1 Arithmetics

1.1 Elementary Rules for Calculations

1.1.1 Numbers

1.1.1.1 Natural, Integer, and Rational Numbers

1. Definitions and Notation

The positive and negative integers, fractions, and zero together are called the *rational numbers*. In relation to these the following notations are used (see 5.2.1, **1.**, p. 327):

- Set of natural numbers: $\mathbb{N}=\{0, 1, 2, 3, \ldots\}$,
- Set of integers: $\mathbb{Z}= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
- Set of rational numbers: $\mathbb{Q}= \{x|x = \frac{p}{q} \text{ with } p \in \mathbb{Z},\ q \in \mathbb{Z} \text{ and } q \neq 0\}$.

The notion of natural numbers arose from enumeration and ordering. The natural numbers are also called the *non-negative integers*.

2. Properties of the Set of Rational Numbers

- The set of rational numbers is infinite.
- The set is *ordered*, i.e., for any two different given numbers a and b one can tell which is the smaller one.
- The set is *dense everywhere*, i.e., between any two different rational numbers a and b $(a < b)$ there is at least one rational number c $(a < c < b)$. Consequently, there is an infinite number of other rational numbers between any two different rational numbers.

3. Arithmetical Operations

The arithmetical operations (addition, subtraction, multiplication and division) can be performed with any two rational numbers, and the result is a rational number. The only exception is *division by zero*, which is not possible: The operation written in the form $a : 0$ is meaningless because it does not have any result: If $a \neq 0$, then there is no rational number b such that $b \cdot 0 = a$ could be fulfilled, and if $a = 0$ then b can be any of the rational numbers. The frequently occurring formula $a : 0 = \infty$ (infinity) does not mean that the division is possible; it is only the notation for the statement: If the denominator approaches zero and, e.g., the numerator does not, then the absolute value (magnitude) of the quotient exceeds any finite limit.

4. Decimal Fractions, Continued Fractions

Every rational number a can be represented as a terminating or periodically infinite decimal fraction or as a finite continued fraction (see 1.1.1.4, p. 3).

5. Geometric Representation

Fixing an *origin* the *zero point* 0, a positive direction the *orientation*, and the unit of length l the *measuring rule*, (see also 2.17.1, p. 115 and **(Fig. 1.1)**), then every rational number a corresponds to a certain point on this line. This point has the coordinate a, and it is a so-called *rational point*. The line is called the *numerical axis*. Because the set of rational numbers is dense everywhere, between two rational points there are infinitely many further rational points.

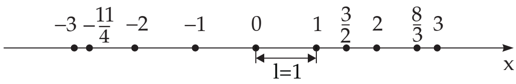

Figure 1.1

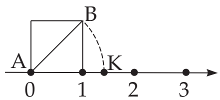

Figure 1.2

I.N. Bronshtein et al., *Handbook of Mathematics*,
DOI 10.1007/978-3-662-46221-8_1

1.1.1.2 Irrational and Transcendental Numbers

The set of rational numbers is not satisfactory for calculus. Even though it is dense everywhere, it does not cover the whole numerical axis. If for example the diagonal AB of the unit square rotates around A so that B goes into the point K, then K does not have any rational coordinate **(Fig. 1.2)**.
The introduction of *irrational numbers* allows to assign a number to every point of the numerical axis. In textbooks there are given exact definitions for irrational numbers, e.g., by nests of intervals. For this survey it is enough to note that the irrational numbers take all the non-rational points of the numerical axis and every irrational number corresponds to a point of the axis, and that every irrational number can be represented as a non-periodic infinite decimal fraction.
First of all, the non-integer real roots of the algebraic equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0 \quad (n > 1,\ \text{integer; integer coefficients}), \tag{1.1a}$$

belong to the irrational numbers. These roots are called *algebraic irrationals*.

■ **A:** The simplest examples of algebraic irrationals are the real roots of $x^n - a = 0$ $(a > 0)$, as numbers of the form $\sqrt[n]{a}$, if they are not rational.

■ **B:** $\sqrt[2]{2} = 1.414\ldots$, $\sqrt[3]{10} = 2.154\ldots$ are algebraic irrationals.

The irrational numbers which are not algebraic irrationals are called *transcendental*.

■ **A:** $\pi = 3.141592\ldots$ $e = 2.718281\ldots$ are transcendental numbers.

■ **B:** The decimal logarithm of the integers, except the numbers of the form 10^n, are transcendental.

The non-integer roots of the quadratic equation

$$x^2 + a_1x + a_0 = 0 \quad (a_1, a_0\ \text{integers}) \tag{1.1b}$$

are called *quadratic irrationals*. They have the form $(a + b\sqrt{D})/c$ $\ (a, b, c$ integers, $c \neq 0$; $D > 0$, square-free number).

■ The division of a line segment a in the ratio of the golden section $x/a = (a - x)/x$ (see 3.5.2.3, **3.**, p. 194) leads to the quadratic equation $x^2 + x - 1 = 0$, if $a = 1$. The solution $x = (\sqrt{5} - 1)/2$ is a quadratic irrational. It contains the irrational number $\sqrt{5}$.

1.1.1.3 Real Numbers

Rational and irrational numbers together form the *set of real numbers*, which is denoted by $\mathbb{R}$.

1. Most Important Properties

The set of real numbers has the following important properties (see also 1.1.1.1, **2.**, p. 1). It is:

- *Infinite.*
- *Ordered.*
- *Dense everywhere.*
- *Closed*, i.e., every point of the numerical axis corresponds to a real number. This statement does not hold for the rational numbers.

2. Arithmetical Operations

Arithmetical operations can be performed with any two real numbers and the result is a real number, too. The only exception is division by zero (see 1.1.1.1, **3.**, p. 1). Raising to a power and also its inverse operation can be performed among real numbers; so it is possible to take an arbitrary root of any positive number; every positive real number has a logarithm for an arbitrary positive basis, except that 1 cannot be a basis.

A further generalization of the notion of numbers leads us to the concept of *complex numbers* (see 1.5, p. 34).

3. Interval of Numbers

A connected set of real numbers with endpoints a and b is called an *interval of numbers with endpoints a and b*, where $a < b$ and a is allowed to be $-\infty$ and b is allowed to be $+\infty$. If the endpoint itself does not belong to the interval, then this end of the interval is *open*, in the opposite case it is *closed*.

An interval is given by its endpoints a and b, putting them in braces: A bracket for a closed end of the interval and a parenthesis for an open one. It is to be distinguished between *open intervals* (a, b), *half-open (half-closed) intervals* $[a, b)$ or $(a, b]$ and *closed intervals* $[a, b]$, according to whether none of the endpoints, one of the endpoints or both endpoints belong to it, respectively. Frequently the notation $]a, b[$ instead of (a, b) for open intervals, and analogously $[a, b[$ instead of $[a, b)$ is used. In the case of graphical representations, in this book the open end of the interval is denoted by a round arrow head, the closed one by a filled point.

1.1.1.4 Continued Fractions

Continued fractions are nested fractions, by which rational and irrational numbers can be represented and approximated even better than by decimal representation (see 19.8.1.1, p. 1002 and ■ **A** and ■ **B** on p. 4).

1. Rational Numbers

The continued fraction of a rational number is finite. Positive rational numbers which are greater than 1 have the form (1.2). For abbreviation the symbol $\frac{p}{q} = [a_0; a_1, a_2, \ldots, a_n]$ is used with $a_k \geq 1 \ (k = 1, 2, \ldots, n)$.

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}\,. \tag{1.2}$$

The numbers a_k are calculated with the help of the *Euclidean algorithm*:

$$\frac{p}{q} = a_0 + \frac{r_1}{q} \quad \left(0 < \frac{r_1}{q} < 1\right), \tag{1.3a}$$

$$\frac{q}{r_1} = a_1 + \frac{r_2}{r_1} \quad \left(0 < \frac{r_2}{r_1} < 1\right), \tag{1.3b}$$

$$\frac{r_1}{r_2} = a_2 + \frac{r_3}{r_2} \quad \left(0 < \frac{r_3}{r_2} < 1\right), \tag{1.3c}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{r_{n-2}}{r_{n-1}} = a_{n-1} + \frac{r_n}{r_{n-1}} \quad \left(0 < \frac{r_n}{r_{n-1}} < 1\right), \tag{1.3d}$$

$$\frac{r_{n-1}}{r_n} = a_n \quad (r_{n+1} = 0)\,. \tag{1.3e}$$

■ $$\frac{61}{27} = 2 + \frac{7}{27} = 2 + \cfrac{1}{3 + \cfrac{6}{7}} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{6}}} = [2; 3, 1, 6]\,.$$

2. Irrational Numbers

Continued fractions of irrational numbers do not break off. They are called infinite continued fractions with $[a_0; a_1, a_2, \ldots]$.

If some numbers a_k are repeated in an infinite continued fraction, then this fraction is called a *periodic continued fraction* or *recurring chain fraction*. Every periodic continued fraction represents a quadratic irrationality, and conversely, every quadratic irrationality has a representation in the form of a periodic continued fraction.

■ The number $\sqrt{2} = 1.4142135\ldots$ is a quadratic irrationality and it has the periodic continued fraction representation $\sqrt{2} = [1; 2, 2, 2, \ldots]$.

3. Aproximation of Real Numbers

If $\alpha = [a_0; a_1, a_2, \ldots]$ is an arbitrary real number, then every finite continued fraction

$$\alpha_k = [a_0; a_1, a_2, \ldots, a_k] = \frac{p}{q} \tag{1.4}$$

represents an approximation of α . The continued fraction α_k is called the k-th approximant of α . It can be calculated by the recursive formula

$$\alpha_k = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}} \quad (k \geq 1; \;\; p_{-1} = 1, p_0 = a_0; q_{-1} = 0, q_0 = 1). \tag{1.5}$$

According to the *Liouville approximation theorem*, the following estimat holds:

$$|\alpha - \alpha_k| = |\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k^2}\,. \tag{1.6}$$

Furthermore, it can be shown that the approximants approach the real number α with increasing accuracy alternatively from above and from below. The approximants converge to α especially fast if the numbers $a_i \;\; (i = 1, 2, \ldots, k)$ in (1.4) have large values. Consequently, the convergence is worst for the numbers $[1; 1, 1, \ldots]$.

■ **A:** From the decimal presentation of π the continued fraction representation $\pi = [3; 7, 15, 1, 292, \ldots]$ follows with the help of (1.3a)–(1.3e). The corresponding approximants (1.5) with the estimate according to (1.6) are: $\alpha_1 = \dfrac{22}{7}$ with $|\pi - \alpha_1| < \dfrac{1}{7^2} \approx 2 \cdot 10^{-2}$, $\alpha_2 = \dfrac{333}{106}$ with $|\pi - \alpha_2| < \dfrac{1}{106^2} \approx 9 \cdot 10^{-5}$, $\alpha_3 = \dfrac{355}{113}$ with $|\pi - \alpha_3| < \dfrac{1}{113^2} \approx 8 \cdot 10^{-5}$. The actual errors are much smaller. They are less than $1.3 \cdot 10^{-3}$ for α_1, $8.4 \cdot 10^{-5}$ for α_2 and $2.7 \cdot 10^{-7}$ for α_3 . The approximants α_1, α_2 and α_3 represent better approximations for π than the decimal representation with the corresponding number of digits.

■ **B:** The formula of the golden section $x/a = (a - x)/x$ (see 1.1.1.2, p. 2, 3.5.2.3, **3.**, p. 194 and 17.3.2.4, **4.**, p. 908) can be represented by the following two continued fractions: $x = a[1; 1, 1, \ldots]$ and $x = \dfrac{a}{2}(1 + \sqrt{5}) = \dfrac{a}{2}(1 + [2; 4, 4, 4, \ldots])$. The approximant α_4 delivers in the first case an accuracy of $0.018\,a$, in the second case of $0.000\,001\,a$.

1.1.1.5 Commensurability

Two numbers a and b are called *commensurable*, i.e., measurable by the same number, if both are an integer multiple of a third number c. From $a = mc, \;\; b = nc \;\; (m, n \in \mathbb{Z})$ it follows that

$$\frac{a}{b} = x \;\; (x \text{ rational}). \tag{1.7}$$

Otherwise a and b are *incommensurable*.

■ **A:** The length of a side and a diagonal of a square are incommensurable because their ratio is the irrational number $\sqrt{2}$.

■ **B:** The lengths of the golden section are incommensurable, because their ratio contains the irrational number $\sqrt{5}$ (see 1.1.1.2, p. 2 and 3.5.2.3, **3.**,p. 194). Therefore the sides and diagonals in a regular pentagon are incommensurable (see ■ in 3.1.5.3, p. 139). Today Hippasos from Metapontum (450 BC) is considered to have discovered the irrational numbers via this example.

1.1.2 Methods for Proof

Mostly three types of proofs are used:
• direct proof,
• indirect proof,

• proof by (mathematical or arithmetical) induction.
Furthermore there are constructive proofs.

1.1.2.1 Direct Proof

The starting point is a theorem which has already been proven (premise p) and the truth of the statement of the new theorem is derived from it (conclusion q). The logical steps mostly used for the conclusions are implication and equivalence (see 5.1, p. 323).

1. Direct Proof by Implication

The *implication* $p \Rightarrow q$ means that the truth of the conclusion follows from the truth of the premise (see " Implication " in the truth table, 5.1.1, p. 323).

■ Prove the inequality $\dfrac{a+b}{2} \geq \sqrt{ab}$ for $a > 0,\ b > 0$. The premise is the well-known binomial formula $(a+b)^2 = a^2 + 2ab + b^2$. By subtracting $4ab$ follows $(a+b)^2 - 4ab = (a-b)^2 \geq 0$. From this inequality the statement is obtained certainly if the investigations are restricted only to the positive square roots because of $a > 0$ and $b > 0$.

2. Direct Proof by Equivalence

The proof will be delivered by *verifying* an equivalent statement. In practice it means that all the arithmetical operations which have to be used for changing p into q must be uniquely invertible.

■ Prove the inequality $1 + a + a^2 + \cdots + a^n < \dfrac{1}{1-a}$ for $0 < a < 1$.
Multiplying by $1-a$ yields $1 - a + a - a^2 + a^2 - a^3 \pm \cdots + a^n - a^{n+1} = 1 - a^{n+1} < 1$.
This last inequality is true because of the assumption $0 < a^{n+1} < 1$. The starting inequality also holds because all the arithmetical operations to be used are uniquely invertible.

1.1.2.2 Indirect Proof or Proof by Contradiction

To prove the statement q: Starting from its *negation* $\bar{q}$, and from $\bar{q}$ arriving at a false statement r, i.e., $\bar{q} \Rightarrow r$ (see also 5.1.1, **7.**, p. 325). In this case $\bar{q}$ must be false, because using the implication a false assumption can result only in a false conclusion (see truth table 5.1.1, p. 323). If $\bar{q}$ is false q must be true.

■ Prove that the number $\sqrt{2}$ is irrational. Suppose, $\sqrt{2}$ is rational. So the equality $\sqrt{2} = \dfrac{a}{b}$ holds for some integers a, b and $b \neq 0$. Assuming that the numbers a, b are *coprime numbers*, i.e., they do not have any common divisor, then follows $(\sqrt{2})^2 = 2 = \dfrac{a^2}{b^2}$ or $a^2 = 2b^2$, therefore, a^2 is an even number, and this is possible only if $a = 2n$ is an even number. Deducing $a^2 = 4n^2 = 2b^2$ holds, hence b must be an even number, too. It is obviously a contradiction to the assumption that a and b are coprimes.

1.1.2.3 Mathematical Induction

Theorems and dependent on natural numbers n are proven with this method. The principle of mathematical induction is the following: If the statement is valid for a natural number n_0, and if from the validity of the statement for a natural number $n \geq n_0$ the validity of the statement follows for $n+1$, then the statement is valid for every natural number $n \geq n_0$. According to these, the steps of the proof are:

1. Basis of the Induction: The truth of the statement is to be shown for $n = n_0$. Mostly $n_0 = 1$ can be choosen.

2. Induction Hypothesis: The statement is valid for an integer n (premise p).

3. Induction Conclusion: Formulation the proposition for $n+1$ (conclusion q).

4. Proof of the Implication: $p \Rightarrow q$.

Steps **3.** and **4.** together are called the *induction step* or *logical deduction from n to $n+1$*.

■ Prove the formula $s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

The steps of the proof by induction are:

1. $n = 1 : \; s_1 = \frac{1}{1\cdot 2} = \frac{1}{1+1}$ is obviously true.

2. Suppose $s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ holds for an $n \ge 1$.

3. Supposing **2.** it is to show: $s_{n+1} = \frac{n+1}{n+2}$.

4. The proof: $s_{n+1} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = s_n + \frac{1}{(n+1)(n+2)} =$
$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$.

1.1.2.4 Constructive Proof

In approximation theory, for instance, the proof of an existence theorem usually follows a *constructive* process, i.e., the steps of the proof give a method of calculation for a result which satisfies the propositions of the existence theorem.

■ The existence of a third-degree interpolation-spline function (see 19.7.1.1, **1.**, p. 996) can be proved in the following way: It is to be shown that the calculation of the coefficients of a spline satisfying the requirements of the existence theorem results in a tridiagonal linear equation system, which has a unique solution (see 19.7.1.1, **2.**, p. 997).

1.1.3 Sums and Products

1.1.3.1 Sums

1. Definition

To briefly denote a sum the *summation sign* $\sum$ is used:

$$a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k. \tag{1.8}$$

With this notation the sum of n summands a_k $(k = 1, 2, \ldots, n)$ is denoted, k is called the *running index* or *summation variable*.

