}\omega\, t} u(x,t)\,dt = U(x,\omega). \tag{15.97}$$

During the transformation of the derivatives the variable x is again kept constant:

$$\mathcal{F}\left\{ \frac{\partial^{(n)} u(x,t)}{\partial t^n} \right\} = (\mathrm{i}\omega)^n \mathcal{F}\{\, u(x,t)\,\} = (\mathrm{i}\omega)^n U(x,\omega). \tag{15.98}$$

The differentiation with respect to x is supposed to be interchangeable with the Fourier integral:

$$\mathcal{F}\left\{ \frac{\partial u(x,t)}{\partial x} \right\} = \frac{\partial}{\partial x}\mathcal{F}\{\, u(x,t)\,\} = \frac{\partial}{\partial x} U(x,\omega)\,. \tag{15.99}$$

In this way an ordinary differential equation is obtained in the image space. Furthermore the boundary and initial conditions are to be transformed into the image space.

2. Solution of the One-Dimensional Wave Equation for a Homogeneous Medium

1. Formulation of the Problem The one-dimensional wave equation with vanishing perturbation term and for a homogeneous medium is:

$$u_{xx} - u_{tt} = 0. \tag{15.100a}$$

Like the three-dimensional wave equation (see 9.2.3.2,p. 590), the equation (15.100a) is a partial differential equation of hyperbolic type. The Cauchy problem is correctly defined by the following initial conditions

$$u(x,0) = f(x) \quad (-\infty < x < \infty), \quad u_t(x,0) = g(x) \quad (0 \le t < \infty). \tag{15.100b}$$

2. Fourier Transformation The Fourier transformation is to be performed with respect to x where the time coordinate is kept constant:

$$\mathcal{F}\{\, u(x,t)\,\} = U(\omega,t). \tag{15.101a}$$

One gets:

$$(\mathrm{i}\omega)^2 U(\omega,t) - \frac{d^2U(\omega,t)}{dt^2} = 0 \qquad \text{with} \tag{15.101b}$$

$$\mathcal{F}\{\, u(x,0)\,\} = U(\omega,0) = \mathcal{F}\{\, f(x)\,\} = F(\omega), \tag{15.101c}$$

$$\mathcal{F}\{\, u_t(x,0)\,\} = U'(\omega,0) = \mathcal{F}\{\, g(x)\,\} = G(\omega). \tag{15.101d}$$

$$\omega^2 U + U'' = 0. \tag{15.101e}$$

The result is an ordinary differential equation with respect to t with the parameter ω of the transform. The general solution of this known differential equation with constant coefficients is

$$U(\omega,t) = C_1 e^{\mathrm{i}\omega t} + C_2 e^{-\mathrm{i}\omega t}. \tag{15.102a}$$

Determining the constants C_1 and C_2 from the initial values

$$U(\omega,0) = C_1 + C_2 = F(\omega)\,, \quad U'(\omega,0) = \mathrm{i}\omega\, C_1 - \mathrm{i}\omega\, C_2 = G(\omega)\,, \tag{15.102b}$$

gives

$$C_1 = \frac{1}{2}[\, F(\omega) + \frac{1}{\mathrm{i}\omega}G(\omega)\,], \quad C_2 = \frac{1}{2}[\, F(\omega) - \frac{1}{\mathrm{i}\omega}G(\omega)\,]. \tag{15.102c}$$

The solution is therefore

$$U(\omega,t) = \frac{1}{2}[\, F(\omega) + \frac{1}{\mathrm{i}\omega}G(\omega)\,]e^{\mathrm{i}\omega t} + \frac{1}{2}[\, F(\omega) - \frac{1}{\mathrm{i}\omega}G(\omega)\,]e^{-\mathrm{i}\omega t}. \tag{15.102d}$$

3. Inverse Transformation Using the shifting theorem

$$\mathcal{F}\{\, f(ax+b)\,\} = 1/a \cdot e^{\mathrm{i}b\omega/a} F(\omega/a), \tag{15.103a}$$

for the inverse transformation of $F(\omega)$, yields

$$\mathcal{F}^{-1}\{\, e^{\mathrm{i}\omega t} F(\omega)\,\} = f(x+t), \quad \mathcal{F}^{-1}[\, e^{-\mathrm{i}\omega t} F(\omega)\,] = f(x-t)\,. \tag{15.103b}$$

Applying the integration rule

$$\mathcal{F}\left\{ \int\limits_{-\infty}^{x} f(\tau)\, d\tau \right\} = \frac{1}{\mathrm{i}\omega} F(\omega) \quad \text{gives} \tag{15.103c}$$

$$\mathcal{F}^{-1}\left\{ \frac{1}{\mathrm{i}\omega} G(\omega) e^{\mathrm{i}\omega t} \right\} = \int\limits_{-\infty}^{x} \mathcal{F}^{-1}\{\, G(\omega)e^{\mathrm{i}\omega t}\}\, d\tau = \int\limits_{-\infty}^{x} g\,(\tau+t)\, d\tau = \int\limits_{-\infty}^{x+t} g\,(z)\, dz \tag{15.103d}$$

after substituting $\tau + t = z$. Analogously to the previous integral

$$\mathcal{F}^{-1}\left\{ -\frac{1}{\mathrm{i}\omega} G(\omega) e^{-\mathrm{i}\omega t} \right\} = -\int\limits_{-\infty}^{x-t} g\,(z)\, dz \tag{15.103e}$$

follows. Finally, the solution in the original space is

$$u(x,t) = \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t) + \int\limits_{x-t}^{x+t} g\,(z)\, dz\,. \tag{15.104}$$

15.4 Z-Transformation

In natural sciences and also in engineering one often has to distinguish between continuous and discrete processes. While continuous processes can be described by differential equations, the discrete processes result mostly in *difference equations*. The solution of differential equations mostly uses Fourier and Laplace transformations, however, to solve difference equations other operator methods have been developed. The best known method is the z-transformation, which is closely related to the Laplace transformation.

