Chapter 9
Fuzzy Multi-objective Bi-level Goal Programming

In Chap. 8, we presented the definitions, solutions, and algorithms for the *fuzzy multi-objective bi-level programming* (FMO-BLP) problems. This chapter still addresses the fuzzy multi-objective bi-level problem but applies a goal programming approach. We call it *fuzzy multi-objective bi-level goal programming* (FMO-BLGP). This chapter will discuss related definitions, solution concepts, and algorithms for the FMO-BLGP problem and will focus on the linear version of the FMO-BLGP problem. First, a fuzzy ranking method is used to give a mathematical definition for a FMO-BLGP problem, and then, based on a fuzzy vectors distance measure definition, a fuzzy bi-level goal programming (FBLGP) model is proposed. An algorithm for solving the FMO-BLGP problem is also developed.

This chapter is organized as follows: the identification of the FMO-BLGP problem is presented in Sect. 9.1, and a fuzzy bi-level goal decision model and related theorems are developed in Sect. 9.2. Section 9.3 proposes a fuzzy bi-level goal-programming algorithm for solving FMO-BLGP problems. In Sect. 9.4, a numerical example is adopted to illustrate the executing procedure of the algorithm and experiments are carried out. Finally, we discuss and analyze the performance of this algorithm in Sect. 9.5.

9.1 Problem Identification

In many real-world bi-level decision applications, the leader or the follower not only have multiple objectives but also have their individual predefined decision targets (called goals) to achieve the objectives through a decision procedure. Therefore, goal programming could be integrated with the FMO-BLP approach to handle the FMO-BLGP problem well.

Example 9.1 In a company, the CEO is the leader, and the heads of branches of the company are the followers in making an annual budget for the company. The CEO has two objectives: maximizing profit and maximizing marketing occupation with two goals \$8M and 80 % of the local market respectively. The branch heads have two objectives: maximizing profit with the goal of “\$4M profit” and maximizing

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customer satisfaction with the goal of "increasing customer satisfaction by 10 % compared to last year". To achieve the two sets of goals, we can establish a FMO-BLP model and develop a goal programming-based algorithm.

9.2 Solution Concepts

Based on the fuzzy ranking method in Definition 7.19, a FMO-BLP problem is defined as:

For $x \in X \subset R^n,\ y \in Y \subset R^m$, $F: X \times Y \to (F^*(R))^s$, and $f: X \times Y \to (F^*(R))^t$,

$$\min_{x \in X} \quad F(x,y) = \left(\tilde{\alpha}_{11}x + \tilde{\beta}_{11}y, \ldots, \tilde{\alpha}_{s1}x + \tilde{\beta}_{s1}y\right) \tag{9.1a}$$

$$\text{s.t.} \quad \tilde{A}_1 x + \tilde{B}_1 y \preceq_\alpha \tilde{b}_1, \tag{9.1b}$$

$$\min_{y \in Y} \quad f(x,y) = \left(\tilde{\alpha}_{12}x + \tilde{\beta}_{12}y, \ldots, \tilde{\alpha}_{t1}x + \tilde{\beta}_{t1}y\right) \tag{9.1c}$$

$$\text{s.t.} \quad \tilde{A}_2 x + \tilde{B}_2 y \preceq_\alpha \tilde{b}_2, \tag{9.1d}$$

where $\tilde{\alpha}_{h1},\ \tilde{\alpha}_{i2} \in (F^*(R))^n,\ \tilde{\beta}_{h1},\ \tilde{\beta}_{i2} \in (F^*(R))^m,\ \tilde{b}_1 \in (F^*(R))^p,\ \tilde{b}_2 \in (F^*(R))^q$, $\tilde{A}_1 = (\tilde{a}_{ij})_{p \times n}$, $\tilde{B}_1 = (\tilde{b}_{ij})_{p \times m}$, $\tilde{A}_2 = (\tilde{e}_{ij})_{q \times n}$, $\tilde{B}_2 = (\tilde{s}_{ij})_{q \times m}$, $\tilde{a}_{ij},\ \tilde{b}_{ij},\ \tilde{e}_{ij},\ \tilde{s}_{ij} \in F^*(R),\ h = 1, \ldots, s,\ i = 1, \ldots, t.$

To build a FMO-BLGP model, a distance measure between two fuzzy vectors needs to be developed to measure the distance between a decision and the predefined goal. To do so, a certain number of λ-cuts is used to approximate a fuzzy number, and a final solution is considered to be reached when solutions under two adjacent λ-cuts are nearly equal. To help implement this strategy, a λ-cut based fuzzy vector distance measure is defined below:

Definition 9.1 Let $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, $\tilde{b} = \left(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\right)$ be two n-dimensional fuzzy vectors, $\phi = \{\alpha \le \lambda_0 < \lambda_1 < \cdots < \lambda_l \le 1\}$ be a division of $[\alpha, 1]$, the distance between $\tilde{a}$ and $\tilde{b}$ under ϕ is defined as:

$$D\left(\tilde{a}, \tilde{b}\right) \triangleq \frac{1}{l+1} \sum_{i=1}^{n} \sum_{j=0}^{l} \left\{ \left| a_{i\lambda_j}^L - b_{i\lambda_j}^L \right| + \left| a_{i\lambda_j}^R - b_{i\lambda_j}^R \right| \right\}, \tag{9.2}$$

where α is a predefined satisfactory degree.

In Definition 9.1, a satisfactory degree α is used to give flexibility to compare two fuzzy vectors. It is possible that two fuzzy vectors might not be compared, that is no ranking relation, by using Definition 9.1. For example, when we compare two fuzzy vectors $\tilde{a}$ and $\tilde{b}$, if some of the left λ-cuts of $\tilde{a}$ are less than those of $\tilde{b}$, while some right λ-cuts of $\tilde{a}$ are larger than those of $\tilde{b}$, there is no ranking relation between $\tilde{a}$ and $\tilde{b}$. To solve the incomparable problem, we can enhance the aspiration levels of the attributes, that is, we can adjust the satisfactory degree α to a point where all incomparable parts are discarded. It can be understood as a risk taken by a decision maker who neglects all values with the possibility of occurrence smaller than α. In such a situation, a solution is supposed to be reached under this aspiration level. So, normally, we take the same α for both objectives and constraints in a bi-level programming problem.

Lemma 9.1 *For any n-dimensional fuzzy vectors* $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, *fuzzy distance D defined in* (9.1a)–(9.1d) *satisfies the following properties*:

1. $D(\tilde{a},\tilde{b}) = 0$, if $\tilde{a}_i = \tilde{b}_i$, $i = 1, 2, \ldots, n$;
2. $D(\tilde{a},\tilde{b}) = D(\tilde{b},\tilde{a})$;
3. $D(\tilde{a},\tilde{b}) \leq D(\tilde{a},\tilde{c}) + D(\tilde{c},\tilde{b})$.