2. Rules of Calculation

1. Sum of Summands Equal to Each Other , i.e., $a_k = a$ for $k = 1, 2, \ldots, n$:

$$\sum_{k=1}^{n} a_k = na. \tag{1.9a}$$

2. Multiplication by a Constant Factor

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k. \tag{1.9b}$$

3. Separating a Sum

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k \quad (1 < m < n). \tag{1.9c}$$

4. Addition of Sums with the Same Length

$$\sum_{k=1}^{n} (a_k + b_k + c_k + \ldots) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k + \sum_{k=1}^{n} c_k + \ldots . \tag{1.9d}$$

5. Renumbering

$$\sum_{k=1}^{n} a_k = \sum_{k=m}^{m+n-1} a_{k-m+1}, \quad \sum_{k=m}^{n} a_k = \sum_{k=l}^{n-m+l} a_{k+m-l}. \tag{1.9e}$$

6. Exchange the Order of Summation in Double Sums

$$\sum_{i=1}^{n}\left(\sum_{k=1}^{m} a_{ik}\right) = \sum_{k=1}^{m}\left(\sum_{i=1}^{n} a_{ik}\right). \tag{1.9f}$$

1.1.3.2 Products

1. Definition

The abbreviated notation for a product is the *product sign* $\prod$:

$$a_1 a_2 \dots a_n = \prod_{k=1}^{n} a_k. \tag{1.10}$$

With this notation a product of n factors a_k $(k = 1, 2, \dots, n)$ is denoted, where k is called the *running index.*

2. Rules of Calculation

1. Product of Coincident Factors , i.e., $a_k = a$ for $k = 1, 2, \dots, n$:

$$\prod_{k=1}^{n} a_k = a^n. \tag{1.11a}$$

2. Factoring out a Constant Factor

$$\prod_{k=1}^{n} (c a_k) = c^n \prod_{k=1}^{n} a_k. \tag{1.11b}$$

3. Separating into Partial Products

$$\prod_{k=1}^{n} a_k = \left(\prod_{k=1}^{m} a_k\right)\left(\prod_{k=m+1}^{n} a_k\right) \quad (1 < m < n). \tag{1.11c}$$

4. Product of Products

$$\prod_{k=1}^{n} a_k b_k c_k \dots = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right)\left(\prod_{k=1}^{n} c_k\right) \dots. \tag{1.11d}$$

5. Renumbering

$$\prod_{k=1}^{n} a_k = \prod_{k=m}^{m+n-1} a_{k-m+1}, \quad \prod_{k=m}^{n} a_k = \prod_{k=l}^{n-m+l} a_{k+m-l}. \tag{1.11e}$$

6. Exchange the Order of Multiplication in Double Products

$$\prod_{i=1}^{n}\left(\prod_{k=1}^{m} a_{ik}\right) = \prod_{k=1}^{m}\left(\prod_{i=1}^{n} a_{ik}\right). \tag{1.11f}$$

1.1.4 Powers, Roots, and Logarithms

1.1.4.1 Powers

The notation a^x is used for the algebraic operation of *raising to a power*. The number a is called the *base*, x is called the *exponent* or *power*, and a^x is called the *power*. Powers are defined as in **Table 1.1**.

For the allowed values of bases and exponents there are the following
Rules of Calculation:

$$a^x\, a^y = a^{x+y}, \quad a^x : a^y = \frac{a^x}{a^y} = a^{x-y}, \tag{1.12}$$

$$a^x\, b^x = (a\, b)^x, \quad a^x : b^x = \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x, \tag{1.13}$$

$$(a^x)^y = (a^y)^x = a^{x\,y}, \tag{1.14}$$

$$a^x = e^{x \ln a} \quad (a > 0). \tag{1.15}$$

Here $\ln a$ is the natural logarithm of a where $e = 2.718281828459\ldots$ is the base. Special powers are

$$(-1)^n = \begin{cases} +1, \text{ if } n \text{ even}, \\ -1, \text{ if } n \text{ odd}, \end{cases} \quad (1.16a) \qquad a^0 = 1 \text{ for any } a \neq 0\,. \quad (1.16b)$$

Table 1.1 Definition of powers

base a	*exponent x*	*power a^x*
arbitrary real, $\neq 0$	0	1
	$n = 1, 2, 3, \ldots$	$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$ (a to the power n)
	$n = -1, -2, -3, \ldots$	$a^n = \dfrac{1}{a^{-n}}$
positive real	rational: $\dfrac{p}{q}$ (p, q integer, $q > 0$)	$a^{\frac{p}{q}} = \sqrt[q]{a^p}$ (q-th root of a to the power p)
	irrational: $\lim\limits_{k\to\infty} \dfrac{p_k}{q_k}$	$\lim\limits_{k\to\infty} a^{\frac{p_k}{q_k}}$
0	positive	0

1.1.4.2 Roots

According to **Table 1.1** the n-th root of a positive number a is the positive number denoted by

$$\sqrt[n]{a} \quad (a > 0, \text{ real}; \; n > 0, \text{ integer}). \tag{1.17a}$$

This operation is called *taking of the root* or *extraction of the root*, a is the *radicand*, n is the *radical* or *index*.

The solution of the equation

$$x^n = a \quad (a \text{ real or complex}\,; \; n > 0\,, \text{ integer}) \tag{1.17b}$$

is often denoted by $x = \sqrt[n]{a}$. But there is no reason to be confused: In this relation the notation denotes all the solutions of the equation, i.e., it represents n different values x_k $(k = 1, 2, \ldots, n)$ to be calculated. In the cace of negative or complex values they are to be determined by (1.140b) (see 1.5.3.6, p. 38).

■ **A:** The equation $x^2 = 4$ has two real solutions, namely $x_{1,2} = \pm 2$.

■ **B:** The equation $x^3 = -8$ has three roots among the complex numbers: $x_1 = 1 + \mathrm{i}\sqrt{3}, x_2 = -2$ and $x_3 = 1 - \mathrm{i}\sqrt{3}$, but only one among the reals.

1.1.4.3 Logarithms

1. Definition

The *logarithm* u of a positive number $x > 0$ to the *base* $b > 0,\ b \neq 1$, is the exponent of the power which has the value x with b in the base. It is denoted by $u = \log_b x$. Consequently the equation

$b^u = x$ (1.18a) yields $\log_b x = u$ (1.18b)

and conversely the second one yields the first one. In particular holds

$$\log_b 1 = 0, \quad \log_b b = 1, \quad \log_b 0 = \begin{cases} -\infty \text{ for } b > 1, \\ +\infty \text{ for } b < 1. \end{cases} \tag{1.18c}$$

The logarithm of negative numbers can be defined only among the complex numbers. The logarithmic functions see 2.6.2, p. 73.
To take the logarithm of a given number means to find its logarithm. To take the logarithm of an expression means it is transformed like (1.19a, 1.19b). The determination of a number or an expression from its logarithm is called *raising to a power.*

2. Some Properties of the Logarithm

a) Every positive number has a logarithm to any positive base, except the base $b = 1$.

b) For $x > 0$ and $y > 0$ the following **Rules of Calculation** are valid for any b (which is allowed to be a base):

$$\log(xy) = \log x + \log y, \quad \log\left(\frac{x}{y}\right) = \log x - \log y, \tag{1.19a}$$

$$\log x^n = n \log x, \quad \text{in particular} \ \log \sqrt[n]{x} = \frac{1}{n} \log x\,. \tag{1.19b}$$

With (1.19a, 1.19b) the logarithm of products and fractions can be calculated as sums or differences of logarithms .

■ Take the logarithm of the expression $\dfrac{3x^2\sqrt[3]{y}}{2zu^3}$: $\log \dfrac{3x^2\sqrt[3]{y}}{2zu^3} = \log\left(3x^2\sqrt[3]{y}\right) - \log\left(2zu^3\right)$
$= \log 3 + 2\log x + \dfrac{1}{3}\log y - \log 2 - \log z - 3\log u.$

Often the reverse transformation is required, i.e., an expression containing logarithms of different amounts is to be rewritten into one, which is the logarithm of one expression.

■ $\log 3 + 2\log x + \dfrac{1}{3}\log y - \log 2 - \log z - 3\log u = \log \dfrac{3x^2\sqrt[3]{y}}{2zu^3}.$

c) Logarithms to different bases are proportional, i.e., the logarithm to a base a can be change into a logarithm to the base b by multiplication:

$$\log_a x = M \log_b x \ \text{ where } \ M = \log_a b = \frac{1}{\log_b a}. \tag{1.20}$$

M is called the *modulus of the transformation.*

1.1.4.4 Special Logarithms

1. The logarithm to the base 10 is called the *decimal* or *Briggsian logarithm*, in formulas:

$$\log_{10} x = \lg x \ \text{ and } \ \log(x10^\alpha) = \alpha + \log x. \tag{1.21}$$

2. The logarithm to the base e is called the *natural* or *Neperian logarithm*, in formulas:

$$\log_e x = \ln x. \tag{1.22}$$

The modulus of transformation to change from the natural logarithm into the decimal one is

$$M = \log e = \frac{1}{\ln 10} = 0.4342944819\,, \tag{1.23}$$

and to change from the decimal into the natural one it is

$$M_1 = \frac{1}{M} = \ln 10 = 2.3025850930\,. \tag{1.24}$$

3. The logarithm to base 2 is called the *binary logarithm*, in formulas:

$$\log_2 x = \mathrm{ld}\, x \quad \text{or} \quad \log_2 x = \mathrm{lb}\, x. \tag{1.25}$$

4. The values of the decimal and natural logarithm can be found in *logarithm tables*. Some time ago the logarithm was used for numerical calculation of powers, and it often made numerical multiplication and division easier. Mostly the decimal logarithm was used. Today pocket calculators and personal computers make these calculations.
Every number given in decimal form (so every real number), which is called in this relation the *antilog*, can be written in the form

$$x = \hat{x}10^k \ \text{ with } \ 1 \le \hat{x} < 10 \tag{1.26a}$$

by factoring out an appropriate power of ten: 10^k with integer k. This form is called the *half-logarithmic representation*. Here $\hat{x}$ is given by the sequence of figures of x, and 10^k is the order of magnitude of x. Then for the logarithm holds

$$\log x = k + \log \hat{x} \quad \text{with} \quad 0 \le \log \hat{x} < 1, \quad \text{i.e.,} \quad \log \hat{x} = 0, \ldots\,. \tag{1.26b}$$

Here k is the so-called *characteristic* and the sequence of figures behind the decimal point of $\log \hat{x}$ is called the *mantissa*. The mantissa can be found in logarithm tables.

■ $\lg 324 = 2.5105$, the characteristic is 2, the mantissa is 5105. Multiplying or dividing this number by 10^n, for example 324000; 3240; 3.24; 0.0324, their logarithms have the same mantissa, here 5105, but different characteristics. That is why the mantissas are given in *logarithm tables*. In order to get the mantissa of a number x first the decimal point has to be moved to the right or to the left to get a number between 1 and 10, and the characteristic of the antilog x is determined by how many digits k the decimal point was moved.

5. Slide rule Beside the logarithm, the *slide rule* was of important practical help in numerical calculations. The slide rule works by the principle of the form (1.19a), so multiplying and dividing is done by adding and subtracting numbers. On the slide rule the scale-segments are denoted according to the logarithm values, so multiplication and division can be performed as addition or subtraction (see Scale and Graph Papers 2.17.1, p. 115).

1.1.5 Algebraic Expressions

1.1.5.1 Definitions

1. Algebraic Expression

One or more algebraic quantities, such as numbers or symbols, are called an *algebraic expression* or *term* if they are connected by the symbols, $+$, $-$, $\cdot$, $:$, $\sqrt{\ }$, etc., as well as by different types of braces for fixing the order of operations.

2. Identity

is an equality relation between two algebraic expressions if for arbitrary values of the symbols in them the equality holds.

3. Equation

is an equality relation between two algebraic expressions if the equality holds only for a few values of the symbols. For instance an equality relation

$$F(x) = f(x) \tag{1.27}$$

between two functions with the same independent variable is considered as an *equation with one variable* if it holds only for certain values of the variable. If the equality is valid for every value of x, it is called an identity, or one says the equality holds identically, written as formula $F(x) \equiv f(x)$.

4. Identical Transformations

are performed in order to change an algebraic expression into another one if the two expression are identically equal. The goal is to have another form, e.g., to get a shorter form or a more convenient form for further calculations. Often it is of interest to have the expression in a form which is especially good for solving an equation, or taking the logarithm, or calculating the derivative or integral of it, etc.

1.1.5.2 Algebraic Expressions in Detail

1. Principal Quantities

Principal quantities are those general numbers (literal symbols) occurring in algebraic expressions, according to which the expressions are classified. They must be fixed in any single case. In the case of functions, the independent variables are the principal quantities. The other quantities not given by numbers are the *parameters* of the expression. In some expressions the parameters are called *coefficients*.

■ So-called coefficients occur e.g. in the cases of polynomials, Fourier series, and linear differential equations, etc.

An expression belongs to a certain class depending on which kind of operations are performed on the principal quantities. Usually, the last letters of the alphabet $x, y, z, u, v, \dots$ are used to denote the principal quantities and the first letters $a, b, c, \dots$ are used for parameters. The letters $m, n, p, \dots$ are usually used for positive integer parameter values, e.g. for indices in summations or in iterations.

2. Integral Rational Expressions

are expressions which contain only addition, subtraction, and multiplication of the principal quantities, including powers of them with non-negative integer exponents.

3. Rational Expressions

contain also division by principal quantities, i.e., division by integral rational expressions, so principal quantities can have negative integers in the exponent.

4. Irrational Expressions

contain roots, i.e., non-integer rational powers of integral rational or rational expressions with respect to their principal quantities, of course.

5. Transcendental Expressions

contain exponential, logarithmic or trigonometric expressions of the principal quantities, i.e., there can be irrational numbers in the exponent of an expression of principal quantities, or an expression of principal quantities can be in the exponent, or in the argument of a trigonometric or logarithmic expression.

1.1.6 Integral Rational Expressions

1.1.6.1 Representation in Polynomial Form

Every integral rational expression can be changed into polynomial form by elementary transformations, as in addition, subtraction, and multiplication of monomials and polynomials.

■ $(-a^3 + 2a^2x - x^3)(4a^2 + 8ax) + (a^3x^2 + 2a^2x^3 - 4ax^4) - (a^5 + 4a^3x^2 - 4ax^4)$
$= -4a^5 + 8a^4x - 4a^2x^3 - 8a^4x + 16a^3x^2 - 8ax^4 + a^3x^2 + 2a^2x^3 - 4ax^4 - a^5 - 4a^3x^2 + 4ax^4$
$= -5a^5 + 13a^3x^2 - 2a^2x^3 - 8ax^4.$

1.1.6.2 Factoring Polynomials

Polynomials often can be decomposed into a product of monomials and polynomials. To do so, *factoring out*, *grouping*, special formulas and special properties of equations can be used.

■ **A:** Factoring out: $8ax^2y - 6bx^3y^2 + 4cx^5 = 2x^2(4ay - 3bxy^2 + 2cx^3)$.

■ **B:** Grouping: $6x^2 + xy - y^2 - 10xz - 5yz = 6x^2 + 3xy - 2xy - y^2 - 10xz - 5yz = 3x(2x + y) - y(2x + y) - 5z(2x + y) = (2x + y)(3x - y - 5z)$.

■ **C:** Using the properties of equations (see also 1.6.3.1, p. 43): $P(x) = x^6 - 2x^5 + 4x^4 + 2x^3 - 5x^2$. **a)** Factoring out x^2. **b)** Realizing that $\alpha_1 = 1$ and $\alpha_2 = -1$ are the roots of the equation $P(x) = 0$ and dividing $P(x)$ by $x^2(x-1)(x+1) = x^4 - x^2$ gives the quotient $x^2 - 2x + 5$. This expression can no longer be decomposed into real factors because $p = -2$, $q = 5$, $p^2/4 - q < 0$, so finally the decomposition is $x^6 - 2x^5 + 4x^4 + 2x^3 - 5x^2 = x^2(x-1)(x+1)(x^2 - 2x + 5)$.

1.1.6.3 Special Formulas

$$(x \pm y)^2 = x^2 \pm 2xy + y^2, \tag{1.28}$$

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz, \tag{1.29}$$

$$\begin{aligned}(x + y + z + \cdots + t + u)^2 &= x^2 + y^2 + z^2 + \cdots + t^2 + u^2 + \\ &+ 2xy + 2xz + \cdots + 2xu + 2yz + \cdots + 2yu + \cdots + 2tu,\end{aligned} \tag{1.30}$$

$$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3. \tag{1.31}$$

The calculation of the expression $(x \pm y)^n$ is done by the help of the binomial formula (see (1.36a)–(1.37a)).

$$(x + y)(x - y) = x^2 - y^2, \tag{1.32}$$

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}, \quad \text{(for integer } n\text{, and } n > 1), \tag{1.33}$$

$$\frac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1} \quad \text{(for odd } n\text{, and } n > 1), \tag{1.34}$$

$$\frac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1} \quad \text{(for even } n\text{, and } n > 1). \tag{1.35}$$

1.1.6.4 Binomial Theorem

1. Power of an Algebraic Sum of Two Summands (First Binomial Formula)

The formula

$$\begin{aligned}(a + b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \\ &+ \cdots + \frac{n(n-1)\ldots(n-m+1)}{m!}a^{n-m}b^m + \cdots + nab^{n-1} + b^n\end{aligned} \tag{1.36a}$$

is called the *binomial theorem*, where a and b are real or complex values and $n = 1, 2, \ldots$. Using the *binomial coefficients* delivers a shorter and more convenient notation:

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \tag{1.36b}$$

or

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k. \tag{1.36c}$$

2. Power of an Algebraic Difference (Second Binomial Formula)

$$\begin{aligned}(a - b)^n &= a^n - na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \\ &+ \cdots + (-1)^m \frac{n(n-1)\ldots(n-m+1)}{m!}a^{n-m}b^m + \cdots + (-1)^n b^n\end{aligned} \tag{1.37a}$$

or

$$(a-b)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^{n-k} b^k. \tag{1.37b}$$

3. Binomial Coefficients

The definition is for non-negative and integer n and k:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad (0 \le k \le n), \tag{1.38a}$$

where $n!$ is the product of the positive integers from 1 to n, and it is called n *factorial*:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n, \text{ and by definition } 0! = 1. \tag{1.38b}$$

The binomial coefficients can easily be seen from the *Pascal triangle* in **Table 1.2**. The first and the last number is equal to one in every row; every other coefficient is the sum of the numbers standing on left and on right in the row above it.

Simple calculations verify the following formulas:

$$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}, \tag{1.39a}$$

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{n} = 1. \tag{1.39b}$$

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}. \tag{1.39c}$$

$$\binom{n+1}{k} = \frac{n+1}{n-k+1}\binom{n}{k}. \tag{1.39d}$$

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}. \tag{1.39e}$$

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}. \tag{1.39f}$$

Table 1.2 Pascal's triangle

n	Coefficients												
0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1
	↑		↑		↑		↑		↑		↑		↑
6	$\binom{6}{0}$		$\binom{6}{1}$		$\binom{6}{2}$		$\binom{6}{3}$		$\binom{6}{4}$		$\binom{6}{5}$		$\binom{6}{6}$
⋮	.	.	.	.	.	.	.	.	.	.	.	.	.

