# 13 Vector Analysis and Vector Fields

## 13.1 Basic Notions of the Theory of Vector Fields

### 13.1.1 Vector Functions of a Scalar Variable

#### 13.1.1.1 Definitions

**1. Vector Function of a Scalar Variable *t***
A vector function of a scalar variable is a vector $\vec{\mathbf{a}}$ whose components are real functions of $t$:

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}(t) = a_x(t)\vec{\mathbf{e}}_x + a_y(t)\vec{\mathbf{e}}_y + a_z(t)\vec{\mathbf{e}}_z. \tag{13.1}$$

The notions of limit, continuity, differentiability are defined componentwise for the vector $\vec{\mathbf{a}}(t)$.

**2. Hodograph of a Vector Function**
Considering the vector function $\vec{\mathbf{a}}(t)$ as a position or radius vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$ of a point $P$, then this function describes a space curve while $t$ varies **(Fig. 13.1)**. This space curve is called the *hodograph* of the vector function $\vec{\mathbf{a}}(t)$.

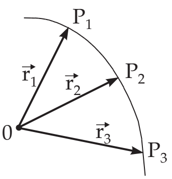


Figure 13.1

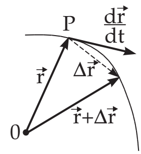


Figure 13.2

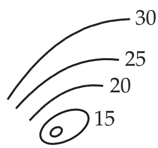


Figure 13.3

#### 13.1.1.2 Derivative of a Vector Function

The derivative of (13.1) with respect to $t$ is also a vector function of $t$:

$$\frac{d\vec{\mathbf{a}}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{a}}(t+\Delta t) - \vec{\mathbf{a}}(t)}{\Delta t} = \frac{da_x(t)}{dt}\vec{\mathbf{e}}_x + \frac{da_y(t)}{dt}\vec{\mathbf{e}}_y + \frac{da_z(t)}{dt}\vec{\mathbf{e}}_z. \tag{13.2}$$

The geometric representation of the derivative $\frac{d\vec{\mathbf{r}}}{dt}$ of the radius vector is a vector pointing in the direction of the tangent of the hodograph at the point $P$ **(Fig. 13.2)**. Its length depends on the choice of the parameter $t$. If $t$ is the time, then the vector $\vec{\mathbf{r}}(t)$ describes the motion of a point $P$ in space (the space curve is its path), and $\frac{d\vec{\mathbf{r}}}{dt}$ has the direction and magnitude of the velocity of this motion. If $t = s$ is the arclength of this space curve, measured from a certain point, then obviously $\left|\frac{d\vec{\mathbf{r}}}{ds}\right| = 1$.

#### 13.1.1.3 Rules of Differentiation for Vectors

$$\frac{d}{dt}\left(\vec{\mathbf{a}} \pm \vec{\mathbf{b}} \pm \vec{\mathbf{c}}\right) = \frac{d\vec{\mathbf{a}}}{dt} \pm \frac{d\vec{\mathbf{b}}}{dt} \pm \frac{d\vec{\mathbf{c}}}{dt}, \tag{13.3a}$$

$$\frac{d}{dt}\left(\varphi\vec{\mathbf{a}}\right) = \frac{d\varphi}{dt}\vec{\mathbf{a}} + \varphi\frac{d\vec{\mathbf{a}}}{dt} \qquad (\varphi \text{ is a scalar function of } t), \tag{13.3b}$$

$$\frac{d}{dt}\left(\vec{\mathbf{a}}\vec{\mathbf{b}}\right) = \frac{d\vec{\mathbf{a}}}{dt}\vec{\mathbf{b}} + \vec{\mathbf{a}}\frac{d\vec{\mathbf{b}}}{dt}, \tag{13.3c}$$



I.N. Bronshtein et al., *Handbook of Mathematics*,
DOI 10.1007/978-3-662-46221-8_13

$$\frac{d}{dt}(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = \frac{d\vec{\mathbf{a}}}{dt} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \frac{d\vec{\mathbf{b}}}{dt} \quad \text{(the factors must not be interchanged),} \tag{13.3d}$$

$$\frac{d}{dt}\vec{\mathbf{a}}[\varphi(t)] = \frac{d\vec{\mathbf{a}}}{d\varphi} \cdot \frac{d\varphi}{dt} \quad \text{(chain rule).} \tag{13.3e}$$

If $|\vec{\mathbf{a}}(t)| = \text{const}$, i.e., $\vec{\mathbf{a}}^2(t) = \vec{\mathbf{a}}(t) \cdot \vec{\mathbf{a}}(t) = \text{const}$, then it follows from (13.3c) that $\vec{\mathbf{a}} \cdot \frac{d\vec{\mathbf{a}}}{dt} = 0$, i.e., $\frac{d\vec{\mathbf{a}}}{dt}$ and $\vec{\mathbf{a}}$ are perpendicular to each other. Examples of this fact:

■ **A:** Radius and tangent vectors of a circle in the plane and

■ **B:** position and tangent vectors of a curve on the sphere. Then the hodograph is a *spherical curve*.

#### 13.1.1.4 Taylor Expansion for Vector Functions

$$\vec{\mathbf{a}}(t+h) = \vec{\mathbf{a}}(t) + h\frac{d\vec{\mathbf{a}}}{dt} + \frac{h^2}{2!}\frac{d^2\vec{\mathbf{a}}}{dt^2} + \cdots + \frac{h^n}{n!}\frac{d^n\vec{\mathbf{a}}}{dt^n} + \cdots . \tag{13.4}$$

The expansion of a vector function in a Taylor series makes sense only if it is convergent. Because the limit is defined componentwise, the convergence can be checked componentwise, so the convergence of this series with vector terms can be determined exactly by the same methods as the convergence of a series with complex terms (see 14.3.2, p. 751). So the convergence of a series with vector terms is reduced to the convergence of a series with scalar terms.

The differential of a vector function $\vec{\mathbf{a}}(t)$ is defined by:

$$d\vec{\mathbf{a}} = \frac{d\vec{\mathbf{a}}}{dt}\Delta t. \tag{13.5}$$

### 13.1.2 Scalar Fields

#### 13.1.2.1 Scalar Field or Scalar Point Function

If a number (scalar value) $U$ is assigned to every point $P$ of a subset of space, then one writes

$$U = U(P) \tag{13.6a}$$

and one calls (13.6a) a *scalar field* (or *scalar function*).

■ Examples of scalar fields are temperature, density, potential, etc., of solids.

A scalar field $U = U(P)$ can also be considered as

$$U = U(\vec{\mathbf{r}}), \tag{13.6b}$$

where $\vec{\mathbf{r}}$ is the position vector of the point $P$ with a given pole 0 (see 3.5.1.1, **6.**, p. 182).

#### 13.1.2.2 Important Special Cases of Scalar Fields

##### 1. Plane Field

One speaks of a plane field, if the function is defined only for the points of a plane in space.

##### 2. Central Field

If a function has the same value at all points $P$ lying at the same distance from a fixed point $C(\vec{\mathbf{r}}_1)$, called the center, then it is called a *central symmetric field* or also a *central or spherical field*. The function $U$ depends only on the distance $\overline{CP} = |\vec{\mathbf{r}}|$:

$$U = f(|\vec{\mathbf{r}}|). \tag{13.7a}$$

■ The field of the intensity of a point-like source, e.g., the field of brightness of a point-like source of light at the pole, can be described with $|\vec{\mathbf{r}}| = r$ as the distance from the light source:

$$U = \frac{c}{r^2} \quad (c \text{ const}). \tag{13.7b}$$

### 3. Axial Field

If the function $U$ has the same value at all points lying at an equal distance from a certain straight line (axis of the field) then the field is called *cylindrically symmetric* or an *axially symmetric field*, or briefly an *axial field*.

## 13.1.2.3 Coordinate Representation of Scalar Fields

If the points of a subset of space are given by their coordinates, e.g., by Cartesian, cylindrical, or spherical coordinates, then the corresponding scalar field (13.6a) is represented, in general, by a function of three variables:

$$U = \Phi(x, y, z), \qquad U = \Psi(\rho, \varphi, z) \qquad \text{or} \qquad U = \chi(r, \vartheta, \varphi). \tag{13.8a}$$

In the case of a plane field, a function with two variables is sufficient. It has the form in Cartesian and polar coordinates:

$$U = \Phi(x, y) \qquad \text{or} \qquad U = \Psi(\rho, \varphi). \tag{13.8b}$$

The functions in (13.8a) and (13.8b), in general, are assumed to be continuous, except, maybe, at some points, curves or surfaces of discontinuity. The functions have the form

**a) for a central field:** $\quad U = U(\sqrt{x^2 + y^2 + z^2}) = U(\sqrt{\rho^2 + z^2}) = U(r) \qquad (13.9a)$

with the origin of the coordinate system as the *pole* of the field,

**b) for an axial field:** $\quad U = U(\sqrt{x^2 + y^2}) = U(\rho) = U(r \sin\vartheta) \qquad (13.9b)$

with the $z$-axis as the axis of the field.

Dealing with central fields is easiest using spherical coordinates, with axial fields using cylindrical coordinates.

## 13.1.2.4 Level Surfaces and Level Lines of a Field

### 1. Level Surface

A level surface is the union of all points $P$ in space where the function (13.6a) has a constant value

$$U = U(P) = \text{const.} \tag{13.10a}$$

Different constants $U_0, U_1, U_2, \ldots$ define different level surfaces. There is a level surface passing through every point except the points where the function is not defined. The level surface equations in the three coordinate systems used so far are:

$$U = \Phi(x, y, z) = \text{const}, \qquad U = \Psi(\rho, \varphi, z) = \text{const}, \qquad U = \chi(r, \vartheta, \varphi) = \text{const}. \tag{13.10b}$$

■ Examples of level surfaces of different fields:

**A:** $U = \vec{\mathbf{c}}\,\vec{\mathbf{r}} = c_x x + c_y y + c_z z$: Parallel planes.

**B:** $U = x^2 + 2y^2 + 4z^2$: Similar ellipsoids in similar positions.

**C:** Central field: Concentric spheres.

**D:** Axial field: Coaxial cylinders.

### 2. Level Lines

Level lines replace level surfaces in plane fields. They satisfy the equation

$$U = \text{const.} \tag{13.11}$$

Level lines are usually drawn for equal intervals of $U$ and each of them is marked by the corresponding value of $U$ **(Fig. 13.3)**.

■ Well-known examples are the isobaric lines on a synoptic map or the contour lines on topographic maps.

In particular cases, level surfaces degenerate into points or lines, and level lines degenerate into separate points.

■ The level lines of the fields a) $U = xy$, b) $U = \dfrac{y}{x^2}$, c) $U = x^2 + y^2 = \rho^2$, d) $U = \dfrac{1}{\rho}$ are represented in **Fig. 13.4**.

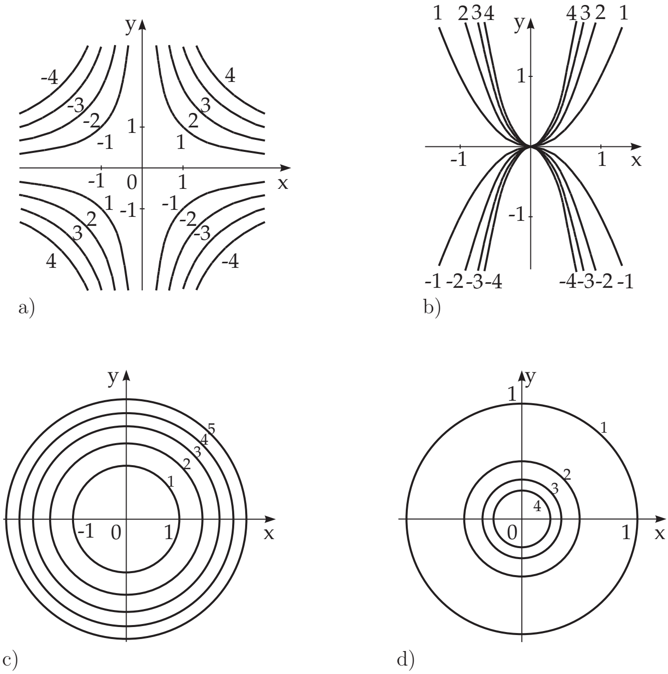


Figure 13.4

## 13.1.3 Vector Fields

### 13.1.3.1 Vector Field or Vector Point Function

If a vector $\vec{\mathbf{V}}$ is assigned to every point $P$ of a subset of space, then it is denoted by

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}(P) \tag{13.12a}$$

and calls (13.12a) a *vector field*.

■ Examples of vector fields are the velocity field of a fluid in motion, a field of force, and a magnetic or electric intensity field.

A vector field $\vec{\mathbf{V}} = \vec{\mathbf{V}}(P)$ can be regarded as a vector function

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}(\vec{\mathbf{r}}), \tag{13.12b}$$

where $\vec{\mathbf{r}}$ is the position vector of the point $P$ with a given pole 0. If all values of $\vec{\mathbf{r}}$ as well as $\vec{\mathbf{V}}$ lie in a plane, then the field is called a plane vector field (see 3.5.2, p. 190).

### 13.1.3.2 Important Cases of Vector Fields

#### 1. Central Vector Field

In a central vector field all vectors $\vec{\mathbf{V}}$ lie on straight lines passing through a fixed point called the *center* **(Fig. 13.5a)**.
Locating the pole at the center, then the field is defined by the formula

$$\vec{\mathbf{V}} = f(\vec{\mathbf{r}})\,\vec{\mathbf{r}}, \tag{13.13a}$$

where all the vectors have the same direction as the radius vector $\vec{\mathbf{r}}$. It often has some advantage to define the field by the formula

$$\vec{\mathbf{V}} = \varphi(\vec{\mathbf{r}})\frac{\vec{\mathbf{r}}}{r} \quad (r = |\vec{\mathbf{r}}|), \tag{13.13b}$$

where $|\varphi(\vec{\mathbf{r}})|$ is the length of the vector $\vec{\mathbf{V}}$ and $\dfrac{\vec{\mathbf{r}}}{r}$ is a unit vector into the direction of $\vec{\mathbf{r}}$.

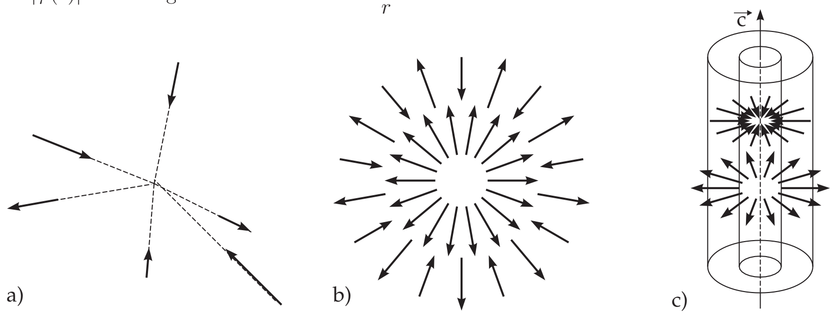


Figure 13.5

#### 2. Spherical Vector Field

A spherical vector field is a special case of a central vector field, where the length of the vector $\vec{\mathbf{V}}$ depends only on the distance $|\vec{\mathbf{r}}|$ **(Fig. 13.5b)**.