15.4.1 Properties of the Z-Transformation

15.4.1.1 Discrete Functions

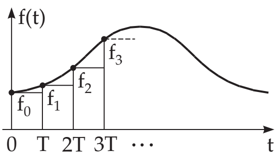

Figure 15.27

If a function $f(t)$ $(0 \le t < \infty)$ is known only at discrete values $t_n = nT$ $(n = 0, 1, 2, \ldots\,;\ T > 0$ is a constant) of the argument, then one writes $f(nT) = f_n$ and forms the sequence $\{f_n\}$. Such a sequence is produced, e.g., in electrotechnics by "scanning" a function $f(t)$ at discrete time periods t_n. Its representation results in a *step function* **(Fig. 15.27)**.
The sequence $\{f_n\}$ and the function $f(nT)$ defined only at discrete points of the argument, which is called a *discrete function*, are equivalent.

15.4.1.2 Definition of the Z-Transformation

1. Original Sequence and Transform

The infinite series

$$F(z) = \sum_{n=0}^{\infty} f_n \left(\frac{1}{z}\right)^n \tag{15.105}$$

is assigned to the sequence $\{f_n\}$. If this series is convergent, then the sequence $\{f_n\}$ is called *z-transformable*, and it is denoted by

$$F(z) = \mathcal{Z}\{f_n\}\,. \tag{15.106}$$

$\{f_n\}$ is called the *original sequence*, $F(z)$ is the *transform*, z denotes a complex variable and $F(z)$ is a complex-valued function.

■ $f_n = 1$ $(n = 0, 1, 2, \ldots)$. The corresponding infinite series is

$$F(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n . \tag{15.107}$$

It represents a geometric series with common ratio $1/z$, which is convergent if $\left|\frac{1}{z}\right| < 1$ and its sum is $F(z) = \frac{z}{z-1}$. It is divergent for $\left|\frac{1}{z}\right| \ge 1$. Therefore, the sequence $\{1\}$ is z-transformable for $\left|\frac{1}{z}\right| < 1$, i.e., for every exterior point of the unit circle $|z| = 1$ in the z plane.

2. Properties

Since the transform $F(z)$ according to (15.105) is a power series of the complex variable $1/z$, the properties of the complex power series (see 14.3.1.3, p. 750) imply the following results:

a) For a z-transformable sequence $\{f_n\}$, there exists a real number R such that the series (15.105) is absolutely convergent for $|z| > 1/R$ and divergent for $|z| < 1/R$. The series is uniformly convergent for $|z| \ge 1/R_0 > 1/R$. R is the radius of convergence of the power series (15.105) of $1/z$. If the series is convergent for every $|z| > 0$, then $R = \infty$. For non z-transformable sequences there is $R = 0$.

b) If $\{f_n\}$ is z-transformable for $|z| > 1/R$, then the corresponding transform $F(z)$ is an analytic function for $|z| > 1/R$ and it is the unique transform of $\{f_n\}$. Conversely, if $F(z)$ is an analytic function for $|z| > 1/R$ and is regular also at $z = \infty$, then there is a unique original sequence $\{f_n\}$ for $F(z)$. Here, $F(z)$ is called regular at $z = \infty$, if $F(z)$ has a power series expansion in the form (15.105) and $F(\infty) = f_0$.

3. Limit Theorems

Analogously to the limit properties of the Laplace transformation ((15.7b), p. 770), the following limit theorems are valid for the z-transformation:

a) If $F(z) = \mathcal{Z}\{f_n\}$ exists, then

$$f_0 = \lim_{z\to\infty} F(z). \tag{15.108}$$

Here z can tend to infinity along the real axis or along any other path. Since the series

$$z\{F(z) - f_0\} = f_1 + f_2\frac{1}{z} + f_3\frac{1}{z^2} + \cdots, \tag{15.109}$$

$$z^2\left\{F(z) - f_0 - f_1\frac{1}{z}\right\} = f_2 + f_3\frac{1}{z} + f_4\frac{1}{z^2} + \cdots, \tag{15.110}$$

$$\vdots \qquad \vdots \qquad \vdots$$

are obviously z transforms, analogously to (15.108) one gets

$$f_1 = \lim_{z\to\infty} z\{F(z) - f_0\}, \qquad f_2 = \lim_{z\to\infty} z^2\left\{F(z) - f_0 - f_1\frac{1}{z}\right\}, \ldots . \tag{15.111}$$

The original sequence $\{f_n\}$ can be determined from its transform $F(z)$ in this way.

b) If $\lim\limits_{n\to\infty} f_n$ exists, then

$$\lim_{n\to\infty} f_n = \lim_{z\to 1+0}(z-1)F(z). \tag{15.112}$$

However the value $\lim\limits_{n\to\infty} f_n$ from (15.112) can be determined only if its existence is guaranteed, since the above statement is not reversible.