Goals set for the objectives of a leader ($\tilde{g}_L$) and a follower ($\tilde{g}_F$) in (9.1a)–(9.1d) are defined as:

$$\tilde{g}_L = (\tilde{g}_{L1}, \tilde{g}_{L2}, \ldots, \tilde{g}_{Ls}), \tag{9.3a}$$

$$\tilde{g}_F = (\tilde{g}_{F1}, \tilde{g}_{F2}, \ldots, \tilde{g}_{Ft}), \tag{9.3b}$$

where $\tilde{g}_{Li}(i = 1, \ldots, s)$ and $\tilde{g}_{Fj}(j = 1, \ldots, t)$ are fuzzy numbers with membership functions of $\mu_{\tilde{g}_{L_i}}$ and $\mu_{\tilde{g}_{F_j}}$ respectively.

Our concern is to make the objectives of both the leader and the follower as near to their goals as possible. Using the distance measure defined in (9.1a)–(9.1d), we transform the FMO-BLGP problem into a FBLGP problem as follows:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \to (F^*(R))^s$, $f: X \times Y \to (F^*(R))^t$,

$$\min_{x \in X} \quad D(F(x, y), \tilde{g}_L) \tag{9.4a}$$

$$\text{s.t.} \quad \tilde{A}_1 x + \tilde{B}_1 y \leq_{\alpha} \tilde{b}_1, \tag{9.4b}$$

$$\min_{y \in Y} \quad D(f(x, y), \tilde{g}_F) \tag{9.4c}$$

$$\text{s.t.} \quad \tilde{A}_2 x + \tilde{B}_2 y \leq_{\alpha} \tilde{b}_2, \tag{9.4d}$$

where $\tilde{A}_1 = (\tilde{a}_{ij})_{p\times n}$, $\tilde{B}_1 = (\tilde{b}_{ij})_{p\times m}$, $\tilde{A}_2 = (\tilde{e}_{ij})_{q\times n}$, $\tilde{B}_2 = (\tilde{s}_{ij})_{q\times m}$, $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij} \in (F^*(R))$, and α is a predefined satisfactory degree.

From Definitions 9.1, we transfer problem (9.4a)–(9.4d) into:

$$\min_{x\in X} \quad \frac{1}{l+1}\sum_{h=1}^{s}\sum_{j=0}^{l}\Big\{\Big|\alpha^L_{h1\lambda_j}x+\beta^L_{h1\lambda_j}y-g^L_{Lh\lambda_j}\Big| + \Big|\alpha^R_{h1\lambda_j}x+\beta^R_{h1\lambda_j}y-g^R_{Lh\lambda_j}\Big|\Big\} \tag{9.5a}$$

$$\text{s.t.} \quad \begin{aligned} &A^L_{1\lambda_j}x+B^L_{1\lambda_j}y\le b^L_{1\lambda_j},\\ &A^R_{1\lambda_j}x+B^R_{1\lambda_j}y\le b^R_{1\lambda_j},\quad j=0,1,\ldots,l, \end{aligned} \tag{9.5b}$$

$$\min_{y\in Y} \quad \frac{1}{l+1}\sum_{i=1}^{t}\sum_{j=0}^{l}\Big\{\Big|\alpha^L_{i2\lambda_j}x+\beta^L_{i2\lambda_j}y-g^L_{Fi\lambda_j}\Big| + \Big|\alpha^R_{i2\lambda_j}x+\beta^R_{i2\lambda_j}y-g^R_{Fi\lambda_j}\Big|\Big\} \tag{9.5c}$$

$$\text{s.t.} \quad \begin{aligned} &A^L_{2\lambda_j}x+B^L_{2\lambda_j}y\le b^L_{2\lambda_j},\\ &A^R_{2\lambda_j}x+B^R_{2\lambda_j}y\le b^R_{2\lambda_j},\quad j=0,1,\ldots,l \end{aligned} \tag{9.5d}$$

where $\phi=\{\alpha\le\lambda_0<\lambda_1<\cdots<\lambda_l\le 1\}$ is a division of $[\alpha,1]$.

For a clear understanding of the idea adopted, we define:

$$V^{L-}_{h1}=\frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha^L_{h1\lambda_j}x+\sum_{j=0}^{l}\beta^L_{h1\lambda_j}y-\sum_{j=0}^{l}g^L_{Lh\lambda_j}\right|-\left(\sum_{j=0}^{l}\alpha^L_{h1\lambda_j}x+\sum_{j=0}^{l}\beta^L_{h1\lambda_j}y-\sum_{j=0}^{l}g^L_{Lh\lambda_j}\right)\right\},$$

$$V^{L+}_{h1}=\frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha^L_{h1\lambda_j}x+\sum_{j=0}^{l}\beta^L_{h1\lambda_j}y-\sum_{j=0}^{l}g^L_{Lh\lambda_j}\right|+\left(\sum_{j=0}^{l}\alpha^L_{h1\lambda_j}x+\sum_{j=0}^{l}\beta^L_{h1\lambda_j}y-\sum_{j=0}^{l}g^L_{Lh\lambda_j}\right)\right\},$$

$$V^{R-}_{h1}=\frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha^R_{h1\lambda_j}x+\sum_{j=0}^{l}\beta^R_{h1\lambda_j}y-\sum_{j=0}^{l}g^R_{Lh\lambda_j}\right|-\left(\sum_{j=0}^{l}\alpha^R_{h1\lambda_j}x+\sum_{j=0}^{l}\beta^R_{h1\lambda_j}y-\sum_{j=0}^{l}g^R_{Lh\lambda_j}y\right)\right\},$$

$$V_{h1}^{R+} = \frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha_{h1\lambda_j}^{R}x + \sum_{j=0}^{l}\beta_{h1\lambda_j}^{R}y - \sum_{j=0}^{l}g_{Lh\lambda_j}^{R}\right| + \left(\sum_{j=0}^{l}\alpha_{h1\lambda_j}^{R}x + \sum_{j=0}^{l}\beta_{h1\lambda_j}^{R}y - \sum_{j=0}^{l}g_{Lh\lambda_j}^{R}\right)\right\}, \tag{9.6}$$

$$h = 1, 2, \ldots, s,$$

$$V_{i2}^{L-} = \frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{L}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{L}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{L}\right| - \left(\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{L}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{L}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{L}\right)\right\},$$

$$V_{i2}^{L+} = \frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{L}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{L}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{L}\right| + \left(\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{L}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{L}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{L}\right)\right\},$$

$$V_{i2}^{R-} = \frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{R}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{R}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{R}\right| + \left(\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{R}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{R}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{R}y\right)\right\},$$

$$V_{i2}^{R+} = \frac{1}{2}\left\{\left|\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{R}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{R}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{R}\right| + \left(\sum_{j=0}^{l}\alpha_{i2\lambda_j}^{R}x + \sum_{j=0}^{l}\beta_{i2\lambda_j}^{R}y - \sum_{j=0}^{l}g_{F_i\lambda_j}^{R}\right)\right\},$$

$$i = 1, 2, \ldots, t,$$

where V_{h1}^{L-} and V_{h1}^{L+} are deviational variables representing the under-achievement and over-achievement of the hth goal for a leader under the left λ-cut respectively. V_{h1}^{R-} and V_{h1}^{R+} are deviational variables representing the under-achievement and over-achievement of the hth goal for a leader under the right λ-cut respectively. V_{i2}^{L-}, V_{i2}^{L+}, V_{i2}^{R-}, and V_{i2}^{R+} are for a follower respectively.