■ Examples are the *Newton* and the *Coulomb force field* of a point-like mass or of a point-like electric charge:

$$\vec{\mathbf{V}} = \frac{c}{r^3}\vec{\mathbf{r}} = \frac{c}{r^2}\frac{\vec{\mathbf{r}}}{r} \quad (c \text{ const}). \tag{13.14}$$

The special case of a plane spherical vector field is called a *circular field*.

#### 3. Cylindrical Vector Field

**a)** All vectors $\vec{\mathbf{V}}$ lie on straight lines intersecting a certain line (called the *axis*) and perpendicular to it, and

**b)** all vectors $\vec{\mathbf{V}}$ at the points lying at the same distance from the axis have equal length, and they are directed either toward the axis or away from it **(Fig. 13.5c)**.

Locating the pole on the axis parallel to the unit vector $\vec{\mathbf{c}}$, then the field has the form

$$\vec{\mathbf{V}} = \varphi(\rho)\frac{\vec{\mathbf{r}}^*}{\rho}, \tag{13.15a}$$

where $\vec{\mathbf{r}}^*$ is the projection of $\vec{\mathbf{r}}$ on a plane perpendicular to the axis:

$$\vec{\mathbf{r}}^* = \vec{\mathbf{c}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{c}}). \tag{13.15b}$$

By intersecting this field with planes perpendicular to the axis, one always gets equal circular fields.

### 13.1.3.3 Coordinate Representation of Vector Fields

#### 1. Vector Field in Cartesian Coordinates

The vector field (13.12a) can be defined by scalar fields $V_1(\vec{\mathbf{r}})$, $V_2(\vec{\mathbf{r}})$, and $V_3(\vec{\mathbf{r}})$ which are the coordinate functions of $\vec{\mathbf{V}}$, i.e., the coefficients of its decomposition into any three non-coplanar base vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$, and $\vec{\mathbf{e}}_3$:

$$\vec{\mathbf{V}} = V_1\vec{\mathbf{e}}_1 + V_2\vec{\mathbf{e}}_2 + V_3\vec{\mathbf{e}}_3. \tag{13.16a}$$

With the coordinate unit vectors $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$ as base vectors and expressing the coefficients $V_1$, $V_2$, $V_3$ in Cartesian coordinates one gets

$$\vec{\mathbf{V}} = V_x(x,y,z)\vec{\mathbf{i}} + V_y(x,y,z)\vec{\mathbf{j}} + V_z(x,y,z)\vec{\mathbf{k}}. \tag{13.16b}$$

So, the vector field can be defined with the help of three scalar functions of three scalar variables.

#### 2. Vector Field in Cylindrical and Spherical Coordinates

In cylindrical and spherical coordinates, the coordinate unit vectors

$$\vec{\mathbf{e}}_\rho,\ \vec{\mathbf{e}}_\varphi,\ \vec{\mathbf{e}}_z\ (= \vec{\mathbf{k}}), \qquad \text{and} \qquad \vec{\mathbf{e}}_r\ (= \frac{\vec{\mathbf{r}}}{r}),\ \vec{\mathbf{e}}_\vartheta,\ \vec{\mathbf{e}}_\varphi \tag{13.17a}$$

are tangents to the coordinate lines at each point **(Fig. 13.6, 13.7)**. In this order they always form a right-handed system. The coefficients are expressed as functions of the corresponding coordinates:

$$\vec{\mathbf{V}} = V_\rho(\rho,\varphi,z)\vec{\mathbf{e}}_\rho + V_\varphi(\rho,\varphi,z)\vec{\mathbf{e}}_\varphi + V_z(\rho,\varphi,z)\vec{\mathbf{e}}_z\,, \tag{13.17b}$$

$$\vec{\mathbf{V}} = V_r(r,\vartheta,\varphi)\vec{\mathbf{e}}_r + V_\varphi(r,\vartheta,\varphi)\vec{\mathbf{e}}_\varphi + V_\vartheta(r,\vartheta,\varphi)\vec{\mathbf{e}}_\vartheta. \tag{13.17c}$$

At transition from one point to the other, the coordinate unit vectors change their directions, but remain mutually perpendicular.

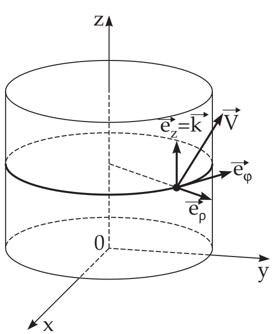


Figure 13.6

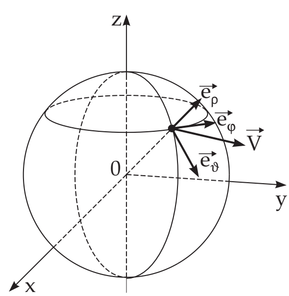


Figure 13.7

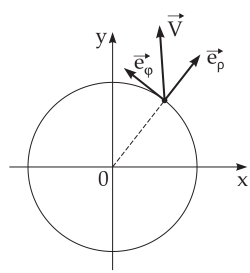


Figure 13.8

### 13.1.3.4 Transformation of Coordinate Systems

See also **Table 13.1**.

#### 1. Cartesian Coordinates in Terms of Cylindrical Coordinates

$$V_x = V_\rho\cos\varphi - V_\varphi\sin\varphi, \qquad V_y = V_\rho\sin\varphi + V_\varphi\cos\varphi, \qquad V_z = V_z. \tag{13.18}$$

**2. Cylindrical Coordinates in Terms of Cartesian Coordinates**

$$V_\rho = V_x \cos\varphi + V_y \sin\varphi, \qquad V_\varphi = -V_x \sin\varphi + V_y \cos\varphi, \qquad V_z = V_z. \tag{13.19}$$

**3. Cartesian Coordinates in Terms of Spherical Coordinates**

$$\begin{aligned} V_x &= V_r \sin\vartheta \cos\varphi - V_\varphi \sin\varphi + V_\vartheta \cos\varphi \cos\vartheta, \\ V_y &= V_r \sin\vartheta \sin\varphi + V_\varphi \cos\varphi + V_\vartheta \sin\varphi \cos\vartheta, \\ V_z &= V_r \cos\vartheta - V_\vartheta \sin\vartheta. \end{aligned} \tag{13.20}$$

**4. Spherical Coordinates in Terms of Cartesian Coordinates**

$$\begin{aligned} V_r &= V_x \sin\vartheta \cos\varphi + V_y \sin\vartheta \sin\varphi + V_z \cos\vartheta, \\ V_\vartheta &= V_x \cos\vartheta \cos\varphi + V_y \cos\vartheta \sin\varphi - V_z \sin\vartheta, \\ V_\varphi &= -V_x \sin\varphi + V_y \cos\varphi. \end{aligned} \tag{13.21}$$

**5. Expression of a Spherical Vector Field in Cartesian Coordinates**

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2 + y^2 + z^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}). \tag{13.22}$$

**6. Expression of a Cylindrical Vector Field in Cartesian Coordinates**

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2 + y^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}}). \tag{13.23}$$

In the case of a spherical vector field, spherical coordinates are most convenient for investigations, i.e., the form $\vec{\mathbf{V}} = V(r)\vec{\mathbf{e}}_r$; and for investigations in cylindrical fields, cylindrical coordinates are most convenient, i.e., the form $\vec{\mathbf{V}} = V(\varphi)\vec{\mathbf{e}}_\varphi$. In the case of a plane field **(Fig. 13.8)**

$$\vec{\mathbf{V}} = V_x(x,y)\vec{\mathbf{i}} + V_y(x,y)\vec{\mathbf{j}} = V_\rho(\rho,\varphi)\vec{\mathbf{e}}_\rho + V_\varphi(\rho,\varphi)\vec{\mathbf{e}}_\varphi, \tag{13.24}$$

holds and for a circular field

$$\vec{\mathbf{V}} = \varphi(\sqrt{x^2 + y^2})(x\vec{\mathbf{i}} + y\vec{\mathbf{j}}) = \varphi(\rho)\vec{\mathbf{e}}_\rho. \tag{13.25}$$

Table 13.1 Relations between the components of a vector in Cartesian, cylindrical, and spherical coordinates

| **Cartesian coordinates** | **Cylindrical coord.** | **Spherical coordinates** |
|---|---|---|
| $\vec{\mathbf{V}} = V_x\vec{\mathbf{e}}_x + V_y\vec{\mathbf{e}}_y + V_z\vec{\mathbf{e}}_z$ | $V_\rho\vec{\mathbf{e}}_\rho + V_\varphi\vec{\mathbf{e}}_\varphi + V_z\vec{\mathbf{e}}_z$ | $V_r\vec{\mathbf{e}}_r + V_\vartheta\vec{\mathbf{e}}_\vartheta + V_\varphi\vec{\mathbf{e}}_\varphi$ |
| $V_x$ | $= V_\rho \cos\varphi - V_\varphi \sin\varphi$ | $= V_r \sin\vartheta \cos\varphi + V_\vartheta \cos\vartheta \cos\varphi - V_\varphi \sin\varphi$ |
| $V_y$ | $= V_\rho \sin\varphi + V_\varphi \cos\varphi$ | $= V_r \sin\vartheta \sin\varphi + V_\vartheta \cos\vartheta \sin\varphi + V_\varphi \cos\varphi$ |
| $V_z$ | $= V_z$ | $= V_r \cos\vartheta - V_\vartheta \sin\vartheta$ |
| $V_x \cos\varphi + V_y \sin\varphi$ | $= V_\rho$ | $= V_r \sin\vartheta + V_\vartheta \cos\vartheta$ |
| $-V_x \sin\varphi + V_y \cos\varphi$ | $= V_\varphi$ | $= V_\varphi$ |
| $V_z$ | $= V_z$ | $= V_r \cos\vartheta - V_\vartheta \sin\vartheta$ |
| $V_x \sin\vartheta \cos\varphi + V_y \sin\vartheta \sin\varphi + V_z \cos\vartheta$ | $= V_\rho \sin\vartheta + V_z \cos\vartheta$ | $= V_r$ |
| $V_x \cos\vartheta \cos\varphi + V_y \cos\vartheta \sin\varphi - V_z \sin\varphi$ | $= V_\rho \cos\vartheta - V_z \sin\vartheta$ | $= V_\vartheta$ |
| $-V_x \sin\varphi + V_y \cos\varphi$ | $= V_\varphi$ | $= V_\varphi$ |

#### 13.1.3.5 Vector Lines

A curve $C$ is called a *line of a vector* or a *vector line* of the vector field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ **(Fig. 13.9)** if the vector $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is a tangent vector of the curve at every point $P$. There is a vector line passing through every point of the field. Vector lines do not intersect each other, except, maybe, at points where the function $\vec{\mathbf{V}}$ is not defined, or where it is the zero vector. The differential equations of the vector lines of a vector field $\vec{\mathbf{V}}$ given in Cartesian coordinates are

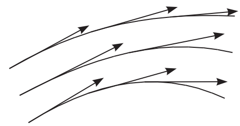

Figure 13.9

**a) in general:** $\dfrac{dx}{V_x} = \dfrac{dy}{V_y} = \dfrac{dz}{V_z}$, (13.26a) **b) for a plane field:** $\dfrac{dx}{V_x} = \dfrac{dy}{V_y}$. (13.26b)

To solve these differential equations see 9.1.1.2, p. 542 or 9.2.1.1, p. 571.

■ **A:** The vector lines of a central field are rays starting at the center of the vector field.

■ **B:** The vector lines of the vector field $\vec{\mathbf{V}} = \vec{\mathbf{c}} \times \vec{\mathbf{r}}$ are circles lying in planes perpendicular to the vector $\vec{\mathbf{c}}$. Their centers are on the axis parallel to $\vec{\mathbf{c}}$.

# 13.2 Differential Operators of Space

## 13.2.1 Directional and Space Derivatives

### 13.2.1.1 Directional Derivative of a Scalar Field

The directional derivative of a scalar field $U = U(\vec{\mathbf{r}})$ at a point $P$ with position vector $\vec{\mathbf{r}}$ in the direction $\vec{\mathbf{c}}$ **(Fig. 13.10)** is defined as the limit of the quotient

$$\frac{\partial U}{\partial \vec{\mathbf{c}}} = \lim_{\varepsilon \to 0} \frac{U(\vec{\mathbf{r}} + \varepsilon \vec{\mathbf{c}}) - U(\vec{\mathbf{r}})}{\varepsilon}. \qquad (13.27)$$

If the derivative of the field $U = U(\vec{\mathbf{r}})$ at a point $\vec{\mathbf{r}}$ in the direction of the unit vector $\vec{\mathbf{c}}^0$ of $\vec{\mathbf{c}}$ is denoted by $\dfrac{\partial U}{\partial \vec{\mathbf{c}}^0}$, then the relation between the derivative of the function with respect to the vector $\vec{\mathbf{c}}$ and with respect to its unit vector $\vec{\mathbf{c}}^0$ at the same point is

$$\frac{\partial U}{\partial \vec{\mathbf{c}}} = |\vec{\mathbf{c}}| \frac{\partial U}{\partial \vec{\mathbf{c}}^0}. \qquad (13.28)$$

The derivative $\dfrac{\partial U}{\partial \vec{\mathbf{c}}^0}$ with respect to the unit vector represents the speed of increase of the function $U$ in the direction of the vector $\vec{\mathbf{c}}^0$ at the point $\vec{\mathbf{r}}$. If $\vec{\mathbf{n}}$ is the normal unit vector to the level surface passing through the point $\vec{\mathbf{r}}$, and $\vec{\mathbf{n}}$ is pointing in the direction of increasing $U$, then $\dfrac{\partial U}{\partial \vec{\mathbf{n}}}$ has the greatest value among all the derivatives at the point with respect to the unit vectors in different directions. Between the directional derivatives with respect to $\vec{\mathbf{n}}$ and with respect to any direction $\vec{\mathbf{c}}^0$ holds the relation

$$\frac{\partial U}{\partial \vec{\mathbf{c}}^0} = \frac{\partial U}{\partial \vec{\mathbf{n}}} \cos(\vec{\mathbf{c}}^0, \vec{\mathbf{n}}) = \frac{\partial U}{\partial \vec{\mathbf{n}}} \cos\varphi = \vec{\mathbf{c}}^0 \cdot \operatorname{grad} U \quad \text{(see (13.34), p. 710)}. \qquad (13.29)$$

Hereafter, directional derivatives always mean the directional derivative with respect to a unit vector.

