Chapter 25
The Heath–Jarrow–Morton Framework

Abstract Interest rate modelling can also be performed by starting from the dynamics of the instantaneous forward rate. As we shall see the dynamics of all other quantities of interest can then be derived from it. This approach has its origin in Ho and Lee (J Finance XLI:1011–1029, 1986) but was most clearly articulated in Heath et al. (Econometrica 60(1):77–105, 1992a), to which we shall subsequently refer as Heath–Jarrow–Morton. In this framework, the condition of no riskless arbitrage results in the drift coefficient of the forward rate dynamics being expressed in terms of the forward rate volatility function. The major weakness in implementing the Heath–Jarrow–Morton approach is that the spot rate dynamics are usually path dependent (non-Markovian). We consider a class of functional forms of the forward rate volatility that allow the model to be reduced to a finite dimensional Markovian system of stochastic differential equations. This class contains some important models considered in the literature.

25.1 Introduction

The interest rate derivative models developed in Chap. 23 took as their starting point the dynamics of the instantaneous spot interest rate. The models we derived there also had the characteristic that the market price of interest rate risk appears in the pricing relationships. We saw in Sect. 23.8 that at least in principle it is possible to remove this dependence of the models on preference related quantities. This can be done by expressing terms involving the market price of interest rate risk in terms of market observed quantities such as the currently observed yield curve and volatilities of traded interest rate dependent instruments. However this procedure for rendering spot rate models preference-free can be tedious and for some model specifications may be computationally intensive.

An alternative interest rate modelling approach, originated by Ho and Lee (1986), is the Heath–Jarrow–Morton approach which starts from the dynamics of the forward rate and requires the specification of the initial term structure and the volatility of the associated forward rate. The dynamics of the spot interest rate are then developed from those of the forward rate. The spot interest rate is also an important economic variable whose assessment determines the evolution of the

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bond prices. Heath et al. (1992b) describe how this framework can be used to price and hedge the entire interest rate derivative book of a financial institution thus offering a consistent approach in managing interest rate exposure. However, the major shortcoming of the Heath–Jarrow–Morton approach is that the spot rate dynamics are not path independent (i.e. it is non-Markovian) and the entire history of the term structure has to be carried thus increasing the computational complexity.

The key unobserved input to this approach to term structure modelling is the aforementioned volatility of the forward rates. Many of the forms of the volatility functions reported in the literature have been chosen for analytical convenience rather than on the basis of empirical evidence. In fact apart from the study of Heath et al. (1990), Flesaker (1993), Amin and Morton (1994), Amin and Ng (1993), Ho et al. (2001), and Bhar et al. (2004), there has not been a great deal of empirical research into the appropriate form of the volatility function to be used in the arbitrage free class of models. This is due to the fact that the non-Markovian nature of the stochastic dynamical system makes difficult application of standard econometric estimation procedures.

The non-Markovian feature also makes difficult the expression for prices of term-structure contingent claims in terms of partial differential equations. In the Heath–Jarrow–Morton approach these prices are expressed as expectation operators, under the equivalent martingale measure, of appropriate payoffs. Nowhere in the existing literature is it stated how to consistently turn this expectation operator into a partial differential equation. It is important to be able to do so in order to apply to the evaluation of interest rate sensitive contingent claims many useful computational techniques, as outlined for example in Wilmott et al. (1993). These techniques are the most appropriate to value various path dependent options such as American, Asian etc., but require an expression of the contingent claim price in terms of partial differential equation operators with appropriate boundary conditions.

The notation used in the original Heath–Jarrow–Morton paper allows for a very general dependence of the forward rate volatility functions on path dependent quantities. For the sake of definiteness we shall assume in this chapter that the path dependence of the forward rate volatility functions arises from dependence on the instantaneous spot interest rate and/or a set of discrete fixed-tenor forward rates. As we shall see this specification allows us to develop a fairly broad class of interest rate derivative models.

With such a specification, the instantaneous spot rate process in the Heath–Jarrow–Morton framework can be expressed in terms of a finite dimensional Markovian system. The dimension of the resultant system of stochastic differential equations is dependent on the exact form of the volatility function and it usually includes variables that at first sight seem not to be readily observable. But we shall show how it is possible to express these in terms of forward rates or yields, which may be observable.

The transformation to the Markovian form also allows easier comparison with other approaches such as, Vasicek (1977), Cox et al. (1985a) and Hull and White (1987, 1990, 1994). This is important in the sense that it allows us to easily reconcile many of the alternative approaches to the modelling of the term structure of interest rates.

25.2 The Basic Structure

We denote as $f(t, T)$ the instantaneous forward rate negotiated at time t for instantaneous borrowing at time $T(> t)$. The starting point of the Heath–Jarrow–Morton model of the term structure of interest rates is the stochastic integral equation for the forward rate[1]

$$f(t,T) = f(0,T) + \int_0^t \alpha(v,T,\omega(v))dv + \int_0^t \sigma(v,T,\omega(v))dW(v), \tag{25.1}$$

for $0 \le t \le T$, where $\alpha(v, T, \omega(v))$ and $\sigma(v, T, \omega(v))$ are the instantaneous drift and the volatility function at time v of the forward rate $f(v, T)$, respectively. The instantaneous drift $\alpha(v, T, \omega(v))$ and volatility function $\sigma(v, T, \omega(v))$ of the forward rate $f(v, T)$ could depend through ω on path dependent quantities, such as the instantaneous spot rate and/or a set of discrete tenor forward rates. Thus the specifications in Eq. (25.1) allow for functional forms of the type $\hat{\alpha}(v, T, r(v), f(v, \tau_1), f(v, \tau_2), \cdots, f(v, \tau_m))$ and $\hat{\sigma}(v, T, r(v), f(v, \tau_1), f(v, \tau_2), \cdots, f(v, \tau_m))$ where $f(t, \tau_i)$ is the instantaneous forward rate at time t applicable at the fixed tenor $\tau_i(> t)$ with m such tenors. The noise term $dW(v)$ is the increment of a standard Wiener process generated by a probability measure $\mathbb{P}$. Note that for expositional simplicity in this section we consider only one noise term affecting the evolution of the forward rate. The stochastic integral equation (25.1), may alternatively be expressed as the stochastic differential equation

$$df(t,T) = \alpha(t,T,\omega(t))dt + \sigma(t,T,\omega(t))dW(t). \tag{25.2}$$

It follows from Eq. (25.1) that the instantaneous spot rate $r(t)(\equiv f(t,t))$ is given by the stochastic integral equation

$$r(t) = f(0,t) + \int_0^t \alpha(v,t,\omega(v))dv + \int_0^t \sigma(v,t,\omega(v))dW(v). \tag{25.3}$$

The corresponding stochastic differential equation for $r(t)$ [see Eq. (22.39)] is

$$dr = \mu_r(t)dt + \sigma(t,t,\omega(t))dW(t), \tag{25.4}$$

[1] We refer the reader to Sect. 22.5 for further discussion on the interpretation of (25.1).

where

$$\mu_r(t) = f_2(0,t) + \alpha(t,t,\omega(t)) + \int_0^t \alpha_2(v,t,\omega(v))dv + \int_0^t \sigma_2(v,t,\omega(v))dW(v), \tag{25.5}$$

where f_2, α_2 and σ_2 denote the partial derivative of f, α and σ respectively, with respect to their second arguments. We recall that the bond price at time t is related to the forward rate by

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \qquad 0 \leq t \leq T. \tag{25.6}$$

By the use of Fubini's theorem for stochastic integrals and application of Ito's lemma (see Sect. 22.5.1) the bond price satisfies the stochastic differential equation

$$dP(t,T) = [r(t) + b(t,T)]P(t,T)dt + \sigma_B(t,T)P(t,T)dW(t), \tag{25.7}$$

where

$$\sigma_B(t,T) \equiv -\int_t^T \sigma(t,s,\omega(t))ds, \tag{25.8}$$

and

$$b(t,T) \equiv -\int_t^T \alpha(t,s,\omega(t))ds + \frac{1}{2}\sigma_B^2(t,T). \tag{25.9}$$

A quantity of interest is the money market account

$$A(t) = \exp\left(\int_0^t r(y)dy\right), \tag{25.10}$$

which is the value at time t of a dollar continuously compounded from 0 to t at the instantaneous spot rate r. This quantity may be used to define the relative bond price

$$Z(t,T) = \frac{P(t,T)}{A(t)}, \qquad (0 \leq t \leq T). \tag{25.11}$$

The fact that $dA = r(t)A(t)dt$ and application of the rule for the quotient of two diffusions (see Sect. 6.6) yields the result that the relative bond price satisfies the stochastic differential equation

$$dZ(t,T) = b(t,T)Z(t,T)dt + \sigma_B(t,T)Z(t,T)dW(t). \tag{25.12}$$

25.3 The Arbitrage Pricing of Bonds

Bonds can be priced using exactly the same hedging portfolio as was used in Chap. 23, namely we use bonds of two different maturities. We know from the arbitrage arguments of Chap. 23 that in order that there not exist riskless arbitrage opportunities between bonds of different maturities then the instantaneous excess bond return, risk adjusted by its volatility must equal the market price of interest rate risk; see Eq. (23.9). The relevant bond dynamics in the current context are given by (25.7), so that here Eq. (23.9) becomes

$$\frac{[r(t)+b(t,T)]-r(t)}{\sigma_B(t,T)} = \begin{array}{c}\textit{market price of}\\ \textit{interest rate risk}\end{array} \equiv -\phi(t), \tag{25.13}$$

which simplifies to

$$b(t,T)+\phi(t)\sigma_B(t,T)=0. \tag{25.14}$$

Using expressions (25.8) and (25.9) this last equation may be written explicitly as

$$\int_t^T \alpha(t,s,\omega(t))ds - \frac{1}{2}\left(\int_t^T \sigma(t,s,\omega(t))ds\right)^2 + \phi(t)\int_t^T \sigma(t,s,\omega(t))ds = 0.$$