For an arbitrary real value α ($\alpha \in \mathbb{R}$) and a non-negative integer k one can define the binomial coefficient $\binom{\alpha}{k}$:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \quad \text{for integer } k \text{ and } k \ge 1, \quad \binom{\alpha}{0} = 1. \tag{1.40}$$

■ $\binom{-\frac{1}{2}}{3} = \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!} = -\frac{5}{16}$.

4. Properties of the Binomial Coefficients

- The binomial coefficients increase until the middle of the binomial formula (1.36b), then decrease.
- The binomial coefficients are equal for the terms standing in symmetric positions with respect to the start and the end of the expression.
- The sum of the binomial coefficients in the binomial formula of degree n is equal to 2^n .
- The sum of the coefficients at the odd positions is equal to the sum of the coefficients at the even positions.

5. Binomial Series

The formula (1.36a) of the binomial theorem can also be extended for negative and fraction exponents. If $|b| < a$, then $(a+b)^n$ has a *convergent infinite series* (see also 21.5, p. 1057):

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots . \tag{1.41}$$

1.1.6.5 Determination of the Greatest Common Divisor of Two Polynomials

It is possible that two polynomials $P(x)$ of degree n and $Q(x)$ of degree m with $n \geq m$ have a common polynomial factor, which contains x. The least common multiple of these factors is the *greatest common divisor* of the polynomials.

■ $P(x) = (x-1)^2(x-2)(x-4)$, $Q(x) = (x-1)(x-2)(x-3)$; the greates common devisor is $(x-1)(x-2)$.

If $P(x)$ and $Q(x)$ do not have any common polynomial factor, they are called *relatively prime* or *coprime*. In this case, their greatest common divisor is a constant.

The greatest common divisor of two polynomials $P(x)$ and $Q(x)$ can be determined by the *Euclidean algorithm* without decomposing them into factors:

1. Division of $P(x)$ by $Q(x) = R_0(x)$ results in the quotient $T_1(x)$ and the remainder $R_1(x)$:

$$P(x) = Q(x)T_1(x) + R_1(x) . \tag{1.42a}$$

2. Division of $Q(x)$ by $R_1(x)$ results in the quotient $T_2(x)$ and the remainder $R_2(x)$:

$$Q(x) = R_1(x)T_2(x) + R_2(x). \tag{1.42b}$$

3. Division of $R_1(x)$ by $R_2(x)$ results in $T_3(x)$ and $R_3(x)$, etc. The greatest common divisor of the two polynomials is the last non-zero remainder $R_k(x)$. This method is known from the arithmetic of natural numbers (see 1.1.1.4, p. 3).

The determination of the greatest common divisor can be used, e. g., when equations musz be solved to separate the roots with higher multiplicity or to apply the Sturm method (see 1.6.3.2, **2.**, p. 44).

1.1.7 Rational Expressions

1.1.7.1 Reducing to the Simplest Form

Every rational expression can be written in the form of a quotient of two coprime polynomials. To do this, only elementary transformations are necessary such as addition, subtraction, multiplication and division of polynomials and fractions and simplification of fractions.

■ Find the most simple form of $\dfrac{3x + \dfrac{2x+y}{z}}{x\left(x^2 + \dfrac{1}{z^2}\right)} - y^2 + \dfrac{x+z}{z}$:

$$\frac{(3xz+2x+y)z^2}{(x^3z^2+x)z} + \frac{-y^2z+x+z}{z} = \frac{3xz^3+2xz^2+yz^2+(x^3z^2+x)(-y^2z+x+z)}{x^3z^3+xz} =$$
$$\frac{3xz^3+2xz^2+yz^2-x^3y^2z^3-xy^2z+x^4z^2+x^2+x^3z^3+xz}{x^3z^3+xz}\,.$$

1.1.7.2 Determination of the Integral Rational Part

A quotient of two polynomials with the same variable x is a *proper fraction* if the degree of the numerator is less than the degree of the denominator. In the opposite case, it is called an *improper fraction.* Every improper fraction can be decomposed into a sum of a proper fraction and a polynomial by dividing the numerator by the denominator, i.e., separating the integral rational part.

■ Determine the integral rational part of $R(x) = \dfrac{3x^4 - 10ax^3 + 22a^2x^2 - 24a^3x + 10a^4}{x^2 - 2ax + 3a^2}$:

$$(3x^4 - 10ax^3 + 22a^2x^2 - 24a^3x + 10a^4) : (x^2 - 2ax + 3a^2) = 3x^2 - 4ax + 5a^2 + \frac{-2a^3x - 5a^4}{x^2 - 2ax - 3a^2}$$

$$\begin{array}{l}
3x^4 - 6ax^3 + 9a^2x^2 \\
\hline
\quad - 4ax^3 + 13a^2x^2 - 24a^3x \\
\quad - 4ax^3 + 8a^2x^2 - 12a^3x \\
\hline
\qquad\quad 5a^2x^2 - 12a^3x + 10a^4 \\
\qquad\quad 5a^2x^2 - 10a^3x + 15a^4 \\
\hline
\qquad\qquad - 2a^3x - 5a^4.
\end{array}$$

$$R(x) = 3x^2 - 4ax + 5a^2 + \frac{-2a^3x - 5a^4}{x^2 - 2ax + 3a^2}.$$

The integral rational part of a rational function $R(x)$ is considered to be as an *asymptotic approximation* for $R(x)$ because for large values of $|x|$, the value of the proper fraction part tends to zero, and $R(x)$ behaves as its polynomial part.

1.1.7.3 Partial Fraction Decomposition

Every proper rational fraction

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0} \quad (n < m) \tag{1.43}$$

with coprime polynomials in the numerator and denominator can be uniquely decomposed into a sum of partial fractions. The coefficients $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$ are real or complex numbers. The partial fractions have the form

$$\frac{A}{(x-\alpha)^k} \quad (1.44a) \qquad \text{and} \qquad \frac{Dx+E}{(x^2+px+q)^m} \quad \text{where} \quad \left(\frac{p}{2}\right)^2 - q < 0\,. \quad (1.44b)$$

In the followings real coefficients are assumed in $R(x)$ in (1.43).
First the leading coefficient b_m of the denominator $Q(x)$ is transformed into 1 by dividing the numerator and the denominator of (1.43) by the original value of b_m. In the case of real coefficients the following three cases are to be distinguished.
In the case of complex coefficients in $R(x)$ only the first two cases can occur, since complex polynomials can be factorized into a product of first degree polynomials. Every proper rational fraction $R(x)$ can be expanded into a sum of fractions of the form (1.44a), where A and α are complex numbers.

1. Partial Fraction Decomposition, Case 1

The denominator $Q(x)$ has m different simple roots $\alpha_1, \ldots, \alpha_m$. Then the expansion has the form

$$\frac{P(x)}{Q(x)} = \frac{a_nx^n + \cdots + a_0}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_m)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_m}{x-\alpha_m} \tag{1.45a}$$

with coefficients

$$A_1 = \frac{P(\alpha_1)}{Q'(\alpha_1)}, \quad A_2 = \frac{P(\alpha_2)}{Q'(\alpha_2)}, \quad \dots, \quad A_m = \frac{P(\alpha_m)}{Q'(\alpha_m)}, \tag{1.45b}$$

where in the numerators of (1.45b) the values of the derivative $\frac{dQ}{dx}$ are taken for $x = \alpha_1,\ x = \alpha_2, \dots$.

■ $\frac{6x^2 - x + 1}{x^3 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$, $\alpha_1 = 0$, $\alpha_2 = +1$ and $\alpha_3 = -1$;
$P(x) = 6x^2 - x + 1$, $Q'(x) = 3x^2 - 1$, $A = \frac{P(0)}{Q'(0)} = -1$, $B = \frac{P(1)}{Q'(1)} = 3$ and $C = \frac{P(-1)}{Q'(-1)} = 4$,
$\frac{P(x)}{Q(x)} = -\frac{1}{x} + \frac{3}{x-1} + \frac{4}{x+1}$.
An other possibility to determine the coefficients $A_1,\ A_2, \dots, A_m$ is the method of comparing coefficients (see **4.**, p. 17).

2. Partial Fraction Decomposition, Case 2

The denominator $Q(x)$ has l multiple real roots $\alpha_1, \alpha_2, \dots, \alpha_l$ with multiplicities $k_1, k_2, \dots, k_l$ respectively. Then the decomposition has the form

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{(x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\dots(x-\alpha_l)^{k_l}} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{(x-\alpha_1)^2} + \dots + \frac{A_{k_1}}{(x-\alpha_1)^{k_1}} \\ &\quad + \frac{B_1}{x-\alpha_2} + \frac{B_2}{(x-\alpha_2)^2} + \dots + \frac{B_{k_2}}{(x-\alpha_2)^{k_2}} + \dots + \frac{L_{k_l}}{(x-\alpha_l)^{k_l}}.\end{aligned} \tag{1.46}$$

■ $\frac{x+1}{x(x-1)^3} = \frac{A_1}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3}$. The coefficients A_1, B_1, B_2, B_3 can be determined by the method of comparing coefficients.

3. Partial Fraction Decomposition, Case 3

If the denominator $Q(x)$ has also complex roots, then its factorization is

$$\begin{aligned}Q(x) &= (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x-\alpha_l)^{k_l} \\ &\quad \cdot(x^2 + p_1x + q_1)^{m_1}(x^2 + 2p_2x + q_2)^{m_2}\cdots(x^2 + p_rx + q_r)^{m_r}\end{aligned} \tag{1.47}$$

according to (1.168), p. 44. Here $\alpha_1, \alpha_2, \dots, \alpha_l$ are the l real roots of polynomial $Q(x)$. Beside these roots $Q(x)$ has r complex conjugate pairs of roots, which are the roots of the quadratic factors $x^2 - p_ix + q_i$ (i =1,2,…,r). The numbers p_i, q_i are real, and $\left(\frac{p_i}{2}\right)^2 - q_i < 0$ holds. In this case the partial fraction decomposition has the form

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0}{(x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x^2+p_1x+q_1)^{m_1}(x^2+p_2x+q_2)^{m_2}\cdots} \\ &= \frac{A_1}{x-\alpha_1} + \frac{A_2}{(x-\alpha_1)^2} + \dots + \frac{A_{k_1}}{(x-\alpha_1)^{k_1}} + \frac{B_1}{x-\alpha_2} + \frac{B_2}{(x-\alpha_2)^2} + \dots + \frac{B_{k_2}}{(x-\alpha_2)^{k_2}} + \cdots \\ &\quad + \frac{C_1x + D_1}{x^2+p_1x+q_1} + \frac{C_2x + D_2}{(x^2+p_1x+q_1)^2} + \dots + \frac{C_{m_1}x + D_{m_1}}{(x^2+p_1x+q_1)^{m_1}} + \\ &\quad + \frac{E_1x + F_1}{x^2+p_2x+q_2} + \frac{E_2x + F_2}{(x^2+p_2x+q_2)^2} + \dots + \frac{E_{m_2}x + F_{m_2}}{(x^2+p_2x+q_2)^{m_2}} + \cdots.\end{aligned} \tag{1.48}$$

■ $\frac{5x^2 - 4x + 16}{(x-3)(x^2-x+1)^2} = \frac{A}{x-3} + \frac{C_1x + D_1}{x^2-x+1} + \frac{C_2x + D_2}{(x^2-x+1)^2}$. The coefficients A, C_1, D_1, C_2, D_2 are to be determined by the method of comparing coefficients.

4. Method of Comparing Coefficients
In order to determine the coefficients $A_1, A_2, \ldots, E_1, F_1 \ldots$ in (1.48) the expression (1.48) has to be multiplied by $Q(x)$, then the result $Z(x)$ is compared with $P(x)$, since $Z(x) \equiv P(x)$. After ordering $Z(x)$ by the powers of x, one gets a system of equations by comparing the coefficients of the corresponding x–powers in $Z(x)$ and $P(x)$. This method is called the *method of comparing coefficients* or *method of undetermined coefficients*.

■ $\dfrac{6x^2 - x + 1}{x^3 - x} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1} = \dfrac{A(x^2-1) + Bx(x+1) + Cx(x-1)}{x(x^2-1)}$.
Comparing the coefficients of the same powers of x, one gets the system of equations $6 = A + B + C$, $-1 = B - C$, $1 = -A$, and its solutions are $A = -1, B = 3, C = 4$.

1.1.7.4 Transformations of Proportions

The equality

$$\frac{a}{b} = \frac{c}{d} \quad (1.49a) \qquad \text{yields} \qquad ad = bc, \quad \frac{a}{c} = \frac{b}{d}, \quad \frac{d}{b} = \frac{c}{a}, \quad \frac{b}{a} = \frac{d}{c} \quad (1.49b)$$

and furthermore

$$\frac{a \pm b}{b} = \frac{c \pm d}{d}, \quad \frac{a \pm b}{a} = \frac{c \pm d}{c}, \quad \frac{a \pm c}{c} = \frac{b \pm d}{d}, \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}. \quad (1.49c)$$

From the equalities of the proportions

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} \quad (1.50a) \qquad \text{it follows that} \qquad \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = \frac{a_1}{b_1}. \quad (1.50b)$$

1.1.8 Irrational Expressions

Every irrational expression can be written in a simpler form by 1. simplifying the exponent, 2. taking out terms from the radical sign and 3. moving the irrationality into the numerator.

1. Simplifying the Exponent The exponent can be simplified if the radicand can be factorized and the index of the radical and the exponents in the radicand have a common factor; the index of the radical and the exponents must be divided by their greatest common divisor.

■ $\sqrt[6]{16(x^{12} - 2x^{11} + x^{10})} = \sqrt[6]{4^2 \cdot x^{5\cdot 2}(x-1)^2} = \sqrt[3]{4x^5(x-1)}$.

2. Moving the Irrationality There are different ways to *move the irrationality* into the numerator.

■ **A:** $\sqrt{\dfrac{x}{2y}} = \sqrt{\dfrac{2xy}{4y^2}} = \dfrac{\sqrt{2xy}}{2y}$. ■ **B:** $\sqrt[3]{\dfrac{x}{4yz^2}} = \sqrt[3]{\dfrac{2xy^2z}{8y^3z^3}} = \dfrac{\sqrt[3]{2xy^2z}}{2yz}$.

■ **C:** $\dfrac{1}{x + \sqrt{y}} = \dfrac{x - \sqrt{y}}{\left(x + \sqrt{y}\right)\left(x - \sqrt{y}\right)} = \dfrac{x - \sqrt{y}}{x^2 - y}$.

■ **D:** $\dfrac{1}{x + \sqrt[3]{y}} = \dfrac{x^2 - x\sqrt[3]{y} + \sqrt[3]{y^2}}{\left(x + \sqrt[3]{y}\right)\left(x^2 - x\sqrt[3]{y} + \sqrt[3]{y^2}\right)} = \dfrac{x^2 - x\sqrt[3]{y} + \sqrt[3]{y^2}}{x^3 + y}$.

3. Simplest Forms of Powers and Radicals Also powers and radicals can be transformed into the simplest form.

■ **A:** $\sqrt[4]{\dfrac{81x^6}{(\sqrt{2} - \sqrt{x})^4}} = \sqrt{\dfrac{9x^3}{(\sqrt{2} - \sqrt{x})^2}} = \dfrac{3x\sqrt{x}}{\sqrt{2} - \sqrt{x}} = \dfrac{3x\sqrt{x}(\sqrt{2} + \sqrt{x})}{2 - x} = \dfrac{3x\sqrt{2x} + 3x^2}{2 - x}$.

■ **B:** $\left(\sqrt{x} + \sqrt[3]{x^2} + \sqrt[4]{x^3} + \sqrt[12]{x^7}\right)\left(\sqrt{x} - \sqrt[3]{x} + \sqrt[4]{x} - \sqrt[12]{x^5}\right) = (x^{1/2} + x^{2/3} + x^{3/4} + x^{7/12})(x^{1/2} - x^{1/3} + x^{1/4} - x^{5/12}) = x + x^{7/6} + x^{5/4} + x^{13/12} - x^{5/6} - x - x^{13/12} - x^{11/12} + x^{3/4} + x^{11/12} + x + x^{5/6} - x^{11/12} - x^{13/12} -$

$x^{7/6}-x=x^{5/4}-x^{13/12}-x^{11/12}+x^{3/4}=\sqrt[4]{x^5}-\sqrt[12]{x^{13}}-\sqrt[12]{x^{11}}+\sqrt[4]{x^3}=x^{3/4}(1-x^{1/6}-x^{1/3}+x^{1/2})=\sqrt[4]{x^3}\,(1-\sqrt[6]{x}-\sqrt[3]{x}+\sqrt{x})$.

1.2 Finite Series

1.2.1 Definition of a Finite Series

The sum

$$s_n=a_0+a_1+a_2+\cdots+a_n=\sum_{i=0}^{n}a_i, \tag{1.51}$$

is called a *finite series*. The summands a_i $(i=0,1,2,\ldots,n)$ are given by certain formulas, they are numbers, and they are the *terms of the series*.