For $(v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s1}^{L-}, v_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}) \in R^{4s}, X' \subseteq X \times R^{4s}, (v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+}) \in R^{4t}$, $Y' \subseteq Y \times R^{4t}$, let

$$\begin{aligned} x &= (x_1, \ldots, x_n) \in X, \\ x' &= \left(x_1, \ldots, x_n, v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s1}^{L-}, \mathrm{v}_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}\right) \in X', \\ y &= (y_1, \ldots, y_m) \in Y, \\ y' &= \left(y_1, \ldots, y_m, v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+}\right) \in Y', \end{aligned}$$

and $v_1, v_2 : X' \times Y' \rightarrow R$.

Associated with problem (9.5a)–(9.5d), we now consider the following bi-level programming problem:

$$\min_{x' \in X'} \quad v_1 = \sum_{h=1}^{s} (v_{h1}^{L-} + v_{h1}^{L+} + v_{h1}^{R-} + v_{h1}^{R+}) \tag{9.7a}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{j=0}^{l} \alpha_{h1\lambda_j}^{L} x + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{L} y + v_{h1}^{L-} - v_{h1}^{L+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}, \\ & \sum_{j=0}^{l} \alpha_{h1\lambda_j}^{R} x + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{R} y + v_{h1}^{R-} - v_{h1}^{R+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}, \\ & v_{h1}^{L-}, v_{h1}^{L+}, v_{h1}^{R-}, v_{h1}^{R+} \geq 0, \\ & v_{h1}^{L-} \cdot v_{h1}^{L+} = 0, v_{h1}^{R-} \cdot v_{h1}^{R+} = 0, \\ & A_{1\lambda_j}^{L} x + B_{1\lambda_j}^{L} y \leq b_{1\lambda_j}^{L}, \\ & A_{1\lambda_j}^{R} x + B_{1\lambda_j}^{R} y \leq b_{1\lambda_j}^{R}, \\ & h = 1, \ldots, s, \quad j = 0, 1, \ldots, l, \end{aligned} \tag{9.7b}$$

$$\min_{y' \in Y'} \quad v_2 = \sum_{i=1}^{t} (v_{i2}^{L-} + v_{i2}^{L+} + v_{i2}^{R-} + v_{i2}^{R+}) \tag{9.7c}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{j=0}^{l} \alpha_{i2\lambda_j}^{L} x + \sum_{j=0}^{l} \beta_{i2\lambda_j}^{L} y + v_{i2}^{L-} - v_{i2}^{L+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{L}, \\ & \sum_{j=1}^{l} \alpha_{i2\lambda_j}^{R} x + \sum_{j=0}^{l} \beta_{i2\lambda_j}^{R} y + v_{i2}^{R-} - v_{i2}^{R+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{R}, \\ & v_{i2}^{L-}, v_{i2}^{L+}, v_{i2}^{R-}, v_{i2}^{R+} \geq 0, \\ & v_{i2}^{L-} \cdot v_{i2}^{L+} = 0, v_{i2}^{R-} \cdot v_{i2}^{R+} = 0, \\ & A_{2\lambda_j}^{L} x + B_{2\lambda_j}^{L} y \leq b_{2\lambda_j}^{L}, \\ & A_{2\lambda_j}^{R} x + B_{2\lambda_j}^{R} y \leq b_{2\lambda_j}^{R}, \\ & i = 1, \ldots, t, \quad j = 0, 1, \ldots, l. \end{aligned} \tag{9.7d}$$

Theorem 9.1 *Let* $(x'^{*}, y'^{*}) = (x^{*}, v_{11}^{L-*}, v_{11}^{L+*}, v_{11}^{R-*}, v_{11}^{R+*}, \ldots, v_{s1}^{L-*}, v_{s1}^{L+*}, v_{s1}^{R-*}, v_{s1}^{R+*}, y^{*}, v_{12}^{L-*}, v_{12}^{L+*}, v_{12}^{R-*}, v_{12}^{R+*}, \ldots, v_{t1}^{L-*}, v_{t1}^{L+*}, v_{t1}^{R-*}, v_{t1}^{R+*})$ *be the optimal solution to problem* (9.7a)–(9.7d)*, then* (x^{*}, y^{*}) *is the optimal solution to problem* (9.5a)–(9.5d).

Proof Let the notations associated with problem (9.5a)–(9.5d) are denoted by:

$$S = \Big\{(x, y)|A_{k\lambda_j}^{L}x + B_{k\lambda_j}^{L}y \le b_{k\lambda_j}^{L}, A_{k\lambda_j}^{R}x + B_{k\lambda_j}^{R}y \le b_{k\lambda_j}^{R}, \\ j = 0, \ldots, l, \quad k = 1, 2\}, \tag{9.8a}$$

$$S(X) = \Big\{x \in X|\exists y \in Y, A_{k\lambda_j}^{L}x + B_{k\lambda_j}^{L}y \le b_{k\lambda_j}^{L}, A_{k\lambda_j}^{R}x \\ + B_{k\lambda_j}^{R}y \le b_{k\lambda_j}^{R}, \quad k = 1, 2, j = 0, \ldots, l\Big\}, \tag{9.8b}$$

$$S(x) = \{y \in Y|(x, y) \in S\}, \tag{9.8c}$$

$$P(x) = \{y \in Y|y \in \text{argmin } \Psi\} \tag{9.8d}$$

where

$$\Psi = \frac{1}{l+1}\sum_{i=1}^{t}\sum_{j=0}^{l}\Big\{\Big|\alpha_{i2\lambda_j}^{L}x + \beta_{i2\lambda_j}^{L} - g_{Lh\lambda_j}^{L}\hat{y} - g_{Fi\lambda_j}^{L}\Big| \\ + \Big|\alpha_{i2\lambda_j}^{R}x + \beta_{i2\lambda_j}^{R}\hat{y} - g_{Fi\lambda_j}^{R}\Big|\Big\}$$

$$IR = \{(x, y)|(x, y) \in S, y \in P(x)\}. \tag{9.8e}$$

Problem (9.5a)–(9.5d) can be written as:

$$\min_{x,y} \quad \frac{1}{l+1}\sum_{h=1}^{s}\sum_{j=0}^{l}\Big\{\Big|\alpha_{h1\lambda_j}^{L}x + \beta_{h1\lambda_j}^{L}y - g_{Lh\lambda_j}^{L}\Big| \\ + \Big|\alpha_{h1\lambda_j}^{R}x + \beta_{h1\lambda_j}^{R}y - g_{Lh\lambda_j}^{R}\Big|\Big\} \tag{9.9a}$$