Keeping t fixed and differentiating with respect to maturity T, the above equation reduces to

$$\alpha(t,T,\omega(t)) - \left(\int_t^T \sigma(t,s,\omega(t))ds\right)\sigma(t,T,\omega(t)) + \phi(t)\sigma(t,T,\omega(t)) = 0,$$

which may be rearranged to

$$\alpha(t,T,\omega(t)) = -\sigma(t,T,\omega(t))\left[\phi(t) - \int_t^T \sigma(t,s,\omega(t))ds\right]. \tag{25.15}$$

Equation (25.15) is the forward rate drift restriction that was first reported by Heath–Jarrow–Morton (Eq. (18) of Heath et al. 1992a). It states that if the bond market is free of riskless arbitrage opportunities then the forward rate drift, the forward rate volatility and the market price of interest rate risk must be tied together as shown by this equation. Heath–Jarrow–Morton show that in fact this condition is both necessary and sufficient for the absence of riskless arbitrage opportunities.

Up to this point Heath–Jarrow–Morton have not done anything conceptually different from the standard arbitrage approach of Chap. 23. However in the Heath–Jarrow–Morton approach Eq. (25.14) is used in a different way. In the standard arbitrage approach, Eq. (25.14) becomes a partial differential equation for the bond price as a function of the assumed driving state variable (usually the instantaneous spot rate). In the Heath–Jarrow–Morton approach, the condition (25.14) becomes the forward rate drift restriction that is used, as we shall see below, to conveniently express the bond price dynamics under an equivalent probability measure. By use of (25.14), the stochastic differential equations (25.7) and (25.12) for $P(t,T)$ and $Z(t,T)$ respectively become

$$dP(t,T) = [r(t) - \phi(t)\sigma_B(t,T)]P(t,T)dt + \sigma_B(t,T)P(t,T)dW(t), \tag{25.16}$$

$$dZ(t,T) = -\phi(t)\sigma_B(t,T)Z(t,T)dt + \sigma_B(t,T)Z(t,T)dW(t). \tag{25.17}$$

At the same time, by substituting (25.15) into (25.3)[2] the stochastic integral equation for $r(t)$ becomes

$$\begin{aligned} r(t) = f(0,t) &+ \int_0^t \sigma(v,t,\omega(v)) \int_v^t \sigma(v,s,\omega(v))ds dv \\ &- \int_0^t \sigma(v,t,\omega(v))\phi(v)dv + \int_0^t \sigma(v,t,\omega(v))dW(v). \end{aligned} \tag{25.18}$$

The key advance in the Heath–Jarrow–Morton approach is that, by an application of Girsanov's theorem to (25.16), Eqs. (25.16)–(25.18), can be written in terms of a different Wiener process generated by an equivalent martingale probability measure $\tilde{\mathbb{P}}$. Thus if we define a new Wiener process $\tilde{W}(t)$ under $\tilde{\mathbb{P}}$ by

$$\tilde{W}(t) = W(t) - \int_0^t \phi(s)ds, \tag{25.19}$$

or in differential form by

$$d\tilde{W}(t) = dW(t) - \phi(t)dt, \tag{25.20}$$

then Eqs. (25.16)–(25.18) become

$$dP(t,T) = r(t)P(t,T)dt + \sigma_B(t,T)P(t,T)d\tilde{W}(t), \tag{25.21}$$

$$dZ(t,T) = \sigma_B(t,T)Z(t,T)d\tilde{W}(t), \tag{25.22}$$

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[2]Note that from (25.15) we have

$$\int_0^t \alpha(v,t,\omega(v))dv = -\int_0^t \sigma(v,t,\omega(v))\phi(v)dv + \int_0^t \sigma(v,t,\omega(v)) \int_v^t \sigma(v,s,\omega(v))ds dv.$$

and

$$r(t)=f(0,t)+\int_0^t \sigma(v,t,\omega(v))\int_v^t \sigma(v,s,\omega(v))dsdv+\int_0^t \sigma(v,t,\omega(v))d\tilde{W}(v). \tag{25.23}$$

Alternatively, Eq. (25.23) can be expressed as the stochastic differential equation

$$\begin{aligned} dr = \Bigg[f_2(0,t) + \frac{\partial}{\partial t}\left(\int_0^t \sigma(v,t,\omega(v))\int_v^t \sigma(v,s,\omega(v))dsdv\right) \\ + \int_0^t \frac{\partial\sigma(v,t,\omega(v))}{\partial t} d\tilde{W}(v)\Bigg] dt + \sigma(t,t,\omega(t))d\tilde{W}(t). \end{aligned} \tag{25.24}$$

It is at times convenient to deal with the ln of the bond price $B(t,T) \equiv \ln P(t,T)$. This quantity, by Ito's lemma, satisfies (under $\tilde{\mathbb{P}}$)

$$dB(t,T) = [r(t) - \frac{1}{2}\sigma_B^2(t,T)]dt + \sigma_B(t,T)d\tilde{W}(t). \tag{25.25}$$

Furthermore, the arbitrage free stochastic integral equation for the forward rate under $\tilde{\mathbb{P}}$ can be written,

$$f(t,T) = f(0,T)+\int_0^t \sigma(v,T,\omega(v))\int_v^T \sigma(v,s,\omega(v))dsdv+\int_0^t \sigma(v,T,\omega(v))d\tilde{W}(v), \tag{25.26}$$

and the corresponding stochastic differential equation as

$$df(t,T) = \sigma(t,T,\omega(t))\int_t^T \sigma(t,s,\omega(t))dsdt + \sigma(t,T,\omega(t))d\tilde{W}(t). \tag{25.27}$$

The essential characteristic of the reformulated stochastic differential and integral equations (25.21)–(25.27) expressed in terms of Brownian motion, under the equivalent probability measure $\tilde{\mathbb{P}}$, is that the empirically awkward market price of risk term, $\phi(t)$, is eliminated from explicit consideration. From the discussion of Girsanov's theorem in Sect. 8.2 [in particular Eqs. (8.38) and (8.42)] we obtain the expression for the Radon–Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^t \phi^2(s)ds + \int_0^t \phi(s)dW(s)\right). \tag{25.28}$$

If we write $\tilde{\mathbb{E}}_t$ to denote mathematical expectation with respect to the equivalent probability measure (i.e. the one associated with $d\tilde{W}(t)$) then from Eq. (25.22)

$$\tilde{\mathbb{E}}_t[dZ(t,T)] = 0.$$

This last equation implies that $Z(t, T)$ is a martingale under $\tilde{\mathbb{P}}$, i.e.

$$Z(t, T) = \tilde{\mathbb{E}}_t(Z(T, T)),$$

or, in terms of the bond price

$$P(t, T) = \tilde{\mathbb{E}}_t\left[\frac{A(t)}{A(T)}\right] = \tilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r(y)dy\right)\right]. \tag{25.29}$$

Equation (25.29) is the fundamental bond pricing equation of the Heath–Jarrow–Morton framework, and it has the same discounted cash flow interpretation as Eq. (23.21). Namely the quantity $\exp\left(-\int_t^T r(y)dy\right)$ should be interpreted as the stochastic discount factor under $\tilde{\mathbb{P}}$ used to discount back to time t the \$1 payoff to be received at time T. We stress that the actual implementation of (25.29) will depend on the form chosen from the forward rate volatility function. At the simplest level, the expectation in Eq. (25.29) could be calculated by numerically simulating Eq. (25.23). Note however that if the volatility function depends on discrete tenor forward rates $f(t, \tau_1), \cdots, f(t, \tau_m)$ then these would have be simulated at the same time. Closed form analytical expressions for the bond price may be obtained with appropriate assumptions on the volatility function as we shall see below.