### 13.2.1.2 Directional Derivative of a Vector Field

The directional derivative of a vector field is defined analogously to the directional derivative of a scalar field. The directional derivative of the vector field $\vec{\mathbf{V}} = \vec{\mathbf{V}}(\vec{\mathbf{r}})$ at a point $P$ with position vector $\vec{\mathbf{r}}$

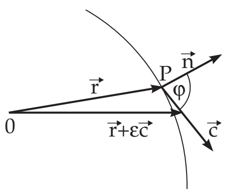


Figure 13.10

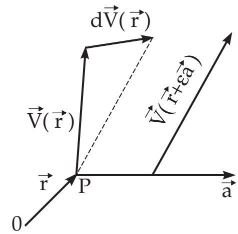


Figure 13.11

**(Fig. 13.11)** with respect to the vector $\vec{\mathbf{a}}$ is defined as the limit of the quotient

$$\frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} = \lim_{\varepsilon \to 0} \frac{\vec{\mathbf{V}}(\vec{\mathbf{r}} + \varepsilon \vec{\mathbf{a}}) - \vec{\mathbf{V}}(\vec{\mathbf{r}})}{\varepsilon} . \qquad (13.30)$$

If the derivative of the vector field $\vec{\mathbf{V}} = \vec{\mathbf{V}}(\vec{\mathbf{r}})$ at a point $\vec{\mathbf{r}}$ in the direction of the unit vector $\vec{\mathbf{a}}^0$ of $\vec{\mathbf{a}}$ is denoted by $\dfrac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}^0}$, then

$$\frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} = |\vec{\mathbf{a}}| \frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}^0} . \qquad (13.31)$$

In Cartesian coordinates, i.e., for $\vec{\mathbf{V}} = V_x \vec{\mathbf{e}}_x + V_y \vec{\mathbf{e}}_y + V_z \vec{\mathbf{e}}_z$, $\vec{\mathbf{a}} = a_x \vec{\mathbf{e}}_x + a_y \vec{\mathbf{e}}_y + a_z \vec{\mathbf{e}}_z$, holds:

$$(\vec{\mathbf{a}} \cdot \operatorname{grad}) \vec{\mathbf{V}} = (\vec{\mathbf{a}} \cdot \operatorname{grad} V_x) \vec{\mathbf{e}_\mathbf{x}} + (\vec{\mathbf{a}} \cdot \operatorname{grad} V_y) \vec{\mathbf{e}_\mathbf{y}} + (\vec{\mathbf{a}} \cdot \operatorname{grad} V_z) \vec{\mathbf{e}_\mathbf{z}} , \qquad (13.32a)$$

in general coordinates:

$$(\vec{\mathbf{a}} \cdot \operatorname{grad}) \vec{\mathbf{V}} = \frac{1}{2} (\operatorname{rot} (\vec{\mathbf{V}} \times \vec{\mathbf{a}}) + \operatorname{grad} (\vec{\mathbf{a}} \cdot \vec{\mathbf{V}}) + \vec{\mathbf{a}} \operatorname{div} \vec{\mathbf{V}} - \vec{\mathbf{V}} \operatorname{div} \vec{\mathbf{a}} - \vec{\mathbf{a}} \times \operatorname{rot} \vec{\mathbf{V}} - \vec{\mathbf{V}} \times \operatorname{rot} \vec{\mathbf{a}} . \qquad (13.32b)$$

### 13.2.1.3 Volume Derivative

Volume derivatives of a scalar field $U = U(\vec{\mathbf{r}})$ or a vector field $\vec{\mathbf{V}}$ at a point $\vec{\mathbf{r}}$ are quantities of three forms, which are obtained as follows:

**1.** Surrounding the point $\vec{\mathbf{r}}$ of the scalar field or of the vector field by a closed surface $\Sigma$. This surface can be represented in parametric form $\vec{\mathbf{r}} = \vec{\mathbf{r}}(u, v) = x(u, v)\vec{\mathbf{e}}_x + y(u, v)\vec{\mathbf{e}}_y + z(u, v)\vec{\mathbf{e}}_z$, so the corresponding vectorial surface element is

$$d\vec{\mathbf{S}} = \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \, du \, dv . \qquad (13.33a)$$

**2.** Evaluating the surface integral over the closed surface $\Sigma$. Here, the following three types of integrals can be considered:

$$\oint\!\!\!\oint_{(\Sigma)} U \, d\vec{\mathbf{S}} , \qquad \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}} , \qquad \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}} . \qquad (13.33b)$$

**3.** Determining the limits (if they exist)

$$\lim_{V \to 0} \frac{1}{V} \oint\!\!\!\oint_{(\Sigma)} U \, d\vec{\mathbf{S}} , \quad \lim_{V \to 0} \frac{1}{V} \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}} , \quad \lim_{V \to 0} \frac{1}{V} \oint\!\!\!\oint_{(\Sigma)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}} . \qquad (13.33c)$$

Here $V$ denotes the volume of the region of space that contains the point with the position vector $\vec{\mathbf{r}}$ inside, and which is bounded by the considered closed surface $\Sigma$.

The limits (13.33c) are called volume derivatives. The *gradient of a scalar field* and the *divergence* and the *rotation* of a vector field can be derived from them in the given order. In the following paragraphs, these notions will be discussed in details (even defining them again.)

## 13.2.2 Gradient of a Scalar Field

The gradient of a scalar field can be defined in different ways.

### 13.2.2.1 Definition of the Gradient

The *gradient* of a function $U$ is a vector $\operatorname{grad} U$, which can be assigned to every point $P$ with the vector $\vec{\mathbf{r}}$ of a scalar field $U = U(\vec{\mathbf{r}})$, having the following properties:

**1.** The direction of $\operatorname{grad} U$ is always perpendicular to the direction of the level surface $U = \text{const}$, passing through the considered points $P$,

**2.** $\operatorname{grad} U$ points always into the direction in which the function $U$ is increasing,

**3.** $|\operatorname{grad} U| = \dfrac{\partial U}{\partial \vec{\mathbf{n}}}$, i.e., the magnitude of $\operatorname{grad} U$ is equal to the directional derivative of $U$ in the *normal direction*.

If the gradient is defined in another way, e.g., as a volume derivative or by the differential operator, then the previous defining properties became consequences of the definition.

### 13.2.2.2 Gradient and Directional Derivative

The directional derivative of the scalar field $U$ with respect to the unit vector $\vec{\mathbf{c}}^0$ is equal to the projection of $\operatorname{grad} U$ onto the direction of the unit vector $\vec{\mathbf{c}}^0$:

$$\frac{\partial U}{\partial \vec{\mathbf{c}}^0} = \vec{\mathbf{c}}^0 \cdot \operatorname{grad} U, \qquad (13.34)$$

i.e., the directional derivative can be calculated as the dot product of the gradient and the unit vector pointing into the required direction.

**Remark**: The directional derivative at certain points in certain directions may also exist if the gradient does not exist there.

### 13.2.2.3 Gradient and Volume Derivative

The *gradient* $U$ of the scalar field $U = U(\vec{\mathbf{r}})$ at a point $\vec{\mathbf{r}}$ can be defined as its *volume derivative*. If the following limit exists, then it is called the *gradient of* $U$ at $\vec{\mathbf{r}}$:

$$\operatorname{grad} U = \lim_{V \to 0} \frac{\displaystyle\oiint_{(\Sigma)} U \, d\vec{\mathbf{S}}}{V}. \qquad (13.35)$$

Here $V$ is the volume of the region of space containing the point belonging to $\vec{\mathbf{r}}$ inside and bounded by the closed surface $\Sigma$. (If the independent variable is not a three-dimensional vector, then the gradient is defined by the differential operator.)

### 13.2.2.4 Further Properties of the Gradient

**1.** The absolute value of the gradient is greater if the level lines or level surfaces drawn as mentioned in 13.1.2.4, **2.**, p. 703, are more dense.

**2.** The gradient is the zero vector ($\operatorname{grad} U = \vec{\mathbf{0}}$) if $U$ has a maximum or minimum at the considered point. The level lines or surfaces degenerate to a point there.

### 13.2.2.5 Gradient of the Scalar Field in Different Coordinates

#### 1. Gradient in Cartesian Coordinates

$$\operatorname{grad} U = \frac{\partial U(x,y,z)}{\partial x}\vec{\mathbf{i}} + \frac{\partial U(x,y,z)}{\partial y}\vec{\mathbf{j}} + \frac{\partial U(x,y,z)}{\partial z}\vec{\mathbf{k}}. \qquad (13.36)$$

**2. Gradient in Cylindrical Coordinates ($x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z$)**

$$\operatorname{grad} U = \operatorname{grad}_\rho U\vec{\mathbf{e}}_\rho + \operatorname{grad}_\varphi U\vec{\mathbf{e}}_\varphi + \operatorname{grad}_z U\vec{\mathbf{e}}_z \quad \text{with} \tag{13.37a}$$

$$\operatorname{grad}_\rho U = \frac{\partial U}{\partial \rho}, \quad \operatorname{grad}_\varphi U = \frac{1}{\rho}\frac{\partial U}{\partial \varphi}, \quad \operatorname{grad}_z U = \frac{\partial U}{\partial z}. \tag{13.37b}$$

**3. Gradient in Spherical Coordinates ($x = r\sin\vartheta\cos\varphi,\ y = r\sin\vartheta\sin\varphi,\ z = r\cos\vartheta$)**

$$\operatorname{grad} U = \operatorname{grad}_r U\vec{\mathbf{e}}_r + \operatorname{grad}_\vartheta U\vec{\mathbf{e}}_\vartheta + \operatorname{grad}_\varphi U\vec{\mathbf{e}}_\varphi \quad \text{with} \tag{13.38a}$$

$$\operatorname{grad}_r U = \frac{\partial U}{\partial r}, \quad \operatorname{grad}_\vartheta U = \frac{1}{r}\frac{\partial U}{\partial \vartheta}, \quad \operatorname{grad}_\varphi U = \frac{1}{r\sin\vartheta}\frac{\partial U}{\partial \varphi}. \tag{13.38b}$$

**4. Gradient in General Orthogonal Coordinates ($\xi, \eta, \zeta$)**

For $\vec{\mathbf{r}}(\xi,\eta,\zeta) = x(\xi,\eta,\zeta)\vec{\mathbf{i}} + y(\xi,\eta,\zeta)\vec{\mathbf{j}} + z(\xi,\eta,\zeta)\vec{\mathbf{k}}$:

$$\operatorname{grad} U = \operatorname{grad}_\xi U\vec{\mathbf{e}}_\xi + \operatorname{grad}_\eta U\vec{\mathbf{e}}_\eta + \operatorname{grad}_\zeta U\vec{\mathbf{e}}_\zeta, \quad \text{where} \tag{13.39a}$$

$$\operatorname{grad}_\xi U = \frac{1}{\left|\frac{\partial \vec{\mathbf{r}}}{\partial \xi}\right|}\frac{\partial U}{\partial \xi}, \quad \operatorname{grad}_\eta U = \frac{1}{\left|\frac{\partial \vec{\mathbf{r}}}{\partial \eta}\right|}\frac{\partial U}{\partial \eta}, \quad \operatorname{grad}_\zeta U = \frac{1}{\left|\frac{\partial \vec{\mathbf{r}}}{\partial \zeta}\right|}\frac{\partial U}{\partial \zeta}. \tag{13.39b}$$

### 13.2.2.6 Rules of Calculations

Assuming in the followings that $\vec{\mathbf{c}}$ and $c$ are constant, the following equalities hold:

$$\operatorname{grad} c = 0, \quad \operatorname{grad}(U_1 + U_2) = \operatorname{grad} U_1 + \operatorname{grad} U_2, \quad \operatorname{grad}(c\,U) = c\operatorname{grad} U. \tag{13.40}$$

$$\operatorname{grad}(U_1 U_2) = U_1 \operatorname{grad} U_2 + U_2 \operatorname{grad} U_1, \quad \operatorname{grad}\varphi(U) = \frac{d\varphi}{dU}\operatorname{grad} U. \tag{13.41}$$

$$\operatorname{grad}(\vec{\mathbf{V}}_1 \cdot \vec{\mathbf{V}}_2) = (\vec{\mathbf{V}}_1 \cdot \operatorname{grad})\vec{\mathbf{V}}_2 + (\vec{\mathbf{V}}_2 \cdot \operatorname{grad})\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_1 \times \operatorname{rot}\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_2 \times \operatorname{rot}\vec{\mathbf{V}}_1. \tag{13.42}$$

$$\operatorname{grad}(\vec{\mathbf{r}} \cdot \vec{\mathbf{c}}) = \vec{\mathbf{c}}. \tag{13.43}$$

**1. Differential of a Scalar Field as the Total Differential of the Function $U$**

$$dU = \operatorname{grad} U \cdot d\vec{\mathbf{r}} = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz. \tag{13.44}$$

**2. Derivative of a Function $U$ along a Space Curve $\vec{\mathbf{r}}(t)$**

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}. \tag{13.45}$$

**3. Gradient of a Central Field**

$$\operatorname{grad} U(r) = U'(r)\frac{\vec{\mathbf{r}}}{r} \quad \text{(spherical field)}, \quad (13.46a) \quad \operatorname{grad} r = \frac{\vec{\mathbf{r}}}{r} \quad \text{(field of unit vectors)}. \quad (13.46b)$$

## 13.2.3 Vector Gradient

The relation (13.32a) inspires the notation

$$\frac{\partial \vec{\mathbf{V}}}{\partial \vec{\mathbf{a}}} = \vec{\mathbf{a}} \cdot \operatorname{grad}(V_x\vec{\mathbf{e}}_x + V_y\vec{\mathbf{e}}_y + V_z\vec{\mathbf{e}}_z) = \vec{\mathbf{a}} \cdot \operatorname{grad}\vec{\mathbf{V}} \tag{13.47a}$$

where grad $\vec{\mathbf{V}}$ is called the *vector gradient*. It follows from the matrix notation of (13.47a) that the vector gradient, as a tensor, can be represented by a matrix:

$$(\vec{\mathbf{a}}\cdot\operatorname{grad})\vec{\mathbf{V}} = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} \\ \frac{\partial V_z}{\partial x} & \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial z} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \quad (13.47b) \qquad \operatorname{grad}\vec{\mathbf{V}} = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} \\ \frac{\partial V_z}{\partial x} & \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial z} \end{pmatrix}. \quad (13.47c)$$

These types of tensors have a very important role in engineering sciences, e.g., for the description of tension and elasticity (see 4.3.2, **4.**, p. 282.

## 13.2.4 Divergence of Vector Fields

### 13.2.4.1 Definition of Divergence

To a vector field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ a scalar field can be assigned which is called its *divergence*. The divergence is defined as a space derivative of the vector field at a point $\vec{\mathbf{r}}$:

$$\operatorname{div}\vec{\mathbf{V}} = \lim_{V\to 0} \frac{\oiint\limits_{(\Sigma)} \vec{\mathbf{V}}\cdot d\vec{\mathbf{S}}}{V}. \quad (13.48)$$

If the vector field $\vec{\mathbf{V}}$ is considered as a stream field, then the divergence can be considered as the fluid output or source, because it gives the amount of fluid given in a unit of volume during a unit of time flowing by the considered point of the vector field $\vec{\mathbf{V}}$. In the case div $\vec{\mathbf{V}} > 0$ the point is called a *source*, in the case div $\vec{\mathbf{V}} < 0$ it is called a *sink*.