1.2.2 Arithmetic Series

1. Arithmetic Series of First Order

is a finite series where the terms form an *arithmetic sequence*, i.e., the difference of two terms standing after each other is a constant:

$$\Delta a_i=a_{i+1}-a_i=d=\text{const}\quad\text{holds,}\quad\text{so}\quad a_i=a_0+id. \tag{1.52a}$$

Thus holds:

$$s_n=a_0+(a_0+d)+(a_0+2d)+\cdots+(a_0+nd) \tag{1.52b}$$

$$s_n=\frac{a_0+a_n}{2}(n+1)=\frac{n+1}{2}(2a_0+nd). \tag{1.52c}$$

2. Arithmetic Series of k-th Order

is a finite series, where the k-th differences $\Delta^k a_i$ of the sequence a_0, a_1, a_2, $\ldots$,a_n are constants. The differences of higher order are calculated by the formula

$$\Delta^\nu a_i=\Delta^{\nu-1}a_{i+1}-\Delta^{\nu-1}a_i\quad(\nu=2,3,\ldots,k). \tag{1.53a}$$

It is convenient to calculate them from the *difference schema* (also difference table or triangle schema):

$$\begin{array}{llllllll}
a_0 & & & & & & & \\
 & \Delta a_0 & & & & & & \\
a_1 & & \Delta^2 a_0 & & & & & \\
 & \Delta a_1 & & \Delta^3 a_0 & & & & \\
a_2 & & \Delta^2 a_1 & & \ddots & \Delta^k a_0 & & \\
 & \Delta a_2 & & \Delta^3 a_1 & & & & \\
a_3 & & \Delta^2 a_2 & & \ddots & \Delta^k a_1 & \ddots & \\
 & \vdots & & \vdots & & \vdots & & \Delta^n a_0 \\
\vdots & & \vdots & & & \Delta^k a_{n-k} & \cdot^{\cdot^{\cdot}} & \\
 & & & \Delta^3 a_{n-3} & \cdot^{\cdot^{\cdot}} & & & \\
 & & \Delta^2 a_{n-2} & & & & & \\
 & \Delta a_{n-1} & & & & & & \\
a_n & & & & & & &
\end{array} \tag{1.53b}$$

The following formulas hold for the terms and the sum:

$$a_i=a_0+\binom{i}{1}\Delta a_0+\binom{i}{2}\Delta^2 a_0+\cdots+\binom{i}{k}\Delta^k a_0\quad(i=1,2,\ldots,n), \tag{1.53c}$$

$$s_n=\binom{n+1}{1}a_0+\binom{n+1}{2}\Delta a_0+\binom{n+1}{3}\Delta^2 a_0+\cdots+\binom{n+1}{k+1}\Delta^k a_0. \tag{1.53d}$$

1.2.3 Geometric Series

The sum (1.51) is called a *geometric series*, if the terms form a *geometric sequence*, i.e., the ratio of two successive terms is a constant:

$$\frac{a_{i+1}}{a_i} = q = \text{const} \quad \text{holds,} \quad \text{so} \quad a_i = a_0 q^i. \tag{1.54a}$$

Thus holds:

$$s_n = a_0 + a_0 q + a_0 q^2 + \cdots + a_0 q^n = a_0 \frac{q^{n+1} - 1}{q - 1} \quad \text{for} \quad q \neq 1, \tag{1.54b}$$

$$s_n = (n+1)a_0 \quad \text{for} \quad q = 1. \tag{1.54c}$$

For $n \to \infty$ (see 7.2.1.1, **2.**, p. 459), there is an *infinite geometric series*, which has a limit if $|q| < 1$, and this limit is called sum s:

$$s = \frac{a_0}{1 - q}\,. \tag{1.54d}$$

1.2.4 Special Finite Series

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}\,, \tag{1.55}$$

$$p + (p+1) + (p+2) + \cdots + (p+n) = \frac{(n+1)(2p+n)}{2}\,, \tag{1.56}$$

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2\,, \tag{1.57}$$

$$2 + 4 + 6 + \cdots + (2n-2) + 2n = n(n+1)\,, \tag{1.58}$$

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}\,, \tag{1.59}$$

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}\,, \tag{1.60}$$

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}\,, \tag{1.61}$$

$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)\,, \tag{1.62}$$

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}\,, \tag{1.63}$$

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} \quad (x \neq 1)\,. \tag{1.64}$$

1.2.5 Mean Values

(See also 16.3.4.1, **1.**, p. 839 and 16.4, p. 848)

1.2.5.1 Arithmetic Mean or Arithmetic Average

The arithmetic mean of the n quantities $a_1, a_2, \ldots, a_n$ is the expression

$$x_A = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k\,. \tag{1.65a}$$

For two values a and b holds:

$$x_A = \frac{a+b}{2}\,. \tag{1.65b}$$

The values a , x_A and b form an arithmetic sequence.

1.2.5.2 Geometric Mean or Geometric Average

The geometric mean of n positive quantities $a_1, a_2, \ldots, a_n$ is the expression

$$x_G = \sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}}. \tag{1.66a}$$

For two positive values a and b holds

$$x_G = \sqrt{ab}\,. \tag{1.66b}$$

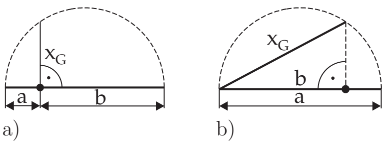

Figure 1.3

The values a, x_G and b form a geometric sequence. If a and b are given line segments, then a segment with length $x_G = \sqrt{ab}$ can be given by the help of one of the constructions shown in **Fig. 1.3a** or in **Fig. 1.3b**.
A special case of the geometric mean is given by dividing a line segment according to the *golden section* (see 3.5.2.3, **3.**, p. 194).

1.2.5.3 Harmonic Mean

The harmonic mean of n quantities $a_1, a_2, \ldots, a_n$ $(a_i \neq 0; i = 1, 2, \ldots, n)$ is the expression

$$x_H = \left[\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\right]^{-1} = \left[\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right]^{-1}. \tag{1.67a}$$

For two values a and b holds

$$x_H = \left[\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)\right]^{-1}, \quad x_H = \frac{2ab}{a+b}. \tag{1.67b}$$

1.2.5.4 Quadratic Mean

The *quadratic mean* of n quantities a_1, a_2,. . . , a_n is the expression

$$x_Q = \sqrt{\frac{1}{n}({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)} = \sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2}\,. \tag{1.68a}$$

For two values a and b holds

$$x_Q = \sqrt{\frac{a^2 + b^2}{2}}\,. \tag{1.68b}$$

The quadratic mean is important in the theory of observational error (see 16.4, p. 848).

1.2.5.5 Relations Between the Means of Two Positive Values

For $x_A = \dfrac{a+b}{2}$, $x_G = \sqrt{ab}$, $x_H = \dfrac{2ab}{a+b}$, $x_Q = \sqrt{\dfrac{a^2+b^2}{2}}$ we have

1. if $\boldsymbol{a < b}$, then

$$a < x_H < x_G < x_A < x_Q < b\,, \tag{1.69a}$$

2. if $\boldsymbol{a = b}$, then

$$a = x_A = x_G = x_H = x_Q = b\,. \tag{1.69b}$$

1.3 Business Mathematics

Business calculations are based on the use of arithmetic and geometric series, on formulas (1.52a)–(1.52c) and (1.54a)–(1.54d). However these applications in banking are so varied and special that a special discipline has developed using specific terminology. So business arithmetic is not confined only to the calculation of the principal by compound interest or the calculation of annuities. It also includes the calculation of interest, repayments, amortization, calculation of instalment payments, annuities, depreciation, effective interest yield and the yield on investment. Basic concepts and formulas for calculations are discussed below. For studying financial mathematics in detail, you will have to consult the relevant literature on the subject (see [1.2], [1.8]).

Insurance mathematics and *risk theory* use the methods of probability theory and mathematical statistics, and they represent a separate discipline, so they don't be discussed here (see [1.4], [1.5]).

1.3.1 Calculation of Interest or Percentage

1.3.1.1 Percentage or Interest

The expression *p percent of K* means $\frac{p}{100}K$, where K denotes the *principal* in business mathematics. The symbol for percent is %, i.e., the following equalities hold:

$$p\% = \frac{p}{100} \qquad \text{or} \qquad 1\% = 0.01. \tag{1.70}$$

1.3.1.2 Increment

If K is raised by $p\%$, the increased value is

$$\tilde{K} = K\left(1 + \frac{p}{100}\right). \tag{1.71}$$

Relating the *increment* $K\frac{p}{100}$ to the new value $\tilde{K}$, the proportion is $K\frac{p}{100} : \tilde{K} = \tilde{p} : 100$, so $\tilde{K}$ contains

$$\tilde{p} = \frac{p \cdot 100}{100 + p} \tag{1.72}$$

percent of increment.

■ If an article has a value of € 200 and a 15% extra charge is added, the final value is € 230. This price contains $\tilde{p} = \dfrac{15 \cdot 100}{115} = 13.04$ percent increment for the user.

1.3.1.3 Discount or Reduction

Reducing the value K by $p\%$ *rebate* yields the reduced value

$$\tilde{K} = K\left(1 - \frac{p}{100}\right). \tag{1.73}$$

Comparing the reduction $K\frac{p}{100}$ to the new value $\tilde{K}$ gives

$$\tilde{p} = \frac{p \cdot 100}{100 - p} \tag{1.74}$$

percent of rebate.

■ If an article has a value € 300, and they give a 10% discount, it will be sold for € 270. This price contains $\tilde{p} = \dfrac{10 \cdot 100}{90} = 11.11$ percent rebate for the buyer.

1.3.2 Calculation of Compound Interest

1.3.2.1 Interest

Interest is either payment for the use of a loan or it is a revenue realized from a receivable. For a principal K, placed for a whole *period of interest* (usually one year),

$$K\frac{p}{100} \tag{1.75}$$

interest is paid at the end of the period of interest. Here p is the *rate of interest for the period of interest*, and one says that $p\%$ interest is paid for the principal K.

1.3.2.2 Compound Interest

Compound interest is computed on the principal and on any interest earned that has not been paid or withdrawn. It is the return on the principal for two or more time periods. The interest of the principal increased by interest is called *compound interest*.
In the following different cases are discussed which depend on how the principal is changing.

1. Single Deposit

Compounded annually the principal K increases after n years up to the final value K_n. At the end of the n-th year this value is:

$$K_n = K\left(1+\frac{p}{100}\right)^n. \tag{1.76}$$

For a briefer notation the substitution $1+\dfrac{p}{100} = q$ is used and q is called the *accumulation factor* or *growth factor*.
Interest may be compounded for any period of time: annually, half-annually, monthly, daily, and so on. Dividing the year into m equal interest periods the interest will be added to the principal K at the end of every period. Then the interest is $K\dfrac{p}{100m}$ for one interest period, and the principal increases after n years with m interest periods up to the value

$$K_{m\cdot n} = K\left(1+\frac{p}{100m}\right)^{m\cdot n}. \tag{1.77}$$

The quantity $\left(1+\dfrac{p}{100}\right)$ is known as the *nominal rate*, and $\left(1+\dfrac{p}{100m}\right)^m$ as the *effective rate.*

■ A principal of € 5000, with a nominal interest 7.2% annually, increases within 6 years **a)** compounded annually to $K_6 = 5000(1+0.072)^6 = €\ 7588.20$, **b)** compounded monthly to $K_{72} = 5000(1+0.072/12)^{72} = €\ 7691.74$.

2. Regular Deposits

Suppose depositing the same amount E in equal intervals. Such an interval must be equal to an interest period. The depositions can be made at the beginning of the interval, or at the end of the interval. At the end of the n-th interest period the balance K_n is

a) Depositing at the Beginning:

$$K_n = Eq\frac{q^n-1}{q-1}. \tag{1.78a}$$

b) Depositing at the End:

$$K_n = E\frac{q^n-1}{q-1}. \tag{1.78b}$$

3. Depositing in the Course of the Year

A year or an interest period is divided into m equal parts. At the beginning or at the end of each of these time periods the same amount E is deposited and bears interest until the end of the year. In this way, after one year the balance K_1 is

a) Depositing at the Beginning:

$$K_1 = E\left[m + \frac{(m+1)p}{200}\right]. \tag{1.79a}$$

b) Depositing at the End:

$$K_1 = E\left[m + \frac{(m-1)p}{200}\right]. \tag{1.79b}$$

In the second year the total K_1 bears interest, and further deposits and interests are added like in the first year, so after n years the balance K_n for midterm deposits and yearly interest payment is:

a) Depositing at the Beginning:

$$K_n = E\left[m + \frac{(m+1)p}{200}\right]\frac{q^n-1}{q-1}. \tag{1.80a}$$

b) Depositing at the End:

$$K_n = E\left[m + \frac{(m-1)p}{200}\right]\frac{q^n-1}{q-1}. \tag{1.80b}$$

■ At a yearly rate of interest $p = 5.2\%$ a depositor deposits € 1000 at the end of every month. After how many years will it reach the balance € 500 000?

From (1.80b), for instance, from $500\,000 = 1000\left[12 + \frac{11\cdot 5.2}{200}\right]\cdot\frac{1.052^n - 1}{0.052}$, follows the answer, $n =$ 22.42 years.

1.3.3 Amortization Calculus

1.3.3.1 Amortization

Amortization is the repayment of credits. The assumptions:
1. For a *debt* S the debtor is charged at $p\%$ interest at the end of an interest period.
2. After N interest period the debt is completely repaid.

The charge of the debtor consists of interest and principal repayment for every interest period. If the interest period is one year, the amount to be paid during the whole year is called an *annuity*.

There are different possibilities for a debtor. For instance, the repayments can be made at the interest date, or meanwhile; the amount of repayment can be different time by time, or it can be constant during the whole term.

1.3.3.2 Equal Principal Repayments

The amortization instalments are paid during the year, but no midterm compound interest is calculated. The following notation should be used:
• S debt (interest payment at the end of a period with $p\%$),
• $T = \dfrac{S}{mN}$ principal repayment ($T = \text{const}$),
• m number of repayments during one interest period,
• N number of interest periods until the debt is fully repaid.
Besides the principal repayments the debtor also has to pay the interest charges:

a) Interest Z_n for the n-th Interest Period:

$$Z_n = \frac{pS}{100}\left[1 - \frac{1}{N}\left(n - \frac{m+1}{2m}\right)\right]. \tag{1.81a}$$

b) Total Interest Z to be Paid for a Debt S, mN Times, During N Interest Periods with an Interest Rate $p\%$:

$$Z = \sum_{n=1}^{N} Z_n = \frac{pS}{100}\left[\frac{N-1}{2} + \frac{m+1}{2m}\right]. \tag{1.81b}$$

■ A debt of € 60 000 has a yearly interest rate of 8%. The principal repayment of € 1000 for 60 months should be paid at the end of the months. How much is the actual interest at the end of each year? The interest for every year is calculated by (1.81a) with $S = 60000$, $p = 8$, $N = 5$ and $m = 12$. They are enumerated in the annexed table.

1. year:	$Z_1 =$	€ 4360
2. year:	$Z_2 =$	€ 3400
3. year:	$Z_3 =$	€ 2440
4. year:	$Z_4 =$	€ 1480
5. year:	$Z_5 =$	€ 520
	$Z =$	€ 12200

The total interest can be calculated also by (1.81b) as $Z = \frac{8 \cdot 60000}{100}\left[\frac{5-1}{2} + \frac{13}{24}\right] = € \, 12\,200$.

1.3.3.3 Equal Annuities

For equal principal repayments $T = \frac{S}{mN}$ the interest payable decreases over the course of time (see the previous example). In contrast to this, in the case of equal *annuities* the same amount is repaid for every interest period. A constant annuity A containing the principal repayment and the interest is repaid, i.e., the charge of the debtor is constant during the whole period of repayment.
With the notation
- S debt (interest payment of $p\%$ at the end of a period),
- A annuity for every interest period (A const),
- a one instalment paid m times per interest period (a const),
- $q = 1 + \frac{p}{100}$ the accumulation factor,

after n interest periods the remaining outstanding debt S_n is:

$$S_n = S\,q^n - a\left[m + \frac{(m-1)p}{200}\right]\frac{q^n-1}{q-1}. \tag{1.82}$$

Here the term Sq^n denotes the value of the debt S after n interest periods with compound interest (see (1.76)). The second term in (1.82) gives the value of the midterm repayments a with compound interest (see (1.80b) with $E = a$). For the annuity holds

$$A = a\left[m + \frac{(m-1)p}{200}\right]. \tag{1.83}$$

Here paying A once means the same as paying a m times. From (1.83) it follows that $A \geq ma$. Because after N interest periods the debt must be completely repaid, from (1.82) for $S_N = 0$ considering (1.83) for the annuity holds:

$$A = S\,q^N\frac{q-1}{q^N-1} = S\frac{q-1}{1-q^{-N}}. \tag{1.84}$$

To solve a problem of business mathematics, from (1.84), any of the quantities A, S, q or N can be expressed, if the others are known.

■ **A:** A loan of € 60 000 bears 8% interest per year, and is to be repaid over 5 years in equal instalments. How much is the yearly annuity A and the monthly instalment a? From (1.84) and (1.83) we get: $A = 60\,000\dfrac{0.08}{1-\dfrac{1}{1.08^5}} = € \, 15\,027.39$, $a = \dfrac{15027.39}{12 + \dfrac{11 \cdot 8}{200}} = € \, 1207.99$.

■ **B:** A loan of $S =$ € 100 000 is to be repaid during $N = 8$ years in equal annuities with an interest rate of 7.5%. At the end of every year € 5000 extra repayment must be made. How much will the monthly instalment be? For the annuity A per year according to (1.84) follows $A = 100\,000\dfrac{0.075}{1-\dfrac{1}{1.075^8}} =$ € 17 072.70. Because A consists of 12 monthly instalments a, and because of the € 5000 extra payment at the end of the year, from (1.83) $A = a\left[12 + \dfrac{11 \cdot 7.5}{200}\right] + 5000 = 17\,072.70$ holds, so the monthly charge is $a =$ € 972.62.

1.3.4 Annuity Calculations

1.3.4.1 Annuities

If a series of payments is made regularly at the same time intervals, in equal or varying amounts, at the beginning or at the end of the interval, it is called *annuity payments*. To distinguish are:

a) Payments on an Account The periodic payments, called *rents*, are paid on an account and bear compound interest. Therefore the formulas of 1.3.2 are to be used.

b) Receipt of Payments The payments of rent are made from capital bearing compound interest. Here the formulas of the annuity calculations in 1.3.3 are to be used, where the annuities are called rents. If no more than the actual interest is paid as a rent, it is called a *perpetual annuity*.

Rent payments (deposits and payoffs) can be made at the interest terms, or at shorter intervals during the period of interest, i.e. in the course of the year.