25.4 Arbitrage Pricing of Bond Options

Suppose we wish to price at time t an option on the bond, for example a European call option on the bond, with the option maturing at T_c. As we saw in Chap. 21 this problem is relevant to the pricing of an interest rate cap. We know from the discussion at the end of the previous section that under the risk-neutral measure $\tilde{\mathbb{P}}$ we can discount the payoff at T_c back to t using the stochastic discount factor

$$\exp\left(-\int_t^{T_c} r(s)ds\right). \tag{25.30}$$

Multiplying the payoff by the discount factor we find that under one realisation of the spot-rate process under $\tilde{\mathbb{P}}$ the option value at t is given by

$$\exp\left(-\int_t^{T_c} r(s)ds\right) \max\left[P(T_c, T) - X, 0\right]. \tag{25.31}$$

The value of the option, $C(t, T_c)$, is then obtained by taking the expectation of this quantity under the risk-neutral measure $\tilde{\mathbb{P}}$ (i.e. forming $\tilde{\mathbb{E}}_t$). Thus we obtain

$$C(t, T_c) = \tilde{\mathbb{E}}_t \left[\exp\left(-\int_t^{T_c} r(s)ds \right) \max[P(T_c, T) - X, 0] \right]. \tag{25.32}$$

In general, if we have some spot interest rate contingent claim with payoff at $t = T_c$ given by $H(r(T_c), T_c)$,[3] then its value at t is given by

$$U(t, T_c) = \tilde{\mathbb{E}}_t \left[\exp\left(-\int_t^{T_c} r(s)ds \right) H(r(T_c), T_c) \right]. \tag{25.33}$$

In order to obtain a pricing partial differential equation for $U(t, T_c)$ we need to obtain the Kolmogorov backward equation associated with Eq. (25.24), the stochastic differential equation for the spot rate process $r(t)$. It is difficult to do this in general because of the non-Markovian term $\int_0^t \frac{\partial \sigma}{\partial t} d\tilde{W}(v)$ that appears in the drift in Eq. (25.24). In Sect. 25.6 we discuss assumptions on σ which allow us to obtain Markovian representations for $r(t)$ and hence obtain pricing partial differential equations.

The reader may wonder why in this section we have not mirrored the argument of Sect. 23.6 and used the hedging argument approach to derive the bond option pricing formula in the present context. The reason is that in Sect. 23.6 there was one underlying factor, $r(t)$, driving the uncertainty of the market, so the option price could be written in the form $C(r, t)$ and Ito's Lemma applied to obtain its dynamics. Similarly in Sect. 24.1 there were two underlying factors, $r(t)$ and $h(t)$, driving the uncertainty of the market and we would write the option price as $C(r, h, t)$. Again application of Ito's Lemma gave us the dynamics for C. In both cases the dynamics of the hedging portfolio could then be obtained. Here we have not so far been so precise about the factors upon which the volatility function $\sigma(t, T, \cdot)$ depends, apart from stating that it could depend on a vector of discrete tenor forward rates and the instantaneous spot rate. In order to mirror the hedging argument approach used in Chap. 23 we need to be more specific about the dynamics of these underlying rates so that we could then obtain the option price dynamics by applying Ito's lemma. This we shall do in a later section, when we discuss the Markovianisation issue. At this point we stress that the expressions (25.29) for the bond price and (25.33) for the interest rate derivative hold for quite general specifications of the volatility function. Of course if we want to implement these expressions, using for example stochastic simulation, then we would need to specify the dynamics (under $\tilde{\mathbb{P}}$) of all stochastic factors entering into the specification of $\sigma(t, T, \cdot)$.

[3] Here we allow the payoff function to depend on the instantaneous spot rate. It could of course depend on various other rates as well.

25.5 Forward-Risk-Adjusted Measure

We saw in Eq. (25.33) that the value of a spot interest rate contingent claim at time t can be written

$$U(t, T_c) = \tilde{\mathbb{E}}_t \left[\exp\left(-\int_t^{T_c} r(s) ds \right) H(r(T_c), T_c) \right], \tag{25.34}$$

where $H(r(T_c), T_c)$ denotes the payoff on the claim at time T_c. Suppose $P(t, T_c)$ represents the price at time t of a pure discount bond maturing at time T_c. Then by Eq. (25.29),

$$P(t, T_c) = \tilde{\mathbb{E}}_t \left[\exp\left(-\int_t^{T_c} r(s) ds \right) \right]. \tag{25.35}$$

We can use the results of Chap. 20 to express the value of the interest rate contingent claim, using $P(t, T_c)$ as the numeraire. By forming the quantity $Y = U/P$ we would obtain [see Eq. (20.14)]

$$U(t, T_c) = P(t, T_c) \mathbb{E}_t^* \left[\frac{U(T_c, T_c)}{P(T_c, T_c)} \right]. \tag{25.36}$$

But in the current notation $U(T_c, T_c) = H(r(T_c), T_c)$ and $P(T_c, T_c) = 1$, hence

$$U(t, T_c) = P(t, T_c) \mathbb{E}_t^* \left[H(r(T_c), T_c) \right]. \tag{25.37}$$

The advantage of (25.37) over (25.34) is that the stochastic discounting term $\exp\left(-\int_t^{T_c} r(s) ds \right)$ which appears in the expectation operation of (25.34) is replaced by the non-stochastic discounting term $P(t, T_c)$ which appears outside the expectation operator of (25.37). The value of this method depends on how easy (or difficult) it is to calculate $\mathbb{E}_t^*$ in (25.37). We saw how this change of measure result was useful in obtaining Merton's bond pricing formula in Sect. 20.3. The measure associated with the $\mathbb{E}_t^*$ operation, which we shall denote as $\mathbb{P}^*$ is known as the T-forward measure. The reason for this nomenclature is that under $\mathbb{P}^*$ the forward rate at time t is the expectation of the instantaneous spot rate at T i.e.

$$f(t, T) = \mathbb{E}_t^* [r(T)]. \tag{25.38}$$

To see this result recall that

$$P(t, T) = \tilde{\mathbb{E}}_t \left[\exp\left(-\int_t^{T} r(s) ds \right) \right].$$

Differentiating this last equation with respect to T we obtain

$$\begin{aligned}\frac{\partial}{\partial T}P(t,T) &= \tilde{\mathbb{E}}_t\left[\frac{\partial}{\partial T}\exp\left(-\int_t^T r(s)ds\right)\right]\\ &= \tilde{\mathbb{E}}_t\left[-\exp\left(-\int_t^T r(s)ds\right)\cdot\frac{\partial}{\partial T}\int_t^T r(s)ds\right]\\ &= -\tilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r(s)ds\right)\cdot r(T)\right]\\ &= -P(t,T)\mathbb{E}_t^*[r(T)].\end{aligned} \tag{25.39}$$

The last line was obtained by applying (25.37) with $H(r(T),T) = r(T)$. Rearranging the last result we obtain

$$-\frac{\partial}{\partial T}\ln P(t,T) = \mathbb{E}_t^*[r(T)]. \tag{25.40}$$

However $f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T)$ hence we have established the result in Eq. (25.38).

For some later applications we need to clarify what form the Radon–Nikodym derivative assumes for the forward risk-adjusted measure. To do this we simply identify V_T with $P(T,T)$ and V_0 with $P(0,T)$ in Eq. (20.20), so that the Radon–Nikodym derivative becomes

$$\xi(0,T) = \frac{P(T,T)}{A(T)P(0,T)} = \frac{1}{A(T)P(0,T)}. \tag{25.41}$$

25.6 Reduction to Markovian Form

The principal difficulty in implementing and estimating Heath–Jarrow–Morton models arises from the non-Markovian noise term in the stochastic integral equation (25.23) for $r(t)$. This manifests itself in the third component of the drift term of the stochastic differential equation (25.24). This component depends on the history of the noise process from time 0 to current time t. Depending upon the specification of the volatility function the second component of the drift term could also depend on the path history up to time t.

Our aim in this section is to consider a class of functional forms of $\sigma(t,T,\cdot)$ that allow the non-Markovian representation of $r(t)$ and $P(t,T)$ to be reduced to a finite dimensional Markovian system of stochastic differential equations. We investigate volatility functions of the forward rate which have the general form of a deterministic function of time and maturity multiplied by a function of the path dependent variable ω, i.e.

$$\sigma(t,T,\omega(t)) = Q(t,T)G(\omega(t)), \quad 0 \le t \le T, \tag{25.42}$$

where G is an appropriately well-behaved function. A useful representation for $Q(t, T)$ would be

$$Q(t,T) = P_n(T-t)e^{-\lambda(T-t)}, \tag{25.43}$$

where $P_n(u)$ is the polynomial

$$P_n(u) = a_0 + a_1 u + \ldots + a_n u^n.$$

This form would allow the term structure of the volatility to exhibit humps as observed in implied forward rate volatilities from cap prices.[4] We recall the discussion of Sect. 22.5.2 when we considered forward rate volatility functions of the form[5]

$$\sigma(t,T,\omega(t)) = \bar{\sigma} e^{-\lambda(T-t)} G(\omega(t)), \tag{25.44}$$

for $\bar{\sigma} > 0$ and λ constant. This structure includes forward rate volatilities for a number of important cases in the literature. Some of these models include

- $\lambda > 0, G(\omega(t)) = 1$ leads to a version of the extended Vasicek model of Hull–White,
- $\lambda > 0, G(\omega(t)) = g(r(t))$ leads to the generalised spot rate model of Ritchken and Sankarasubramanian (1995).
- $\lambda > 0, G(\omega(t)) = \sqrt{r(t)}$ leads to an extended version of the CIR model,
- $\lambda > 0, G(\omega(t)) = g(r(t), f(t,\tau))$ leads to a version of the model of Chiarella and Kwon (1999).