### 13.2.4.2 Divergence in Different Coordinates

#### 1. Divergence in Cartesian Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \quad (13.49a) \qquad \text{with} \quad \vec{\mathbf{V}}(x,y,z) = V_x\vec{\mathbf{i}} + V_y\vec{\mathbf{j}} + V_z\vec{\mathbf{k}}. \quad (13.49b)$$

The scalar field div $\vec{\mathbf{V}}$ can be represented as the dot product of the nabla operator $\nabla$ and the vector $\vec{\mathbf{V}}$ as

$$\operatorname{div}\vec{\mathbf{V}} = \nabla\cdot\vec{\mathbf{V}} \quad (13.49c)$$

and it is translation and rotation invariant, i.e., scalar invariant (see 4.3.3.2, p. 283).

#### 2. Divergence in Cylindrical Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{1}{\rho}\frac{\partial(\rho V_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z} \quad (13.50a) \quad \text{with} \quad \vec{\mathbf{V}}(\rho,\varphi,z) = V_\rho\vec{\mathbf{e}}_\rho + V_\varphi\vec{\mathbf{e}}_\varphi + V_z\vec{\mathbf{e}}_z. \quad (13.50b)$$

#### 3. Divergence in Spherical Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta V_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial\varphi} \quad (13.51a)$$

$$\text{with} \quad \vec{\mathbf{V}}(r,\vartheta,\varphi) = V_r\vec{\mathbf{e}}_r + V_\vartheta\vec{\mathbf{e}}_\vartheta + V_\varphi\vec{\mathbf{e}}_\varphi. \quad (13.51b)$$

#### 4. Divergence in General Orthogonal Coordinates

$$\operatorname{div}\vec{\mathbf{V}} = \frac{1}{\mathrm{D}}\left\{\frac{\partial}{\partial\xi}\left(\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|V_\xi\right) + \frac{\partial}{\partial\eta}\left(\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|\left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|V_\eta\right) + \frac{\partial}{\partial\zeta}\left(\left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|V_\zeta\right)\right\} \quad (13.52a)$$

with $\vec{\mathbf{r}}(\xi,\eta,\zeta) = x(\xi,\eta,\zeta)\vec{\mathbf{i}} + y(\xi,\eta,\zeta)\vec{\mathbf{j}} + z(\xi,\eta,\zeta)\vec{\mathbf{k}},$ (13.52b)

$$\mathrm{D} = \left|\left(\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right)\right| = \left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right| \quad (13.52c) \text{ and } \vec{\mathbf{V}}(\xi,\eta,\zeta) = V_\xi\vec{\mathbf{e}}_\xi + V_\eta\vec{\mathbf{e}}_\eta + V_\zeta\vec{\mathbf{e}}_\zeta. \quad (13.52d)$$

### 13.2.4.3 Rules for Evaluation of the Divergence

$$\operatorname{div}\vec{\mathbf{c}} = 0, \quad \operatorname{div}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{div}\vec{\mathbf{V}}_1 + \operatorname{div}\vec{\mathbf{V}}_2, \quad \operatorname{div}(c\vec{\mathbf{V}}) = c\operatorname{div}\vec{\mathbf{V}}. \quad (13.53)$$

$$\operatorname{div}(U\vec{\mathbf{V}}) = U\operatorname{div}\vec{\mathbf{V}} + \vec{\mathbf{V}}\cdot\operatorname{grad}U \qquad \left(\text{especially } \operatorname{div}(r\vec{\mathbf{c}}) = \frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{c}}}{r}\right). \quad (13.54)$$

$$\operatorname{div}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = \vec{\mathbf{V}}_2\cdot\operatorname{rot}\vec{\mathbf{V}}_1 - \vec{\mathbf{V}}_1\cdot\operatorname{rot}\vec{\mathbf{V}}_2\,. \quad (13.55)$$

### 13.2.4.4 Divergence of a Central Field

$$\operatorname{div}\vec{\mathbf{r}} = 3, \quad \operatorname{div}\varphi(r)\vec{\mathbf{r}} = 3\varphi(r) + r\varphi'(r). \quad (13.56)$$

## 13.2.5 Rotation of Vector Fields

### 13.2.5.1 Definitions of the Rotation

#### 1. Definition

The *rotation* or *curl* of a vector field $\vec{\mathbf{V}}$ at the point $\vec{\mathbf{r}}$ is a vector denoted by $\operatorname{rot}\vec{\mathbf{V}}$, $\operatorname{curl}\vec{\mathbf{V}}$ or with the nabla operator $\nabla\times\vec{\mathbf{V}}$, and defined as the negative space derivative of the vector field:

$$\operatorname{rot}\vec{\mathbf{V}} = -\lim_{V\to 0}\frac{\displaystyle\oiint_{(\Sigma)}\vec{\mathbf{V}}\times d\vec{\mathbf{S}}}{V} = \lim_{V\to 0}\frac{\displaystyle\oiint_{(\Sigma)} d\vec{\mathbf{S}}\times\vec{\mathbf{V}}}{V}. \quad (13.57)$$

#### 2. Definition

The vector field of the rotation of the vector field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ can be defined in the following way:

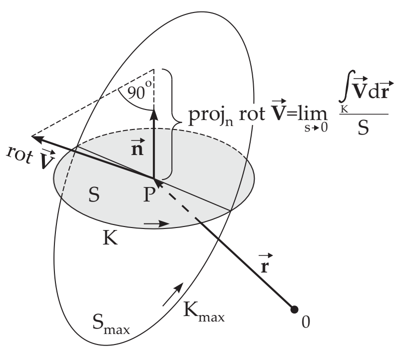


Figure 13.12

**a)** Putting a small surface sheet $S$ **(Fig. 13.12)** through the point $\vec{\mathbf{r}}$ and describing this surface sheet by a vector $\vec{\mathbf{S}}$ whose direction is the direction of the surface normal $\vec{\mathbf{n}}$ and its absolute value is equal to the area of this surface region. The boundary of this surface is denoted by $C$.

**b)** Evaluating the integral $\displaystyle\oint_{(C)}\vec{\mathbf{V}}\cdot d\vec{\mathbf{r}}$ along the closed boundary curve $C$ of the surface (the sense of the curve is positive looking to the surface from the direction of the surface normal (see **Fig. 13.12**).

**c)** Determining the limit (if it exists) $\displaystyle\lim_{S\to 0}\frac{1}{S}\oint_{(C)}\vec{\mathbf{V}}\cdot d\vec{\mathbf{r}}$, while the position of the surface sheet remains unchanged.

**d)** Changing the position of the surface sheet in order to get a maximum value of the limit. The surface area in this position is $S_{\max}$ and the corresponding boundary curve is $C_{\max}$.

**e)** Determining the vector $\operatorname{rot}\vec{\mathbf{r}}$ at the point $\vec{\mathbf{r}}$, whose absolute value is equal to the maximum value found above and its direction coincides with the direction of the surface normal of the corresponding surface. Then one gets:

$$\left|\operatorname{rot}\vec{\mathbf{V}}\right| = \lim_{S_{\max}\to 0} \frac{\oint\limits_{(C_{\max})} \vec{\mathbf{V}}\cdot d\vec{\mathbf{r}}}{S_{\max}}. \tag{13.58a}$$

The projection of $\operatorname{rot}\vec{\mathbf{V}}$ onto the surface normal $\vec{\mathbf{n}}$ of a surface with area $S$, i.e., the component of the vector $\operatorname{rot}\vec{\mathbf{V}}$ in an arbitrary direction $\vec{\mathbf{n}} = \vec{\mathbf{l}}$ is

$$\vec{\mathbf{l}}\cdot\operatorname{rot}\vec{\mathbf{V}} = \operatorname{rot}_l\vec{\mathbf{V}} = \lim_{S\to 0} \frac{\oint\limits_{(C)} \vec{\mathbf{V}}\cdot d\vec{\mathbf{r}}}{S}. \tag{13.58b}$$

The vector lines of the field $\operatorname{rot}\vec{\mathbf{V}}$ are called the *curl lines of the vector field* $\vec{\mathbf{V}}$.

### 13.2.5.2 Rotation in Different Coordinates

#### 1. Rotation in Cartesian Coordinates

$$\operatorname{rot}\vec{\mathbf{V}} = \vec{\mathbf{i}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \vec{\mathbf{j}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \vec{\mathbf{k}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}. \tag{13.59a}$$

The vector field $\operatorname{rot}\vec{\mathbf{V}}$ can be represented as the cross product of the nabla operator $\nabla$ and the vector $\vec{\mathbf{V}}$:

$$\operatorname{rot}\vec{\mathbf{V}} = \nabla \times \vec{\mathbf{V}}. \tag{13.59b}$$

#### 2. Rotation in Cylindrical Coordinates

$$\operatorname{rot}\vec{\mathbf{V}} = \operatorname{rot}_\rho\vec{\mathbf{V}}\vec{\mathbf{e}}_\rho + \operatorname{rot}_\varphi\vec{\mathbf{V}}\vec{\mathbf{e}}_\varphi + \operatorname{rot}_z\vec{\mathbf{V}}\vec{\mathbf{e}}_z \quad \text{with} \tag{13.60a}$$

$$\operatorname{rot}_\rho\vec{\mathbf{V}} = \frac{1}{\rho}\frac{\partial V_z}{\partial\varphi} - \frac{\partial V_\varphi}{\partial z}, \quad \operatorname{rot}_\varphi\vec{\mathbf{V}} = \frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial\rho}, \quad \operatorname{rot}_z\vec{\mathbf{V}} = \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}(\rho V_\varphi) - \frac{\partial V_\rho}{\partial\varphi}\right\}. \tag{13.60b}$$

#### 3. Rotation in Spherical Coordinates

$$\operatorname{rot}\vec{\mathbf{V}} = \operatorname{rot}_r\vec{\mathbf{V}}\vec{\mathbf{e}}_r + \operatorname{rot}_\vartheta\vec{\mathbf{V}}\vec{\mathbf{e}}_\vartheta + \operatorname{rot}_\varphi\vec{\mathbf{V}}\vec{\mathbf{e}}_\varphi \quad \text{with} \tag{13.61a}$$

$$\left.\begin{aligned} \operatorname{rot}_r\vec{\mathbf{V}} &= \frac{1}{r\sin\vartheta}\left\{\frac{\partial}{\partial\vartheta}(\sin\vartheta V_\varphi) - \frac{\partial V_\vartheta}{\partial\varphi}\right\}, \\ \operatorname{rot}_\vartheta\vec{\mathbf{V}} &= \frac{1}{r\sin\vartheta}\frac{\partial V_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rV_\varphi), \\ \operatorname{rot}_\varphi\vec{\mathbf{V}} &= \frac{1}{r}\left\{\frac{\partial}{\partial r}(rV_\vartheta) - \frac{\partial V_r}{\partial\vartheta}\right\}. \end{aligned}\right\} \tag{13.61b}$$

#### 4. Rotation in General Orthogonal Coordinates

$$\operatorname{rot}\vec{\mathbf{V}} = \operatorname{rot}_\xi\vec{\mathbf{V}}\vec{\mathbf{e}}_\xi + \operatorname{rot}_\eta\vec{\mathbf{V}}\vec{\mathbf{e}}_\eta + \operatorname{rot}_\zeta\vec{\mathbf{V}}\vec{\mathbf{e}}_\zeta \quad \text{with} \tag{13.62a}$$

$$\left.\begin{aligned}
\operatorname{rot}_\xi \vec{\mathbf{V}} &= \frac{1}{\mathrm{D}}\left|\frac{\partial \vec{\mathbf{r}}}{\partial \xi}\right|\left[\frac{\partial}{\partial \eta}\left(\left|\frac{\partial \vec{\mathbf{r}}}{\partial \zeta}\right| V_\zeta\right) - \frac{\partial}{\partial \zeta}\left(\left|\frac{\partial \vec{\mathbf{r}}}{\partial \eta}\right| V_\eta\right)\right],\\
\operatorname{rot}_\eta \vec{\mathbf{V}} &= \frac{1}{\mathrm{D}}\left|\frac{\partial \vec{\mathbf{r}}}{\partial \eta}\right|\left[\frac{\partial}{\partial \zeta}\left(\left|\frac{\partial \vec{\mathbf{r}}}{\partial \xi}\right| V_\xi\right) - \frac{\partial}{\partial \xi}\left(\left|\frac{\partial \vec{\mathbf{r}}}{\partial \zeta}\right| V_\zeta\right)\right],\\
\operatorname{rot}_\zeta \vec{\mathbf{V}} &= \frac{1}{\mathrm{D}}\left|\frac{\partial \vec{\mathbf{r}}}{\partial \zeta}\right|\left[\frac{\partial}{\partial \xi}\left(\left|\frac{\partial \vec{\mathbf{r}}}{\partial \eta}\right| V_\eta\right) - \frac{\partial}{\partial \eta}\left(\left|\frac{\partial \vec{\mathbf{r}}}{\partial \xi}\right| V_\xi\right)\right],
\end{aligned}\right\} \tag{13.62b}$$

$$\vec{\mathbf{r}}(\xi,\eta,\zeta) = x(\xi,\eta,\zeta)\vec{\mathbf{i}} + y(\xi,\eta,\zeta)\vec{\mathbf{j}} + z(\xi,\eta,\zeta)\vec{\mathbf{k}}; \qquad \mathrm{D} = \left|\frac{\partial \vec{\mathbf{r}}}{\partial \xi}\right| \cdot \left|\frac{\partial \vec{\mathbf{r}}}{\partial \eta}\right| \cdot \left|\frac{\partial \vec{\mathbf{r}}}{\partial \zeta}\right|. \tag{13.62c}$$

#### 13.2.5.3 Rules for Evaluating the Rotation

$$\operatorname{rot}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{rot}\vec{\mathbf{V}}_1 + \operatorname{rot}\vec{\mathbf{V}}_2, \quad \operatorname{rot}(c\vec{\mathbf{V}}) = c\operatorname{rot}\vec{\mathbf{V}}. \tag{13.63}$$

$$\operatorname{rot}(U\vec{\mathbf{V}}) = U\operatorname{rot}\vec{\mathbf{V}} + \operatorname{grad}U \times \vec{\mathbf{V}}. \tag{13.64}$$

$$\operatorname{rot}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = (\vec{\mathbf{V}}_2 \cdot \operatorname{grad})\vec{\mathbf{V}}_1 - (\vec{\mathbf{V}}_1 \cdot \operatorname{grad})\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_1 \operatorname{div}\vec{\mathbf{V}}_2 - \vec{\mathbf{V}}_2 \operatorname{div}\vec{\mathbf{V}}_1. \tag{13.65}$$

#### 13.2.5.4 Rotation of a Potential Field

This also follows from the Stokes theorem (see 13.3.3.2, p. 725) that the rotation of a potential field is identically zero:

$$\operatorname{rot}\vec{\mathbf{V}} = \operatorname{rot}(\operatorname{grad}\mathrm{U}) = \vec{\mathbf{0}}. \tag{13.66}$$

This also follows from (13.59a) for $\vec{\mathbf{V}} = \operatorname{grad}U$, if the assumptions of the Schwarz interchanging theorem are fulfilled (see 6.2.2.2, **1.**, p. 448).