1.3.4.2 Future Amount of an Ordinary Annuity

The date of the interest calculations and the payments should coincide. The interest is calculated at $p\%$ compound interest, and the payments (rents) on the account are always the same, R. The *future value of the ordinary annuity* R_n, i.e., the amount to which the regular deposits increase after n periods amounts to:

$$R_n = R\frac{q^n - 1}{q - 1} \quad \text{with} \quad q = 1 + \frac{p}{100}\,. \tag{1.85}$$

The *present value of an ordinary annuity* R_0 is the amount which should be paid at the beginning of the first interest period (one time) to reach the final value R_n with compound interest during n periods:

$$R_0 = \frac{R_n}{q^n} \qquad \text{with} \quad q = 1 + \frac{p}{100}\,. \tag{1.86}$$

■ A man claims € 5000 at the end of every year for 10 years from a firm. Before the first payment the firm declares bankruptcy. Only the present value of the ordinary annuity R_0 can be asked from the administration of the bankrupt's estate. With an interest of 4% per year the man gets:

$$R_0 = \frac{1}{q^n}R\frac{q^n - 1}{q - 1} = R\frac{1 - q^{-n}}{q - 1} = 5000\frac{1 - 1.04^{-10}}{0.04} = € \; 40\,554.48.$$

1.3.4.3 Balance after n Annuity Payments

For ordinary annuity payments capital K is at our disposal bearing $p\%$ interest. After every interest period an amount r is paid. The balance K_n after n interest periods, i.e., after n rent payments, is:

$$K_n = Kq^n - R_n = Kq^n - r\frac{q^n - 1}{q - 1} \quad \text{with} \quad q = 1 + \frac{p}{100}\,. \tag{1.87a}$$

Conclusions from (1.87a):

$$r = K\frac{p}{100} \tag{1.87b}$$

Consequently $K_n = K$ holds, so the capital does not change. This is the case of *perpetual annuity*.

$$r > K\frac{p}{100} \tag{1.87c}$$

The capital will be completely used up after N rent payments. From (1.87a) it follows for $K_N = 0$:

$$K = \frac{r}{q^N}\frac{q^N - 1}{q - 1}\,. \tag{1.87d}$$

If midterm interest is calculated and midterm rents are paid, and the original interest period is divided into m equal intervals, then in the formulas (1.85)–(1.87a) n is replaced by mn and accordingly $q = 1 + \frac{p}{100}$ by $q = 1 + \frac{p}{100m}$.

■ What amount must be deposited monthly at the end of the month for 20 years, from which a rent of € 2000 should be paid monthly for 20 years, and the interest period is one month with an interest rate of 0.5%.
From (1.87d) follows for $n = 20 \cdot 12 = 240$ the sum K which is necessary for the required payments: $K = \frac{2000}{1.005^{240}} \frac{1.005^{240} - 1}{0.005} =$ € 279 161.54. The necessary monthly deposits R are given by (1.85): $R_{240} = 279\,161.54 = R\frac{1.005^{240} - 1}{0.005}$, i.e., $R =$ € 604.19.

1.3.5 Depreciation

1.3.5.1 Methods of Depreciation

Depreciation is the term most often used to indicate that assets have declined in service potential in a given year either due to obsolescence or physical factors. Depreciation is a method whereby the *original (cost) value* at the beginning of the reporting year is reduced to the *residual value* at year-end. The following concepts are used:
- A depreciation base,
- N useful life (given in years),
- R_n residual value after n years ($n \le N$),
- a_n $(n = 1, 2, \ldots, N)$ depreciation rate in the n-th year.

The methods of depreciation differ from each other depending on the *amortization rate*:
- *straight-line method*, i.e., equal yearly rates,
- *decreasing-charge method*, i.e., decreasing yearly rates.

1.3.5.2 Straight-Line Method

The yearly depreciations are constant, i.e., for amortization rates a_n and the remaining value R_n after n years follows:

$$a_n = \frac{A - R_N}{N} = a, \qquad (1.88) \qquad\qquad R_n = A - n\frac{A - R_N}{N} \quad (n = 1, 2, \ldots, N). \quad (1.89)$$

Substituting $R_N = 0$, then the value of the given thing is reduced to zero after N years, i.e., it is totally depreciated.

■ The purchase price of a machine is $A =$ € 50 000. In 5 years it should be depreciated to a value $R_5 =$ € 10 000.
Linear depreciation according to (1.88) and (1.89) yields the annexed *amortization schedule*:

Year	Depreciation base	Depreciation expense	Residual value	Cumulated depr. in % of the depr. base
1	50 000	8000	42 000	16.0
2	42 000	8000	34 000	19.0
3	34 000	8000	26 000	23.5
4	26 000	8000	18 000	30.8
5	18 000	8000	10 000	44.4

It shows that the percentage of accumulated depreciation with respect to the actual initial value is increasing.

1.3.5.3 Arithmetically Declining Balance Depreciation

In this case the depreciation is not constant. It is decreasing yearly by the same amount d, by the so-called *multiple*. For depreciation in the n-th year follows:

$$a_n = a_1 - (n-1)d \quad (n = 2, 3, \ldots, N+1;\ a_1 \text{ and } d \text{ are given}). \qquad (1.90)$$

Considering the equality $A - R_N = \sum\limits_{n=1}^{N} a_n$ from the previous equation it follows that:

$$d = \frac{2[Na_1 - (A - R_N)]}{N(N-1)}. \qquad (1.91)$$

For $d = 0$ follows the special case of straight-line depreciation. If $d > 0$, it follows from (1.91) that

$$a_1 > \frac{A - R_N}{N} = a, \tag{1.92}$$

where a is the depreciation rate for straight-line depreciation. The first depreciation rate a_1 of the arithmetically-declining balance depreciation must satisfy the following inequality:

$$\frac{A - R_N}{N} < a_1 < 2\frac{A - R_N}{N}. \tag{1.93}$$

■ A machine of € 50 000 purchase price is to be depreciated to the value of € 10 000 within 5 years by arithmetically declining depreciation. In the first year € 15 000 should be depreciated.
The annexed depreciation schedule is calculated by the given formulas, and it shows that with the exception of the last rate the percentage of depreciation is fairly equal.

Year	Depretiation base	Depreciation expense	Residual value	Depreciation in % of depr. base
1	50 000	15 000	35 000	30.0
2	35 000	11 500	23 500	32.9
3	23 500	8 000	15 500	34.0
4	15 500	4 500	11 000	29.0
5	11 000	1 000	10 000	9.1

1.3.5.4 Digital Declining Balance Depreciation

Digital depreciation is a special case of arithmetically declining depreciation. Here it is required that the last depreciation rate a_N should be equal to the multiple d. From $a_N = d$ it follows that

$$d = \frac{2(A - R_N)}{N(N+1)}, \tag{1.94a}$$

$$a_1 = Nd,\ a_2 = (N-1)d,\ \ldots, a_N = d. \tag{1.94b}$$

■ The purchase price of a machine is € $A = 50\,000$. This machine is to be depreciated in 5 years to the value $R_5 =$ € 10 000 by digital depreciation.
The annexed depreciation schedule, calculated by the given formulas, shows that the percentage of the depreciation is fairly equal.

Year	Depreciation base	Depreciation expense	Residual value	Depreciation in % of the depr. base
1	50 000	$a_1 = 5d = 13\,335$	36 665	26.7
2	36 665	$a_2 = 4d = 10\,668$	25 997	29.1
3	25 997	$a_3 = 3d = 8\,001$	17 996	30.8
4	17 996	$a_4 = 2d = 5\,334$	12 662	29.6
5	12 662	$a_5 = d = 2\,667$	9 995	21.1

1.3.5.5 Geometrically Declining Balance Depreciation

Consider geometrically declining depreciation where $p\%$ of the actual value is depreciated every year. For the residual value R_n after n years holds:

$$R_n = A\left(1 - \frac{p}{100}\right)^n \quad (n = 1, 2, \ldots). \tag{1.95}$$

Usually A (the acquisition cost) is given. The useful life of the asset is N years long. If from the quantities R_N, p and N, two is given, the third one can be calculated by the formula (1.95).

■ **A:** A machine with a purchase value € 50 000 is to be geometrically depreciated yearly by 10%. After how many years will its value drop below € 10 000 for the first time? Based on (1.95), yields
$$N = \frac{\ln(10\,000/50\,000)}{\ln(1 - 0.1)} = 15.27 \text{ years.}$$

■ **B:** For a purchase price of $A =$ € 1000 the residual value R_n should be represented for $n = 1, 2, \ldots, 10$ years by a) straight-line, b) arithmetically declining, c) geometrically declining depreciation. The results are shown in **Fig. 1.4**.

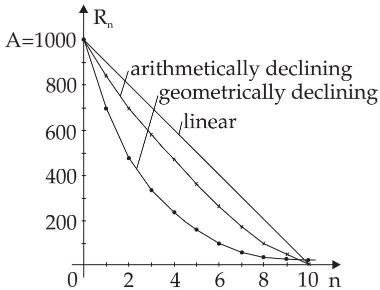

Figure 1.4

1.3.5.6 Depreciation with Different Types of Depreciation Account

Since in the case of geometrically declining depreciation the residual value cannot become equal to zero for a finite n, it is reasonable after a certain time, e.g., after m years, to switch over to straight-line depreciation. m is to be determined to an amount that from this time on the geometrically declining depreciation rate is smaller than the straight-line depreciation rate. From this requirement it follows that:

$$m > N - \frac{100}{p}. \tag{1.96}$$

Here m is the last year of geometrically declining depreciation and N is the last year of linear depreciation when the residual value becomes zero.

■ A machine with a purchase value of € 50 000 is to be depreciated to zero within 15 years, for m years by geometrically declining depreciation with 14% of the residual value, then with the straight-line method. From (1.96) follows $m > 15 - \frac{100}{14} = 7.76$, i.e., after $m = 8$ years it is reasonable to switch over to straight-line depreciation.

1.4 Inequalities

1.4.1 Pure Inequalities

1.4.1.1 Definitions

1. Inequalities

Inequalities are comparisons of two real algebraic expressions represented by one of the following signs:

Type I	$>$	(“greater”)	Type II	$<$	(“smaller”)
Type III	$\neq$	(“not equal”)	Type IIIa	$<>$	(“greater or smaller”)
Type IV	$\geq$	(“greater or equal”)	Type IVa	$\not<$	(“not smaller”)
Type V	$\leq$	(“smaller or equal”)	Type Va	$\not>$	(“not greater”)

The notation III and IIIa, IV and IVa, and V and Va have the same meaning, so they can be replaced by each other. The notation III can also be used for those types of quantities for which the notions of “greater” or “smaller” cannot be defined, for instance for complex numbers or vectors, but in this case it cannot be replaced by IIIa.

2. Identical Inequalities, Inequalities of the Same and of the Opposite Sense, Equivalent Inequalities

1. Identical Inequalities are valid for arbitrary values of the letters contained in them.

2. Inequalities of the Same Sense belong to the same type from the first two, i.e., both belong to type I or both belong to type II.

3. Inequalities of the Opposite Sense belong to different types of the first two, i.e., one to type I, the other to type II.

4. Equivalent Inequalities are inequalities if they are valid exactly for the same values of the unknowns contained in them.

3. Solution of Inequalities

Similarly to equalities, inequalities can contain unknown quantities which are usually denoted by the last letters of the alphabet. The *solution of an inequality* or a system of inequalities means the determination of the limits for the unknowns between which they can change, keeping the inequality or system of inequalities true.

Solutions can be looked for any kind of inequality; mostly *pure inequalities* of type I and II are to be solved.

1.4.1.2 Properties of Inequalities of Type I and II

1. Change the Sense of the Inequality

$$\text{If} \quad a > b \text{ holds,} \quad \text{then} \quad b < a \quad \text{is valid,} \tag{1.97a}$$

$$\text{if} \quad a < b \text{ holds,} \quad \text{then} \quad b > a \quad \text{is valid.} \tag{1.97b}$$

2. Transitivity

$$\text{If} \quad a > b \quad \text{and} \quad b > c \text{ hold,} \quad \text{then} \quad a > c \quad \text{is valid;} \tag{1.98a}$$

$$\text{if} \quad a < b \quad \text{and} \quad b < c \text{ hold,} \quad \text{then} \quad a < c \quad \text{is valid.} \tag{1.98b}$$

3. Addition and Subtraction of a Quantity

$$\text{If} \quad a > b \text{ holds,} \quad \text{then} \quad a \pm c > b \pm c \quad \text{is valid;} \tag{1.99a}$$

$$\text{if} \quad a < b \text{ holds,} \quad \text{then} \quad a \pm c < b \pm c \quad \text{is valid.} \tag{1.99b}$$

By adding or subtracting the same amount to the both sides of inequality, the sense of the inequality does not change.

4. Addition of Inequalities

$$\text{If} \quad a > b \quad \text{and} \quad c > d \text{ hold,} \quad \text{then} \quad a + c > b + d \quad \text{is valid;} \tag{1.100a}$$

$$\text{if} \quad a < b \quad \text{and} \quad c < d \text{ hold,} \quad \text{then} \quad a + c < b + d \quad \text{is valid.} \tag{1.100b}$$

Two inequalities of the same sense can be added.

5. Subtraction of Inequalities

$$\text{If} \quad a > b \quad \text{and} \quad c < d \text{ hold,} \quad \text{then} \quad a - c > b - d \quad \text{is valid;} \tag{1.101a}$$

$$\text{if} \quad a < b \quad \text{and} \quad c > d \text{ hold,} \quad \text{then} \quad a - c < b - d \quad \text{is valid.} \tag{1.101b}$$

Inequalities of the opposite sense can be subtracted; the result keeps the sense of the first inequality. Subtracting inequalities of the same sense is not allowed.

6. Multiplication and Division of an Inequality by a Quantity

$$\text{If} \quad a > b \quad \text{and} \quad c > 0 \text{ hold,} \quad \text{then} \quad ac > bc \quad \text{and} \quad \frac{a}{c} > \frac{b}{c} \text{ are valid,} \tag{1.102a}$$

$$\text{if} \quad a < b \quad \text{and} \quad c > 0 \text{ hold,} \quad \text{then} \quad ac < bc \quad \text{and} \quad \frac{a}{c} < \frac{b}{c} \text{ are valid,} \tag{1.102b}$$

$$\text{if} \quad a > b \quad \text{and} \quad c < 0 \text{ hold,} \quad \text{then} \quad ac < bc \quad \text{and} \quad \frac{a}{c} < \frac{b}{c} \text{ are valid,} \tag{1.102c}$$

$$\text{if} \quad a < b \quad \text{and} \quad c < 0 \text{ hold,} \quad \text{then} \quad ac > bc \quad \text{and} \quad \frac{a}{c} > \frac{b}{c} \text{ are valid.} \tag{1.102d}$$

Multiplication or division of both sides of an inequality by a positive value does not change the sense of the inequality. Multiplication or division by a negative value changes the sense of the inequality.

7. Inequalities and Reciprocal Values

$$\text{If} \quad 0 < a < b \quad \text{or} \quad a < b < 0 \text{ hold,} \quad \text{then} \quad \frac{1}{a} > \frac{1}{b} \quad \text{is valid.} \tag{1.103}$$

1.4.2 Special Inequalities

1.4.2.1 Triangle Inequality for Real Numbers

For arbitrary real numbers $a, b, a_1, a_2, \ldots, a_n$, there are the inequalities

$$|a+b| \le |a| + |b|\,; \qquad |a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|\,. \tag{1.104}$$

The absolute value of the sum of two or more real numbers is less than or equal to the sum of their absolute values. The equality holds only if the summands have the same sign.

1.4.2.2 Triangle Inequality for Complex Numbers

For n complex numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}$

$$\left|\sum_{k=1}^{n} z_k\right| = |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| = \sum_{k=1}^{n} |z_k|. \tag{1.105}$$

1.4.2.3 Inequalities for Absolute Values of Differences of Real and Complex Numbers

For arbitrary real numbers $a, b \in \mathbb{R}$, there are the inequalities

$$||a| - |b|| \le |a - b| \le |a| + |b|. \tag{1.106}$$

The absolute value of the difference of two real numbers is less than or equal to the sum of their absolute values, but greater than or equal to the absolute value of the difference of their absolute values. For two arbitrary complex numbers $z_1, z_2 \in \mathbb{C}$

$$||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|\,. \tag{1.107}$$

1.4.2.4 Inequality for Arithmetic and Geometric Means

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n} \quad \text{for} \quad a_i > 0\,. \tag{1.108}$$

The arithmetic mean of n positive numbers is greater than or equal to their geometric mean. Equality holds only if all the n numbers are equal.

1.4.2.5 Inequality for Arithmetic and Quadratic Means

$$\left|\frac{a_1 + a_2 + \cdots + a_n}{n}\right| \le \sqrt{\frac{{a_1}^2 + {a_2}^2 + \cdots + {a_n}^2}{n}}\,. \tag{1.109}$$

The absolute value of the arithmetic mean of numbers is less than or equal to their quadratic mean.

1.4.2.6 Inequalities for Different Means of Real Numbers

For the harmonic, geometric, arithmetic, and quadratic means of two positive real numbers a and b with $a < b$ the following inequalities hold (see also 1.2.5.5, p. 20):

$$a < x_H < x_G < x_A < x_Q < b. \tag{1.110a}$$

Here

$$x_A = \frac{a+b}{2}\,,\; x_G = \sqrt{ab}\,,\; x_H = \frac{2ab}{a+b}\,,\; x_Q = \sqrt{\frac{a^2+b^2}{2}}\,. \tag{1.110b}$$

1.4.2.7 Bernoulli's Inequality

For every real number $a \ge -1$ and integer $n \ge 1$ holds

$$(1+a)^n \ge 1 + n\,a\,. \tag{1.111}$$

The equality holds only for $n = 1$, or $a = 0$.

1.4.2.8 Binomial Inequality

For arbitrary real numbers $a, b \in \mathbb{R}$ holds

$$|a\,b| \le \frac{1}{2}(a^2 + b^2)\,. \tag{1.112}$$

1.4.2.9 Cauchy-Schwarz Inequality

1. Cauchy-Schwarz Inequality for Real Numbers

The Cauchy-Schwarz inequality holds for arbitrary real numbers $a_i, b_j \in \mathbb{R}$:

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \le \sqrt{a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2}\sqrt{b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2} \tag{1.113a}$$

or

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2)(b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2). \tag{1.113b}$$

For two finite sequences of n real numbers, the sum of the pairwise products is less than or equal to the product of the square roots of the sums of the squares of these numbers. Equality holds only if $a_1 : b_1 = a_2 : b_2 = \cdots = a_n : b_n$.

If $n = 3$ and $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are considered as vectors in a Cartesian coordinate system, then the Cauchy-Schwarz inequality means that the absolute value of the scalar product of two vectors is less than or equal to the product of absolute values of these vectors. If $n > 3$, then this statement can be extended for vectors in n-dimensional Euclidean space.

2. Cauchy-Schwarz Inequality for Complex Numbers

Considering that for complex numbers $|z|^2 = z^*z$ (z^* is the complex conjugate of z), the inequality (1.113b) is valid also for arbitrary complex numbers $z_i, w_j \in \mathbb{C}$:

$(z_1w_1 + z_2w_2 + \cdots + z_nw_n)^*(z_1w_1 + z_2w_2 + \cdots + z_nw_n)$
$\le (z_1{}^*z_1 + z_2{}^*z_2 + \cdots + z_n{}^*z_n)(w_1{}^*w_1 + w_2{}^*w_2 + \cdots + w_n{}^*w_n)$.