Our aim is, under the volatility specification of (25.44), to express Eqs. (25.24) and (25.27) as a Markovian system of stochastic differential equations. By considering the drift term of the stochastic differential equation (25.27), under the volatility specification of (25.44), we obtain

$$\begin{aligned}
\sigma(t,T,\omega(t)) \int_t^T \sigma(t,s,\omega(t))ds &= \bar{\sigma}^2 G^2(\omega(t)) e^{-\lambda(T-t)} \int_t^T e^{-\lambda(s-t)} ds \\
&= \bar{\sigma}^2 G^2(\omega(t)) e^{-\lambda(T-t)} \frac{\left(e^{-\lambda(T-t)} - 1\right)}{-\lambda} \\
&= \sigma^2(t,T,\omega(t)) \frac{\left(e^{\lambda(T-t)} - 1\right)}{\lambda}.
\end{aligned}$$

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[4] Ritchken and Chuang (1999) assume $P_n(u) = P_1(T-t) = (a_0 + a_1(T-t))$.

[5] It is possible to carry through the discussion of this subsection with Eq. (25.44) generalised to $\sigma(t,T,\omega(t)) = \bar{\sigma} e^{-\int_t^T \lambda(s)ds} G(\omega(t))$.

Thus, we express the stochastic differential equation (25.27) for the forward rate as

$$df(t,T) = \left[\sigma^2(t,T,\omega(t))\frac{e^{\lambda(T-t)}-1}{\lambda}\right]dt + \sigma(t,T,\omega(t))d\tilde{W}(t). \tag{25.45}$$

Similarly, we consider the stochastic differential equation (25.24) for r(t) and in particular the first integral term

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\int_0^t \sigma(v,t,\omega(v))\int_v^t \sigma(v,s,\omega(v))dsdv\right)\\&=\int_0^t\left[\sigma_2(v,t,\omega(v))\int_v^t \sigma(v,s,\omega(v))ds + \sigma^2(v,t,\omega(v))\right]dv,\end{aligned} \tag{25.46}$$

where the notation σ_2 represents the partial derivative of σ with respect to its second argument. Given the expression for σ in (25.44), we have that,

$$\sigma_2(v,t,\omega(v)) = -\lambda\sigma(v,t,\omega(v)). \tag{25.47}$$

Thus, the right hand side of (25.46) reduces to

$$\int_0^t\left[-\lambda\sigma(v,t,\omega(v))\int_v^t \sigma(v,s,\omega(v))ds + \sigma^2(v,t,\omega(v))\right]dv. \tag{25.48}$$

By using (25.47) the other two terms of (25.24) can be expressed as

$$\begin{aligned}&\left[\int_0^t \sigma_2(v,t,\omega(v))d\tilde{W}(v)\right]dt + \sigma(t,t,\omega(t))d\tilde{W}(t)\\&=\left[-\lambda\int_0^t \sigma(v,t,\omega(v))d\tilde{W}(v)\right]dt + \sigma(t,t,\omega(t))d\tilde{W}(t).\end{aligned}$$

Thus, the stochastic differential equation (25.24) becomes

$$\begin{aligned}dr = &\left[f_2(0,t) - \lambda\int_0^t \sigma(v,t,\omega(v))\int_v^t \sigma(v,s,\omega(v))dsdv\right.\\&\left.+\int_0^t \sigma^2(v,t,\omega(v))dv - \lambda\int_0^t \sigma(v,t,\omega(v))d\tilde{W}(v)\right]dt\\&+\sigma(t,t,\omega(t))d\tilde{W}(t).\end{aligned} \tag{25.49}$$

We note from the stochastic integral equation (25.23) that

$$r(t) - f(0,t) = \int_0^t \sigma(v,t,\omega(v)) \int_v^t \sigma(v,s,\omega(v))dsdv + \int_0^t \sigma(v,t,\omega(v))d\tilde{W}(v). \tag{25.50}$$

Then by using (25.50), the stochastic differential equation (25.49) for the spot rate is simplified to

$$dr = [f_2(0,t) + \lambda f(0,t) + \psi(t) - \lambda r(t)]\,dt + \sigma(t,t,\omega(t))d\tilde{W}(t), \tag{25.51}$$

where we define the subsidiary variable $\psi(t)$ as

$$\psi(t) = \int_0^t \sigma^2(v,t,\omega(v))dv. \tag{25.52}$$

The subsidiary variable $\psi(t)$ defined in Eq. (25.52) plays a central role in allowing us to transform the original non-Markovian dynamics to Markovian form. Similar subsidiary variables appear in the reduction to Markovian forms of Cheyette (1992), Ritchken and Sankarasubramanian (1995), Bhar and Chiarella (1997a), Inui and Kijima (1998), and Chiarella and Kwon (1999). It is clear from (25.52) that $\psi(t)$ may be interpreted as a variable summarising the path history of the forward rate volatility.

25.7 Some Special Models

At this point the stochastic differential equation (25.51) is still non-Markovian because the integral in the drift term involves the history of the path dependent forward rate volatility. To proceed any further we need to consider specific functional forms for $G(\omega(t))$ in the volatility specifications (25.44).

25.7.1 The Hull–White Extended Vasicek Model

If $G(\omega(t)) = 1$ then the subsidiary variable (25.52) becomes a time function, i.e.

$$\psi(t) = \int_0^t \sigma^2(v,t,\omega(v))dv = \int_0^t \bar{\sigma}^2 e^{-2\lambda(t-v)}dv = \frac{\bar{\sigma}^2}{2\lambda}(1 - e^{-2\lambda t}).$$

Thus by setting

$$\theta(t) = f_2(0,t) + \lambda f(0,t) + \frac{\bar{\sigma}^2}{2\lambda}(1 - e^{-2\lambda t}), \tag{25.53}$$

the stochastic differential equation (25.51) for $r(t)$ finally becomes

$$dr = [\theta(t) - \lambda r(t)]dt + \bar{\sigma}\, d\tilde{W}(t), \tag{25.54}$$

which is a sought Markovian representation. Clearly this is the extended Vasicek model with the long run mean allowed to be time varying.

Furthermore, note that the expression we have obtained for $\theta(t)$ is the same as the one we obtained in Sect. 23.7 when we worked directly from the expression for the bond price obtained from the continuous arbitrage approach—which takes the spot rate process as the driving dynamics. It is also worth pointing out that by setting $\lambda = 0$ we obtain the continuous time specification of the Ho–Lee model. We have already seen how to price European options in this framework in Sect. 23.7. To price American options in this framework, the option pricing equation

$$\frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 C}{\partial r^2} + [\theta(t) - \lambda r]\frac{\partial C}{\partial r} + \frac{\partial C}{\partial t} - rC = 0, \tag{25.55}$$

must be solved subject to the boundary conditions for an American option, see Chiarella and El-Hassan (1996) for details.

25.7.2 The General Spot Rate Model

If $G(\omega(t))$ is a function of the spot interest rate $r(t)$, i.e. $G(\omega(t)) = g(r(t))$, then we need to separately handle the non-Markovian term appearing in the drift of Eq. (25.51). The subsidiary variable (25.52) now becomes

$$\psi(t) = \int_0^t \bar{\sigma}^2 e^{-2\lambda(t-v)} g^2(r(v))dv. \tag{25.56}$$

By differentiating (25.56) we have that

$$d\psi = [\bar{\sigma}^2 g^2(r(t)) - 2\lambda\psi(t)]dt. \tag{25.57}$$

We are now dealing with the two-dimensional Markovian system

$$\begin{aligned} dr &= [f_2(0,t) + \lambda f(0,t) + \psi(t) - \lambda r(t)]dt + \bar{\sigma}\, g(r(t))d\tilde{W}(t), \\ d\psi &= [\bar{\sigma}^2 g^2(r(t)) - 2\lambda\psi(t)]dt. \end{aligned} \tag{25.58}$$

The representation (25.58) was obtained by Ritchken and Sankarasubramanian (1995). If we define the partial differential operator $\mathscr{K}$ by

$$\mathscr{K} = \frac{1}{2}\bar{\sigma}^2 g^2(r)\frac{\partial^2}{\partial r^2} + [f_2(0,t) + \lambda f(0,t) + \psi - \lambda r(t)]\frac{\partial}{\partial r} + [\bar{\sigma}^2 g^2(r) - 2\lambda\psi]\frac{\partial}{\partial \psi},$$

then the Kolmogorov equation for the transition probability density π is

$$\mathscr{K}\pi + \frac{\partial \pi}{\partial t} = 0,$$

and derivative instruments are priced according to the partial differential equation

$$\mathscr{K}V + \frac{\partial V}{\partial t} - rV = 0,$$

(note $V = P$ for bond price, $V = C$ for option price) subject to appropriate boundary conditions, e.g. $V(r, T, T) = 1$, for bonds, $V(r, T_c, T) = \max[0, P(r, T_c, T) - K]$, for European call options, etc. To evaluate American options we need to employ numerical methods. Chiarella and El-Hassan (1998) have found the method of lines to be very effective in this context.