■ For $\vec{\mathbf{r}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$ with $r = |\vec{\mathbf{r}}| = \sqrt{x^2+y^2+z^2}$ holds: $\operatorname{rot}\vec{\mathbf{r}} = \vec{\mathbf{0}}$ and $\operatorname{rot}(\varphi(r)\vec{\mathbf{r}}) = \vec{\mathbf{0}}$, where $\varphi(r)$ is a differentiable function of $r$.

### 13.2.6 Nabla Operator, Laplace Operator

#### 13.2.6.1 Nabla Operator

The symbolic vector $\nabla$ is called the *nabla operator*. Its use simplifies the representation of and calculations with space differential operators. In Cartesian coordinates holds

$$\nabla = \frac{\partial}{\partial x}\vec{\mathbf{i}} + \frac{\partial}{\partial y}\vec{\mathbf{j}} + \frac{\partial}{\partial z}\vec{\mathbf{k}}. \tag{13.67}$$

The components of the nabla operator are considered as partial differential operators, i.e., the symbol $\dfrac{\partial}{\partial x}$ means partial differentiation with respect to $x$, where the other variables are considered as constants.

The formulas for *spatial differential operators* in Cartesian coordinates can be obtained by formal multiplication of this vector operator by the scalar $U$ or by the vector $\vec{\mathbf{V}}$. For instance, in the case of the operators *gradient, vector gradient, divergence*, and *rotation*:

$$\operatorname{grad}U = \nabla U \qquad \text{(gradient of } U \quad \text{(see 13.2.2, p. 710))}, \tag{13.68a}$$

$$\operatorname{grad}\vec{\mathbf{V}} = \nabla\vec{\mathbf{V}} \qquad \text{(vector gradient of } \vec{\mathbf{V}} \quad \text{(see 13.2.3, p. 711))}, \tag{13.68b}$$

$$\operatorname{div}\vec{\mathbf{V}} = \nabla\cdot\vec{\mathbf{V}} \qquad \text{(divergence of } \vec{\mathbf{V}} \quad \text{(see 13.2.4, p. 712))}, \tag{13.68c}$$

$$\operatorname{rot}\vec{\mathbf{V}} = \nabla\times\vec{\mathbf{V}} \qquad \text{(rotation or curl of } \vec{\mathbf{V}} \quad \text{(see 13.2.5, p. 713))}. \tag{13.68d}$$

### 13.2.6.2 Rules for Calculations with the Nabla Operator

**1.** If $\nabla$ stands in front of a linear combination $\sum a_i X_i$ with constants $a_i$ and with point functions $X_i$, then, independently of whether they are scalar or vector functions, we have the formula:

$$\nabla(\sum a_i X_i) = \sum a_i \nabla X_i. \tag{13.69}$$

**2.** If $\nabla$ is applied to a product of scalar or vector functions, then it has to be applied to each of these functions after each other and the results are to be added. There is a $\downarrow$ above the symbol of the function submitted to the operation

$$\nabla(XYZ) = \nabla(\overset{\downarrow}{X}YZ) + \nabla(X\overset{\downarrow}{Y}Z) + \nabla(XY\overset{\downarrow}{Z}), \quad \text{i.e.,} \tag{13.70}$$

$$\nabla(XYZ) = (\nabla X)YZ + X(\nabla Y)Z) + XY(\nabla Z).$$

Then the products have to be transformed according to vector algebra so as the operator $\nabla$ is applied to only one factor with the sign $\downarrow$. Having performed the computation one omits that sign.

■ **A:** $\operatorname{div}(U\vec{\mathbf{V}}) = \nabla(U\vec{\mathbf{V}}) = \nabla(\overset{\downarrow}{U}\vec{\mathbf{V}}) + \nabla(U\overset{\downarrow}{\vec{\mathbf{V}}}) = \vec{\mathbf{V}} \cdot \nabla U + U\nabla \cdot \vec{\mathbf{V}} = \vec{\mathbf{V}} \cdot \operatorname{grad} U + U \operatorname{div} \vec{\mathbf{V}}$.

■ **B:** $\operatorname{grad}(\vec{\mathbf{V}}_1\vec{\mathbf{V}}_2) = \nabla(\vec{\mathbf{V}}_1\vec{\mathbf{V}}_2) = \nabla(\overset{\downarrow}{\vec{\mathbf{V}}_1}\vec{\mathbf{V}}_2) + \nabla(\vec{\mathbf{V}}_1\overset{\downarrow}{\vec{\mathbf{V}}_2})$. Because $\vec{\mathbf{b}}(\vec{\mathbf{a}}\vec{\mathbf{c}}) = (\vec{\mathbf{a}}\vec{\mathbf{b}})\vec{\mathbf{c}} + \vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ follows: $\operatorname{grad}(\vec{\mathbf{V}}_1\vec{\mathbf{V}}_2) = (\vec{\mathbf{V}}_2\nabla)\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2 \times (\nabla \times \vec{\mathbf{V}}_1) + (\vec{\mathbf{V}}_1\nabla)\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_1 \times (\nabla \times \vec{\mathbf{V}}_2)$

$= (\vec{\mathbf{V}}_2 \operatorname{grad})\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2 \times \operatorname{rot} \vec{\mathbf{V}}_1 + (\vec{\mathbf{V}}_1 \operatorname{grad})\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_1 \times \operatorname{rot} \vec{\mathbf{V}}_2$.

### 13.2.6.3 Vector Gradient

The vector gradient $\operatorname{grad} \vec{\mathbf{V}}$ is represented by the nabla operator as

$$\operatorname{grad} \vec{\mathbf{V}} = \nabla \vec{\mathbf{V}}. \tag{13.71a}$$

The expression occurring in the vector gradient $(\vec{\mathbf{a}} \cdot \nabla)\vec{\mathbf{V}}$ (see (13.32b), p. 709) has the form:

$$2(\vec{\mathbf{a}} \cdot \nabla)\vec{\mathbf{V}} = \operatorname{rot}(\vec{\mathbf{V}} \times \vec{\mathbf{a}}) + \operatorname{grad}(\vec{\mathbf{a}}\vec{\mathbf{V}}) + \vec{\mathbf{a}} \operatorname{div} \vec{\mathbf{V}} - \vec{\mathbf{V}} \operatorname{div} \vec{\mathbf{a}} - \vec{\mathbf{a}} \times \operatorname{rot} \vec{\mathbf{V}} - \vec{\mathbf{V}} \times \operatorname{rot} \vec{\mathbf{a}}. \tag{13.71b}$$

In particular one gets for $\vec{\mathbf{r}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$:

$$(\vec{\mathbf{a}} \cdot \nabla)\vec{\mathbf{r}} = \vec{\mathbf{a}}. \tag{13.71c}$$

### 13.2.6.4 Nabla Operator Applied Twice

For every field $\vec{\mathbf{V}}$:

$$\nabla(\nabla \times \vec{\mathbf{V}}) = \operatorname{div} \operatorname{rot} \vec{\mathbf{V}} \equiv 0, \tag{13.72}$$

$$\nabla \times (\nabla U) = \operatorname{rot} \operatorname{grad} U \equiv \vec{\mathbf{0}}, \tag{13.73}$$

$$\nabla(\nabla U) = \operatorname{div} \operatorname{grad} U = \Delta U. \tag{13.74}$$

### 13.2.6.5 Laplace Operator

#### 1. Definition

The dot product of the nabla operator with itself is called the *Laplace operator*:

$$\Delta = \nabla \cdot \nabla = \nabla^2. \tag{13.75}$$

The Laplace operator is not a vector. It prescribes the summation of the second partial derivatives. It can be applied to scalar functions as well as to vector functions. The application to a vector function, componentwise, results in a vector.

The Laplace operator is an *invariant*, i.e., it does not change during translation and/or rotation of the coordinate system.

### 2. Formulas for the Laplace Operator in Different Coordinates

Here the Laplace operator is applied to the scalar point function $U(\vec{\mathbf{r}})$. Then the result is a scalar. The application of it for vector functions $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ results in a vector $\Delta\vec{\mathbf{V}}$ with components $\Delta V_x$, $\Delta V_y$, $\Delta V_z$.

**1. Laplace Operator in Cartesian Coordinates**

$$\Delta U(x,y,z) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}. \tag{13.76}$$

**2. Laplace Operator in Cylindrical Coordinates**

$$\Delta U(\rho,\varphi,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\varphi^2} + \frac{\partial^2 U}{\partial z^2}. \tag{13.77}$$

**3. Laplace Operator in Spherical Coordinates**

$$\Delta U(r,\vartheta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 U}{\partial\varphi^2}. \tag{13.78}$$

**4. Laplace Operator in General Orthogonal Coordinates**

$$\Delta U(\xi,\eta,\zeta) = \frac{1}{\mathrm{D}}\left[\frac{\partial}{\partial\xi}\left(\frac{\mathrm{D}}{\left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|^2}\frac{\partial U}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{\mathrm{D}}{\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|^2}\frac{\partial U}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{\mathrm{D}}{\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|^2}\frac{\partial U}{\partial\zeta}\right)\right] \quad \text{with} \tag{13.79a}$$

$$\vec{\mathbf{r}}(\xi,\eta,\zeta) = x(\xi,\eta,\zeta)\vec{\mathbf{i}} + y(\xi,\eta,\zeta)\vec{\mathbf{j}} + z(\xi,\eta,\zeta)\vec{\mathbf{k}}, \tag{13.79b}$$

$$\mathrm{D} = \left|\frac{\partial\vec{\mathbf{r}}}{\partial\xi}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\eta}\right|\cdot\left|\frac{\partial\vec{\mathbf{r}}}{\partial\zeta}\right|. \tag{13.79c}$$

### 3. Special Relations between the Nabla Operator and Laplace Operator

$$\nabla(\nabla\cdot\vec{\mathbf{V}}) = \operatorname{grad}\operatorname{div}\vec{\mathbf{V}}, \tag{13.80}$$

$$\nabla\times(\nabla\times\vec{\mathbf{V}}) = \operatorname{rot}\operatorname{rot}\vec{\mathbf{V}}, \tag{13.81}$$

$$\nabla(\nabla\cdot\vec{\mathbf{V}}) - \nabla\times(\nabla\times\vec{\mathbf{V}}) = \Delta\vec{\mathbf{V}}, \quad \text{where} \tag{13.82}$$

$$\Delta\vec{\mathbf{V}} = (\nabla\cdot\nabla)\vec{\mathbf{V}} = \Delta V_x\vec{\mathbf{i}} + \Delta V_y\vec{\mathbf{j}} + \Delta V_z\vec{\mathbf{k}} = \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)\vec{\mathbf{i}}$$
$$+ \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)\vec{\mathbf{j}} + \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)\vec{\mathbf{k}}. \tag{13.83}$$

## 13.2.7 Review of Spatial Differential Operations

### 13.2.7.1 Rules of Calculation for Spatial Differential Operators

$U, U_1, U_2$ and $F$ are scalar functions; $c$ is a constant; $\vec{\mathbf{V}}, \vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2$ are vector functions:

$$\operatorname{grad}(U_1 + U_2) = \operatorname{grad}U_1 + \operatorname{grad}U_2. \tag{13.84}$$

$$\operatorname{grad}(cU) = c\operatorname{grad}U. \tag{13.85}$$

$$\operatorname{grad}(U_1U_2) = U_1\operatorname{grad}U_2 + U_2\operatorname{grad}U_1. \tag{13.86}$$

$$\operatorname{grad}F(U) = F'(U)\operatorname{grad}U. \tag{13.87}$$

$$\operatorname{div}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{div}\vec{\mathbf{V}}_1 + \operatorname{div}\vec{\mathbf{V}}_2. \tag{13.88}$$

$$\operatorname{div}(c\vec{\mathbf{V}}) = c\operatorname{div}\vec{\mathbf{V}}. \tag{13.89}$$

$$\operatorname{div}(U\vec{\mathbf{V}}) = \vec{\mathbf{V}} \cdot \operatorname{grad} U + U \operatorname{div} \vec{\mathbf{V}}. \quad (13.90)$$

$$\operatorname{rot}(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) = \operatorname{rot} \vec{\mathbf{V}}_1 + \operatorname{rot} \vec{\mathbf{V}}_2. \quad (13.91)$$

$$\operatorname{rot}(c\vec{\mathbf{V}}) = c \operatorname{rot} \vec{\mathbf{V}}. \quad (13.92)$$

$$\operatorname{rot}(U\vec{\mathbf{V}}) = U \operatorname{rot} \vec{\mathbf{V}} - \vec{\mathbf{V}} \times \operatorname{grad} U. \quad (13.93)$$

$$\operatorname{div} \operatorname{rot} \vec{\mathbf{V}} \equiv 0. \quad (13.94)$$

$$\operatorname{rot} \operatorname{grad} U \equiv \vec{\mathbf{0}} \quad \text{(zero vector)}. \quad (13.95)$$

$$\operatorname{div} \operatorname{grad} U = \Delta U. \quad (13.96)$$

$$\operatorname{rot} \operatorname{rot} \vec{\mathbf{V}} = \operatorname{grad} \operatorname{div} \vec{\mathbf{V}} - \Delta \vec{\mathbf{V}}. \quad (13.97)$$

$$\operatorname{div}(\vec{\mathbf{V}}_1 \times \vec{\mathbf{V}}_2) = \vec{\mathbf{V}}_2 \cdot \operatorname{rot} \vec{\mathbf{V}}_1 - \vec{\mathbf{V}}_1 \cdot \operatorname{rot} \vec{\mathbf{V}}_2. \quad (13.98)$$