$$\text{s.t.} \quad (x, y) \in IR. \tag{9.9b}$$

Similarly, we denote those for problem (9.7a)–(9.7d) by:

$$S' = \Big\{(x', y')|A^L_{(k\lambda_j)}x + B^L_{(k\lambda_j)}y \le b^L_{(k\lambda_j)}, A^R_{(k\lambda_j)}x + B^R_{(k\lambda_j)}y \le b^R_{(k\lambda_j)}, \quad k = 1, 2, \quad j = 0, 1, \ldots, l,$$
$$\sum_{j=0}^{l} \alpha^L_{h1\lambda_j}x + \sum_{j=0}^{l} \beta^L_{h1\lambda_j}y + v^{L-}_{h1} - v^{L+}_{h1} = \sum_{j=0}^{l} g^L_{Lh\lambda_j},$$
$$\sum_{j=0}^{l} \Big(\alpha^R_{h1\lambda_j}\Big)x + \sum_{j=0}^{l} \beta^R_{h1\lambda_j}y + v^{R-}_{h1} - v^{R+}_{h1} = \sum_{j=0}^{l} g^R_{Lh\lambda_j},$$
$$v^{L-}_{h1}, v^{L+}_{h1}, v^{R-}_{h1}, v^{R+}_{h1} \ge 0,$$
$$v^{L-}_{h1} \cdot v^{L+}_{h1} = 0, v^{R-}_{h1} \cdot v^{R+}_{h1} = 0, h = 1, \ldots, s,$$
$$\sum_{j=0}^{l} \alpha^L_{i2\lambda_j}x + \sum_{j=0}^{l} \beta^L_{i2\lambda_j}y + v^{L-}_{i2} - v^{L+}_{i2} = \sum_{j=0}^{l} g^L_{Fi\lambda_j},$$
$$\sum_{j=0}^{l} \alpha^R_{i2\lambda_j}x + \sum_{j=0}^{l} \beta^R_{i2\lambda_j}y + v^{R-}_{i2} - v^{R+}_{i2} = \sum_{j=0}^{l} g^R_{Fi\lambda_j},$$

$$\begin{aligned} &v^{L-}_{i2}, v_i 2^{L+}, v^{R-}_{i2}, v^{R+}_{i2} \ge 0, \\ &v^{L-}_{i2} \cdot v^{L+}_{i2} = 0, v^{R-}_{i2} \cdot v^{R+}_{i2} = 0,\ i = 1, \ldots, t\}, \end{aligned} \tag{9.10}$$

$$S(X') = \{x' \in X' \exists y' \in Y', (x', y') \in S'\}, \tag{9.11}$$

$$S(x') = \{y' \in Y'(x', y') \in S'\}, \tag{9.12}$$

$$P(x') = \left\{ y' \in Y'|y' \in \operatorname{argmin}\left[\sum_{i=1}^{t}\left(\hat{v}^{L-}_{i2} + \hat{v}^{L+}_{i2} + \hat{v}^{R-}_{i2} + \hat{v}^{R+}_{i2}\right)\right]\right\}, \tag{9.13}$$

$$IR' = \{(x', y')|(x', y') \in S', y' \in P(x')\}. \tag{9.14}$$

Problem (9.7a)–(9.7d) can be written as

$$\min_{(x', y')} \quad \left\{ \sum_{h=1}^{s} \left(v^{L-}_{h_1} + v^{L+}_{h1} + v^{R-}_{h1} + v^{R+}_{h1} \right) \ :\ (x', y') \in IR' \right\} \tag{9.15}$$

As (x'^*, y'^*) is the optimal solution to problem (9.7a)–(9.7d), from (9.15), it can be seen that, for any $(x', y') \in IR'$, we have

$$\sum_{h=1}^{s} \left(v^{L-}_{h_1} + v^{L+}_{h1} + v^{R-}_{h1} + v^{R+}_{h1} \right) \ge \sum_{h=1}^{s} \left(v^{L-*}_{h_1} + v^{L+*}_{h1} + v^{R-*}_{h1} + v^{R+*}_{h1} \right).$$

It follows from the definitions of v_{h1}^{L-} and v_{h1}^{L+} that

$$v_{h1}^{L-} + v_{h1}^{L+} = \left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{L} x + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{L} y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}\right|,$$
$$v_{h1}^{L-*} + v_{h1}^{L+*} = \left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{L} x^* + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{L} y^* - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}\right|,$$

for $h = 1, \ldots, s$. Similarly, we have

$$v_{h1}^{R-} + v_{h1}^{R+} = \left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{R} x + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{R} y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}\right|,$$
$$v_{h1}^{R-*} + v_{h1}^{R+*} = \left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{R} x^* + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{R} y^* - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}\right|,$$

for $h = 1, \ldots, s$. So, for any $(x', y') \in IR'$, we can obtain

$$\begin{aligned} &\left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{L} x' + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{L} y' - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}\right| \\ &\quad + \left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{R} x' + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{R} y' - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}\right| \\ &\geq \left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{L} x^* + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{L} y^* - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}\right| \\ &\quad + \left|\sum_{j=0}^{l} \alpha_{h1\lambda_j}^{R} x^* + \sum_{j=0}^{l} \beta_{h1\lambda_j}^{R} y^* - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}\right|. \end{aligned} \tag{9.16}$$

We now prove that the projection of S' onto the $X \times Y$ space, denoted by $S'|_{X\times Y}$ is equal to S.

On the one hand, for any $(x, y) \in S'|_{X\times Y}$, from the constraints: $A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \leq b_{k\lambda_j}^{L}, A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \leq b_{j\lambda_j}^{R}$, $k = 1, 2, j = 0, \ldots, l$, we have $(x, y) \in S$, so $S'|_{X\times Y} \subseteq S$.

On the other hand, for any $(x, y) \in S$, by (9.6), we can always find $v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}$, $v_{11}^{R+}, \ldots, v_{s1}^{L-}, v_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}, v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+}$, which satisfies the constraints of (9.7b) and (9.7d). Together with the inequalities of $A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \leq b_{k\lambda_j}^{L}$, and $A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \leq b_{k\lambda_j}^{R}$, $k = 1, 2, \quad j = 0, 1, \ldots, l$, requested by S, we have $x, v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s1}^{L-}, v_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}, y, v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots,$ $v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+} \in S'$, thus $(x, y) \in S'|_{X\times Y}$, $S \subseteq S'|_{X\times Y}$.