Note that in the special case of $g(r(t)) = \sqrt{r(t)}$, we are dealing with the extended CIR model

$$\begin{aligned} dr &= [f_2(0,t) + \lambda f(0,t) + \psi(t) - \lambda r(t)]dt + \bar{\sigma}\sqrt{r(t)}d\tilde{W}(t), \\ d\psi &= [\bar{\sigma}^2 r(t) - 2\lambda\psi(t)]dt. \end{aligned} \tag{25.59}$$

25.7.3 The Forward Rate Dependent Volatility Model

In this case, we further generalise the form of the volatility function to include forward interest rates. Here $G(\omega(t))$ can be a function of the spot interest rate, $r(t)$, and of the forward interest rate, $f(t,\tau)$ of a fixed maturity τ, so that $G(\omega(t)) = g(r(t), f(t,\tau))$. For example, $f(t,\tau)$ could be some long-term forward rate. The intuition behind such a specification is that not only the spot interest rate but also a fixed maturity forward interest rate influence the evolution of the term structure. The particular forward rate to be used may depend on the application under consideration. This approach may be considered to be equivalent in some sense to the Brennan and Schwartz (1979) model where a short-term rate and a long-term rate are used to explain the evolution of the term structure. We need to determine the additional state variables necessary to make the system Markovian although with a higher dimension. The associated subsidiary variable (25.52) is given by

$$\psi(t) = \int_0^t \bar{\sigma}^2 e^{-2\lambda(t-v)} g^2(r(v), f(v,\tau))dv. \tag{25.60}$$

By differentiating (25.60), we obtain the stochastic differential equation for $\psi(t)$ as

$$\begin{aligned} d\psi &= [\sigma^2(t,t,\omega(t)) - 2\lambda\psi(t)]dt \\ &= [\bar{\sigma}^2 g^2(r(t), f(t,\tau)) - 2\lambda\psi(t)]dt \end{aligned} \tag{25.61}$$

We have now reduced the non-Markovian stochastic dynamics to a three dimensional Markovian stochastic dynamical system consisting of the stochastic differential equation for the spot rate $r(t)$ [recall (25.51)]

$$dr = [f_2(0,t) + \lambda f(0,t) + \psi(t) - \lambda r(t)]\, dt + \bar{\sigma} g(r(t), f(t,\tau)) d\tilde{W}(t), \tag{25.62}$$

the stochastic differential equation for the discrete forward rate $f(t,\tau)$ [recall (25.45)], namely,

$$\begin{aligned} df(t,\tau) &= \sigma^2(t,\tau,\omega(t)) \frac{(e^{\lambda(\tau-t)} - 1)}{\lambda} dt + \sigma(t,\tau,\omega(t)) d\tilde{W}(t), \\ &= \bar{\sigma}^2 g^2(r(t), f(t,\tau)) e^{-2\lambda(\tau-t)} \frac{(e^{\lambda(\tau-t)} - 1)}{\lambda} dt \\ &\quad + \bar{\sigma} e^{-\lambda(\tau-t)} g(r(t), f(t,\tau)) d\tilde{W}(t), \end{aligned} \tag{25.63}$$

and the stochastic differential equations (25.61) for $\psi(t)$. Finally we recall that the dynamics of the forward rate to any maturity T, $f(t,T)$ is given by (25.45) and so are determined once $r(t)$ and $f(t,\tau)$ are determined. These latter quantities are driven by the three stochastic differential equations (25.61)–(25.63) which together form the Markovian representation. The price of any derivative instrument would then have to depend on $r(t)$ and $f(t,\tau)$. Thus a bond of maturity T would have a price at time t denoted by $P(t,T,r(t), f(t,\tau))$, and this price is also driven by the three-dimensional Markovian stochastic differential equation system referred to above.

25.7.3.1 Interpreting the Subsidiary Variable $\psi(t)$

However, it would perhaps be more satisfying to relate $\psi(t)$ to the market rates $r(t)$ and $f(t,\tau)$. Indeed it turns out that such a relationship does exist for the forward rate volatility function assumed in Eq. (25.44) for the generalised case of $G(\omega(t)) = g(r(t), f(t,\tau))$.

Proposition 25.1 *The subsidiary integrated square variance quantity $\psi(t)$ defined in Eq. (25.60) is related to the rates $r(t)$ and $f(t,\tau)$ via*

$$\psi(t) = \lambda\alpha\,(t,\tau)\,[r\,(t) - f\,(0,t)] - \lambda e^{-\lambda(t-\tau)}\alpha\,(t,\tau)\,[f\,(t,\tau) - f\,(0,\tau)]\,, \tag{25.64}$$

where $\alpha(t,\tau) \equiv \dfrac{e^{-\lambda t}}{e^{-\lambda\tau} - e^{-\lambda t}}$.

For the proof of Proposition 25.1 see Appendix 25.1.

An important consequence of Proposition 25.1 is that it allows us to reduce by one the dimension of the stochastic dynamic system (25.61)–(25.63) to the two-dimensional one consisting of the stochastic differential equations (25.62) and (25.63) with $\psi(t)$ being defined by Eq. (25.64). This reduction in dimension is quite significant if we seek to solve for derivative prices in this framework by use of partial differential equations or lattice based methods as in Bhar et al. (2000), since then we need to deal only with two rather than three spatial variables in the partial differential operator. The reduction is less significant, though still useful, when using Monte-Carlo simulation. This is so since Monte Carlo simulation requires the simulation of the one Wiener increment, $d\tilde{W}(t)$. The generation of $\psi(t)$ by Eq. (25.64) rather than discretising Eq. (25.61) should lead to some computational efficiency.

A consequence of Proposition 25.1 is that we are able to express the forward rate to any maturity T in terms of the two rates $r(t)$ and $f(t,\tau)$.

Proposition 25.2 *The forward rate $f(t,T)$ to any maturity T is given by*

$$\begin{aligned} &f(t,T) - f(0,T) = \\ &\quad - e^{-2\lambda(T-\tau)}\frac{\alpha(\tau,t)}{\alpha(T,t)}\,[f(t,\tau) - f(0,\tau)] + e^{-2\lambda(T-t)}\frac{\alpha(t,\tau)}{\alpha(T,\tau)}\,[r(t) - f(0,t)]\,. \end{aligned} \tag{25.65}$$

For the proof of Proposition 25.2 see Appendix 25.2.

25.7.3.2 The Term Structure of Interest Rates

We recall the Heath–Jarrow–Morton approach of defining a money market account (25.10) and showing that the relative bond price

$$Z(t,T) = \frac{P(t,T)}{A(t)},$$

is a martingale, so that the bond price can be written

$$P(t,T) = \tilde{\mathbb{E}}_t\left[\frac{A(t)}{A(T)}\right] = \tilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r(y)dy\right)\right]. \tag{25.66}$$

Here, $\tilde{\mathbb{E}}_t$ is the expectation taken with respect to the probability distribution generated by the stochastic differential system (25.62) and (25.63). We use

$$\pi\left(r(t^*), f(t^*, \tau) \mid r(t), f(t, \tau)\right),$$

to denote the transition probability density function between t and t^* $(t \le t^*)$. This quantity satisfies the Kolmogorov backward partial differential equation, which for our case is given by,

$$\mathscr{K}\pi + \frac{\partial \pi}{\partial t} = 0,$$

where the operator $\mathscr{K}$ is the infinitesimal generator of the diffusion process for $f(t, \tau)$, $r(t)$ driven by the stochastic differential equations (25.62) and (25.63). It turns out that $\mathscr{K}$ is given by (see Appendix 25.3),

$$\begin{aligned}\mathscr{K}\pi \equiv\ & \sigma_1^2 \frac{\left(e^{\lambda(\tau-t)} - 1\right)}{\lambda} \frac{\partial \pi}{\partial f} + [f_2(0,t) + \lambda f(0,t) + \psi - \lambda r] \frac{\partial \pi}{\partial r} \\ & + \frac{1}{2}\sigma_1^2 \frac{\partial^2 \pi}{\partial f^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \pi}{\partial r^2} + \sigma_1 \sigma_r \frac{\partial^2 \pi}{\partial f \partial r},\end{aligned} \tag{25.67}$$

where $\sigma_1(t) = \sigma(t, \tau, \omega(t))$ and $\sigma_r(t) = \sigma(t, t, \omega(t))$. By application of the Feynman–Kac formula to Eq. (25.66) we find that the bond price $P(t, T, r, f)$ satisfies the partial differential equation,

$$\frac{\partial P}{\partial t} + \mathscr{K} P - r(t)P = 0, \tag{25.68}$$

subject to the terminal condition

$$P(T, T, r, f) = 1,$$

and the boundary conditions

$$\begin{aligned} P(t, T, \infty, f) &= 0, \qquad (f \ge 0), \\ P(t, T, r, \infty) &= 0, \qquad (r \ge 0). \end{aligned}$$

The further boundary conditions $P(t, T, 0, f)$ and $P(t, T, r, 0)$ may be obtained by an extrapolation procedure to be discussed in Bhar et al. (2000). Note that in subsequent discussion we set

$$D(t) \equiv f_2(0, t) + \lambda f(0, t).$$

A consequence of Proposition 25.2 is that it turns out to be possible to obtain an analytical expression for the bond price. In fact we may state the following proposition:

Proposition 25.3 *The price of bonds driven by the Markovian stochastic differential equation system* (25.62) *and* (25.63) *can be expressed as*

$$P(t,T,r,f) = \frac{P(0,T)}{P(0,t)} \exp\left[-\beta(t,T)\,(r(t) - f(0,t)) - \frac{1}{2}\beta^2(t,T)\psi(t)\right]. \tag{25.69}$$

where $\beta(t,T) = \frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right)$, *and* $\psi(t)$ *is defined in Eq.* (25.64).

For the proof of Proposition 25.3 see Appendix 25.4.