3. Cauchy-Schwarz Inequality for Convergent Infinite Series and Integrals

An analogous statement to (1.113b) is the Cauchy-Schwarz inequality for convergent infinite series and for certain integrals:

$$\left(\sum_{n=1}^{\infty} a_nb_n\right)^2 \le \left(\sum_{n=1}^{\infty} a_n{}^2\right)\left(\sum_{n=1}^{\infty} b_n{}^2\right), \tag{1.114}$$

$$\left[\int_a^b f(x)\,\varphi(x)\,dx\right]^2 \le \left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [\varphi(x)]^2\,dx\right). \tag{1.115}$$

1.4.2.10 Chebyshev Inequality

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real positive numbers, then the following inequalities hold:

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \le \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \tag{1.116a}$$

$$\text{for} \quad a_1 \le a_2 \le \ldots \le a_n \quad \text{and} \quad b_1 \le b_2 \le \ldots \le b_n,$$
$$\text{or} \quad a_1 \ge a_2 \ge \ldots \ge a_n \quad \text{and} \quad b_1 \ge b_2 \ge \ldots \ge b_n,$$

and

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \ge \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \tag{1.116b}$$

$$\text{for} \quad a_1 \le a_2 \le \ldots \le a_n \quad \text{and} \quad b_1 \ge b_2 \ge \ldots \ge b_n.$$

For two finite sequences with n positive numbers, the product of the arithmetic means of these sequences is less than or equal to the aritmetic mean of the pairwise products if both sequences are increasing or both are decreasing; but the inequality is valid in the opposite sense if one of the sequences is increasing and the other one is decreasing.

1.4.2.11 Generalized Chebyshev Inequality

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real positive numbers, then the following inequalities hold:

$$\sqrt[k]{\frac{{a_1}^k + {a_2}^k + \cdots + {a_n}^k}{n}} \; \sqrt[k]{\frac{{b_1}^k + {b_1}^k + \cdots + {b_n}^k}{n}} \leq \sqrt[k]{\frac{(a_1 b_1)^k + (a_2 b_2)^k + \cdots + (a_n b_n)^k}{n}} \quad (1.117a)$$

$$\text{for} \quad a_1 \leq a_2 \leq \ldots \leq a_n \quad \text{and} \quad b_1 \leq b_2 \leq \ldots \leq b_n$$
$$\text{or} \quad a_1 \geq a_2 \geq \ldots \geq a_n \quad \text{and} \quad b_1 \geq b_2 \geq \ldots \geq b_n$$

and

$$\sqrt[k]{\frac{{a_1}^k + {a_2}^k + \cdots + {a_n}^k}{n}} \; \sqrt[k]{\frac{{b_1}^k + {b_1}^k + \cdots + {b_n}^k}{n}} \geq \sqrt[k]{\frac{(a_1 b_1)^k + (a_2 b_2)^k + \cdots + (a_n b_n)^k}{n}} \quad (1.117b)$$

$$\text{for} \quad a_1 \leq a_2 \leq \ldots \leq a_n \quad \text{and} \quad b_1 \geq b_2 \geq \ldots \geq b_n .$$

1.4.2.12 Hölder Inequality

1. Hölder Inequality for Series

If p and q are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ is fulfilled, and if $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are arbitrary $2n$ complex numbers, then the following inequality holds:

$$\sum_{k=1}^{n} |x_k y_k| \leq \left[\sum_{k=1}^{n} |x_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{n} |y_k|^q\right]^{\frac{1}{q}} . \quad (1.118a)$$

This inequality is also valid for countable infinite pairs of numbers:

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty} |y_k|^q\right]^{\frac{1}{q}} , \quad (1.118b)$$

where from the convergence of the series on the right-hand side the convergence of the left-hand side follows.

2. Hölder Inequality for Integrals

If $f(x)$ and $g(x)$ are two measurable functions on the measure space $(\mathrm{X}, \mathcal{A}, \mu)$ (see 12.9.2, p. 695), then the following inequality holds:

$$\int_X |f(x)g(x)| d\mu \leq \left[\int_X |f(x)|^p \, d\mu\right]^{\frac{1}{p}} \left[\int_X |g(x)|^q \, d\mu\right]^{\frac{1}{q}} . \quad (1.118c)$$

1.4.2.13 Minkowski Inequality

1. Minkowski Inequality for Series

If $p \geq 1$ holds, and $\{x_k\}_{k=1}^{k=\infty}$ and $\{y_k\}_{k=1}^{\infty}$ with $x_k, y_k \in \mathbb{C}$ are two sequences of numbers, then holds:

$$\left[\sum_{k=1}^{\infty} |x_k + y_k|^p\right]^{\frac{1}{p}} \leq \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{\infty} |y_k|^p\right]^{\frac{1}{p}} . \quad (1.119a)$$

2. Minkowski Inequality for Integrals

If $f(x)$ and $g(x)$ are two measurable functions on the measure space $(\mathrm{X}, \mathcal{A}, \mu)$ (see 12.9.2, p. 695), then holds:

$$\left[\int_X |f(x)+g(x)|^p d\mu\right]^{\frac{1}{p}} \le \left[\int_X |f(x)|^p d\mu\right]^{\frac{1}{p}} + \left[\int_X |g(x)|^p d\mu\right]^{\frac{1}{p}}. \quad (1.119b)$$

1.4.3 Solution of Linear and Quadratic Inequalities

1.4.3.1 General Remarks

During the solution of an inequality it is transformed into equivalent inequalities step by step. Similarly to the solution of an equation the same expression can be added to both sides; formally, it may seem that a summand is brought from one side to the other, changing its sign. Furthermore one can multiply or divide both sides of an inequality by a non-zero expression, where the inequality keeps its sense if this expression has a positive value, and changes its sense if this expression has a negative value. An inequality of first degree can always be transformed into the form

$$ax > b. \quad (1.120)$$

The simplest form of an inequality of second degree is

$$x^2 > m \quad (1.121a) \qquad \text{or} \qquad x^2 < m \quad (1.121b)$$

and in the general case it has the form

$$ax^2 + bx + c > 0 \quad (1.122a) \qquad \text{or} \qquad ax^2 + bx + c < 0. \quad (1.122b)$$

1.4.3.2 Linear Inequalities

The linear inequality of first degree (1.120) has the solution

$$x > \frac{b}{a} \text{ for } a > 0 \quad (1.123a) \qquad \text{and} \qquad x < \frac{b}{a} \text{ for } a < 0. \quad (1.123b)$$

■ $5x + 3 < 8x + 1, \quad 5x - 8x < 1 - 3, \quad -3x < -2, \quad x > \frac{2}{3}.$

1.4.3.3 Quadratic Inequalities

Inequalities of second degree in the form

$$x^2 > m \quad (1.124a) \qquad \text{and} \qquad x^2 < m \quad (1.124b)$$

have solutions

a) $x^2 > m$: For $m \ge 0$ the solution is $x > \sqrt{m}$ and $x < -\sqrt{m}$ $(|x| > \sqrt{m})$, (1.125a)

for $m < 0$ the inequality obviously holds for any x. (1.125b)

b) $x^2 < m$: For $m > 0$ the solution is $-\sqrt{m} < x < \sqrt{m}$ $(|x| < \sqrt{m})$, (1.126a)

for $m \le 0$ there is no solution. (1.126b)

1.4.3.4 General Case for Inequalities of Second Degree

$$ax^2 + bx + c > 0 \quad (1.127a) \qquad \text{or} \qquad ax^2 + bx + c < 0. \quad (1.127b)$$

First dividing the inequality by a. If $a < 0$ then the sense of the inequality changes, but in any case it will have the form

$$x^2 + px + q < 0 \quad (1.127c) \qquad \text{or} \qquad x^2 + px + q > 0. \quad (1.127d)$$

By completing the square it follows that

$$\left(x+\frac{p}{2}\right)^2 < \left(\frac{p}{2}\right)^2 - q \qquad (1.127e) \qquad \text{or} \qquad \left(x+\frac{p}{2}\right)^2 > \left(\frac{p}{2}\right)^2 - q. \qquad (1.127f)$$

Denoting $x+\dfrac{p}{2}$ by z and $\left(\dfrac{p}{2}\right)^2 - q$ by m, the inequalities

$$z^2 < m \qquad (1.128a) \qquad \text{or} \qquad z^2 > m \qquad (1.128b)$$

can be obtained. Solving these inequalities yields the values for x.

■ **A:** $-2x^2+14x-20>0,\ x^2-7x+10<0,\ \left(x-\frac{7}{2}\right)^2<\frac{9}{4},\ -\frac{3}{2}<x-\frac{7}{2}<\frac{3}{2},$
$-\frac{3}{2}+\frac{7}{2}<x<\frac{3}{2}+\frac{7}{2}.$
The solution is $2<x<5$.

■ **B:** $x^2+6x+15>0,\ (x+3)^2>-6.$ The inequality holds identically.

■ **C:** $-2x^2+14x-20<0,\ \left(x-\frac{7}{2}\right)^2>\frac{9}{4},\ x-\frac{7}{2}>\frac{3}{2}$ and $x-\frac{7}{2}<-\frac{3}{2}.$
The solution intervals are $x>5$ and $x<2$.

1.5 Complex Numbers

1.5.1 Imaginary and Complex Numbers

1.5.1.1 Imaginary Unit

The imaginary unit is denoted by i, which represents a number different from any real number, and whose square is equal to -1. In electronics, instead of i the letter j is usually used to avoid accidently confusing it with the intensity of current, also denoted by i. The introduction of the *imaginary unit* leads to the *generalization of the notion of numbers* to the *complex numbers*, which play a very important role in algebra and analysis. The complex numbers have several interpretations in geometry and physics.

1.5.1.2 Complex Numbers

The *algebraic form of a complex number* is

$$z = a + \mathrm{i}\,b. \qquad (1.129a)$$

When a and b take all possible real values, then one gets all possible complex numbers z. The number a is the *real part*, the number b is the *imaginary part* of the number z:

$$a = \mathrm{Re}(z), \quad b = \mathrm{Im}(z). \qquad (1.129b)$$

For $b=0$ it is $z=a$, so the real numbers form a subset of the complex numbers. For $a=0$ it is $z=\mathrm{i}\,b$, which is a "pure imaginary number".
The total set of complex numbers is denoted by $\mathbb{C}$.

Remark: Functions $w=f(z)$ with complex variable $z=x+\mathrm{i}\,y$ will be discussed in function theory (see 14.1, p. 731 ff).

1.5.2 Geometric Representation

1.5.2.1 Vector Representation

Similarly to the representation of the real numbers on the numerical axis, the complex numbers can be represented as points in the so-called Gaussian number plane: A number $z=a+\mathrm{i}\,b$ is represented by the point whose abscissa is a and ordinate is b **(Fig. 1.5)**. The real numbers are on the axis of abscissae which is also called the real axis, the pure imaginary numbers are on the axis of ordinates which is also called the imaginary axis. On this plane every point is given uniquely by its *position vector* or *radius vector* (see 3.5.1.1, **6.**, p. 181), so every complex number corresponds to a vector which starts at the origin and is directed to the point defined by the complex number. So, complex numbers can be represented as points or as vectors **(Fig. 1.6)**.

1.5.2.2 Equality of Complex Numbers

Two complex numbers are equal by definition if their *real parts* and *imaginary parts* are equal to each other. From a geometric viewpoint, two complex numbers are equal if the position vectors corresponding to them are equal. In the opposite case the complex numbers are not equal. The notions "greater" and "smaller" are meaningless for complex numbers.

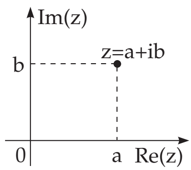

Figure 1.5

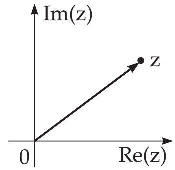

Figure 1.6

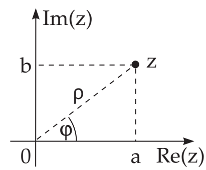

Figure 1.7

1.5.2.3 Trigonometric Form of Complex Numbers

The form

$$z = a + \mathrm{i}\, b \tag{1.130a}$$

is called the algebraic form of the complex number. Using polar coordinates yields the *trigonometric form of the complex numbers* **(Fig. 1.7)**:

$$z = \rho(\cos\varphi + \mathrm{i}\ \sin\varphi). \tag{1.130b}$$

The length of the position vector of a point $\rho = |z|$ is called the *absolute value* or the *magnitude of the complex number*, the angle φ, given in radian measure, is called the *argument of the complex number* and is denoted by $\arg z$:

$$\rho = |z|, \quad \varphi = \arg z = \omega + 2k\pi \ \text{ with } \ 0 \le \rho < \infty, \quad -\pi < \omega \le +\pi, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{1.130c}$$

One calls φ the *principal value of the argument of the complex number.*
The relations between ρ, φ and a, b for a point are the same as between the Cartesian and polar coordinates of a point (see 3.5.2.2, p. 192):

$$a = \rho\cos\varphi, \tag{1.131a}$$
$$b = \rho\sin\varphi, \tag{1.131b}$$
$$\rho = \sqrt{a^2 + b^2}, \tag{1.131c}$$

$$\varphi = \begin{cases} \arccos\dfrac{a}{\rho} & \text{for } b \ge 0,\ \rho > 0, \\ -\arccos\dfrac{a}{\rho} & \text{for } b < 0,\ \rho > 0, \\ \text{undefined} & \text{for } \rho = 0 \end{cases} \tag{1.131d}$$

$$\varphi = \begin{cases} \arctan\dfrac{b}{a} & \text{for } a > 0, \\ +\dfrac{\pi}{2} & \text{for } a = 0,\ b > 0, \\ -\dfrac{\pi}{2} & \text{for } a = 0,\ b < 0, \\ \arctan\dfrac{b}{a} + \pi & \text{for } a < 0,\ b \ge 0, \\ \arctan\dfrac{b}{a} - \pi & \text{for } a < 0,\ b < 0. \end{cases} \tag{1.131e}$$

The complex number $z = 0$ has absolute value equal to zero; its argument $\arg 0$ is undefined.

1.5.2.4 Exponential Form of a Complex Number

The representation

$$z = \rho e^{\mathrm{i}\varphi} \tag{1.132a}$$

is called the *exponential form of the complex number*, where ρ is the magnitude and φ is the argument. The *Euler relation* is the formula

$$e^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\,\sin\varphi\,. \tag{1.132b}$$

■ Representation of a complex number in three forms:

a) $z = 1 + \mathrm{i}\sqrt{3}$ (algebraic form), **b)** $z = 2\left(\cos\dfrac{\pi}{3} + \mathrm{i}\,\sin\dfrac{\pi}{3}\right)$ (trigonometric form),

c) $z = 2\,e^{\mathrm{i}\frac{\pi}{3}}$ (exponential form), considering the principal value of it.

Without restriction to the principal value holds the representation

d) $z = 1+\mathrm{i}\sqrt{3} = 2\exp\left[\mathrm{i}\left(\dfrac{\pi}{3} + 2k\pi\right)\right] = 2\left[\cos\left(\dfrac{\pi}{3} + 2k\pi\right) + \mathrm{i}\,\sin\left(\dfrac{\pi}{3} + 2k\pi\right)\right] \quad (k = 0, \pm 1, \pm 2, \ldots)\,.$

1.5.2.5 Conjugate Complex Numbers

Two complex numbers z and z^* are called *conjugate complex numbers* if their real parts are equal and their imaginary parts differ only in sign:

$$\mathrm{Re}(z^*) = \mathrm{Re}(z)\,, \quad \mathrm{Im}(z^*) = -\mathrm{Im}(z)\,. \tag{1.133a}$$

The geometric interpretation of points corresponding to the conjugate complex numbers are points symmetric with respect to the real axis. Conjugate complex numbers have the same absolute value, their arguments differ only in sign:

$$z = a + \mathrm{i}\,b = \rho(\cos\varphi + \mathrm{i}\,\sin\varphi) = \rho e^{\mathrm{i}\varphi}, \tag{1.133b}$$

$$z^* = a - \mathrm{i}\,b = \rho(\cos\varphi - \mathrm{i}\,\sin\varphi) = \rho e^{-\mathrm{i}\varphi}. \tag{1.133c}$$

Instead of z^* one often uses the notation $\overline{z}$ for the conjugate of z.

1.5.3 Calculation with Complex Numbers

1.5.3.1 Addition and Subtraction

Addition and subtraction of two or more complex numbers given in algebraic form is defined by the formula

$$\begin{aligned} z_1 + z_2 - z_3 + \cdots &= (a_1 + \mathrm{i}\,b_1) + (a_2 + \mathrm{i}\,b_2) - (a_3 + \mathrm{i}\,b_3) + \cdots \\ &= (a_1 + a_2 - a_3 + \cdots) + \mathrm{i}\,(b_1 + b_2 - b_3 + \cdots)\,. \end{aligned} \tag{1.134}$$

The calculation can be done in the same way as doing with usual binomials. As a geometric interpretation of addition and subtraction can be considered the addition and subtraction of the corresponding vectors **(Fig. 1.8)**. For these the usual rules for vector calculations are to be used (see 3.5.1.1, p. 181). For z and z^*, $z + z^*$ is always real, and $z - z^*$ is pure imaginary.