### 13.2.7.2 Expressions of Vector Analysis in Cartesian, Cylindrical, and Spherical Coordinates (see Table 13.2)

Table 13.2 Expressions of vector analysis in Cartesian, cylindrical, and spherical coordinates

| | Cartesian coordinates | Cylindrical coordinates | Spherical coordinates |
|---|---|---|---|
| $d\vec{\mathbf{s}} = d\vec{\mathbf{r}}$ | $\vec{\mathbf{e}}_x dx + \vec{\mathbf{e}}_y dy + \vec{\mathbf{e}}_z dz$ | $\vec{\mathbf{e}}_\rho d\rho + \vec{\mathbf{e}}_\varphi \rho d\varphi + \vec{\mathbf{e}}_z dz$ | $\vec{\mathbf{e}}_r dr + \vec{\mathbf{e}}_\vartheta r d\vartheta + \vec{\mathbf{e}}_\varphi r \sin\vartheta d\varphi$ |
| grad$U$ | $\vec{\mathbf{e}}_x \frac{\partial U}{\partial x} + \vec{\mathbf{e}}_y \frac{\partial U}{\partial y} + \vec{\mathbf{e}}_z \frac{\partial U}{\partial z}$ | $\vec{\mathbf{e}}_\rho \frac{\partial U}{\partial \rho} + \vec{\mathbf{e}}_\varphi \frac{1}{\rho}\frac{\partial U}{\partial \varphi} + \vec{\mathbf{e}}_z \frac{\partial U}{\partial z}$ | $\vec{\mathbf{e}}_r \frac{\partial U}{\partial r} + \vec{\mathbf{e}}_\vartheta \frac{1}{r}\frac{\partial U}{\partial \vartheta} + \vec{\mathbf{e}}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial U}{\partial \varphi}$ |
| div$\vec{\mathbf{V}}$ | $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$ | $\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$ | $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(V_\vartheta \sin\vartheta)$ <br> $+\frac{1}{r\sin\vartheta}\frac{\partial V_\varphi}{\partial \varphi}$ |
| rot$\vec{\mathbf{V}}$ | $\vec{\mathbf{e}}_x \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$ <br> $+\vec{\mathbf{e}}_y \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$ <br> $+\vec{\mathbf{e}}_z \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$ | $\vec{\mathbf{e}}_\rho \left(\frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)$ <br> $+\vec{\mathbf{e}}_\varphi \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)$ <br> $+\vec{\mathbf{e}}_z \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\varphi) - \frac{1}{\rho}\frac{\partial V_\rho}{\partial \varphi}\right)$ | $\vec{\mathbf{e}}_r \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial \vartheta}(V_\varphi \sin\vartheta) - \frac{\partial V_\vartheta}{\partial \varphi}\right]$ <br> $+\vec{\mathbf{e}}_\vartheta \frac{1}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial V_r}{\partial \varphi} - \frac{\partial}{\partial r}(r V_\varphi)\right]$ <br> $+\vec{\mathbf{e}}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}(r V_\vartheta) - \frac{\partial V_r}{\partial \vartheta}\right]$ |
| $\Delta U$ | $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$ | $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2}$ <br> $+\frac{\partial^2 U}{\partial z^2}$ | $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right)$ <br> $+\frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial U}{\partial \vartheta}\right)$ <br> $+\frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2 U}{\partial \varphi^2}$ |

### 13.2.7.3 Fundamental Relations and Results (see Table 13.3)

Table 13.3 Fundamental relations for spatial differential operators

| Operator | Symbol | Relation | Argument | Result | Meaning |
|---|---|---|---|---|---|
| Gradient | $\operatorname{grad} U$ | $\nabla U$ | scalar | vector | maximal increase |
| Vector gradient | $\operatorname{grad} \vec{\mathbf{V}}$ | $\nabla \vec{\mathbf{V}}$ | vector | tensor second order | |
| Divergence | $\operatorname{div} \vec{\mathbf{V}}$ | $\nabla \cdot \vec{\mathbf{V}}$ | vector | scalar | source, sink |
| Rotation | $\operatorname{rot} \vec{\mathbf{V}}$ | $\nabla \times \vec{\mathbf{V}}$ | vector | vector | curl |
| Laplace operator | $\Delta U$ | $(\nabla \cdot \nabla) U$ | scalar | scalar | potential field source |
| Laplace operator | $\Delta \vec{\mathbf{V}}$ | $(\nabla \cdot \nabla) \vec{\mathbf{V}}$ | vector | vector | |

## 13.3 Integration in Vector Fields

Integration in vector fields is usually performed in Cartesian, cylindrical or in spherical coordinate systems. Usually one integrates along curves, surfaces, or volumes. The line, surface, and volume elements needed for these calculations are collected in **Table 13.4**.

Table 13.4 Line, surface, and volume elements in Cartesian, cylindrical, and spherical coordinates

| | Cartesian coordinates | Cylindrical coordinates | Spherical coordinates |
|---|---|---|---|
| $d\vec{\mathbf{r}}$ | $\vec{\mathbf{e}}_x dx + \vec{\mathbf{e}}_y dy + \vec{\mathbf{e}}_z dz$ | $\vec{\mathbf{e}}_\rho d\rho + \vec{\mathbf{e}}_\varphi \rho d\varphi + \vec{\mathbf{e}}_z dz$ | $\vec{\mathbf{e}}_r dr + \vec{\mathbf{e}}_\vartheta r d\vartheta + \vec{\mathbf{e}}_\varphi r \sin\vartheta d\varphi$ |
| $d\vec{\mathbf{S}}$ | $\vec{\mathbf{e}}_x dydz + \vec{\mathbf{e}}_y dxdz + \vec{\mathbf{e}}_z dxdy$ | $\vec{\mathbf{e}}_\rho \rho d\varphi dz + \vec{\mathbf{e}}_\varphi d\rho dz + \vec{\mathbf{e}}_z \rho d\rho d\varphi$ | $\vec{\mathbf{e}}_r r^2 \sin\vartheta d\vartheta d\varphi$<br>$+\vec{\mathbf{e}}_\vartheta r \sin\vartheta dr d\varphi$<br>$+\vec{\mathbf{e}}_\varphi r dr d\vartheta d\varphi$ |
| $dv^*$ | $dxdydz$ | $\rho d\rho d\varphi dz$ | $r^2 \sin\vartheta dr d\vartheta d\varphi$ |
| | $\vec{\mathbf{e}}_x = \vec{\mathbf{e}}_y \times \vec{\mathbf{e}}_z$<br>$\vec{\mathbf{e}}_y = \vec{\mathbf{e}}_z \times \vec{\mathbf{e}}_x$<br>$\vec{\mathbf{e}}_z = \vec{\mathbf{e}}_x \times \vec{\mathbf{e}}_y$ | $\vec{\mathbf{e}}_\rho = \vec{\mathbf{e}}_\varphi \times \vec{\mathbf{e}}_z$<br>$\vec{\mathbf{e}}_\varphi = \vec{\mathbf{e}}_z \times \vec{\mathbf{e}}_\rho$<br>$\vec{\mathbf{e}}_z = \vec{\mathbf{e}}_\rho \times \vec{\mathbf{e}}_\varphi$ | $\vec{\mathbf{e}}_r = \vec{\mathbf{e}}_\vartheta \times \vec{\mathbf{e}}_\varphi$<br>$\vec{\mathbf{e}}_\vartheta = \vec{\mathbf{e}}_\varphi \times \vec{\mathbf{e}}_r$<br>$\vec{\mathbf{e}}_\varphi = \vec{\mathbf{e}}_r \times \vec{\mathbf{e}}_\vartheta$ |
| | $\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ | $\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ | $\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ |
| | The indices $i$ and $j$ take the place of $x, y, z$ or $\rho, \varphi, z$ or $r, \vartheta, \varphi$. | | |
| * | The volume is denoted here by $v$ to avoid confusion with the absolute value of the vector function $\lvert\vec{\mathbf{V}}\rvert = V$. | | |

### 13.3.1 Line Integral and Potential in Vector Fields

#### 13.3.1.1 Line Integral in Vector Fields

**1. Definition** The scalar-valued curvilinear integral or line integral of a vector function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ along a rectifiable curve $\overset{\frown}{AB}$ **(Fig. 13.13)** is the scalar value

$$P = \int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.99a}$$

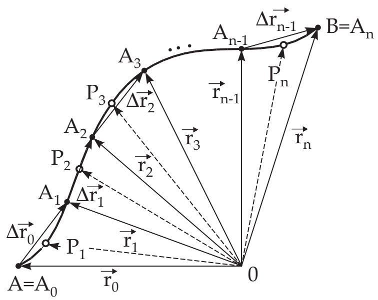


Figure 13.13

**2. Evaluation of this Integral in Five Steps**

**a)** Dividing the path $\overset{\frown}{AB}$ **(Fig. 13.13)** by division points $A_1(\vec{\mathbf{r}}_1),\ A_2(\vec{\mathbf{r}}_2), \dots, A_{n-1}(\vec{\mathbf{r}}_{n-1})$ $(A = A_0,\ B = A_n)$ into $n$ small arcs which are approximated by the vectors $\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_{i-1} = \Delta\vec{\mathbf{r}}_{i-1}$.

**b)** Choosing arbitrarily the points $P_i$ with position vectors $\vec{\xi}_i$ lying inside or at the boundary of each small arc.

**c)** Calculating the dot product of the value of the function $\vec{\mathbf{V}}(\vec{\xi}_i)$ at these chosen points with the corresponding $\Delta\vec{\mathbf{r}}_{i-1}$.

**d)** Taking the sum of all the $n$ products.

**e)** Calculating the limit of the sums got in this way $\sum\limits_{i=1}^{n} \tilde{\vec{\mathbf{V}}}(\vec{\xi}_i) \cdot \Delta\vec{\mathbf{r}}_{i-1}$ for $|\Delta\vec{\mathbf{r}}_{i-1}| \to 0$, while $n \to \infty$ obviously.

If this limit exists independently of the choice of the points $A_i$ and $P_i$, then it is called the line integral

$$\int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}} = \lim_{\substack{|\Delta\vec{\mathbf{r}}_{i-1}|\to 0\\ n\to\infty}} \sum_{i=1}^{n} \tilde{\vec{\mathbf{V}}}(\vec{\xi}_i) \cdot \Delta\vec{\mathbf{r}}_{i-1}. \tag{13.99b}$$

A sufficient condition for the existence of the line integral (13.99a,b) is that the vector function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ and the curve $\overset{\frown}{AB}$ are continuous and the curve has a tangent varying continuously. A vector function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is continuous if its components, the three scalar functions, are continuous.

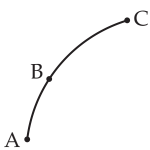


Figure 13.14

### 13.3.1.2 Interpretation of the Line Integral in Mechanics

If $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is a field of force, i.e., $\vec{\mathbf{V}}(\vec{\mathbf{r}}) = \vec{\mathbf{F}}(\vec{\mathbf{r}})$, then the line integral (13.99a) represents the work done by $\vec{\mathbf{F}}$ while a particle $m$ moves along the path $\overset{\frown}{AB}$ **(Fig. 13.13,13.14)**.

### 13.3.1.3 Properties of the Line Integral

$$\int\limits_{\overset{\frown}{ABC}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} + \int\limits_{\overset{\frown}{BC}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} \qquad \textbf{(Fig. 13.14)}. \tag{13.100}$$

$$\int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = -\int\limits_{\overset{\frown}{BA}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.101}$$

$$\int\limits_{\overset{\frown}{AB}} \left[\vec{\mathbf{V}}(\vec{\mathbf{r}}) + \vec{\mathbf{W}}(\vec{\mathbf{r}})\right] \cdot d\vec{\mathbf{r}} = \int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} + \int\limits_{\overset{\frown}{AB}} \vec{\mathbf{W}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.102}$$

$$\int\limits_{\overset{\frown}{AB}} c\vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = c\int\limits_{\overset{\frown}{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} \qquad \text{(c const)}. \tag{13.103}$$

### 13.3.1.4 Line Integral in Cartesian Coordinates

In Cartesian coordinates the following formula holds:

$$\int_{\widehat{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \int_{\widehat{AB}} (V_x\,dx + V_y\,dy + V_z\,dz)\,. \tag{13.104}$$

### 13.3.1.5 Integral Along a Closed Curve in a Vector Field

A line integral is called a *contour integral* if the path of integration is a closed curve. If the scalar value of the integral is denoted by $P$ and the closed curve is denoted by $C$, then the following notation is used:

$$P = \oint_{(C)} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.105}$$

### 13.3.1.6 Conservative Field or Potential Field

#### 1. Definition

If the value $P$ of the line integral (13.99a) in a vector field depends only on the initial point $A$ and the endpoint $B$, and is independent of the path between them, then this field is called a *conservative field* or a *potential field*.

The value of the contour integral in a conservative field is always equal to zero:

$$\int_{(C)} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = 0. \tag{13.106}$$

A conservative field is always irrotational:

$$\operatorname{rot} \vec{\mathbf{V}} = \vec{\mathbf{0}}, \tag{13.107}$$

and conversely, this equality is a sufficient condition for a vector field to be conservative. Of course, it is to be supposed that the partial derivatives of the field function $\vec{\mathbf{V}}$ are continuous with respect to the corresponding coordinates, and the domain of $\vec{\mathbf{V}}$ is simply connected. This condition, also called the *integrability condition* (see 8.3.4.2, p. 521), has the following form in Cartesian coordinates

$$\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}\,, \quad \frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y}\,, \quad \frac{\partial V_z}{\partial x} = \frac{\partial V_x}{\partial z}\,. \tag{13.108}$$

#### 2. Potential of a Conservative Field,

or its potential function or briefly its potential is the scalar function

$$U(\vec{\mathbf{r}}) = \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.109a}$$

In a conservative field it is calculated with a fixed initial point $A(\vec{\mathbf{r}}_0)$ and a variable endpoint $B(\vec{\mathbf{r}})$ as the line integral

$$U(\vec{\mathbf{r}}) = \int_{\widehat{AB}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}. \tag{13.109b}$$

**Remark**: In physics, the potential $U^*(\vec{\mathbf{r}})$ of a function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ at the point $\vec{\mathbf{r}}$ is often considered with the opposite sign:

$$U^*(\vec{\mathbf{r}}) = -\int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{V}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = -U(\vec{\mathbf{r}}). \tag{13.110}$$

### 3. Relations between Gradient, Line Integral, and Potential

If the relation $\vec{\mathbf{V}}(\vec{\mathbf{r}}) = \operatorname{grad} U(\vec{\mathbf{r}})$ holds, then $U(\vec{\mathbf{r}})$ is the potential of the field $\vec{\mathbf{V}}(\vec{\mathbf{r}})$, and conversely, $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is a conservative or potential field. In physics often the negative sign is used corresponding to (13.110).

### 4. Calculation of the Potential in a Conservative Field

If the function $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is given in Cartesian coordinates $\vec{\mathbf{V}} = V_x\vec{\mathbf{i}} + V_y\vec{\mathbf{j}} + V_z\vec{\mathbf{k}}$, then for the total differential of its potential function $U$

$$dU = V_x\,dx + V_y\,dy + V_z\,dz \tag{13.111a}$$

holds. Here, the coefficients $V_x$, $V_y$, $V_z$ must fulfill the integrability condition (13.108). The determination of $U$ follows from the equation system

$$\frac{\partial U}{\partial x} = V_x, \quad \frac{\partial U}{\partial y} = V_y, \quad \frac{\partial U}{\partial z} = V_z. \tag{13.111b}$$

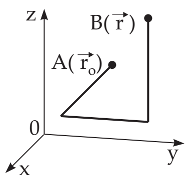


Figure 13.15

In practice, the calculation of the potential can be done by performing the integration along three straight line segments parallel to the coordinate axes and connected to each other **(Fig. 13.15)**:

$$U = \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}} = U(x_0, y_0, z_0) + \int_{x_0}^{x} V_x(x, y_0, z_0)\,dx + \int_{y_0}^{y} V_y(x, y, z_0)\,dy + \int_{z_0}^{z} V_z(x, y, z)\,dz\,. \tag{13.112}$$

## 13.3.2 Surface Integrals

### 13.3.2.1 Vector of a Plane Sheet

The vector representation of the surface integral of general type (see 8.3.4.2, p. 537) requires to assign a vector $\vec{\mathbf{S}}$ to a plane surface region $S$, which is perpendicular to this region and its absolute value is equal to the area of $S$. **Fig. 13.16a** shows the case of a plane sheet. The positive direction in $S$ is given by defining the positive sense along a closed curve $C$ according to the *right-hand law* (also called *right-screw rule*): Looking from the initial point of the vector into the direction of its final point, then the *positive sense* is the clockwise direction. By this choice of orientation of the boundary curve one fixes the exterior side of this surface region, i.e., the side on which the vector lies. This definition works in the case of any surface region bounded by a closed curve **(Fig. 13.16b,c)**.