The bond pricing equation (25.69) has precisely the same form as the one derived by Ritchken and Sankarasubramanian (1995) who (in current notation) assumed a form for the volatility function in Eq. (25.42) with $G(\omega(t)) = g(r(t))$ which is independent of the forward rate $f(t,\tau)$. In fact the results in Propositions 25.2 and 25.3 can be considerably generalised. Chiarella and Kwon (1999) have shown that (25.69) holds in precisely the same form even when the forward rate volatility depends on a set of discrete forward rates $f(t,\tau_1), f(t,\tau_2), \ldots, f(t,\tau_r)$ where $t \le \tau_1 < \tau_2 < \cdots < \tau_r \le T$. Of course, under these different specifications the history variable $\psi(t)$ will evolve differently but the functional relationship remains the same.

25.7.3.3 Pricing European Bond Options

Consider an option written on the bond of maturity T. We suppose the option matures at time $T_c(< T)$ and denote its price by $C(t,T,r,f)$. This price satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + \mathscr{K}C - rC = 0, \quad (0 \le t \le T_c). \tag{25.70}$$

If we are dealing with a European call option with strike price E then the terminal condition for (25.70) is

$$C(T_c,T,r,f) = [P(T_c,T,r,f) - E]^+ .$$

The boundary conditions at infinity are

$$C(t,T,\infty,f) = 0, \qquad f \ge 0,$$
$$C(t,T,r,\infty) = 0, \qquad r \ge 0.$$

We recall that the bond prices at option maturity for any given values of $r(T_c)$, $f(T_c,\tau)$ can be obtained directly from Eq. (25.69) without the need to solve the bond pricing partial differential equation (25.68). In Bhar et al. (2000), we discuss the solution of the partial differential equation (25.70) by means of the alternating directions implicit (ADI) method.

An alternative approach to pricing the European option is to use the result (also derived by Heath–Jarrow–Morton) that

$$C(t, T, r, f) = \tilde{\mathbb{E}}_t \left[\exp\left(-\int_t^{T_c} r(y)dy \right) [P(T_c, T, r(T_c), f(T_c, \tau)) - E]^+ \right]. \tag{25.71}$$

The expectation in Eq. (25.71) could be approximated by simulating an appropriate number of times the stochastic differential equation system (25.62) and (25.63) from t to T_c.

25.8 Heath–Jarrow–Morton Multi-Factor Models

In our previous discussion, we focussed on the case where only one noise factor was impinging on the forward rate curve. However Heath–Jarrow–Morton framework allows for the possibility of multiple noise sources, i.e.,

$$f(t, T) = f(0, T) + \int_0^t \alpha(v, T, \cdot)dv + \sum_{i=1}^{n} \int_0^t \sigma_i(v, T, \cdot)dW_i(v), \tag{25.72}$$

where the n noise terms dW_i are the increments of independent Wiener processes and the $\sigma_i(t, T, \cdot)$ are the volatility functions associated with each noise term. The manipulations leading to (25.15) in Sect. 25.2 are identical in the multiple noise case and merely involve a little more algebra. Thus setting $T = t$ in (25.72) we have

$$r(t) = f(0, t) + \int_0^t \alpha(v, t, \cdot)dv + \sum_{i=1}^{n} \int_0^t \sigma_i(v, t, \cdot)dW_i(v). \tag{25.73}$$

Substituting (25.72) into (25.6) and following the same procedure that yielded (25.7) we find that the stochastic differential equation for the bond price now becomes

$$dP(t, T) = [r(t) + b(t, T, \cdot)]P(t, T)dt + \sum_{i=1}^{n} a_i(t, T, \cdot)P(t, T)dW_i(t), \tag{25.74}$$

where

$$a_i(t, T, \cdot) = -\int_t^T \sigma_i(t, v, \cdot)dv, \tag{25.75}$$

and

$$b(t,T,\cdot) = -\int_t^T \alpha(t,v,\cdot)dv + \frac{1}{2}\sum_{i=1}^{n} a_i^2(t,T,\cdot). \qquad (25.76)$$

The process for the relative bond price

$$Z(t,T) = \frac{P(t,T)}{A(t)},$$

is easily found to be

$$dZ(t,T) = b(t,T,\cdot)Z(t,T)dt + \sum_{i=1}^{n} a_i(t,T,\cdot)Z(t,T)dW_i(t). \qquad (25.77)$$

Now by forming a portfolio of bonds of $(n+1)$ different maturities and holding these in proportions that ensure the existence of no riskless arbitrage opportunities results in the condition (for interpretation compare with (10.5) that gives the expected excess return condition in the multi-factor case)

$$[r(t) + b(t,T,\cdot)] - r(t) = -\sum_{i=1}^{n} \phi_i(t)a_i(t,T,\cdot), \qquad (25.78)$$

where $\phi_i(t)$ is the market price of risk associated with the ith noise factor. The term on the left-hand side of Eq. (25.78) is the expected excess return on the bond, the term on the right hand side is the sum of the risk-premia $(\phi_i a_i)$ for bearing the risk associated with each source of uncertainty $(W_i(t))$. Equation (25.78) simplifies to

$$b(t,T,\cdot) + \sum_{i=1}^{n} \phi_i(t)a_i(t,T,\cdot) = 0, \qquad (25.79)$$

which is the multifactor analogue of the forward rate drift restriction (25.14). By use of (25.75) Eq. (25.79) reads

$$\int_t^T \alpha(t,v,\cdot)dv = \sum_{i=1}^{n}\left[\frac{1}{2}a_i^2(t,T,\cdot) + \phi_i(t)a_i(t,T,\cdot)\right]. \qquad (25.80)$$

Differentiating this last equation with respect to maturity T we find that

$$\alpha(t,T,\cdot) = -\sum_{i=1}^{n}\sigma_i(t,T,\cdot)\left[\phi_i(t) - \int_t^T \sigma_i(t,v,\cdot)dv\right], \qquad (25.81)$$

which is the forward rate drift restriction in the multi-factor case. Thus substituting (25.81) into (25.73), (25.74) and (25.77) the stochastic differential equations for $r(t)$, $P(t,T)$ and $Z(t,T)$ become respectively, in the arbitrage free environment,

$$\begin{aligned} dr = \Bigg[& f_2(0,t) + \frac{\partial}{\partial t}\sum_{i=1}^{n}\int_0^t \sigma_i(v,t,\cdot)\int_v^t \sigma_i(v,y,\cdot)dydv + \sum_{i=1}^{n}\int_0^t \frac{\partial \sigma_i}{\partial t}(v,t,\cdot)dW_i(v) \\ & - \sum_{i=1}^{n}\phi_i(t)\sigma_i(t,t,\cdot) - \sum_{i=1}^{n}\int_0^t \phi_i(v)\frac{\partial \sigma_i}{\partial t}(v,t,\cdot)dv \Bigg] dt + \sum_{i=1}^{n}\sigma_i(t,t,\cdot)dW_i(t) \end{aligned}$$

$$dP(t,T) = \left[r(t) - \sum_{i=1}^{n}\phi_i(t)a_i(t,T,\cdot) \right] P(t,T)dt + \sum_{i=1}^{n} a_i(t,T,\cdot)P(t,T)dW_i(t),$$

$$dZ(t,T) = -\sum_{i=1}^{n}\phi_i(t)a_i(t,T,\cdot)Z(t,T)dt + \sum_{i=1}^{n} a_i(t,T,\cdot)Z(t,T)dW_i(t).$$

We then form the new set of processes

$$\tilde{W}_i(t) = W_i(t) - \int_0^t \phi_i(s)ds, \quad (i = 1, 2, \cdots, n).$$

By use of Girsanov's theorem these become Wiener processes under the equivalent measure $\tilde{\mathbb{P}}$. Thus the forgoing set of equations become

$$dP(t,T) = r(t)P(t,T)dt + \sum_{i=1}^{n} a_i(t,T,\cdot)P(t,T)d\tilde{W}_i(t), \tag{25.82}$$

$$dZ(t,T) = \sum_{i=1}^{n} a_i(t,T,\cdot)Z(t,T)d\tilde{W}_i(t), \tag{25.83}$$

$$\begin{aligned} dr = \Bigg[& f_2(0,t) + \frac{\partial}{\partial t}\sum_{i=1}^{n}\int_0^t \sigma_i(v,t,\cdot)\int_v^t \sigma_i(v,y,\cdot)dydv \\ & + \sum_{i=1}^{n}\int_0^t \frac{\partial \sigma_i}{\partial t}(v,t,\cdot)d\tilde{W}_i(v) \Bigg] dt + \sum_{i=1}^{n}\sigma_i(t,t,\cdot)d\tilde{W}_i(t). \end{aligned} \tag{25.84}$$

Again under $\tilde{\mathbb{P}}$ the relative bond price $Z(t,T)$ is a martingale, so that

$$Z(t,T) = \tilde{\mathbb{E}}_t\left[Z(T,T)\right], \tag{25.85}$$

which in terms of the bond price becomes

$$P(t,T) = \tilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r(y)dy\right)\right], \tag{25.86}$$

which of course is the same as (25.29). The difference here is the $\tilde{\mathbb{E}}_t$ is generated by (25.84) which in its turn is driven by the n independent noise terms $\tilde{W}_i(t)$. Thus if we were to use simulation to directly evaluate (25.86) we would need to simulate n independent sequences of normal random variables in order to simulate a path for $r(t)$. Furthermore if the $\sigma_i(t, T)$ depend on other factors, such as discrete tenor forward rates, then the processes for these (under $\tilde{\mathbb{P}}$) would have to be simulated as well.