■ $f_n = (-1)^n \ (n = 0, 1, 2, \ldots)$. Then $\mathcal{Z}\{f_n\} = \dfrac{z}{z+1}$ and $\lim\limits_{z\to 1+0}(z-1)\dfrac{z}{z+1} = 0$, but $\lim\limits_{n\to\infty}(-1)^n$ does not exist.

15.4.1.3 Rules of Calculations

In applications of the z-transformation it is very important to know how certain operations defined on the original sequences affect the transforms, and conversely. For the sake of simplicity here the notation $F(z) = \mathcal{Z}\{f_n\}$ for $|z| > 1/R$ is used.

1. Translation

Forward and backward translations are distinguished.

1. First Shifting Theorem: $\mathcal{Z}\{f_{n-k}\} = z^{-k}F(z) \qquad (k = 0, 1, 2, \ldots)$, (15.113)

here $f_{n-k} = 0$ is defined for $n - k < 0$.

2. Second Shifting Theorem: $\mathcal{Z}\{f_{n+k}\} = z^k\left[F(z) - \sum\limits_{\nu=0}^{k-1} f_\nu \left(\dfrac{1}{z}\right)^\nu\right] \qquad (k = 1, 2, \ldots)$. (15.114)

2. Summation

For $|z| > \max\left(1, \dfrac{1}{R}\right)$

$$\mathcal{Z}\left\{\sum_{\nu=0}^{n-1} f_\nu\right\} = \frac{1}{z-1}F(z). \tag{15.115}$$

3. Differences

For the *differences*

$$\Delta f_n = f_{n+1} - f_n, \quad \Delta^m f_n = \Delta(\Delta^{m-1} f_n) \qquad (m = 1, 2, \ldots\,;\ \Delta^0 f_n = f_n) \tag{15.116}$$

the following equalities hold

$$\begin{aligned} \mathcal{Z}\{\Delta f_n\} &= (z-1)F(z) - z f_0, \\ \mathcal{Z}\{\Delta^2 f_n\} &= (z-1)^2 F(z) - z(z-1) f_0 - z\Delta f_0, \\ \vdots \quad &= \quad \vdots \\ \mathcal{Z}\{\Delta^k f_n\} &= (z-1)^k F(z) - z \sum_{\nu=0}^{k-1} (z-1)^{k-\nu-1} \Delta^\nu f_0. \end{aligned} \tag{15.117}$$

4. Damping

For an arbitrary complex number $\lambda \neq 0$ and $|z| > \dfrac{|\lambda|}{R}$:

$$\mathcal{Z}\{\lambda^n f_n\} = F\left(\frac{z}{\lambda}\right). \tag{15.118}$$

5. Convolution

The *convolution* of two sequences $\{f_n\}$ and $\{g_n\}$ is the operation

$$f_n * g_n = \sum_{\nu=0}^{n} f_\nu\, g_{n-\nu}. \tag{15.119}$$

If the z-transformed functions $\mathcal{Z}\{f_n\} = F(z)$ for $|z| > 1/R_1$ and $\mathcal{Z}\{g_n\} = G(z)$ for $|z| > 1/R_2$ exist, then

$$\mathcal{Z}\{f_n * g_n\} = F(z)G(z) \tag{15.120}$$

for $|z| > \max\left(\dfrac{1}{R_1}, \dfrac{1}{R_2}\right)$. Relation (15.120) is called the *convolution theorem* of the z-transformation. It corresponds to the rules of multiplying two power series.

6. Differentiation of the Transform

$$\mathcal{Z}\{n f_n\} = -z \frac{dF(z)}{dz}. \tag{15.121}$$

Higher-order derivatives of $F(z)$ can be determined by the repeated application of (15.121).

7. Integration of the Transform

Under the assumption $f_0 = 0$,

$$\mathcal{Z}\left\{\frac{f_n}{n}\right\} = \int_z^\infty \frac{F(\xi)}{\xi}\, d\xi. \tag{15.122}$$

15.4.1.4 Relation to the Laplace Transformation

Describing a discrete function $f(t)$ (see 15.4.1.1, p. 794) as a step function, then

$$f(t) = f(nT) = f_n \qquad \text{for} \quad nT \le t < (n+1)T \quad (n = 0, 1, 2, \ldots\,;\ T > 0,\ T\ \text{const}) \tag{15.123}$$

holds. Using the Laplace transformation (see 15.2.1.1, **1.**, p. 770) for this piecewise constant function, for $T = 1$ yields:

$$\mathcal{L}\{f(t)\} = F(p) = \sum_{n=0}^{\infty} \int_n^{n+1} f_n e^{-pt}\, dt = \sum_{n=0}^{\infty} f_n \frac{e^{-np} - e^{-(n+1)p}}{p} = \frac{1 - e^{-p}}{p} \sum_{n=0}^{\infty} f_n e^{-np}. \tag{15.124}$$

The infinite series in (15.124) is called the *discrete Laplace transformation* and is denoted by $\mathcal{D}$:

$$\mathcal{D}\{f(t)\} = \mathcal{D}\{f_n\} = \sum_{n=0}^{\infty} f_n e^{-np}. \tag{15.125}$$

After the substitution of $e^p = z$ in (15.125) $\mathcal{D}\{f_n\}$ represents a series with powers of $1/z$, which is a so-called *Laurent series* (see 14.3.4, p. 752). The substitution $e^p = z$ suggested the name of the z transformation. With this substitution from (15.124) one finally gets the following relations between the Laplace and z-transformation in the case of step functions:

$$pF(p) = \left(1 - \frac{1}{z}\right) F(z) \quad (15.126a) \qquad \text{or} \quad p\mathcal{L}\{f(t)\} = \left(1 - \frac{1}{z}\right) \mathcal{Z}\{f_n\}. \quad (15.126b)$$

In this way the relations of z-transforms of step functions (see **Table 21.15**, p. 1128) can be transformed into relations of Laplace transforms of step functions (see **Table 21.13**, p. 1109), and conversely.