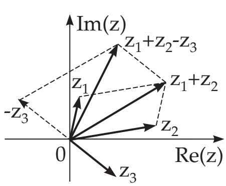

Figure 1.8

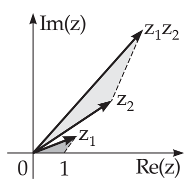

Figure 1.9

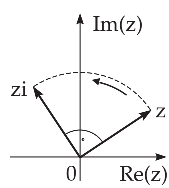

Figure 1.10

1.5.3.2 Multiplication

The multiplication of two complex numbers z_1 and z_2 given in algebraic form is defined by the following formula

$$z_1 z_2 = (a_1 + \mathrm{i}\, b_1)(a_2 + \mathrm{i}\, b_2) = (a_1 a_2 - b_1 b_2) + \mathrm{i}\,(a_1 b_2 + b_1 a_2)\,. \tag{1.135a}$$

For numbers given in trigonometric form holds

$$\begin{aligned} z_1 z_2 &= [\rho_1(\cos\varphi_1 + \mathrm{i}\,\sin\varphi_1)][\rho_2(\cos\varphi_2 + \mathrm{i}\,\sin\varphi_2)] \\ &= \rho_1\rho_2[\cos(\varphi_1+\varphi_2) + \mathrm{i}\,\sin(\varphi_1+\varphi_2)], \end{aligned} \tag{1.135b}$$

i.e., the absolute value of the product is equal to the product of the absolute values of the factors, and the argument of the product is equal to the sum of the arguments of the factors. The exponential form of the product is

$$z_1 z_2 = \rho_1\rho_2 e^{\mathrm{i}(\varphi_1+\varphi_2)}. \tag{1.135c}$$

The geometric interpretation of the product of two complex numbers z_1 and z_2 is a vector (**Fig. 1.9**). It is generated by rotation of the vector corresponding to z_1 by the argument of the vector z_2 (clockwise or counterclockwise according to the sign of this argument), and the length of the vector will be stretched by $|z_2|$.

The product $z_1 z_2$ can also be represented with similar triangles **(Fig. 1.9)**. The multiplication of a complex number z by i means a rotation by $\pi/2$ and the absolute value does not change **(Fig. 1.10)**. For z and z^*:

$$zz^* = \rho^2 = |z|^2 = a^2 + b^2. \tag{1.136}$$

1.5.3.3 Division

Division is defined as the inverse operation of multiplication. For complex numbers given in algebraic form holds

$$\frac{z_1}{z_2} = \frac{a_1 + \mathrm{i}\, b_1}{a_2 + \mathrm{i}\, b_2} = \frac{a_1 a_2 + b_1 b_2}{{a_2}^2 + {b_2}^2} + \mathrm{i}\,\frac{a_2 b_1 - a_1 b_2}{{a_2}^2 + {b_2}^2}\,. \tag{1.137a}$$

For complex numbers given in trigonometric form holds

$$\frac{z_1}{z_2} = \frac{\rho_1(\cos\varphi_1 + \mathrm{i}\,\sin\varphi_1)}{\rho_2(\cos\varphi_2 + \mathrm{i}\,\sin\varphi_2)} = \frac{\rho_1}{\rho_2}[\cos(\varphi_1-\varphi_2) + \mathrm{i}\,\sin(\varphi_1-\varphi_2)]\,, \tag{1.137b}$$

i.e., the absolute value of the quotient is equal to the ratio of the absolute values of the dividend and the divisor; the argument of the quotient is equal to the difference of the arguments.

For the exponential form follows

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{\mathrm{i}(\varphi_1-\varphi_2)}. \tag{1.137c}$$

In the geometric representation the vector corresponding to z_1/z_2 can be generated by a rotation of the vector representing z_1 by $-\arg z_2$, and then by a contraction by $|z_2|$.

Remark: Division by zero is impossible.

1.5.3.4 General Rules for the Basic Operations

Calculations with complex numbers $z = a + \mathrm{i}\, b$ are to be done in the same way as doing with ordinary binomials, but considering $\mathrm{i}^2 = -1$. Dividing a complex number by a complex number first the imaginary part of the denominator has to be removed by multiplying the numerator and the denominator of the fraction by the complex conjugate of the divisor. This is possible because

$$(a + \mathrm{i}\, b)(a - \mathrm{i}\, b) = a^2 + b^2 \tag{1.138}$$

is a real number.

■ $\dfrac{(3-4\mathrm{i}\,)(-1+5\mathrm{i}\,)^2}{1+3\mathrm{i}} + \dfrac{10+7\mathrm{i}}{5\mathrm{i}} = \dfrac{(3-4\mathrm{i}\,)(1-10\mathrm{i}\,-25)}{1+3\mathrm{i}} + \dfrac{(10+7\mathrm{i}\,)\mathrm{i}}{5\mathrm{i}\,\mathrm{i}} = \dfrac{-2(3-4\mathrm{i}\,)(12+5\mathrm{i}\,)}{1+3\mathrm{i}} +$

$$\frac{7-10\mathrm{i}}{5} = \frac{-2(56-33\mathrm{i}\,)(1-3\mathrm{i}\,)}{(1+3\mathrm{i}\,)(1-3\mathrm{i}\,)} + \frac{7-10\mathrm{i}}{5} = \frac{-2(-43-201\mathrm{i}\,)}{10} + \frac{7-10\mathrm{i}}{5} = \frac{1}{5}(50+191\mathrm{i}\,) = 10+38.2\mathrm{i}\,.$$

1.5.3.5 Taking Powers of Complex Numbers

The n-th power of a complex number could be calculated using the binomial formula, but it would be very inconvenient. For practical reasons the trigonometric form is to be used and the so-called *de Moivre formula*:

$$[\rho(\cos\varphi + \mathrm{i}\,\sin\varphi)]^n = \rho^n(\cos n\varphi + \mathrm{i}\,\sin n\varphi)\,, \tag{1.139a}$$

i.e., the absolute value is raised to the n-th power, and the argument is multiplied by n. In particular, holds:

$$\mathrm{i}^{\,2} = -1, \quad \mathrm{i}^{\,3} = -\mathrm{i}\,, \quad \mathrm{i}^{\,4} = +1 \qquad (1.139\mathrm{b}) \qquad \text{in general} \qquad \mathrm{i}^{\,4n+k} = \mathrm{i}^{\,k}\,. \qquad (1.139\mathrm{c})$$

1.5.3.6 Taking the n-th Root of a Complex Number

Taking of the n-th root is the inverse operation of taking powers. For $z = \rho(\cos\varphi + \mathrm{i}\,\sin\varphi) \neq 0$ the notation

$$z^{1/n} = \sqrt[n]{z} \quad (n > 0,\ \text{integer}), \tag{1.140a}$$

is the shorthand notation for the n different values

$$\omega_k = \sqrt[n]{\rho}\left(\cos\frac{\varphi + 2k\pi}{n} + \mathrm{i}\,\sin\frac{\varphi + 2k\pi}{n}\right),$$
$$(k = 0, 1, 2, \ldots, n-1). \tag{1.140b}$$

While addition, subtraction, multiplication, division, and taking a power with integer exponent have unique results, taking the n-th root has n different solutions ω_k.
The geometric interpretations of the points ω_k are the vertices of a regular n-gon whose center is at the origin. In **Fig. 1.11** the six values of $\sqrt[6]{z}$ are represented.

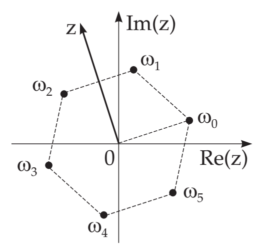

Figure 1.11

1.6 Algebraic and Transcendental Equations

1.6.1 Transforming Algebraic Equations to Normal Form

1.6.1.1 Definition

The variable x in the equality

$$F(x) = f(x) \tag{1.141}$$

is called the unknown if the equality is valid only for certain values $x_1, x_2, \ldots, x_n$ of the variable, and these values are called the *solutions* or the *roots* of the equation. Two equations are considered equivalent if they have exactly the same roots.
An equation is called an *algebraic equation* if the functions $F(x)$ and $f(x)$ are algebraic, i.e., they are rational or irrational expressions; of course one of them can be constant. Every algebraic equation can be transformed into the *normal form*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1.142}$$

by algebraic transformations. The roots of the original equation occur among the roots of the normal form, but under certain circumstances some are superfluous. The *leading coefficient* a_n is frequently transformed to the value 1.
The exponent n is called the *degree of the equation*.

■ Determine the normal form of the equation $\dfrac{x-1+\sqrt{x^2-6}}{3(x-2)} = 1 + \dfrac{x-3}{x}$. The transformations

step by step are:
$x(x-1+\sqrt{x^2-6}) = 3x(x-2)+3(x-2)(x-3)$, $\quad x^2-x+x\sqrt{x^2-6} = 3x^2-6x+3x^2-15x+18$, $\quad x\sqrt{x^2-6} = 5x^2-20x+18$, $\quad x^2(x^2-6) = 25x^4-200x^3+580x^2-720x+324$, $\quad 24x^4-200x^3+586x^2-720x+324=0$. The result is an equation of fourth degree in normal form.

1.6.1.2 System of n Algebraic Equations

Every *system of algebraic equations* can be transformed to normal form, i.e., into a system of polynomial equations:

$$P_1(x,y,z,\ldots)=0\,,\;\; P_2(x,y,z,\ldots)=0\,,\;\;\ldots\,,\;\; P_n(x,y,z,\ldots)=0\,. \tag{1.143}$$

The P_i $(i=1,2,\ldots,n)$ are polynomials in $x,y,z,\ldots$.

■ Determine the normal form of the equation system: 1. $\dfrac{x}{\sqrt{y}}=\dfrac{1}{z}$, 2. $\dfrac{x-1}{y-1}=\sqrt{z}$, 3. $xy=z$.
The normal form is: 1. $x^2z^2-y=0$, 2. $x^2-2x+1-y^2z+2yz-z=0$, 3. $xy-z=0$.

1.6.1.3 Extraneous Roots

After transforming an algebraic equation into the normal form (1.142) it can happen that the equation $P(x)=0$ has some roots which are not solutions of the original equation (1.141). The roots of the equation $P(x)=0$ must be substituted into the original equation to check whether they are really solutions of (1.141).

Extraneous solutions can emerge if not invertible transformations are performed:

1. Vanishing denominator If the equation has the form

$$\frac{P(x)}{Q(x)}=0 \tag{1.144a}$$

with polynomials $P(x)$ and $Q(x)$, then the normal form of (1.144a) after multiplying by the denominator $Q(x)$ is:

$$P(x)=0. \tag{1.144b}$$

The roots of (1.144b) are the same as the roots of (1.144a), except the ones which are roots both of the numerator and of the denominator, i.e. which satisfy $P(x)=0$ and $Q(x)=0$. If $x=\alpha$ is a root of the denominator, then in the case $x=\alpha$ the multiplication by $Q(x)$ is a multiplication by zero. Every time when a non-identical transformation is performed, the checking of the solutions is necessary (see also 1.6.3.1, p. 43).

■ $\dfrac{x^3}{x-1}=\dfrac{1}{x-1}$. The corresponding normal form is $x^4-x^3-x+1=0$. $x_1=1$ is a solution of the normal form, but it is not a solution of the original equation, since the fractions are not defined for $x=1$.

2. Irrational equations If the original equation contains radicals, the normal form is usually achieved by powering. E.g. squaring is not an identical transformation (since it is not invertible).

■ $\sqrt{x+7}+1=2x$ or $\sqrt{x+7}=2x-1$. By squaring both sides of the second form of the equation its normal form is $4x^2-5x-6=0$, and the roots are $x_1=2$ and $x_2=-3/4$. The root $x_1=2$ is a solution of the original equation, but the root $x_2=-3/4$ is not.

1.6.2 Equations of Degree at Most Four

1.6.2.1 Equations of Degree One (Linear Equations)

1. Normal Form

$$ax+b=0\;\;(a\neq 0)\,. \tag{1.145}$$

2. Number of Solutions
There is a unique solution

$$x_1 = -\frac{b}{a}\,. \qquad (1.146)$$

1.6.2.2 Equations of Degree Two (Quadratic Equations)

1. Normal Form

$$ax^2 + bx + c = 0 \quad (a \neq 0) \qquad (1.147a)$$

or divided by a:

$$x^2 + px + q = 0\,. \qquad (1.147b)$$

2. Number of Real Solutions of a Real Equation Depending on the sign of the discriminant

$$D = 4ac - b^2 \ \text{ for } \ (1.147a) \ \text{ or } \ D = q - \frac{p^2}{4} \ \text{ for } \ (1.147b)\,, \qquad (1.148)$$

holds:
• for $D < 0$, there are two real solutions (two real roots),
• for $D = 0$, there is one real solution (two coincident roots),
• for $D > 0$, there is no real solution (two complex roots).

3. Properties of the Roots of a Quadratic Equation If x_1 and x_2 are the roots of the quadratic equation (1.147a) or (1.147b), then the following equalities hold:

$$x_1 + x_2 = -\frac{b}{a} = -p\,, \quad x_1 \cdot x_2 = \frac{c}{a} = q\,. \qquad (1.149)$$

4. Solution of Quadratic Equations
Method 1: Factorization of

$$ax^2 + bx + c = a(x-\alpha)(x-\beta) \quad (1.150a) \qquad \text{or} \quad x^2 + px + q = (x-\alpha)(x-\beta)\,, \quad (1.150b)$$

if it is successful, immediately gives the roots

$$x_1 = \alpha\,, \quad x_2 = \beta\,. \qquad (1.151)$$

■ $x^2 + x - 6 = 0,\ x^2 + x - 6 = (x+3)(x-2)\,, x_1 = -3,\ x_2 = 2.$

Method 2: Using the solution formula in the cases $D \le 0$:
a) For (1.147a) the solutions are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (1.152a) \qquad \text{or} \quad x_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac}}{a}\,. \quad (1.152b)$$

If b is an even integer the second formula is to be used.
b) For (1.147b) the solutions are

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}\,. \qquad (1.153)$$

1.6.2.3 Equations of Degree Three (Cubic Equations)

1. Normal Form

$$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0) \qquad (1.154a)$$

or after dividing by a and substituting $y = x + \dfrac{b}{3a}$ there is

$$y^3 + 3py + 2q = 0 \quad \text{or in reduced form} \quad y^3 + p^*y + q^* = 0\,, \qquad (1.154b)$$

where

$$q^* = 2q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \ \text{ and } \ p^* = 3p = \frac{3ac - b^2}{3a^2}. \tag{1.154c}$$

2. Number of Real Solutions Depending on the sign of the discriminant

$$D = q^2 + p^3 \tag{1.155}$$

holds:

• for $D > 0$, one real solution (one real and two complex roots),
• for $D < 0$, three real solutions (three different real roots),
• for $D = 0$, one real solution (one real root with multiplicity three) in the case $p = q = 0$; or two real solutions (a single and a double real root) in the case $p^3 = -q^2 \neq 0$.

3. Properties of the Roots of a Cubic Equation If x_1, x_2, and x_3 are the roots of the cubic equation (1.154a), then the following equalities hold:

$$x_1 + x_2 + x_3 = -\frac{b}{a}, \quad x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a}, \quad x_1x_2x_3 = -\frac{d}{a}. \tag{1.156}$$

4. Solution of a Cubic Equation

Method 1: If it is possible to decompose the left-hand side into a product of linear terms

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma) \tag{1.157a}$$

one immediately gets the roots

$$x_1 = \alpha, \quad x_2 = \beta, \quad x_3 = \gamma. \tag{1.157b}$$

■ $x^3 + x^2 - 6x = 0,\ x^3 + x^2 - 6x = x(x+3)(x-2);\quad x_1 = 0,\ x_2 = -3,\ x_3 = 2.$

Method 2: Using the Formula of Cardano. By substituting $y = u + v$ the equation (1.154b) has the form

$$u^3 + v^3 + (u + v)(3uv + 3p) + 2q = 0. \tag{1.158a}$$

This equation is obviously satisfied if

$$u^3 + v^3 = -2q \quad \text{and} \quad uv = -p \tag{1.158b}$$

hold. Writing (1.158b) in the form

$$u^3 + v^3 = -2q, \qquad u^3v^3 = -p^3, \tag{1.158c}$$

there are two unknowns u^3 and v^3, the sum and product of which are known. Therefore using the Vieta root theorem (see 1.6.3.1, **3.**, p. 44) the solutions of the quadratic equation

$$w^2 - (u^3 + v^3)w + u^3v^3 = w^2 + 2qw - p^3 = 0 \tag{1.158d}$$

can be calculated:

$$w_1 = u^3 = -q + \sqrt{q^2 + p^3}, \qquad w_2 = v^3 = -q - \sqrt{q^2 + p^3}, \tag{1.158e}$$

so for the solution y of (1.154b) the *Cardano formula* results in

$$y = u + v = \sqrt[3]{-q + \sqrt{q^2 + p^3}} + \sqrt[3]{-q - \sqrt{q^2 + p^3}}. \tag{1.158f}$$

Since the third root of a complex number means three different numbers (see (1.140b), p. 38) there are nine different cases, but because of $uv = -p$, the solutions are reduced to the following three:

$$y_1 = u_1 + v_1 \text{ (if possible, consider the real third roots } u_1 \text{ and } v_1 \text{ such that } u_1v_1 = -p), \tag{1.158g}$$

$$y_2 = u_1\left(-\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\right) + v_1\left(-\frac{1}{2} - \frac{\mathrm{i}}{2}\sqrt{3}\right), \tag{1.158h}$$

$$y_3 = u_1\left(-\frac{1}{2} - \frac{\mathrm{i}}{2}\sqrt{3}\right) + v_1\left(-\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\right). \tag{1.158i}$$

■ $y^3+6y+2=0$ with $p=2,\ q=1$ and $q^2+p^3=9$ and $u=\sqrt[3]{-1+3}=\sqrt[3]{2}=1.2599$, $v=\sqrt[3]{-1-3}=\sqrt[3]{-4}=-1.5874$. The real root is $y_1=u+v=-0.3275$, the complex roots are $y_{2,3}=-\frac{1}{2}(u+v)\pm \mathrm{i}\frac{\sqrt{3}}{2}(u-v)=0.1638\pm \mathrm{i}\cdot 2.4659$.

Method 3: For a real equation, the *auxiliary values* given in **Table 1.3** can be used. With p from (1.154c)

$$r=\pm\sqrt{|p|} \tag{1.159}$$

is substituted where the sign of r is the same as the sign of q. Next, using **Table 1.3**, one can determine the value of the auxiliary variable φ and with it the roots y_1, y_2 and y_3 depending on the signs of p and $D=q^2+p^3$.