25.9 Relating Heath–Jarrow–Morton to Hull–White Two-Factor Models

We have already seen in Sect. 25.7.1 that the Hull–White extended Vasicek model can be derived (far more simply) in the Heath–Jarrow–Morton framework once an appropriate form for the volatility function is chosen. In this section we show that the Hull–White two-factor model can be obtained as a special case of the multi-factor Heath–Jarrow–Morton model.

First we recall the Hull–White two-factor model Hull and White (1994), as summarised by Rebonato (1998). The instantaneous spot rate is assumed to follow the process

$$dr = [\theta(t) + h(t) - ar(t)]dt + \gamma_1 dz_1, \tag{25.87}$$

where the additional term $h(t)$ in the drift satisfies

$$dh = -b\, h(t)dt + \gamma_2 dz_2. \tag{25.88}$$

Here z_1, z_2 are correlated Wiener processes, i.e.

$$\mathbb{E}[dz_1\ dz_2] = \rho dt.$$

First we note that we may reexpress (25.87) and (25.88) in terms of the independent Wiener processes w_1, w_2 as

$$dr = [\theta(t) + h(t) - ar(t)]dt + \gamma_1 \sqrt{1-\rho^2}\, dw_1 + \gamma_1 \rho\, dw_2, \tag{25.89}$$

$$dh = -b\, h(t)\, dt + \gamma_2 dw_2. \tag{25.90}$$

We consider a two-factor Heath–Jarrow–Morton model with volatility specifications

$$\sigma_i(t, T) = \bar{\sigma}_i e^{-\lambda_i(T-t)}, \tag{25.91}$$

where $\bar{\sigma}_i$ and λ_i are constants, for $i = 1, 2$. Accordingly, the forward rate dynamics are expressed as

$$f(t,T) = f(0,T) + \int_0^t \alpha(v,T)dv + \int_0^t \sigma_1(v,T)dW_1(v) + \int_0^t \sigma_2(v,T)dW_2(v),$$

which under the risk-neutral measure (see Sect. 25.8) becomes

$$f(t,T) = f(0,T) + \sum_{i=1}^{2}\int_0^t \sigma_i(v,T)\int_v^T \sigma_i(v,y)dy\,dv + \sum_{i=1}^{2}\int_0^t \sigma_i(v,T)d\tilde{W}_i(v).$$

The spot rate process under the risk-neutral measure satisfies

$$r(t) = f(0,t) + \sum_{i=1}^{2}\int_0^t \sigma_i(v,t)\int_v^t \sigma_i(v,y)dy\,dv + \sum_{i=1}^{2}\int_0^t \sigma_i(v,t)d\tilde{W}_i(v), \tag{25.92}$$

or

$$\begin{aligned} dr = \Bigg[& f_2(0,t) + \frac{\partial}{\partial t}\sum_{i=1}^{2}\int_0^t \sigma_i(v,t)\int_v^t \sigma_i(v,y)dy\,dv \\ & + \sum_{i=1}^{2}\int_0^t \frac{\partial \sigma_i}{\partial t}(v,t)d\widetilde{W_i}(v)\Bigg]dt + \sum_{i=1}^{2}\sigma_i(t,t)d\tilde{W}_i(t). \end{aligned} \tag{25.93}$$

The volatility functions (25.91) have the property

$$\frac{\partial \sigma_i(t,T)}{\partial T} = -\lambda_i \sigma_i(t,T).$$

Thus the stochastic differential equation (25.93) for the spot rate becomes

$$dr = \left[f_2(0,t) + \frac{\partial}{\partial t}\sum_{i=1}^{2} S_i(t) - \sum_{i=1}^{2}\lambda_i x_i(t)\right]dt + \sum_{i=1}^{2}\bar{\sigma}_i\,d\tilde{W}_i(t), \tag{25.94}$$

where we set

$$S_i(t) = \bar{\sigma}_i^2 \int_0^t e^{-\lambda_i(t-v)}\int_v^t e^{-\lambda_i(y-v)}dy\,dv,$$

and

$$x_i(t) = \int_0^t \bar{\sigma}_i e^{-\lambda_i(t-v)}d\tilde{W}_i(v),$$

where $i = 1, 2$. The variables $x_i(t)$ satisfy the stochastic differential equations

$$\begin{aligned} dx_i &= \bar{\sigma}_i e^{-\lambda_i(t-t)} d\tilde{W}_i(t) - \int_0^t \lambda_i \bar{\sigma}_i e^{-\lambda_i(t-v)} d\tilde{W}_i(v) dt \\ &= -\lambda_i x_i(t) dt + \bar{\sigma}_i d\tilde{W}_i(t). \end{aligned} \tag{25.95}$$

The system (25.94), (25.95) for $r(t)$, $x_1(t)$, and $x_2(t)$ is the Markovian system that generates the probability distribution for $\tilde{\mathbb{E}}_t$ (i.e. this is the system we would simulate if we use Monte-Carlo simulation to evaluate $\tilde{\mathbb{E}}_t$). To obtain the link with the Hull–White two-factor model note that with the volatility functions (25.91) the stochastic integral equation for $r(t)$, see Eq. (25.92), becomes

$$r(t) = f(0,t) + \sum_{i=1}^{2} S_i(t) + x_1(t) + x_2(t). \tag{25.96}$$

This last equation may be used to eliminate $x_1(t)$ in (25.94), thus

$$x_1(t) = r(t) - f(0,t) - \sum_{i=1}^{2} S_i(t) - x_2(t),$$

which upon substitution into (25.94) yields

$$dr = [f_2(0,t) + \lambda_1 f(0,t) + \mathbf{S}(t) + (\lambda_1 - \lambda_2)x_2(t) - \lambda_1 r(t)]dt + \sum_{i=1}^{2} \bar{\sigma}_i d\tilde{W}_i(t), \tag{25.97}$$

where we have defined

$$\mathbf{S}(t) = \sum_{i=1}^{2} \left[\frac{\partial}{\partial t} S_i(t) + \lambda_1 S_i(t) \right], \tag{25.98}$$

and $x_2(t)$ is driven by Eq. (25.95) with $i = 2$, viz

$$dx_2 = -\lambda_2 x_2(t) dt + \bar{\sigma}_2 d\tilde{W}_2(t). \tag{25.99}$$

To fully obtain the correspondence with the Hull–White two-factor model in Eqs. (25.89), (25.90) set

$$\begin{aligned} \theta(t) &= f_2(0,t) + \lambda_1 f(0,t) + \mathbf{S}(t), \\ a &= \lambda_1, \\ b &= \lambda_2, \end{aligned} \tag{25.100}$$

so that we are now dealing with the system

$$dr = [\theta(t) + (a-b)x_2(t) - ar(t)]dt + \bar{\sigma}_1 d\tilde{W}_1(t) + \bar{\sigma}_2 d\tilde{W}_2(t), \quad (25.101)$$

$$dx_2 = -b\, x_2(t)dt + \bar{\sigma}_2 d\tilde{W}_2(t). \quad (25.102)$$

If we set

$$h(t) = (a-b)x_2(t)$$

then Eq. (25.102) becomes

$$dh = -bh(t)dt + (a-b)\bar{\sigma}_2 d\tilde{W}_2(t). \quad (25.103)$$

The system (25.101) and (25.103) for $r(t)$ and $h(t)$ is equivalent to the Hull–White two-factor system (25.89), (25.90) if we set

$$\bar{\sigma}_1 = \gamma_1\sqrt{1-\rho^2}, \quad \bar{\sigma}_2 = \gamma_1\rho \quad \text{and} \quad (a-b)\bar{\sigma}_2 = \gamma_2, \quad (25.104)$$

from which the parameters of the Hull–White two-factor model may be related to the parameters of the two-factor Heath–Jarrow–Morton model via

$$\rho = \frac{\bar{\sigma}_2}{\sqrt{\bar{\sigma}_1^2 + \bar{\sigma}_2^2}}, \quad \gamma_1 = \sqrt{\bar{\sigma}_1^2 + \bar{\sigma}_2^2}, \quad \gamma_2 = (\lambda_1 - \lambda_2)\bar{\sigma}_2, \quad b = \lambda_2. \quad (25.105)$$

It is certainly instructive to understand how the Hull–White class of models can be derived within the Heath–Jarrow–Morton framework. However, the biggest advantage is that the $\theta(t)$ function in the stochastic differential equation (25.101) for $r(t)$ is automatically calibrated to the initially observed forward curve $f(0,t)$.

25.10 The Covariance Structure Implied by the Heath–Jarrow–Morton Model

Another important issue to be considered is the analysis of the statistical properties of the evolution of the forward rates and yields under the jump-diffusion framework. As we have mentioned before one factor models allow for only parallel shifts of the yield curve, so bond prices and forward rates of all maturities are perfectly correlated. Multi dimensional models, on the other hand, impose a correlation structure between forward rates of different maturities which based on empirical studies shows an exponentially decaying behavior. Rebonato (1998) provides an interesting discussion on forward rate correlations and examines the patterns observed in financial markets. Here we seek to understand the effect on the forward rate correlation structure implied by the assumptions concerning in the forward rate dynamics.