So, we can prove that

$$S'|_{X\times Y} = S. \tag{9.17}$$

Similarly, we have

$$S(x)'|_{X\times Y} = S(x), \tag{9.18}$$

$$S(X)'|_{X\times Y} = S(X). \tag{9.19}$$

Also, from $\sum_{j=0}^{l} \alpha^{L}_{i2\lambda_j} x + \sum_{j=0}^{l} \beta^{L}_{i2\lambda_j} y + v^{L-}_{i2} - v^{L+}_{i2} = \sum_{j=0}^{l} g^{L}_{Fi\lambda_j}$, and $v^{L-}_{i2} \cdot v^{L+}_{i2} = 0$, we have

$$v^{L-}_{i2} \mp v^{L+}_{i2} = \left| \sum_{j=0}^{l} \alpha^{L}_{i2\lambda_j} x + \sum_{j=0}^{l} \beta^{L}_{i2\lambda_j} y - \sum_{j=0}^{l} g^{L}_{Fi\lambda_j} \right| \tag{9.20}$$

for $i = 1, \ldots, t$. Similarly, we have

$$v^{R-}_{i2} \mp v^{R+}_{i2} = \left| \sum_{j=0}^{l} \alpha^{R}_{i2\lambda_j} x + \sum_{j=0}^{l} \beta^{R}_{i2\lambda_j} y - \sum_{j=0}^{l} g^{R}_{Fi\lambda_j} \right| \tag{9.21}$$

for $i = 1, \ldots, t$. Thus, we obtain

$$P(x') = \{y' \in Y' | y' \in \arg\min \Psi'\} \tag{9.22}$$

where

$$\Psi' = \sum_{i=1}^{t} \sum_{j=0}^{l} \left\{ \left| \alpha^{L}_{i2\lambda_j} x + \beta^{L}_{i2\lambda_j} - g^{L}_{Lh\lambda_j} \hat{y} - g^{L}_{Fi\lambda_j} \right| \right.$$
$$\left. + \left| \alpha^{R}_{i2\lambda_j} x + \beta^{R}_{i2\lambda_j} \hat{y} - g^{R}_{Fi\lambda_j} \right|, \hat{y} \in S(x') \right\}.$$

From (9.17) and (9.22), we obtain

$$p(x')|_{X\times Y} = p(x). \tag{9.23}$$

From (9.8e), (9.14), (9.17), and (9.23), we obtain

$$IR'|_{X\times Y} = IR, \tag{9.24}$$

which means that, in $X \times Y$ space, the leaders of problem (9.5a)–(9.5d) and (9.7a)–(9.7d) have the same optimizing space.

Thus, from (9.16) and (9.24), for any $(x, y) \in IR$, we have

$$\frac{1}{l+1}\sum_{h=1}^{s}\sum_{j=0}^{l}\left\{\left|\alpha_{h1\lambda_j}^{L}x+\beta_{h1\lambda_j}^{L}y-g_{Lh\lambda_j}^{L}\right|+\left|\alpha_{h1\lambda_j}^{R}x+\beta_{h1\lambda_j}^{R}y-g_{Lh\lambda_j}^{R}\right|\right\}$$

$$\geq\frac{1}{l+1}\sum_{h=1}^{s}\sum_{j=0}^{l}\left(\left|\alpha_{h1\lambda_j}^{L}x^*+\beta_{h1\lambda_j}^{L}y^*-g_{Lh\lambda_j}^{L}\right|+\left|\alpha_{h1\lambda_j}^{R}x^*+\beta_{h1\lambda_j}^{R}y^*-g_{Lh\lambda_j}^{R}\right|\right).$$

Consequently, (x^*, y^*) is the optimal solution of problem (9.5a)–(9.5d). □

Adopting the weighting method, (9.7a)–(9.7d) can be further transferred into (9.25a)–(9.25d):

$$\min_{x'\in X'} \quad v_1^- + v_1^+ \tag{9.25a}$$

$$\begin{aligned}\text{s.t.}\quad & \alpha_1 x+\beta_1 y+v_1^- - v_1^+ = \sum_{h=1}^{s}\sum_{j=0}^{l}\left(g_{Lh\lambda_j}^{L}+g_{Lh\lambda_j}^{R}\right)\\ & v_1^-, v_1^+ \geq 0,\\ & v_1^-\cdot v_1^+ = 0,\\ & A_{1\lambda_j}^{L}x+B_{1\lambda_j}^{L}y\leq b_{1\lambda_j}^{L},\\ & A_{1\lambda_j}^{R}x+B_{1\lambda_j}^{R}y\leq b_{1\lambda_j}^{R},\\ & j=0,1,\ldots,l,\end{aligned} \tag{9.25b}$$

$$\min_{y'\in Y'} \quad v_2^- + v_2^+ \tag{9.25c}$$

$$\begin{aligned}\text{s.t.}\quad & \alpha_2 x+\beta_2 y+v_2^- - v_2^+ = \sum_{i=1}^{t}\sum_{j=0}^{l}(g_{Fi\lambda_j}^{L}+g_{Fi\lambda_j}^{R})\\ & v_2^-, v_2^+ \geq 0,\\ & v_2^-\cdot v_2^+ = 0,\\ & A_{2\lambda_j}^{L}x+B_{2\lambda_j}^{L}y\leq b_{2\lambda_j}^{L},\\ & A_{2\lambda_j}^{R}x+B_{2\lambda_j}^{R}y\leq b_{2\lambda_j}^{R},\\ & j=0,1,\ldots,l,\end{aligned} \tag{9.25d}$$

where $x' = (x_1,\ldots,x_n,v_1^-,v_1^+), y' = (y_1,\ldots,y_m,v_2^-,v_2^+), v_1^- = \sum_{h=1}^{s}\left(v_{L1}^{L-}+v_{L1}^{R-}\right)$, $v_1^+ = \sum_{h=1}^{s}\left(v_{h1}^{L+}+v_{h1}^{R+}\right)$, $v_2^- = \sum_{i=1}^{t}\left(v_{i2}^{L-}+v_{i2}^{R-}\right)$, $v_2^+ = \sum_{i=1}^{t}\left(v_{i2}^{L+}+v_{i2}^{R+}\right)$, $\alpha_1 = \sum_{h=1}^{s}\sum_{j=0}^{l}\left(\alpha_{h1\lambda_j}^{L}+\alpha_{h1\lambda_j}^{R}\right)$, $\beta_1 = \sum_{h=1}^{s}\sum_{j=0}^{l}\left(\beta_{h1\lambda_j}^{L}+\beta_{h1\lambda_j}^{R}\right)$, $\alpha_2 = \sum_{i=1}^{t}\sum_{j=0}^{l}\left(\alpha_{i2\lambda_j}^{L}+\alpha_{i2\lambda_j}^{R}\right)$, $\beta_2 = \sum_{i=1}^{t}\sum_{j=0}^{l}\left(\beta_{i2\lambda_j}^{L}+\beta_{i2\lambda_j}^{R}\right)$.

In this formula, v_1^- and v_1^+ are deviational variables representing the under-achievement and over-achievement of goals for a leader, and v_2^- and v_2^+ are deviational variables representing the under-achievement and over-achievement of goals for a follower respectively.