15.4.1.5 Inverse of the Z-Transformation

The inverse of the z-transformation is to find the corresponding unique original sequence $\{f_n\}$ from its transform $F(z)$:

$$\mathcal{Z}^{-1}\{F(z)\} = \{f_n\}. \quad (15.127)$$

There are different possibilities for the inverse transformation.

1. Using Tables

If the function $F(z)$ is not given in tables, then one can try to transform it to a function which is given in **Table 21.15**.

2. Laurent Series of $F(z)$

Using the definition (15.105), p. 794 the inverse transform can be determined directly if a series expansion of $F(z)$ with respect to $1/z$ is known or if it can be determined.

3. Taylor Series of $F\left(\frac{1}{z}\right)$

Since $F\left(\frac{1}{z}\right)$ is a series of increasing powers of z, from (15.105) and using the Taylor formula follows

$$f_n = \frac{1}{n!} \frac{d^n}{dz^n} F\left(\frac{1}{z}\right)\Big|_{z=0} \quad (n = 0, 1, 2, \ldots). \quad (15.128)$$

4. Application of Limit Theorems

Using the limits (15.108) and (15.111), p. 795, the original sequence $\{f_n\}$ can be directly determined from its transform $F(z)$.

■ $F(z) = \frac{2z}{(z-2)(z-1)^2}$. Using the previous four methods:

1. Partial fraction decomposition (see 1.1.7.3, p. 15) of $F(z)/z$ yields functions which are contained in **Table 21.15**.

$$\frac{F(z)}{z} = \frac{2}{(z-2)(z-1)^2} = \frac{A}{z-2} + \frac{B}{(z-1)^2} + \frac{C}{z-1}. \quad \text{So}$$

$$F(z) = \frac{2z}{z-2} - \frac{2z}{(z-1)^2} - \frac{2z}{z-1} \quad \text{and therefore} \quad f_n = 2(2^n - n - 1) \quad \text{for } n \geq 0.$$

2. By division $F(z)$ gets a series with decreasing powers of z:

$$F(z) = \frac{2z}{z^3 - 4z^2 + 5z - 2} = 2\frac{1}{z^2} + 8\frac{1}{z^3} + 22\frac{1}{z^4} + 52\frac{1}{z^5} + 114\frac{1}{z^6} + \ldots. \quad (15.129)$$

From this expression one gets $f_0 = f_1 = 0,\ f_2 = 2,\ f_3 = 8,\ f_4 = 22,\ f_5 = 52,\ f_6 = 114,\ \ldots$, but not a closed expression is obtained for the general term f_n.

3. For formulating $F\left(\frac{1}{z}\right)$ and its required derivatives, (see (15.128)) it is advisable to consider the partial fraction decomposition of $F(z)$

$$\left.\begin{aligned}
F\left(\frac{1}{z}\right) &= \frac{2}{1-2z} - \frac{2z}{(1-z)^2} - \frac{2}{1-z}, && \text{i.e.} \quad F\left(\frac{1}{z}\right) = 0 \quad \text{for } z=0,\\
\frac{dF\left(\frac{1}{z}\right)}{dz} &= \frac{4}{(1-2z)^2} - \frac{4z}{(1-z)^3} - \frac{4}{(1-z)^2}, && \text{i.e.} \quad \frac{dF\left(\frac{1}{z}\right)}{dz} = 0 \quad \text{for } z=0,\\
\frac{d^2F\left(\frac{1}{z}\right)}{dz^2} &= \frac{16}{(1-2z)^3} - \frac{12z}{(1-z)^4} - \frac{12}{(1-z)^3}, && \text{i.e.} \quad \frac{d^2F\left(\frac{1}{z}\right)}{dz^2} = 4 \quad \text{for } z=0,\\
\frac{d^3F\left(\frac{1}{z}\right)}{dz^3} &= \frac{96}{(1-2z)^4} - \frac{48z}{(1-z)^5} - \frac{48}{(1-z)^4}, && \text{i.e.} \quad \frac{d^3F\left(\frac{1}{z}\right)}{dz^3} = 48 \quad \text{for } z=0,\\
&\vdots && \vdots
\end{aligned}\right\} \tag{15.130}$$

from which $f_0,\ f_1,\ f_2,\ f_3,\ \ldots$ are easily obtained considering (15.128).