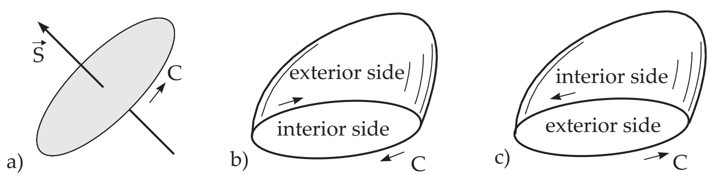


Figure 13.16

### 13.3.2.2 Evaluation of the Surface Integral

The evaluation of a surface integral in scalar or vector fields is independent of whether the surface $S$ is bounded by a closed curve or is itself a closed surface. The evaluation is performed in five steps:

**a)** Dividing the surface region $S$ on the exterior side defined by the orientation of the boundary curve **(Fig. 13.17)** into $n$ arbitrary elementary surfaces $\Delta S_i$ so that each of these surface elements can be approximated by a plane surface element. Assigning the vector $\Delta\vec{\mathbf{S}}_i$ to every surface element $\Delta S_i$ as given in (13.33a). In the case of a closed surface, the positive direction is defined so that the exterior side is where $\Delta\vec{\mathbf{S}}_{\mathbf{i}}$ should start.

**b)** Choosing an arbitrary point $P_i$ with the position vector $\vec{\mathbf{r}}_{\mathbf{i}}$ inside or on the boundary of each surface element.

**c)** Producing the products $U(\vec{\mathbf{r}}_{\mathbf{i}})\,\Delta\vec{\mathbf{S}}_{\mathbf{i}}$ in the case of a scalar field and the product $\vec{\mathbf{V}}(\vec{\mathbf{r}}_{\mathbf{i}})\cdot\Delta\vec{\mathbf{S}}_{\mathbf{i}}$ or $\vec{\mathbf{V}}(\vec{\mathbf{r}}_{\mathbf{i}})\times\Delta\vec{\mathbf{S}}_{\mathbf{i}}$ in the case of a vector field.

**d)** Taking the sum of all these products.

**e)** Evaluating the limit while the diameters of $\Delta S_i$ tend to zero, i.e., $|\Delta\vec{\mathbf{S}}_{\mathbf{i}}| \to 0$ for $n \to \infty$. So, the surface elements tend to zero in the sense given in 8.4.1, **1.**, p. 524, for double integrals.
If this limit exists independently of the partition and of the choice of the points $\vec{\mathbf{r}}_{\mathbf{i}}$, then one calls it the surface integral of $\vec{\mathbf{V}}$ on the given surface.

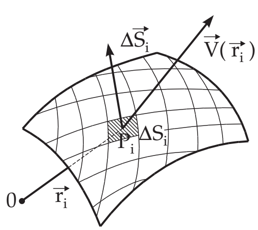

Figure 13.17

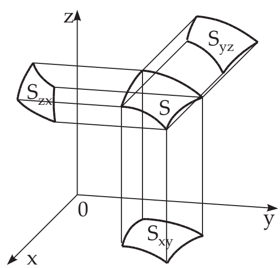

Figure 13.18

### 13.3.2.3 Surface Integrals and Flow of Fields

#### 1. Vector Flow of a Scalar Field

$$\vec{\mathbf{P}} = \lim_{\substack{|\Delta\vec{\mathbf{S}}_i|\to 0\\ n\to\infty}} \sum_{i=1}^{n} U(\vec{\mathbf{r}}_i)\,\Delta\vec{\mathbf{S}}_i = \int_{(S)} U(\vec{\mathbf{r}})\,d\vec{\mathbf{S}}. \tag{13.113}$$

#### 2. Scalar Flow of a Vector Field

$$Q = \lim_{\substack{|\Delta\vec{\mathbf{S}}_i|\to 0\\ n\to\infty}} \sum_{i=1}^{n} \vec{\mathbf{V}}(\vec{\mathbf{r}}_i)\cdot\Delta\vec{\mathbf{S}}_i = \int_{(S)} \vec{\mathbf{V}}(\vec{\mathbf{r}})\cdot d\vec{\mathbf{S}}. \tag{13.114}$$

#### 3. Vector Flow of a Vector Field

$$\vec{\mathbf{R}} = \lim_{\substack{|\Delta\vec{\mathbf{S}}_i|\to 0\\ n\to\infty}} \sum_{i=1}^{n} \vec{\mathbf{V}}(\vec{\mathbf{r}}_i)\times\Delta\vec{\mathbf{S}}_i = \int_{(S)} \vec{\mathbf{V}}(\vec{\mathbf{r}})\times d\vec{\mathbf{S}}. \tag{13.115}$$

### 13.3.2.4 Surface Integral in Cartesian Coordinates as Surface Integrals of Second Type

$$\int_{(S)} U\,d\vec{\mathbf{S}} = \iint_{(S_{yz})} U\,dy\,dz\,\vec{\mathbf{i}} + \iint_{(S_{zx})} U\,dz\,dx\,\vec{\mathbf{j}} + \iint_{(S_{xy})} U\,dx\,dy\,\vec{\mathbf{k}}. \tag{13.116}$$

$$\int_{(S)} \vec{\mathbf{V}}\cdot d\vec{\mathbf{S}} = \iint_{(S_{yz})} V_x\,dy\,dz + \iint_{(S_{zx})} V_y\,dz\,dx + \iint_{(S_{xy})} V_z\,dx\,dy. \tag{13.117}$$

$$\int\limits_{(S)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}} = \iint\limits_{(S_{yz})} (V_z\vec{\mathbf{j}} - V_y\vec{\mathbf{k}})\,dy\,dz + \iint\limits_{(S_{zx})} (V_x\vec{\mathbf{k}} - V_z\vec{\mathbf{i}})\,dz\,dx + \iint\limits_{(S_{xy})} (V_y\vec{\mathbf{i}} - V_x\vec{\mathbf{j}})\,dx\,dy. \quad (13.118)$$

The existence theorems for these integrals can be given similarly to those in 8.5.2, **4.**, p. 537.
In the formulas above, each of the integrals is taken over the projection $S$ on the corresponding coordinate plane **(Fig. 13.18)**, where one of the variables $x$, $y$ or $z$ should be expressed by the others from the equation of $S$.

**Remark:** Integrals over a closed surface are denoted by

$$\oint\limits_{(S)} U\,d\vec{\mathbf{S}} = \oiint\limits_{(S)} U\,d\vec{\mathbf{S}}, \qquad \oint\limits_{(S)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}} = \oiint\limits_{(S)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}}, \qquad \oint\limits_{(S)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}} = \oiint\limits_{(S)} \vec{\mathbf{V}} \times d\vec{\mathbf{S}}. \quad (13.119)$$

■ **A:** Calculate the integral $\vec{\mathbf{P}} = \int\limits_{(S)} xyz\,d\vec{\mathbf{S}}$, where the surface is the plane region $x+y+z=1$ bounded by the coordinate planes. The upward side is the positive side:

$\vec{\mathbf{P}} = \iint\limits_{(S_{yz})} (1-y-z)yz\,dy\,dz\,\vec{\mathbf{i}} + \iint\limits_{(S_{zx})} (1-x-z)xz\,dz\,dx\,\vec{\mathbf{j}} + \iint\limits_{(S_{xy})} (1-x-y)xy\,dx\,dy\,\vec{\mathbf{k}}$;

$\iint\limits_{(S_{yz})} (1-y-z)yz\,dy\,dz = \int_0^1 \int_0^{1-z} (1-y-z)yz\,dy\,dz = \frac{1}{120}$. We get the two further integrals analogously. The result is: $\vec{\mathbf{P}} = \frac{1}{120}(\vec{\mathbf{i}} + \vec{\mathbf{j}} + \vec{\mathbf{k}})$.

■ **B:** Calculate the integral $Q = \int\limits_{(S)} \vec{\mathbf{r}} \cdot d\vec{\mathbf{S}} = \iint\limits_{(S_{yz})} x\,dy\,dz + \iint\limits_{(S_{zx})} y\,dz\,dx + \iint\limits_{(S_{xy})} z\,dx\,dy$ over the same plane region as in **A**: $\iint\limits_{(S_{yz})} x\,dy\,dz = \int_0^1 \int_0^{1-z} (1-y-z)\,dy\,dz = \frac{1}{6}$. Both other integrals are calculated similarly. The result is: $Q = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

■ **C:** Calculate the integral $\vec{\mathbf{R}} = \int\limits_{(S)} \vec{\mathbf{r}} \times d\vec{\mathbf{S}} = \int\limits_{(S)} (x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}) \times (dy\,dz\,\vec{\mathbf{i}} + dz\,dx\,\vec{\mathbf{j}} + dx\,dy\,\vec{\mathbf{k}})$, where the surface region is the same as in **A**: Performing the computations gives $\vec{\mathbf{R}} = \vec{\mathbf{0}}$.

## 13.3.3 Integral Theorems

### 13.3.3.1 Integral Theorem and Integral Formula of Gauss

#### 1. Integral Theorem of Gauss or the Divergence Theorem

The *integral theorem of Gauss* gives the relation between a volume integral of the divergence of $\vec{\mathbf{V}}$ over a volume $v$, and a surface integral over the surface $S$ surrounding this volume. The orientation of the surface (see 8.5.2.1, p. 535) is defined so that the exterior side is the positive one. The vector function $\vec{\mathbf{V}}$ should be continuous, their first partial derivatives should exist and be continuous. The integral theorem of Gauss reads as follows:

$$\oiint\limits_{(S)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}} = \iiint\limits_{(v)} \operatorname{div} \vec{\mathbf{V}}\,dv, \quad (13.120a)$$

i.e., the scalar flow of the field $\vec{\mathbf{V}}$ through a closed surface $S$ is equal to the integral of divergence of $\vec{\mathbf{V}}$ over the volume $v$ bounded by $S$. In Cartesian coordinates one gets:

$$\oiint_{(S)} (V_x\,dy\,dz + V_y\,dz\,dx + V_z\,dx\,dy) = \iiint_{(v)} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\,dy\,dz\,. \tag{13.120b}$$

### 2. Integral Formula of Gauss

In the planar case, the integral theorem of Gauss restricted to the $x, y$ plane becomes the *integral formula of Gauss*. It represents the correspondence between a line integral and the corresponding surface integral. The integral formula of Gauss reads as follows:

$$\iint_{(B)} \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] dx\,dy = \oint_{(C)} [P(x,y)\,dx + Q(x,y)\,dy]\,. \tag{13.121}$$

$B$ denotes a plane region which is bounded by $C$. $P$ and $Q$ are continuous functions with continuous first partial derivatives.

### 3. Sector Formula

The *sector formula* is an important special case of the Gauss integral formula to calculate the area of plane regions. For $Q = x$, $P = -y$ it follows that

$$F = \iint_{(B)} dx\,dy = \frac{1}{2}\oint_{(C)} [x\,dy - y\,dx]. \tag{13.122}$$

## 13.3.3.2 Integral Theorem of Stokes

The *integral theorem of Stokes* gives the relation between a surface integral over an oriented surface region $S$, in which the vector field $\vec{\mathbf{V}}$ is defined, and the integral along the closed boundary curve $C$ of the surface $S$. The sense of the curve $C$ is chosen so that the sense of traverse forms a *right-screw* with the surface normal (see 13.3.2.1, p. 722). The vector function $\vec{\mathbf{V}}$ should be continuous and it should have continuous first partial derivatives. The integral theorem of Stokes reads as follows:

$$\iint_{(S)} \operatorname{rot}\vec{\mathbf{V}}\cdot d\vec{\mathbf{S}} = \oint_{(C)} \vec{\mathbf{V}}\cdot d\vec{\mathbf{r}}, \tag{13.123a}$$

i.e., the vector flow of the rotation through a surface $S$ bounded by the closed curve $C$ is equal to the contour integral of the vector field $\vec{\mathbf{V}}$ along the curve $C$.
In Cartesian coordinates

$$\iint_{(S)} \left[\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) dy\,dz + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) dz\,dx + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy\right]$$
$$= \oint_{(C)} (V_x\,dx + V_y\,dy + V_z\,dz) \tag{13.123b}$$

holds. In the planar case, the integral theorem of Stokes, just as that of Gauss, becomes into the integral formula (13.121) of Gauss.

## 13.3.3.3 Integral Theorems of Green

The Green integral theorems give relations between volume and surface integrals. They are the applications of the Gauss theorem for the function $\vec{\mathbf{V}} = U_1 \operatorname{grad} U_2$, where $U_1$ and $U_2$ are scalar field functions and $v$ is the volume surrounded by the surface $S$. The following theorems hold:

**1.** $$\iiint_{(v)} (U_1\Delta U_2 + \operatorname{grad} U_2\cdot\operatorname{grad} U_1)\,dv = \oiint_{(S)} U_1 \operatorname{grad} U_2\cdot d\vec{\mathbf{S}}, \tag{13.124}$$

**2.** $\displaystyle\iiint\limits_{(v)} (U_1 \Delta U_2 - U_2 \Delta U_1)\, dv = \oiint\limits_{(S)} (U_1 \operatorname{grad} U_2 - U_2 \operatorname{grad} U_1) \cdot d\vec{\mathbf{S}}.$ (13.125)

In particular for $U_1 = 1, U_2 = U$

**3.** $\displaystyle\iiint\limits_{(v)} \Delta U\, dv = \oiint\limits_{(S)} \operatorname{grad} U \cdot d\vec{\mathbf{S}}$ (13.126)

holds. In Cartesian coordinates the third Green theorem has the following form (compare (13.120b)):

$$\iiint\limits_{(v)} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) dv = \oiint\limits_{(S)} \left( \frac{\partial U}{\partial x}\, dy\, dz + \frac{\partial U}{\partial y}\, dz\, dx + \frac{\partial U}{\partial z}\, dx\, dy \right). \qquad (13.127)$$

■ **A:** Calculating the line integral $I = \oint\limits_{(C)} (x^2 y^3\, dx + dy + z\, dz)$ with a circle $C$ as the intersection curve of the cylinder $x^2 + y^2 = a^2$ and the plane $z = 0$. With the Stokes theorem (13.123a) one gets:

$I = \oint\limits_{(C)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{r}} = \iint\limits_{(S)} \operatorname{rot} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}} = -\iint\limits_{(S^*)} 3x^2 y^2\, dx\, dy = -3 \int\limits_{\varphi=0}^{2\pi} \int\limits_{r=0}^{a} r^5 \cos^2\varphi \sin^2\varphi\, dr\, d\varphi = -\frac{a^6}{8}\pi$ with $\operatorname{rot} \vec{\mathbf{V}} = -3x^2 y^2 \vec{\mathbf{k}}$, $d\vec{\mathbf{S}} = \vec{\mathbf{k}}\, dx\, dy$ and the circle $S^*$: $x^2 + y^2 \le a^2$.