The non-linear conditions of $v_1^- \cdot v_1^+ = 0$, and $v_2^- \cdot v_2^+ = 0$ need not to be maintained if the Kuhn-Tucker approach together with the simplex algorithm are adopted, since only equivalence at an optimum is wanted. Further explanation can be found from (Charnes and Cooper 1961). Thus, the problem (9.25a)–(9.25d) is further transformed into:

$$\min_{(x, v_1^-, v_1^+) \in X'} \quad v_1 = v_1^- + v_1^+ \tag{9.26a}$$

$$\begin{aligned} \text{s.t.} \quad & \alpha_1 x + \beta_1 y + v_1^- - v_1^+ = \sum_{h=1}^{s} \sum_{j=0}^{l} \left(g_{Lh\lambda_j}^L + g_{Lh\lambda_j}^R \right), \\ & v_1^-, v_1^+ \geq 0, \\ & v_1^- \cdot v_1^+ = 0, \\ & A_{1\lambda_j}^L x + B_{1\lambda_j}^L y \leq b_{1\lambda_j}^L, \\ & A_{1\lambda_j}^R x + B_{1\lambda_j}^R y \leq b_{1\lambda_j}^R, \\ & j = 0, 1, \ldots, l, \end{aligned} \tag{9.26b}$$

$$\min_{(y, v_2^-, v_2^+) \in Y'} \quad v_2 = v_2^- + v_2^+ \tag{9.26c}$$

$$\begin{aligned} \text{s.t.} \quad & \alpha_2 x + \beta_2 y + v_2^- - v_2^+ = \sum_{h=1}^{s} \sum_{j=0}^{l} \left(g_{Lh\lambda_j}^L + g_{Lh\lambda_j}^R \right), \\ & v_2^-, v_2^+ \geq 0, \\ & v_2^- \cdot v_2^+ = 0, \\ & A_{2\lambda_j}^L x + B_{2\lambda_j}^L y \leq b_{2\lambda_j}^L, \\ & A_{2\lambda_j}^R x + B_{2\lambda_j}^R y \leq b_{2\lambda_j}^R, \\ & j = 0, 1, \ldots, l, \end{aligned} \tag{9.26d}$$

Problem (9.26a)–(9.26d) is a standard linear bi-level problem that can be solved by the Kuhn-Tucker approach.

Based on the definitions and theorems for the FMO-BLP problem, we will present a solution algorithm for such a problem in the next section.

9.3 Fuzzy Bi-level Goal-Programming Algorithm

Based on the analysis above, the fuzzy bi-level goal-programming algorithm is detailed as:

Algorithm 9.1: Fuzzy Bi-level Goal-programming Algorithm

[Begin]

Step 1: (Input) Obtain relevant coefficients which include:

(1) Coefficients of (9.1);

(2) Coefficients of (9.3);

(3) Satisfactory degreeα;

(4) $\varepsilon > 0$.

Step 2: (Initialize) Let $k = 1$, which is the counter to record current loop. In (9.5), where $\lambda_j \in [\alpha, 1]$, let $\lambda_0 = \alpha$ and $\lambda_1 = 1$ respectively, then each objective will be transferred into four non-fuzzy objective functions, and each fuzzy constraint is converted into four non-fuzzy constraints.

Step 3: (Compute) By introducing auxiliary variables v_1^-, v_1^+, v_2^-, and v_2^+, we obtain the format of problem (9.26). The solution $(x, v_1^-, v_1^+, y, v_2^-, v_2^+)$ of (9.26) is obtained by the Kuhn-Tucker approach.

Step 4: (Compare)

If $k = 1$,

then $(x, v_1^-, v_1^+, y, v_2^-, v_2^+)_1 = (x, v_1^-, v_1^+, y, v_2^-, v_2^+)_2$;

go to Step 5;

else if $||(x, v_1^-, v_1^+, y, v_2^-, v_2^+)_2 - (x, v_1^-, v_1^+, y, v_2^-, v_2^+)_1|| < \varepsilon$,

go to Step 7.

Step 5: (Split) Suppose that there are $(L+1)$ nodes $\lambda_j, j = 0,1,\ldots,L$ in the interval $[\alpha, 1]$, insert L new nodes $\delta_t, (t = 1,2,\ldots,L)$ in $[\alpha, 1]$ such that $\delta_t = (\lambda_{t-1} + \lambda_t)/2$.

Step 6: (Loop) $k = k + 1$.

Step 7: (Output) $(x, y)_2$ is obtained as the final solution.

[End]

9.4 A Numerical Example and Experiments

In this section, we apply the fuzzy bi-level goal-programming algorithm proposed in Sect. 9.3 on a numerical example to illustrate its operation. Experiments are then carried out on some numerical examples with different scales to test the algorithm's performance.

9.4.1 A Numerical Example

To illustrate the fuzzy bi-level goal-programming algorithm, we consider the following FMO-BLP problem.

Example 9.2 **Step 1:** (Input the relevant coefficients).

1. Coefficients of (9.1a)–(9.1d).
   Suppose that the problem has one leader and one follower with two objectives F_1 and F_2 for the leader and f_1 and f_2 for the follower respectively. This FMO-BLP problem is as follows:

$$\begin{aligned}
\max_{x \in X} \quad & F_1(x,y) = \tilde{6}x + \tilde{3}y \\
\max_{x \in X} \quad & F_2(x,y) = -\tilde{3}x + \tilde{6}y \\
\text{s.t.} \quad & -\tilde{1}x + \tilde{3}y \leq \widetilde{21}, \\
& \min_{y \in Y} \quad f_1(x,y) = \tilde{4}x + \tilde{3}y \\
& \min_{y \in Y} \quad f_2(x,y) = \tilde{3}x + \tilde{1}y \\
& \text{s.t.} \quad -\tilde{1}x - \tilde{3}y \leq \widetilde{27},
\end{aligned}$$

   where $x \in R, y \in R$, and $X = \{x | x \geq 0\}$, $Y = \{y | y \geq 0\}$.
   The membership functions for this FMO-BLP problem are as follows:

$$\mu_{\tilde{6}}(x) = \begin{cases} 0, & x < 5 \\ \frac{x^2-25}{11}, & 5 \leq x < 6 \\ 1, & x = 6 \\ \frac{64-x^2}{28}, & 6 < x \leq 8 \\ 0, & x > 8 \end{cases}, \quad \mu_{\tilde{3}}(x) = \begin{cases} 0 & x < 2 \\ \frac{x^2-4}{5} & 2 \leq x < 3 \\ 1 & x = 3, \\ \frac{25-x^2}{16} & 3 < x \leq 5 \\ 0 & x > 5 \end{cases}$$

$$\mu_{-\tilde{3}}(x) = \begin{cases} 0, & x < -5 \\ \frac{16-x^2}{7}, & -4 \leq x < -3 \\ 1, & x = -3 \\ \frac{25-x^2}{16}, & -3 < x \leq -1 \\ 0, & x > -1 \end{cases}, \quad \mu_{\tilde{4}}(x) = \begin{cases} 0, & x < 3 \\ \frac{x^2-9}{7}, & 3 \leq x < 4 \\ 1, & x = 4, \\ \frac{36-x^2}{20}, & 4 < x \leq 6 \\ 0, & x > 6 \end{cases}$$