Table 1.3 Auxiliary values for the solution of equations of degree three

$p<0$		$p>0$
$q^2+p^3\le 0$	$q^2+p^3>0$	
$\cos\varphi=\frac{q}{r^3}$	$\cosh\varphi=\frac{q}{r^3}$	$\sinh\varphi=\frac{q}{r^3}$
$y_1=-2r\cos\frac{\varphi}{3}$ $y_2=+2r\cos\left(60^\circ-\frac{\varphi}{3}\right)$ $y_3=+2r\cos\left(60^\circ+\frac{\varphi}{3}\right)$	$y_1=-2r\cosh\frac{\varphi}{3}$ $y_2=r\cosh\frac{\varphi}{3}+\mathrm{i}\sqrt{3}\,r\sinh\frac{\varphi}{3}$ $y_3=r\cosh\frac{\varphi}{3}-\mathrm{i}\sqrt{3}\,r\sinh\frac{\varphi}{3}$	$y_1=-2r\sinh\frac{\varphi}{3}$ $y_2=r\sinh\frac{\varphi}{3}+\mathrm{i}\sqrt{3}\,r\cosh\frac{\varphi}{3}$ $y_3=r\sinh\frac{\varphi}{3}-\mathrm{i}\sqrt{3}\,r\cosh\frac{\varphi}{3}$

■ $y^3-9y+4=0.\quad p=-3,\ q=2,\ q^2+p^3<0,\ r=\sqrt{3},\ \cos\varphi=\frac{2}{3\sqrt{3}}=0.3849,\ \varphi=67^\circ 22'$.
$y_1=-2\sqrt{3}\cos 22^\circ 27'=-3.201,\ y_2=2\sqrt{3}\cos(60^\circ-22^\circ 27')=2.747,\ y_3=2\sqrt{3}\cos(60^\circ+22^\circ 27')=0.455$.
Checking: $y_1+y_2+y_3=0.001$ which can be considered 0 for the accuracy of our calculations.

Method 4: Numerical approximate solution, see 19.1.2, p. 952; numerical approximate solution by the help of a nomogram, see 2.19, p. 128.

1.6.2.4 Equations of Degree Four

1. Normal Form

$$ax^4+bx^3+cx^2+dx+e=0 \quad (a\neq 0)\,. \tag{1.160}$$

If all the coefficients are real, this equation has 0 or 2 or 4 real solutions.

2. Special Forms If $b=d=0$ holds, the roots of the *biquadratic equation*

$$ax^4+cx^2+e=0 \tag{1.161a}$$

can be calculated by the formulas

$$x_{1,2,3,4}=\pm\sqrt{y},\quad y=\frac{-c\pm\sqrt{c^2-4ae}}{2a}. \tag{1.161b}$$

For $a=e$ and $b=d$, the roots of the equation

$$ax^4+bx^3+cx^2+bx+a=0 \tag{1.161c}$$

can be calculated by the formulas

$$x_{1,2,3,4}=\frac{y\pm\sqrt{y^2-4}}{2},\quad y=\frac{-b\pm\sqrt{b^2-4ac+8a^2}}{2a}. \tag{1.161d}$$

3. Solution of a General Equation of Degree Four

Method 1: If somehow the left-hand side of the equation can be factorized

$$ax^4 + bx^3 + cx^2 + dx + e = 0 = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \tag{1.162a}$$

then the roots can be immediately determined:

$$x_1 = \alpha\,, \quad x_2 = \beta\,, \quad x_3 = \gamma\,, \quad x_4 = \delta. \tag{1.162b}$$

■ $x^4 - 2x^3 - x^2 + 2x = 0,\; x(x^2-1)(x-2) = x(x-1)(x+1)(x-2);$
$x_1 = 0,\; x_2 = 1,\; x_3 = -1,\; x_4 = 2.$

Method 2: The roots of the equation (1.162a) for $a = 1$ coincide with the roots of the equation

$$x^2 + (b+A)\frac{x}{2} + \left(y + \frac{by-d}{A}\right) = 0, \tag{1.163a}$$

where $A = \pm\sqrt{8y + b^2 - 4c}$ and y is one of the real roots of the equation of third degree

$$8y^3 - 4cy^2 + (2bd - 8e)y + e(4c - b^2) - d^2 = 0 \tag{1.163b}$$

with $B = \dfrac{b^3}{8} - \dfrac{bc}{2} \neq 0$. The case $B = 0$ gives by the help of the substitution $x = u - \dfrac{b}{4}$ a biquadratic equation of the form (1.161a) for u with $a = 1$.

Method 3: Approximate solution, see 19.1.2, p. 952.

1.6.2.5 Equations of Higher Degree

It is impossible to give a formula or a finite sequence of formulas which produce the roots of an equation of degree five or higher (see also 19.1.2.2,**2.**, p. 954).

1.6.3 Equations of Degree n

1.6.3.1 General Properties of Algebraic Equations

1. Roots

The left-hand side of the equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0 \tag{1.164a}$$

is a polynomial $P_n(x)$ of degree n, and a solution of (1.164a) is a root of the polynomial $P_n(x)$. If α is a root of the polynomial, then $P_n(x)$ is divisible by $(x-\alpha)$. Generally

$$P_n(x) = (x-\alpha)P_{n-1}(x) + P_n(\alpha). \tag{1.164b}$$

Here $P_{n-1}(x)$ is a polynomial of degree $n-1$. If $P_n(x)$ is divisible by $(x-\alpha)^k$, but it is not divisible by $(x-\alpha)^{k+1}$ then α is called a *root of order* k of the equation $P_n(x) = 0$. In this case α is a common root of the polynomial $P_n(x)$ and its derivatives to order $(k-1)$.

2. Fundamental Theorem of Algebra

Every equation of degree n whose coefficients are real or complex numbers has n real or complex roots, where the roots of higher order are counted by their multiplicity. Denoting the roots of $P(x)$ by $\alpha, \beta, \gamma, \ldots$ and they have multiplicity $k, l, m, \ldots$, then the *product representation of the polynomial* is

$$P(x) = (x-\alpha)^k(x-\beta)^l(x-\gamma)^m \ldots\,. \tag{1.165a}$$

The solution of the equation $P(x) = 0$ can be simplified by reducing the equation to another one, which has the same roots, but only with multiplicity one (if possible). In order to get this, the polynomial is to be composed into a product of two factors

$$P(x) = Q(x)T(x), \tag{1.165b}$$

such that

$$T(x) = (x-\alpha)^{k-1}(x-\beta)^{l-1}\ldots, \quad Q(x) = (x-\alpha)(x-\beta)\ldots\,. \tag{1.165c}$$

Because the roots of the polynomial $P(x)$ with higher multiplicity are the roots of its derivative $P'(x)$, too, $T(x)$ is the greatest common devisor of the polynomial $P(x)$ and its derivative $P'(x)$ (see 1.1.6.5, p.14). Dividing $P(x)$ by $T(x)$ yields the polynomial $Q(x)$ which has all the roots of $P(x)$, and each root occurs with multiplicity one.

3. Theorem of Vieta About Roots

The relations between the n roots $x_1, x_2, \ldots, x_n$ and the coefficients of the equation (1.164a) are:

$$\begin{aligned} x_1 + x_2 + \ldots + x_n &= \sum_{i=1}^{n} x_i = -a_{n-1}, \\ x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n &= \sum_{\substack{i,j=1\\ i<j}}^{n} x_ix_j = a_{n-2}, \\ x_1x_2x_3 + x_1x_2x_4 + \ldots + x_{n-2}x_{n-1}x_n &= \sum_{\substack{i,j,k=1\\ i<j<k}}^{n} x_ix_jx_k = -a_{n-3}, \\ \ldots \\ x_1x_2\ldots x_n &= (-1)^n a_0. \end{aligned} \tag{1.166}$$

1.6.3.2 Equations with Real Coefficients

1. Complex Roots

Polynomial equations with real coefficients can also have complex roots but only pairwise conjugate complex numbers, i.e., if $\alpha = a + \mathrm{i}\,b$ is a root, then $\beta = a - \mathrm{i}\,b$ is also a root, and it has the same multiplicity. The expressions $p = -(\alpha + \beta) = -2a$ and $q = \alpha\beta = a^2 + b^2$ satisfy the unequation $\left(\frac{p}{2}\right)^2 - q < 0$, so that

$$(x - \alpha)(x - \beta) = x^2 + px + q \tag{1.167}$$

holds. Substituting the product corresponding to (1.167) for every pair of factors in (1.165a), one gets a decomposition of the polynomial with real coefficients into *real factors*.

$$\begin{aligned} P(x) = {} & (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x-\alpha_l)^{k_l} \\ & \cdot(x^2+p_1x+q_1)^{m_1}(x^2+p_2x+q_2)^{m_2}\cdots(x^2+p_rx+q_r)^{m_r}. \end{aligned} \tag{1.168}$$

Here $\alpha_1, \alpha_2, \ldots, \alpha_l$ are the l real roots of the polynomial $P(x)$. It also has r pairs of conjugate complex roots, which are the roots of the quadratic factors $x^2 + p_ix + q_i$ $(i = 1, 2, \ldots, r)$. The numbers α_j $(j = 1, 2, \ldots, l)$, p_i and q_i $(i = 1, 2, \ldots, r)$ are real and the inequalities $\left(\frac{p_i}{2}\right)^2 - q_i < 0$ hold.

2. Number of Roots of an Equation with Real Coefficients

According to (1.167) every equation of odd degree has at least one real root. The number of further real roots of (1.164a) between two arbitrary real numbers $a < b$, can be determined in the following way:

a) Separate the Multiple Roots: Separating the multiple roots of $P(x) = 0$, yields an equation which has all the roots of the original equation, but only with multiplicity one. Then the form mentioned in the case of the fundamental theorem must be produced.
For practical reasons it is a good idea to start with the determination of the *Sturm chain* (the *Sturm functions* (1.169)). This is almost the same as the Euclidean algorithm for determining the greatest common devisor, but it gives some further information. If P_m is not a constant then $P(x)$ has multiple roots, which must be separated. Therefore in the following it can be assumed that $P(x) = 0$ has no multiple roots.

b) Creating the Sequence of Sturm Functions:

$$P(x), P'(x), P_1(x), P_2(x), \ldots, P_m = \text{const.} \tag{1.169}$$

Here $P(x)$ is the left-hand side of the equation, $P'(x)$ is the first derivative of $P(x)$, $P_1(x)$ is the remainder on division of $P(x)$ by $P'(x)$, but with the opposite sign, $P_2(x)$ is the remainder on division of $P'(x)$ by $P_1(x)$ similarly with the opposite sign, etc.; P_m = const is the last non-zero remainder, but it must be a constant, otherwise $P(x)$ and $P'(x)$ have common devisors, and $P(x)$ has multiple roots. In order to simplify the calculations the remainders can be multiplied by positive numbers, what does not change the result.

c) Theorem of Sturm: If A is the number of changes in sign, i.e. the number of changes from "+" to "−" and vice versa, in the sequence (1.169) for $x = a$, and B is the number of changes in sign in the sequence (1.169) for $x = b$, then the difference $A - B$ is equal to the number of real roots of $P(x) = 0$ in the interval $[a, b]$. If in the sequence some numbers are equal to zero, then they should not be considered in the sign change count.

■ Determination of the number of roots of the equation $x^4 - 5x^2 + 8x - 8 = 0$ in the interval $[0, 2]$. The calculations by the *Sturm functions* are: $P(x) = x^4 - 5x^2 + 8x - 8$; $P'(x) = 4x^3 - 10x + 8$; $P_1(x) = 5x^2 - 12x + 16$; $P_2(x) = -3x + 284$; $P_3 = -1$. Substituting $x = 0$ results in the sequence $-8, +8, +16, +284, -1$ with two changes in sign, substituting $x = 2$ results in $+4, +20, +12, +278, -1$ with one change in sign, so $A - B = 2 - 1 = 1$, i.e., between 0 and 2 there is one root.

d) Descartes Rule: The number of positive roots of the equation $P(x) = 0$ is not greater than the number of changes of sign in the sequence of coefficients of the polynomial $P(x)$, and these two numbers can differ from each other only by an even number.

■ What can be told about the roots of the equation $x^4 + 2x^3 - x^2 + 5x - 1 = 0$? The coefficients in the equation have signs $+, +, -, +, -$, i.e., there are three changes of sign. By the rule of Descartes the equation has either three or one roots. Because on replacing x by $-x$ the roots of the equation change their signs, and on replacing x by $x + h$ the roots are shifted by h, the number of negative roots, or the roots greater than h can be estimated by the help of the rule of Descartes. In the given example replacing x by $-x$ yields $x^4 - 2x^3 - x^2 - 5x - 1 = 0$, i.e., the equation has at most one negative root. Replacing x by $x + 1$ yields $x^4 + 6x^3 + 11x^2 + 13x + 6 = 0$, i.e., every positive root of the equation (one or three) is smaller than 1.

3. Solution of Equations of Degree *n*

Usually equations with $n > 4$ can be solved only approximately. In practice, approximate methods are also used to get solutions of equations of degree three or four (see 19.1.2.3, p. 954).
In order to determine certain real roots of an algebraic equation the general numerical procedures for non-linear equations can be used (see 19.1, p. 949). In order to determine all roots, including the complex roots of an algebraic equation of degree n the Brodetsky-Smeal method can be used (see [1.7], [19.31]). In order to determine complex roots one can use the Bairstow method (see [19.31]).

1.6.4 Reducing Transcendental Equations to Algebraic Equations

1.6.4.1 Definition

An equation $F(x) = f(x)$ is transcendental if at least one of the functions $F(x)$ or $f(x)$ is not algebraic.

■ **A:** $3^x = 4^{x-2} \cdot 2^x$, ■ **B:** $2\log_5(3x-1) - \log_5(12x+1) = 0$, ■ **C:** $3\cosh x = \sinh x + 9$,

■ **D:** $2^{x-1} = 8^{x-2} - 4^{x-2}$, ■ **E:** $\sin x = \cos^2 x - \frac{1}{4}$, ■ **F:** $x\cos x = \sin x$.

In some cases it is possible to reduce the solution of a transcendental equation to the solution of an algebraic equation, for instance by appropriate substitutions. In general, transcendental equations can be solved only approximately. In the following sections some special transcendental equations are discussed which can be reduced to algebraic equations.

1.6.4.2 Exponential Equations

Exponential equations can be reduced to algebraic equations in the following two cases, if the unknown x or a polynomial $P(x)$ is only in the exponent of some quantities $a, b, c, \ldots$:

a) If the powers $a^{P_1(x)}$, $b^{P_2(x)}, \ldots$ are connected by multiplication or division, then the logarithm can be taken on an arbitrary base.

■ $3^x = 4^{x-2} \cdot 2^x$; $x \log 3 = (x-2)\log 4 + x \log 2$; $x = \dfrac{2\log 4}{\log 4 - \log 3 + \log 2}$.

b) If $a, b, c, \ldots$ are integer (or rational) powers of the same number k, i.e., $a = k^n$, $b = k^m$, $c = k^l, \ldots$, holds, then by substituting $y = k^x$ one can get an algebraic equation for y, and after solving it follows the solution $x = \dfrac{\log y}{\log k}$.

■ $2^{x-1} = 8^{x-2} - 4^{x-2}$; $\dfrac{2^x}{2} = \dfrac{2^{3x}}{64} - \dfrac{2^{2x}}{16}$. Substitution of $y = 2^x$ results in $y^3 - 4y^2 - 32y = 0$ and $y_1 = 8$, $y_2 = -4$, $y_3 = 0$; $2^{x_1} = 8$, $2^{x_2} = -4$, $2^{x_3} = 0$, so $x_1 = 3$ follows. There are no further real roots.

1.6.4.3 Logarithmic Equations

Logarithmic equations can be reduced to algebraic equations in the following two cases, if the unknown x or a polynomial $P(x)$ is only under the logarithm sign:

a) If the equation contains only the logarithm of the same expression, then by introducing this as a new unknown, one can solve the equation with respect to it. The original unknown can be determined by using the logarithm.

■ $m[\log_a P(x)]^2 + n = a\sqrt{[\log_a P(x)]^2 + b}$. The substitution $y = \log_a P(x)$ yields the equation $my^2 + n = a\sqrt{y^2 + b}$. After solving for y one gets the solution for x from the equation $P(x) = a^y$.

b) If the equation is a linear combination of logarithms of polynomials of x, on the same base a, with integer coefficients m, n, ..., i.e., it has the form $m \log_a P_1(x) + n \log_a P_2(x) + \ldots = 0$, then the left-hand side can be written as the logarithm of a rational expression. (The original equation may contain rational coefficients and rational expressions under the logarithm, or logarithms with different bases, if the bases are rational powers of each other.)

■ $2\log_5(3x-1) - \log_5(12x+1) = 0$, $\log_5 \dfrac{(3x-1)^2}{12x+1} = \log_5 1$, $\dfrac{(3x-1)^2}{12x+1} = 1$; $x_1 = 0$, $x_2 = 2$. Substituting $x_1 = 0$ in the original equation gives negative values in the logarithm, i.e., this logarithm is a complex value, so $x = 0$ is not a solution.

1.6.4.4 Trigonometric Equations

Trigonometric equations can be reduced to algebraic equations if the unknown x or the expression $nx+a$ with integer n is only in the argument of the trigonometric functions. After using the trigonometric formulas (see 2.7.2, p.81) the equation will contain only one unique function containing x, and after replacing it by y an algebraic equation arises. The solution for x is obtained from the solutions for y, naturally taking the multi-valuedness of the solution into consideration.

■ $\sin x = \cos^2 x - \dfrac{1}{4}$ or $\sin x = 1 - \sin^2 x - \dfrac{1}{4}$. Substituting $y = \sin x$ yields $y^2 + y - \dfrac{3}{4} = 0$ and $y_1 = \dfrac{1}{2}$, $y_2 = -\dfrac{3}{2}$. The result y_2 gives no real solution, because $|\sin x| \le 1$ for all real x; from $y_1 = \dfrac{1}{2}$

follows $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{5\pi}{6} + 2k\pi$ with $k = 1, 2, 3, \ldots$.

1.6.4.5 Equations with Hyperbolic Functions

Equations with hyperbolic functions can be reduced to algebraic equations if the unknown x is only in the argument of the hyperbolic functions. Rewriting the hyperbolic functions as exponential expressions, then substituting $y = e^x$ and $\frac{1}{y} = e^{-x}$, and the result is an algebraic equation for y. After solving this the solution is $x = \ln y$.

■ $3\cosh x = \sinh x + 9$; $\frac{3(e^x + e^{-x})}{2} = \frac{e^x - e^{-x}}{2} + 9$; $e^x + 2e^{-x} - 9 = 0$; $y + \frac{2}{y} - 9 = 0$, $y^2 - 9y + 2 = 0$; $y_{1,2} = \frac{9 \pm \sqrt{73}}{2}$; $x_1 = \ln \frac{9 + \sqrt{73}}{2} \approx 2.1716$, $x_2 = \ln \frac{9 - \sqrt{73}}{2} \approx -1.4784$.