4. Application of the limit theorems (see 15.4.1.2, **3.**, p. 795) gives:

$$f_0 = \lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{2z}{z^3-4z^2+5z-2} = 0, \tag{15.131a}$$

$$f_1 = \lim_{z\to\infty} z(F(z)-f_0) = \lim_{z\to\infty} \frac{2z^2}{z^3-4z^2+5z-2} = 0, \tag{15.131b}$$

$$f_2 = \lim_{z\to\infty} z^2\left(F(z)-f_0-f_1\frac{1}{z}\right) = \lim_{z\to\infty} \frac{2z^3}{z^3-4z^2+5z-2} = 2, \tag{15.131c}$$

$$f_3 = \lim_{z\to\infty} z^3\left(F(z)-f_0-f_1\frac{1}{z}-f_2\frac{1}{z^2}\right) = \lim_{z\to\infty} z^3\left(\frac{2z}{z^3-4z^2+5z-2}-\frac{2}{z^2}\right) = 8,\ \ldots \tag{15.131d}$$

where the *Bernoulli–l'Hospital rule* is applied (see 2.1.4.8, **2.**, p. 56). The original sequence $\{f_n\}$ can be determined successively.

15.4.2 Applications of the Z-Transformation

15.4.2.1 General Solution of Linear Difference Equations

A linear difference equation of order k with constant coefficients has the form

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = g_n \quad (n = 0,1,2\ldots). \tag{15.132}$$

Here k is a natural number. The coefficients a_i $(i = 0,1,\ldots,k)$ are given real or complex numbers and they do not depend on n. Here a_0 and a_k are non-zero numbers. The sequence $\{g_n\}$ is given, and the sequence $\{y_n\}$ is to be determined.

To determine a particular solution of (15.132) the values $y_0, y_1, \ldots, y_{k-1}$ have to be previously given. Then the next value y_k can be determined for $n = 0$ from (15.132). Next one gets y_{k+1} for $n = 1$ from $y_1,\ y_2,\ \ldots,\ y_k$ and from (15.132). In this way all values y_n can be calculated recursively. However a general solution can be given for the values y_n with the z-transformation, using the second shifting theorem (15.114) applied for (15.132):

$$a_k z^k\left[Y(z) - y_0 - y_1 z^{-1} - \cdots - y_{k-1} z^{-(k-1)}\right] + \cdots + a_1 z[Y(z) - y_0] + a_0 Y(z) = G(z). \tag{15.133}$$

Here one denotes $Y(z) = \mathcal{Z}\{y_n\}$ and $G(z) = \mathcal{Z}\{g_n\}$. Substituting $a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0 = p(z)$, the solution of the so-called transformed equation (15.133) is

$$Y(z) = \frac{1}{p(z)} G(z) + \frac{1}{p(z)} \sum_{i=0}^{k-1} y_i \sum_{j=i+1}^{k} a_j z^{j-i}. \tag{15.134}$$

As in the case of solving linear differential equations with the Laplace transformation, there is the similar advantage of the z-transformation that initial values are included in the transformed equation, so the solution contains them automatically. The required solution $\{y_n\} = \mathcal{Z}^{-1}\{Y(z)\}$ follows from (15.134) by the inverse transformation discussed in 15.4.1.5, p. 797.

15.4.2.2 Second-Order Difference Equations (Initial Value Problem)

The linear second-order difference equation has the form

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = g_n\,, \tag{15.135}$$

where y_0 and y_1 are given as initial values. Using the second shifting theorem for (15.135) the transformed equation is

$$z^2\left[Y(z) - y_0 - y_1\frac{1}{z}\right] + a_1 z[Y(z) - y_0] + a_0 Y(z) = G(z). \tag{15.136}$$

Substituting $z^2 + a_1 z + a_0 = p(z)$, the transform is

$$Y(z) = \frac{1}{p(z)}G(z) + y_0\frac{z(z+a_1)}{p(z)} + y_1\frac{z}{p(z)}\,. \tag{15.137}$$

If the roots of the polynomial $p(z)$ are α_1 and α_2, then $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, otherwise a_0 is zero, and then the difference equation could be reduced to a first-order one. By partial fraction decomposition and applying **Table 21.15** for the z-transformation one gets from

$$\frac{z}{p(z)} = \begin{cases} \dfrac{1}{\alpha_1 - \alpha_2}\left(\dfrac{z}{z-\alpha_1} - \dfrac{z}{z-\alpha_2}\right) & \text{for } \alpha_1 \neq \alpha_2, \\ \dfrac{z}{(z-\alpha_1)^2} & \text{for } \alpha_1 = \alpha_2, \end{cases}$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{p(z)}\right\} = \{p_n\} = \begin{cases} \dfrac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2, \\ n\alpha_1^{n-1} & \text{for } \alpha_1 = \alpha_2. \end{cases} \tag{15.138a}$$

Since $p_0 = 0$, by the second shifting theorem there is

$$\mathcal{Z}^{-1}\left\{\frac{z^2}{p(z)}\right\} = \mathcal{Z}^{-1}\left\{z\,\frac{z}{p(z)}\right\} = \{p_{n+1}\} \tag{15.138b}$$

and by the first shifting theorem

$$\mathcal{Z}^{-1}\left\{\frac{1}{p(z)}\right\} = \mathcal{Z}^{-1}\left\{\frac{1}{z}\,\frac{z}{p(z)}\right\} = \{p_{n-1}\}. \tag{15.138c}$$

Substituting here $p_{-1} = 0$, based on the convolution theorem one gets the original sequence with

$$y_n = \sum_{\nu=0}^{n} p_{\nu-1} g_{n-\nu} + y_0(p_{n+1} + a_1 p_n) + y_1 p_n\,. \tag{15.138d}$$