$$\mu_{\tilde{1}}(x) = \begin{cases} 0, & x < 0.5 \\ \frac{x^2-0.25}{0.75}, & 0.5 \leq x < 1 \\ 1, & x = 1 \\ \frac{4-x^2}{3}, & 1 < x \leq 2 \\ 0, & x > 2 \end{cases}, \quad \mu_{-\tilde{1}}(x) = \begin{cases} 0, & x < -2 \\ \frac{4-x^2}{3}, & -2 \leq x < -1 \\ 1, & x = -1 \\ \frac{x^2-0.25}{0.75}, & -1 < x \leq -0.5 \\ 0, & x > -0.5 \end{cases},$$

$$\mu_{\widetilde{21}}(x) = \begin{cases} 0, & x < 19 \\ \frac{x^2-361}{80}, & 19 \leq x < 21 \\ 1, & x = 21 \\ \frac{625-x^2}{184}, & 21 < x \leq 25 \\ 0, & x > 25 \end{cases}, \quad \mu_{\widetilde{27}}(x) = \begin{cases} 0, & x < 25 \\ \frac{x^2-625}{104}, & 25 \leq x < 27 \\ 1, & x = 27 \\ \frac{961-x^2}{232}, & 27 < x \leq 31 \\ 0, & x > 31 \end{cases}.$$

2. Suppose the membership functions of the fuzzy goals set for the leader are:

$$\mu_{\widetilde{g_{L1}}}(x) = \begin{cases} 0, & x < 15 \\ \frac{x^2-225}{175}, & 15 \leq x < 20 \\ 1, & x = 20 \\ \frac{900-x^2}{500}, & 20 < x \leq 30 \\ 0, & x > 30 \end{cases}, \quad \mu_{\widetilde{g_{L2}}}(x) = \begin{cases} 0, & x < 4 \\ \frac{x^2-16}{48}, & 4 \leq x < 8 \\ 1, & x = 8 \\ \frac{225-x^2}{161}, & 8 < x \leq 15 \\ 0, & x > 15 \end{cases}.$$

The membership functions of the fuzzy goals set for the follower are:

$$\mu_{\widetilde{g_{F1}}}(x) = \begin{cases} 0, & x < 10 \\ \frac{x^2-100}{225}, & 10 \leq x < 15 \\ 1, & x = 15 \\ \frac{400-x^2}{175}, & 15 < x \leq 20 \\ 0, & x > 20 \end{cases}, \quad \mu_{\widetilde{g_{F2}}}(x) = \begin{cases} 0, & x < 7 \\ \frac{x^2-49}{32}, & 7 \leq x < 9 \\ 1, & x = 9 \\ \frac{121-x^2}{40}, & 9 < x \leq 11 \\ 0, & x > 11 \end{cases}.$$

3. Satisfactory degree: $\alpha = 0.2$.
4. $\varepsilon = 0.15$.

Step 2: (Initialize) Let $k = 1$. Associated with this example, we have

$$\begin{aligned}
&\max_{x \in X} \left|\sqrt{11\lambda + 25}x + \sqrt{5\lambda + 4}y - \sqrt{175\lambda + 225}\right| \\
&\quad + \left|\sqrt{64 - 2811\lambda}x - \sqrt{25 - 16\lambda}y - \sqrt{900 - 500\lambda}\right| \\
&\max_{x \in X} \left|-\sqrt{16 - 7\lambda}x + \sqrt{11\lambda + 25}y - \sqrt{48\lambda + 16}\right| \\
&\quad + \left|-\sqrt{8\lambda + 1}x + \sqrt{64 - 28\lambda}y - \sqrt{225 - 161\lambda}\right| \\
&\text{s.t.} \quad -\sqrt{4 - 2\lambda}x + \sqrt{5\lambda + 4}y \le \sqrt{80\lambda + 361}, \\
&\quad -\sqrt{-0.75\lambda + 0.25}x + \sqrt{25 - 16\lambda}y \le \sqrt{625 - 184\lambda}, \\
&\quad \min_{y \in Y} \quad |\sqrt{7\lambda + 9}x + \sqrt{5\lambda + 4}y + \sqrt{225\lambda + 100}| + |\sqrt{36 - 20\lambda}x \\
&\quad - \sqrt{25 - 16\lambda}y - \sqrt{400 - 175\lambda}| \\
&\quad \min_{y \in Y} \quad -|\sqrt{5\lambda + 4}x + \sqrt{0.75\lambda + 0.25}y - \sqrt{32\lambda + 49}| + | \\
&\quad - \sqrt{25 - 16\lambda}x + \sqrt{4 - 3\lambda}y - \sqrt{121 - 40\lambda}| \\
&\quad \text{s.t.} \quad \sqrt{0.75\lambda + 0.25}x + \sqrt{5\lambda + 4}y \le \sqrt{104\lambda + 625}, \\
&\quad \sqrt{4 - 3\lambda}x + \sqrt{25 - 16\lambda}y \le \sqrt{901 - 232\lambda},
\end{aligned}$$

where $\lambda \in [0.2, 1]$.

Referring to the algorithm, only $\lambda_0 = 0.2$ and $\lambda_1 = 1$ are considered initially. Thus four non-fuzzy objective functions and four non-fuzzy constraints for the leader and the follower are generated respectively:

$$\begin{aligned}
&\max_{x \in X} \quad \frac{1}{4}\Big\{\left|\sqrt{27.2}x + \sqrt{5}y - \sqrt{260}\right| + |6x + 3y - 20| \\
&\quad + \left|\sqrt{58.4}x + \sqrt{21.8}y - 20\sqrt{2}\right| + |6x + 3y - 20| \\
&\quad + \left|-\sqrt{14.6}x + \sqrt{27.2}y - \sqrt{25.6}\right| + |-3x + 6y - 8| + | \\
&\quad -\sqrt{2.6}x + \sqrt{58.4}y - \sqrt{192.8}| + | - 3x + 6y - 8|\Big\} \\
&\text{s.t.} \quad -\sqrt{3.4}x + \sqrt{5}y \le \sqrt{377}, \\
&\quad -x + 3y \le 21, \\
&\quad -\sqrt{0.4}x + \sqrt{5}y \le \sqrt{645.8}, \\
&\quad -x + 3y \le 21, \\
&\quad \min_{y \in Y} \quad \frac{1}{4}\{|3x + 2y - 12.04| + |4x + 3y - 19.1| + |6x - 5y - 7.4| \\
&\qquad + |4x - 3y - 10.63| + |-2x + 0.5y - 18.3| \\
&\qquad + |-3x + y - 15| + |-5x + 2y - 9| + |-3x + y - 9|\} \\
&\quad \text{s.t.} \quad \sqrt{0.4}x + \sqrt{5}y \le \sqrt{645.8}, \\
&\quad x + 3y \le 27, \\
&\quad \sqrt{3.4}x + \sqrt{21.8}y \le \sqrt{914.6}, \\
&\quad x + 3y \le 27.
\end{aligned}$$