■ **B:** Determine the flux $I = \oint\limits_{(S)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}}$ in the drift space $\vec{\mathbf{V}} = x^3 \vec{\mathbf{i}} + y^3 \vec{\mathbf{j}} + z^3 \vec{\mathbf{k}}$ through the surface $S$ of the sphere $x^2 + y^2 + z^2 = a^2$. The theorem of Gauss yields:

$I = \oint\limits_{(S)} \vec{\mathbf{V}} \cdot d\vec{\mathbf{S}} = \iiint\limits_{(v)} \operatorname{div} \vec{\mathbf{V}}\, dv = 3 \iiint\limits_{(v)} (x^2 + y^2 + z^2)\, dx\, dy\, dz = 3 \int\limits_{\varphi=0}^{2\pi} \int\limits_{\vartheta=0}^{\pi} \int\limits_{r=0}^{a} r^4 \sin\vartheta\, dr\, d\vartheta\, d\varphi = \frac{12}{5} a^5 \pi.$

■ **C:** Heat conduction equation: The change in time of the heat $Q$ of a space region $v$ containing no heat source is given by $\displaystyle\frac{dQ}{dt} = \iiint\limits_{(v)} c\varrho \frac{\partial T}{\partial t}\, dv$ (specific heat-capacity $c$, density $\varrho$, temperature $T$), while the corresponding time-dependent change of the heat flow through the surface $S$ of $v$ is given by $\displaystyle\frac{dQ}{dt} = \oiint\limits_{(S)} \lambda \operatorname{grad} T \cdot d\vec{\mathbf{S}}$ (thermal conductivity $\lambda$). Applying the theorem of Gauss for the surface integral (13.120a) one gets from $\displaystyle\iiint\limits_{(v)} \left[ c\varrho \frac{\partial T}{\partial t} - \operatorname{div}(\lambda \operatorname{grad} T) \right] dv = 0$ the heat conduction equation $\displaystyle c\lambda \frac{\partial T}{\partial t} = \operatorname{div}(\lambda \operatorname{grad} T)$, which has the form $\displaystyle\frac{\partial T}{\partial t} = a^2 \Delta T$ in the case of a homogeneous solid ($c$, $\varrho$, $\lambda$ constants).

# 13.4 Evaluation of Fields

## 13.4.1 Pure Source Fields

A field $\vec{\mathbf{V}}_1$ is called a *pure source field* or an *irrotational source field* when its rotation is equal to zero everywhere. If the *divergence* is $q(\vec{\mathbf{r}})$, then

$$\operatorname{div} \vec{\mathbf{V}}_1 = q(\vec{\mathbf{r}}), \qquad \operatorname{rot} \vec{\mathbf{V}}_1 \equiv \vec{\mathbf{0}} \qquad (13.128)$$

holds. In this case, the field has a potential $U$, which is defined at every point $P$ by the *Poisson differential equation* (see 13.5.2, p. 729)

$$\vec{\mathbf{V}}_1 = \operatorname{grad} U, \qquad \operatorname{div}\operatorname{grad} U = \Delta U = q(\vec{\mathbf{r}}), \tag{13.129a}$$

where $\vec{\mathbf{r}}$ is the position vector of $P$. (In physics most often $\vec{\mathbf{V}}_1 = -\operatorname{grad} U$ is used.) The evaluation of $U$ comes from

$$U(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\iiint \frac{\operatorname{div}\vec{\mathbf{V}}(\vec{\mathbf{r}}^*)\,dv(\vec{\mathbf{r}}^*)}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}^*|}. \tag{13.129b}$$

The integration is taken over the whole of space **(Fig. 13.19)**. The divergence of $\vec{\mathbf{V}}$ must be differentiable and be decreasing sufficiently rapidly for large distances.

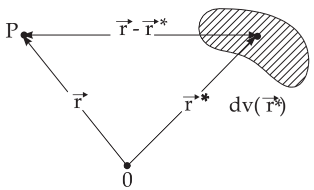


Figure 13.19

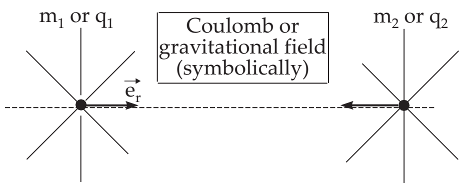


Figure 13.20

## 13.4.2 Pure Rotation Field or Zero-Divergence Field

A *pure rotation* (or *curl*) *field* or a *solenoidal field* is a vector field $\vec{\mathbf{V}}_2$ whose divergence is equal to zero everywhere; this field is free of sources. With $\vec{\mathbf{w}}(\vec{\mathbf{r}})$ as the *rotation density*

$$\operatorname{div}\vec{\mathbf{V}}_2 \equiv 0, \qquad \operatorname{rot}\vec{\mathbf{V}}_2 = \vec{\mathbf{w}}(\vec{\mathbf{r}}) \tag{13.130a}$$

hold. The rotation density $\vec{\mathbf{w}}(\vec{\mathbf{r}})$ cannot be arbitrary; it must satisfy the equation $\operatorname{div}\vec{\mathbf{w}} = 0$. With the approach

$$\vec{\mathbf{V}}_2(\vec{\mathbf{r}}) = \operatorname{rot}\vec{\mathbf{A}}(\vec{\mathbf{r}}), \qquad \operatorname{div}\vec{\mathbf{A}} = 0, \quad \text{i.e.,} \quad \operatorname{rot}\operatorname{rot}\vec{\mathbf{A}} = \vec{\mathbf{w}} \tag{13.130b}$$

follows according to (13.97)

$$\operatorname{grad}\operatorname{div}\vec{\mathbf{A}} - \Delta\vec{\mathbf{A}} = \vec{\mathbf{w}}, \quad \text{i.e.,} \quad \Delta\vec{\mathbf{A}} = -\vec{\mathbf{w}}. \tag{13.130c}$$

So, $\vec{\mathbf{A}}(\vec{\mathbf{r}})$ formally satisfies the Poisson differential equation (see (13.135a), p. 729) just as the potential $U$ of an irrotational field $\vec{\mathbf{V}}_1$ and that is why it is called a *vector potential*. For every point $P$, then

$$\vec{\mathbf{V}}_2 = \operatorname{rot}\vec{\mathbf{A}} \qquad \text{holds with} \quad \vec{\mathbf{A}} = \frac{1}{4\pi}\iiint \frac{\vec{\mathbf{w}}(\vec{\mathbf{r}}^*)}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}^*|}\,dv(\vec{\mathbf{r}}^*). \tag{13.130d}$$

The meaning of $\vec{\mathbf{r}}$ is the same as in (13.129b); the integration is taken over the whole of space.

## 13.4.3 Vector Fields with Point-Like Sources

### 13.4.3.1 Coulomb Field of a Point-Like Charge

The *Coulomb field* is an example of an irrotational field, which is also solenoidal, except at the location of the point charge $q$, the point source **(Fig. 13.20)**. For the Coulomb force

$$\vec{\mathbf{F}}_{\mathrm{C}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,\vec{\mathbf{e}}_r = \frac{q_1}{4\pi\varepsilon_0} q_2 \frac{\vec{\mathbf{r}}}{r^3} = e q_2 \frac{\vec{\mathbf{r}}}{r^3}, \qquad e = \frac{q_1}{4\pi\varepsilon_0} \tag{13.131a}$$

holds. This force affects attractively for electric charges $q_1, q_2$ with different signs and repulsively for charges with equal signs. $\varepsilon_0$ is the electric constant (see **Table 21.2**, p. 1053), $e$ is the intensity or source strength of the source. The electric field strength and the electrostatic potential, generated in the space around the charge $q_1$ and affecting to the charge $q_2$ are given as

$$\vec{\mathbf{E}}_{\mathrm{C}} = \frac{\vec{\mathbf{F}}_{\mathrm{C}}}{q_2} = \frac{e}{r^3}\vec{\mathbf{r}} = -\operatorname{grad} U\,, \quad U = \frac{e}{r}\,. \tag{13.131b}$$

$U$ denotes the electrostatic potential of the field. The scalar flow in accordance with the theorem of Gauss (see (13.120a), p. 724) is equal to $4\pi e$ or 0, depending on whether the surface $S$ encloses the point source or not:

$$\oint_{(S)} \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \begin{cases} 4\pi e, & \text{if } S \text{ encloses the point source,} \\ 0, & \text{otherwise.} \end{cases} \tag{13.131c}$$

Because of the irrotationality of the electrostatic field

$$\operatorname{rot} \vec{\mathbf{E}}_{\mathrm{C}} \equiv \vec{\mathbf{0}}\,. \tag{13.131d}$$

#### 13.4.3.2 Gravitational Field of a Point Mass

The field of gravity of a point mass or the Newton field is a second example of an irrotational and at the same time solenoidal field, except at the point of the center of mass. For the Newton mass attraction

$$\vec{\mathbf{F}}_{\mathrm{N}} = \gamma \frac{m_1 m_2}{r^2}\, \vec{\mathbf{e}}_r \tag{13.132}$$

holds, where $\gamma$ is the gravitational constant (see **Table 21.2**, p. 1053). Every relation valid for the Coulomb field is valid analogously also for the Newton field.

### 13.4.4 Superposition of Fields

#### 13.4.4.1 Discrete Source Distribution

Analogously to superposition of fields in physics, vector fields superpose each other. The *superposition law* is: If the vector fields $\vec{\mathbf{V}}_\nu$ have the potentials $U_\nu$, then the vector field

$$\vec{\mathbf{V}} = \Sigma \vec{\mathbf{V}}_\nu \ \text{ has the potential } \ U = \Sigma U_\nu. \tag{13.133a}$$

For $n$ discrete point sources with source strength $e_\nu$ ($\nu = 1, 2, \ldots, n$), whose fields are superposed, the resulting field can be determined by the algebraic sum of the potentials $U_\nu$:

$$\vec{\mathbf{V}}(\vec{\mathbf{r}}) = -\operatorname{grad} \sum_{\nu=1}^{n} U_\nu \qquad \text{with} \qquad U_\nu = \frac{e_\nu}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_\nu|}. \tag{13.133b}$$

Here, the vector $\vec{\mathbf{r}}$ is again the position vector of the point under consideration, $\vec{\mathbf{r}}_\nu$ are the position vectors of the sources.

If there is an irrotational field $\vec{\mathbf{V}}_1$ and a zero-divergence field $\vec{\mathbf{V}}_2$ together and they are everywhere continuous, then

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2 = -\frac{1}{4\pi}\left[\operatorname{grad} \iiint \frac{q(\vec{\mathbf{r}}^*)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}^*|}\, dv(\vec{\mathbf{r}}^*) - \operatorname{rot} \iiint \frac{\vec{\mathbf{w}}(\vec{\mathbf{r}}^*)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}^*|}\, dv(\vec{\mathbf{r}}^*)\right]. \tag{13.133c}$$

If the vector field is extended to infinity, then the decomposition of $\vec{\mathbf{V}}(\vec{\mathbf{r}})$ is unique if $|\vec{\mathbf{V}}(\vec{\mathbf{r}})|$ decreases sufficient rapidly for $r = |\vec{\mathbf{r}}| \to \infty$. The integration is taken over the whole of space.

#### 13.4.4.2 Continuous Source Distribution

If the sources are distributed continuously along lines, surfaces, or in domains of space, then, instead of the finite source strength $e_\nu$, there are infinitesimals corresponding to the density of the source distributions, and instead of the sums, we have integrals over the domain. In the case of a continuous space distribution of source strength, the divergence is $q(\vec{\mathbf{r}}) = \operatorname{div} \vec{\mathbf{V}}$.

Similar statements are valid for the potential of a field defined by rotation. In the case of a continuous space rotation distribution, the " *rotation density* " is defined by $\vec{\mathbf{w}}(\vec{\mathbf{r}}) = \operatorname{rot} \vec{\mathbf{V}}$.

#### 13.4.4.3 Conclusion

A vector field is determined uniquely by its sources and rotations in space if all these sources and rotations located inside a finite space.

## 13.5 Differential Equations of Vector Field Theory

### 13.5.1 Laplace Differential Equation

The problem to determine the potential $U$ of a vector field $\vec{\mathbf{V}}_1 = \operatorname{grad} U$ containing no sources, leads to the equation according to (13.128) with $q(\vec{\mathbf{r}}) = 0$

$$\operatorname{div} \vec{\mathbf{V}}_1 = \operatorname{div} \operatorname{grad} U = \Delta U = 0, \tag{13.134a}$$

i.e., to the *Laplace differential equation*. In Cartesian coordinates holds:

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{13.134b}$$

Every function satisfying this differential equation and which is continuous and possesses continuous first and second order partial derivatives is called a *Laplace* or *harmonic function* (see also 14.1.2.2,**2.**, p. 732).

There are to distinguish three basic types of boundary value problems:

**1.** Boundary value problem (for an interior domain) or *Dirichlet problem*: A function $U(x, y, z)$ is determined, which is harmonic inside a given space or plane domain and takes the given values at the boundary of this domain.

**2.** Boundary value problem (for an interior domain) or *Neumann problem*: A function $U(x, y, z)$ is determined, which is harmonic inside a given domain and whose normal derivative $\frac{\partial U}{\partial n}$ takes the given values at the boundary of this domain.

**3.** Boundary value problem (for an interior domain): A function $U(x, y, z)$ is determined, which is harmonic inside a given domain and the expression $\alpha U + \beta \frac{\partial U}{\partial n}$ $(\alpha, \beta \text{ const}, \quad \alpha^2 + \beta^2 \neq 0)$ takes the given values at the boundary of this domain.

### 13.5.2 Poisson Differential Equation

The problem to determine the potential $U$ of a vector field $\vec{\mathbf{V}}_1 = \operatorname{grad} U$ with given divergence, leads to the equation according to (13.128) with $q(\vec{\mathbf{r}}) \neq 0$

$$\operatorname{div} \vec{\mathbf{V}}_{\mathbf{1}} = \operatorname{div} \operatorname{grad} U = \Delta U = q(\vec{\mathbf{r}}) \neq 0, \tag{13.135a}$$

i.e., to the *Poisson differential equation*. Since in Cartesian coordinates:

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}, \tag{13.135b}$$

the Laplace differential equation (13.134b) is a special case of the Poisson differential equation (13.135b). The solution is the Newton potential (for point masses) or the Coulomb potential (for point charges)

$$U = -\frac{1}{4\pi} \iiint \frac{q(\vec{\mathbf{r}}^*)\, dv(\vec{\mathbf{r}}^*)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}^*|}. \tag{13.135c}$$

The integration is taken over the whole of space. $U(\vec{\mathbf{r}})$ tends to zero sufficiently rapidly for increasing $|\vec{\mathbf{r}}|$ values.

One can discuss the same three boundary value problems for the Poisson differential equation as for the solution of the Laplace differential equation in 13.5.1. The first and the third boundary value problems can be solved uniquely; for the second one there are to prescribe further special conditions (see [9.5]).