Since $p_{-1} = p_0 = 0$, this relation and (15.138a) imply that in the case of $\alpha_1 \neq \alpha_2$ it follows

$$y_n = \sum_{\nu=2}^{n} g_{n-\nu}\frac{\alpha_1^{\nu-1} - \alpha_2^{\nu-1}}{\alpha_1 - \alpha_2} + y_0\left(\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} + a_1\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right) + y_1\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}. \tag{15.138e}$$

This form can be further simplified, since $a_1 = -(\alpha_1 + \alpha_2)$ and $a_0 = \alpha_1\alpha_2$ (see the root theorems of Vieta, 1.6.3.1, 3., p. 44), so

$$y_n = \sum_{\nu=2}^{n} g_{n-\nu}\frac{\alpha_1^{\nu-1} - \alpha_2^{\nu-1}}{\alpha_1 - \alpha_2} - y_0 a_0\frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2} + y_1\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\,. \tag{15.138f}$$

In the case of $\alpha_1 = \alpha_2$ similarly

$$y_n = \sum_{\nu=2}^{n} g_{n-\nu}(\nu-1)\alpha_1^{\nu-2} - y_0 a_0 (n-1)\alpha_1^{n-2} + y_1 n \alpha_1^{n-1}. \tag{15.138g}$$

In the case of second-order difference equations the inverse transformation of the transform $Y(z)$ can be performed without partial fraction decomposition using correspondences such as, e.g.,

$$\mathcal{Z}^{-1}\left\{\frac{z}{z^2 - 2az\cosh b + a^2}\right\} = a^{n-1}\frac{\sinh bn}{\sinh n} \tag{15.139}$$

and the second shifting theorem. By substituting $a_1 = -2a\cosh b$, and $a_0 = a^2$ the original sequence of (15.137) becomes:

$$y_n = \frac{1}{\sinh b}\left[\sum_{\nu=2}^{n} g_{n-\nu}a^{\nu-2}\sinh(\nu-1)b - y_0 a^n \sinh(n-1)b + y_1 a^{n-1}\sinh nb\right]. \tag{15.140}$$

This formula is useful in numerical computations especially if a_0 and a_1 are complex numbers.

Remark: Notice that the hyperbolic functions are also defined for complex variables.

15.4.2.3 Second-Order Difference Equations (Boundary Value Problem)

It often happens in applications that the values y_n of a difference equation are needed only for a finite number of indices $0 \le n \le N$. In the case of a second-order difference equation (15.135) both *boundary values* y_0 and y_N are usually given. To solve this boundary value problem one starts with the solution (15.138f) of the corresponding initial value problem, where instead of the unknown value y_1 it is to introduce y_N. Substituting $n = N$ into (15.138f), y_1 can be obtained which depends on y_0 and y_N:

$$y_1 = \frac{1}{\alpha_1^N - \alpha_2^N}\left[y_0 a_0(\alpha_1^{N-1} - \alpha_2^{N-1}) + y_N(\alpha_1 - \alpha_2) - \sum_{\nu=2}^{N}(\alpha_1^{\nu-1} - \alpha_2^{\nu-1})g_{N-\nu}\right]. \tag{15.141}$$

Substituting this value into (15.138f)

$$\begin{aligned} y_n = {} & \frac{1}{\alpha_1 - \alpha_2}\sum_{\nu=2}^{n}(\alpha_1^{\nu-1} - \alpha_2^{\nu-1})g_{n-\nu} - \frac{1}{\alpha_1 - \alpha_2}\frac{\alpha_1^n - \alpha_2^n}{\alpha_1^N - \alpha_2^N}\sum_{\nu=2}^{N}(\alpha_1^{\nu-1} - \alpha_2^{\nu-1})g_{N-\nu} \\ & + \frac{1}{\alpha_1^N - \alpha_2^N}[\,y_0(\alpha_1^N\alpha_2^n - \alpha_1^n\alpha_2^N) + y_N(\alpha_1^n - \alpha_2^n)]. \end{aligned} \tag{15.142}$$

The solution (15.142) makes sense only if $\alpha_1^N - \alpha_2^N \ne 0$ holds. Otherwise, the boundary value problem has no general solution, but analogously to the boundary value problems of differential equations eigenvalues and eigenfunctions emerge.

15.5 Wavelet Transformation

15.5.1 Signals

If a physical object emits an effect which spreads out and can be described mathematically, e.g., by a function or a number sequence, then it is called a *signal*.

Signal analysis means to characterize a signal by a quantity that is typical for the signal. This means mathematically: The function or the number sequence, which describes the signal, will be mapped into another function or number sequence, from which the typical properties of the signal can be clearly seen. For such mappings, of course, some informations can also be lost.

The reverse operation of signal analysis, i.e., the reconstruction of the original signal, is called *signal synthesis*.

The connection between signal analysis and signal synthesis can be well represented by an example of Fourier transformation: A signal $f(t)$ (t denotes time) is characterized by the frequency ω. Then, formula (15.143a) describes the signal analysis, and formula (15.143b) describes the signal synthesis:

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt \qquad (15.143a) \quad \text{and} \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\,d\omega. \qquad (15.143b)$$

15.5.2 Wavelets

The Fourier transformation has no localization property, i.e., if a signal changes at one position, then the transform changes everywhere without the possibility that the position of the change could be recognized "at a glance". The basis of this fact is that the Fourier transformation decomposes a signal into *plane waves*. These are described by trigonometric functions, which oscillate with the same period for arbitrary long time. However, for wavelet transformations there is an almost freely chosen function ψ, the *wavelet* (small localized wave), that is shifted and compressed for analysing a signal.