Step 3: (Compute) By introducing auxiliary variables v_1^-, v_1^+, v_2^-, and v_2^+, we have

$$\begin{aligned}
&\min_{x,v_1^-,v_1^+} \quad v_1^- + v_1^+ \\
&\text{s.t.} \quad 3.083x + 20.076y + v_1^- - v_1^+ = 54.73, \\
&\quad -1.8x + 2.2y \le 19.4, \\
&\quad -x + 3y \le 21, \\
&\quad -0.6x + 4.7y \le 24.3, \\
&\quad -x + 3y \le 21 \\
&\quad \min_{y,v_2^-,v_2^+} \quad v_2^- + v_2^+ \\
&\quad \text{s.t.} \quad 16.498x + 8.205y + v_2^- - v_2^+ = 51.337, \\
&\quad\quad 0.6x + 2.2y \le 25.4, \\
&\quad\quad x + 3y \le 7, \\
&\quad\quad 1.8x + 4.7y \le 30.2, \\
&\quad\quad x + 3y \le 27.
\end{aligned}$$

Using the Kuhn-Tucker approach, the current solution is (1.901, 0, 0, 2.434, 0, 0).

Step 4: (Compare) Because k = 1, go to Step 5.

Step 5: (Split) By inserting a new node $\lambda_1 = (0.2 + 1)/2$, there are a total three nodes of $\lambda_0 = 0.2$, $\lambda_1 = 0.6$ and $\lambda_2 = 1$. Then a total of six non-fuzzy objective functions for the leader and follower, together with six non-fuzzy constraints for the leader and follower respectively, are generated.

Step 6: (Loop) $k = 1 + 1 = 2$, go to Step 3, and a current solution of (2.011, 0, 0, 2.356, 0, 0) is obtained. As $|2.011 - 1.901| + |2.356 - 2.434| = 0.188 > \varepsilon = 0.15$, the algorithm continues until the solution of (1.957, 0, 0, 2.388, 0, 0) is obtained. The computing results are listed in Table 9.1.

Step 7: (Output) As $|1.957 - 1.872| + |2.388 - 2.2.446| = 0.14 < \varepsilon = 0.15$, (x^*, y^*) = (1.957, 2.388) is the final solution of this example. The objectives for the leader and follower under (x^*, y^*) = (1.957, 2.388) are:

$$\begin{cases}
F_1(x^*, y^*) = F_1(1.957, 2.388) = \tilde{6} \cdot 1.957 + \tilde{3} \cdot 2.388, \\
F_2(x^*, y^*) = F_2(1.957, 2.388) = -\tilde{3} \cdot 1.957 + \tilde{6} \cdot 2.388, \\
f_1(x^*, y^*) = f_1(1.957, 2.388) = \tilde{4} \cdot 1.957 + \tilde{3} \cdot 2.388, \\
f_2(x^*, y^*) = f_2(1.957, 2.388) = \tilde{3} \cdot 1.957 + \tilde{1} \cdot 2.388.
\end{cases}$$

Table 9.1 Summary of the running solution

k	x	y	$v_{1\lambda}^+$	$v_{1\lambda}^-$	$v_{2\lambda}^+$	$v_{2\lambda}^-$
1	1.901	2.434	0	0	0	0
2	2.011	2.356	0	0	0	0
3	1.872	2.466	0	0	0	0
4	1.957	2.388	0	0	0	0

9.4.2 Experiments and Evaluation

The fuzzy bi-level goal-programming algorithm was implemented by Visual Basic 6.0, and run on a desktop computer with CPU Pentium 4 2.8 GHz, RAM 1G, Windows XP. To test the performance of the proposed algorithm, the following experiments are carried out.

1. To test the efficiency of the proposed algorithm, we employ ten numerical examples and enlarge the problem scales by changing the numbers of decision variables, objective functions and constraints for both leaders and followers from two to ten simultaneously. For each of these examples, the final solution has been obtained within 5 s.
2. To test the performance of the fuzzy distance measure in Definition 9.1, we adjust the satisfactory degree values from 0 to 0.5 on the ten numerical examples again. At the same time, we change some of the fuzzy coefficients in the constraints by moving the points whose membership values equal 0 by 10 % from the left and right respectively. Experiments reveal that, when a satisfactory degree is set as 0, the average solution will change by about 6 % if some of the constraint coefficients are moved as discussed above. When we increase satisfactory degrees, the average solution change decreases. For the point in which satisfactory degrees are equal to 0.5, the average solution change is 0.

From Experiment (1), we can see that the proposed algorithm is quite efficient. The reason is the fact that final solutions can be reached by solving corresponding linear bi-level programming problems, which can be handled by the Kuhn-Tucker approach.

From Experiment (2), we can see that if we change some coefficients of fuzzy numbers within a small range, solutions will be less sensitive to this change under a higher satisfactory degree. The reason is that, when the satisfactory degree is set to 0, every λ-cut of fuzzy coefficients from 0 to 1 will be considered. Thus, the decision maker can certainly be influenced by minor information.

For a decision-making process involved with fuzzy parameters, decision makers may sometimes make small adjustment on the uncertain information about the preference or circumstances. If the change occurs to the minor information, that is with smaller satisfactory degrees, there should normally be no significant change to the final solution. For example, when estimating future profit, the manufacturer may adjust the possibility of five thousand dollars profit from 2 to 3 %, while the possibility of one hundred thousand dollars profit remains 100 %. In such a situation, there should be no outstanding change for his or her final decision on the device investment. Therefore, to increase the satisfactory degrees is an acceptable strategy for a feasible solution.

From the above analysis, the advantages and disadvantages of the proposed fuzzy bi-level goal-programming algorithm are as follows:

1. This algorithm is quite efficient, as it adopts strategies to transform a non-linear bi-level decision problem into a linear decision problem.
2. When pursuing optimality, the negative effect from conflicting objectives can be avoided and a leader can finally reach his or her satisfactory solution by setting goals for the objectives.
3. The information of the original fuzzy numbers is considered adequately by using a certain number of λ-cuts to approximate the final precise solution.
4. In some situations, this algorithm might suffer from expensive calculation, as the size of λ-cuts will increase exponentially with respect to iteration counts.

9.5 Summary

In a bi-level decision model, the leader and/or the follower may have more than one objective to achieve. This kind of bi-level decision problem is studied by goal programming in this chapter. Meanwhile, we take into consideration the situation where coefficients to formulate a bi-level decision model are not precisely known to us. A fuzzy set method is applied to handle these coefficients, and a fuzzy bi-level goal-programming algorithm is proposed to solve the FMO-BLP problems. A numerical example is then adopted to explain this algorithm. Experiments reveal that the algorithm is quite effective and efficient in solving the FMO-BLP problems.