Examples are the Haar wavelet **(Fig. 15.28a)** and the Mexican hat **(Fig. 15.28b)**.

■ **A Haar wavelet:**

$$\psi = \begin{cases} 1 & \text{if} \quad 0 \le x < \frac{1}{2}, \\ -1 & \text{if} \quad \frac{1}{2} \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \qquad (15.144)$$

■ **B Mexican hat:**

$$\psi(x) = -\frac{d^2}{dx^2} e^{-x^2/2} \qquad (15.145)$$

$$= (1 - x^2) e^{-x^2/2}. \qquad (15.146)$$

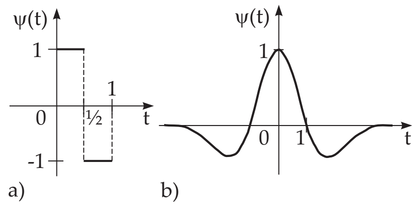

Figure 15.28

Generally, it holds that every function ψ comes into consideration as a wavelet if it is quadratically integrable and its Fourier transform $\Psi(\omega)$ according to (15.143a) results in a positive finite integral

$$\int_{-\infty}^{\infty} \frac{|\Psi(\omega)|}{|\omega|}\,d\omega. \qquad (15.147)$$

Concerning wavelets, the following properties and definitions are essential:

1. For the mean value of the wavelet:

$$\int_{-\infty}^{\infty} \psi(t)\,dt = 0. \qquad (15.148)$$

2. The following integral is called the k-th moment of a wavelet ψ:

$$\mu_k = \int_{-\infty}^{\infty} t^k \psi(t)\,dt. \qquad (15.149)$$

The smallest positive integer n such that $\mu_n \neq 0$, is called the *order* of the wavelet ψ.

■ For the Haar wavelet (15.144), $n = 1$, and for the Mexican hat (15.146), $n = 2$.

3. When $\mu_k = 0$ for every k, ψ has infinite order. Wavelets with bounded support always have finite order.

4. A wavelet of order n is orthogonal to every polynomial of degree $\le n - 1$.

15.5.3 Wavelet Transformation

For a wavelet $\psi(t)$ a family of curves can be formed with parameter a:

$$\psi_a(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t}{a}\right) \qquad (a \neq 0). \qquad (15.150)$$

In the case of $|a| > 0$ the initial function $\psi(t)$ is compressed. In the case of $a < 0$ there is an additional reflection. The factor $1/\sqrt{|a|}$ is a scaling factor.

The functions $\psi_a(t)$ can also be shifted by a second parameter b. Then a two-parameter family of curves arises:

$$\psi_{a,b} = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right) \qquad (a, b \text{ real}; \quad a \neq 0). \tag{15.151}$$

The real shifting parameter b characterizes the first moment, while parameter a gives the deviation of the function $\psi_{a,b}(t)$. The function $\psi_{a,b}(t)$ is called a *basis function* in connection to the *wavelet transformation.*

The wavelet transformation of a function $f(t)$ is defined as:

$$\mathcal{L}_\psi f(a,b) = c\int\limits_{-\infty}^{\infty} f(t)\psi_{a,b}(t)\,dt = \frac{c}{\sqrt{|a|}}\int\limits_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right)\,dt. \tag{15.152a}$$

For the inverse transformation:

$$f(t) = c\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mathcal{L}_\psi f(t)\psi_{a,b}(t)\frac{1}{a^2}\,da\,db. \tag{15.152b}$$

Here c is a constant dependent on the special wavelet ψ.

■ Using the Haar wavelets (15.146) gives

$$\psi\left(\frac{t-b}{a}\right) = \begin{cases} 1 & \text{if } b \le t < b + a/2, \\ -1 & \text{if } b + a/2 \le t < b + a, \\ 0 & \text{otherwise} \end{cases}$$

and therefore

$$\begin{aligned} \mathcal{L}_\psi f(a,b) &= \frac{1}{\sqrt{|a|}}\left(\int_b^{b+a/2} f(t)\,dt - \int_{b+a/2}^{b+a} f(t)\,dt\right) \\ &= \frac{\sqrt{|a|}}{2}\left(\frac{2}{a}\int_b^{b+a/2} f(t)\,dt - \frac{2}{a}\int_{b+a/2}^{b+a} f(t)\,dt\right). \end{aligned} \tag{15.153}$$

The value $\mathcal{L}_\psi f(a,b)$ given in (15.153) represents the difference of the mean values of a function $f(t)$ over two neighboring intervals of length $\frac{|a|}{2}$, connected at the point b.

Remarks:

1. The *dyadic wavelet transformation* has an important role in applications. As basis functions are used the functions

$$\psi_{i,j}(t) = \frac{1}{\sqrt{2^i}}\psi\left(\frac{t-2^i j}{2^i}\right), \tag{15.154}$$

i.e., different basis functions can be generated from one wavelet $\psi(t)$ by doubling or halving the width and shifting by an integer multiple of the width.

2. A wavelet $\psi(t)$ is called an *orthogonal wavelet*, if the basis functions given in (15.154) form an orthogonal system.

3. The Daubechies wavelets have especially good numerical properties. They are orthogonal wavelets with compact support, i.e., they are different from zero only on a bounded subset of the time scale.

They do not have a closed form representation (see [15.9]).

15.5.4 Discrete Wavelet Transformation

15.5.4.1 Fast Wavelet Transformation

The integral representation (15.152b) is very redundant, and so the double integral can be replaced by a double sum without loss of information. Considering this idea at the concrete application of the wavelet transformation one needs

1. an efficient algorithm of the transformation, which leads to the concept of *multi-scale analysis*, and

2. an efficient algorithm of the inverse transformation, i.e., an efficient way to reconstruct signals from their wavelet transformations, which leads to the concept of *frames*.

For more details about these concepts see [15.9], [15.1].

Remark: The great success of wavelets in many different applications, such as
- calculation of physical quantities from measured sequences
- pattern and voice recognition
- data compression in news transmission

is based on "fast algorithms". Analogously to the **FFT** (**F**ast **F**ourier **T**ransformation, see 19.6.4.2, p. 993) one talks here about **FWT** (**F**ast **W**avelet **T**ransformation).

15.5.4.2 Discrete Haar Wavelet Transformation

An example of a discrete wavelet transformation is the Haar wavelet transformation: The values f_i $(i = 1, 2, \dots, N)$ are given from a signal. The detailed values d_i $(i = 1, 2, \dots, N/2)$ are calculated as:

$$s_i = \frac{1}{\sqrt{2}}(f_{2i-1} + f_{2i}), \quad d_i = \frac{1}{\sqrt{2}}(f_{2i-1} - f_{2i}). \tag{15.155}$$

The values d_i are to be stored while the rule (15.155) is applied to the values s_i, i.e., in (15.155) the values f_i are replaced by the values s_i. This procedure is continued, sequentially so that finally from

$$s_i^{(n+1)} = \frac{1}{\sqrt{2}}\left(s_{2i-1}^{(n)} + s_{2i}^{(n)}\right), \quad d_i^{(n+1)} = \frac{1}{\sqrt{2}}\left(s_{2i-1}^{(n)} - s_{2i}^{(n)}\right) \tag{15.156}$$

a sequence of detailed vectors is formed with components $d_i^{(n)}$. Every detailed vector contains information about the properties of the signals.

Remark: For large values of N the discrete wavelet transformation converges to the integral wavelet transformation (15.152a).

15.5.5 Gabor Transformation

Time-frequency analysis is the characterization of a signal with respect to the contained frequencies and time periods when these frequencies appear. Therefore, the signal is divided into time segments (windows) and a Fourier transform is used. It is called a **W**indowed **F**ourier **T**ransformation (**WFT**).

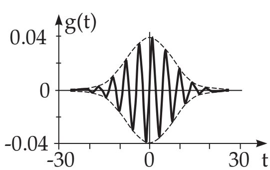

Figure 15.29

The window function should be chosen so that a signal is considered only in the window. Gabor applied the window function

$$g(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} \tag{15.157}$$

(Fig. 15.29). This choice can be explained as $g(t)$, with the "total unit mass", is concentrated at the point $t = 0$ and the width of the window can be considered as a constant (about 2σ).

The *Gabor transformation* of a function $f(t)$ then has the form

$$\mathcal{G}f(\omega, s) = \int_{-\infty}^{\infty} f(t)g(t-s)e^{-\mathrm{i}\omega t}\,dt. \tag{15.158}$$

This determines, with which complex amplitude the dominant wave (fundamental harmonic) $e^{\mathrm{i}\omega t}$ occurs during the time interval $[s-\sigma, s+\sigma]$ in f, i.e., if the frequency ω occurs in this interval, then it has the amplitude $|\mathcal{G}f(\omega, s)|$.

15.6 Walsh Functions

15.6.1 Step Functions

Orthogonal systems of functions have an important role in the approximation theory of functions. For instance, special polynomials or trigonometric functions are used since they are smooth, i.e., they are differentiable sufficiently many times in the considered interval. However, there are problems, e.g., the transition of points of a rough picture, when smooth functions are not suitable for the mathematical description, but *step functions*, piecewise constant functions are more appropriate. Walsh functions are very simple step functions. They take only two function values $+1$ and -1. These two function values correspond to two states, so the Walsh functions can be implemented by computers very easily.

15.6.2 Walsh Systems

Analogously to trigonometric functions also periodic step functions can be considered. The interval $I = [0, 1)$ is used as a period interval and it is divided into 2^n equally long subintervals. Suppose S_n is the set of periodic step functions with period 1 over such an interval. The different step functions belonging to S_n can be considered as vectors of a finite dimensional vector space, since every function $g \in S_n$ is defined by its values $g_0, g_1, g_2, \ldots, g_{2^n-1}$ in the subintervals and it can be considered as a vector:

$$\underline{\mathbf{g}}^{\mathrm{T}} = (g_0, g_1, g_2, \ldots, g_{2^n-1}). \tag{15.159}$$

The Walsh functions belonging to S_n form an orthogonal basis with respect to a suitable scalar product in this space. The basis vectors can be enumerated in many different ways, so one can get many different Walsh systems, which actually contain the same functions. There are three of them which should be mentioned: Walsh–Kronecker functions, Walsh–Kaczmarz functions and Walsh–Paley functions.

The *Walsh transformation* is constructed analogously to the Fourier transformation, where the role of the trigonometric functions is taken by the Walsh functions. One gets, e.g., Walsh series, Walsh polynomials, Walsh sine and Walsh cosine transformations, Walsh integral, and analogously to the fast Fourier transformation there is a Fast Walsh Transformation. For an introduction in the theory and applications of Walsh functions see [15.6].