25.10.1 The Covariance Structure of the Forward Rate Changes

Under the risk neutral measure, the changes of the forward rate follow the dynamics

$$df(t,T) = \sum_{i=1}^{n} \sigma_i(t,T,\omega(t))\zeta_i(t,T,\omega(t))dt + \sum_{i=1}^{n} \sigma_i(t,T,\omega(t))d\tilde{W}_i(t). \tag{25.106}$$

Thus

$$\tilde{\mathbb{E}}_0[df(t,T)] = \sum_{i=1}^{n} \sigma_i(t,T,\omega(t))\zeta_i(t,T,\omega(t))dt. \tag{25.107}$$

Denote T_1 and T_2 two different maturities, then the covariance of the changes on the forward rate is calculated as

$$\begin{aligned}
&\mathrm{cov}_0[df(t,T_1), df(t,T_2)] \\
&= \tilde{\mathbb{E}}_0[(df(t,T_1) - \tilde{\mathbb{E}}_0[df(t,T_1)])(df(t,T_2) - \tilde{\mathbb{E}}_0[df(t,T_2)])] \\
&= \tilde{\mathbb{E}}_0\left[\sum_{i=1}^{n} \sigma_i(t,T_1,\omega(t))d\tilde{W}_i(t) \cdot \sum_{i=1}^{n} \sigma_i(t,T_2,\omega(t))d\tilde{W}_i(t)\right].
\end{aligned}$$

From the independence of the Wiener increments it readily follows that

$$\mathrm{cov}_0[df(t,T_1), df(t,T_2)] = \sum_{i=1}^{n} \sigma_i(t,T_1,\omega(t))\sigma_i(t,T_2,\omega(t))dt, \tag{25.108}$$

and the variance of the forward rate changes $df(t,T_h)$ $(h = 1,2)$ as

$$\mathrm{var}_0[df(t,T_h)] = \sum_{i=1}^{n} \sigma_i^2(t,T_h,\omega(t))dt. \tag{25.109}$$

The correlation coefficient between the forward rates changes $df(t,T_1)$ and $df(t,T_2)$ is then evaluated as

$$\rho(t,T_1,T_2) = \frac{\mathrm{cov}_0[df(t,T_1), df(t,T_2)]}{\sqrt{\mathrm{var}_0[df(t,T_1)]}\sqrt{\mathrm{var}_0[df(t,T_2)]}}, \tag{25.110}$$

where $\text{cov}_0[df(t,T_1), df(t,T_2)]$ and $\text{var}_0[df(t,T_h)]$, $(h = 1, 2)$ are defined above. To demonstrate these results, we assume the volatility functions are of the form

$$\sigma_i(s,t) = \sigma_{0i} e^{-\kappa_{\sigma i}(t-s)}, \qquad i = 1, \dots, n, \tag{25.111}$$

where the σ_{0i}, $\kappa_{\sigma i}$ are constant. Then the covariance (25.108) between $df(t,T_1)$ and $df(t,T_2)$ is calculated as

$$\text{cov}_0[df(t,T_1), df(t,T_2)] = \sum_{i=1}^{n} \sigma_{0i}^2 e^{-\kappa_{\sigma i}(T_1+T_2-2t)}, \tag{25.112}$$

and the correlation coefficient between the forward rates changes $df(t,T_1)$ and $df(t,T_2)$ is evaluated as

$$\rho(t,T_1,T_2) = \frac{\sum_{i=1}^{n} \sigma_{0i}^2 e^{-\kappa_{\sigma i}(T_1+T_2-2t)}}{\sqrt{\text{var}_0[df(t,T_1)]}\sqrt{\text{var}_0[df(t,T_2)]}}, \tag{25.113}$$

where the variance of the forward rate changes $df(t,T_h)$ $(h = 1, 2)$ is

$$\text{var}_0[df(t,T_h)] = \sum_{i=1}^{n} \sigma_{0i}^2 e^{-2\kappa_{\sigma i}(T_h-t)}. \tag{25.114}$$

25.10.2 The Covariance Structure of the Forward Rate

The forward rate under the risk neutral measure is given by

$$\begin{aligned} f(t,T) = f(0,T) &+ \sum_{i=1}^{n} \int_0^t \sigma_i(s,T,\omega(s))\zeta_i(s,T,\omega(s))ds \\ &+ \sum_{i=1}^{n} \int_0^t \sigma_i(s,T,\omega(s))d\tilde{W}_i(s). \end{aligned} \tag{25.115}$$

Thus

$$\tilde{\mathbb{E}}_0[f(t,T)] = f(0,T) + \sum_{i=1}^{n} \int_0^t \sigma_i(s,T,\omega(s))\zeta_i(s,T,\omega(s))ds. \tag{25.116}$$

Denote T_1 and T_2 two maturities then the covariance of the forward rates $f(t, T_1)$ and $f(t, T_2)$ is calculated as

$$\begin{aligned}
&\text{cov}_0[f(t, T_1), f(t, T_2)] \\
&= \tilde{\mathbb{E}}_0[(f(t, T_1) - \tilde{\mathbb{E}}_0[f(t, T_1)])(f(t, T_2) - \tilde{\mathbb{E}}_0[f(t, T_2)])] \\
&= \tilde{\mathbb{E}}_0\left[\sum_{i=1}^{n}\int_0^t \sigma_i(s, T_1, \omega(s))d\tilde{W}_i(s) \cdot \sum_{i=1}^{n}\int_0^t \sigma_i(s, T_2, \omega(s))d\tilde{W}_i(s)\right].
\end{aligned} \tag{25.117}$$

Using the result

$$\mathbb{E}_0[\int_0^t \sigma_i(s, T_1)d\tilde{W}_i(s)\int_0^t \sigma_i(s, T_2)d\tilde{W}_i(s)] = \int_0^t \sigma_i(s, T_1)\sigma_i(s, T_2)ds,$$

and the covariance is given by

$$\text{cov}_0[f(t, T_1), f(t, T_2)] = \sum_{i=1}^{n}\int_0^t \sigma_i(s, T_1)\sigma_i(s, T_2)ds. \tag{25.118}$$

Considering again the volatility functions of the form

$$\sigma_i(s, t) = \sigma_{0i}e^{-\kappa_{\sigma i}(t-s)}, \qquad i = 1, \ldots, n, \tag{25.119}$$

then

$$\int_0^t \sigma_i(s, T_1)\sigma_i(s, T_2)ds = \frac{\sigma_{0i}^2 e^{-\kappa_{\sigma i}(T_1+T_2)}}{2\kappa_{\sigma i}}(e^{2\kappa_{\sigma i}t} - 1). \tag{25.120}$$

The covariance between the forward rates $f(t, T_1)$ and $f(t, T_2)$ becomes

$$\text{cov}_0[f(t, T_1), f(t, T_2)] = \sum_{i=1}^{n}\frac{\sigma_{0i}^2 e^{-\kappa_{\sigma i}(T_1+T_2)}}{2\kappa_{\sigma i}}(e^{2\kappa_{\sigma i}t} - 1), \tag{25.121}$$

and the correlation coefficient $\rho(t, T_1, T_2)$ between the forward rates $f(t, T_1)$ and $f(t, T_2)$ is evaluated as

$$\frac{\sum_{i=1}^{n}\frac{\sigma_{0i}^2 e^{-\kappa_{\sigma i}(T_1+T_2)}}{2\kappa_{\sigma i}}(e^{2\kappa_{\sigma i}t} - 1)}{\sqrt{\text{var}_0[f(t, T_1)]}\sqrt{\text{var}_0[f(t, T_2)]}}, \tag{25.122}$$

where the variance of the forward rate $f(t, T_h)$ $(h = 1, 2)$ is

$$\text{var}_0[f(t, T_h)] = \sum_{i=1}^{n}\frac{\sigma_{0i}^2 e^{-2\kappa_{\sigma i}T_h}}{2\kappa_{\sigma i}}(e^{2\kappa_{\sigma i}t} - 1). \tag{25.123}$$

25.11 Appendix

Appendix 25.1 Proof of Proposition 25.1

Recall that $r(t)$ satisfies the stochastic integral equation (25.23) and $f(t,\tau)$ satisfies the stochastic integral equation (25.26) with T set equal to τ. We assume the forward rate volatility specifications

$$\sigma\left(v,t,\omega(v)\right)=\bar{\sigma}e^{-\lambda(t-v)}g\left(r(v),f(v,\tau)\right)$$

and set

$$\begin{aligned}\sigma^{*}\left(v,t,\omega(v)\right)&=\sigma\left(v,t,\omega(v)\right)\int_{v}^{t}\sigma\left(v,s,\omega(v)\right)ds\\&=\bar{\sigma}^{2}e^{-\lambda(t-v)}g\left(r(v),f(v,\tau)\right)\int_{v}^{t}e^{-\lambda(s-v)}g\left(r(v),f(v,\tau)\right)ds\\&=\bar{\sigma}^{2}g^{2}\left(r(v),f(v,\tau)\right)e^{-\lambda(t-v)}\left(\frac{1-e^{-\lambda(t-v)}}{\lambda}\right).\end{aligned}$$

Note that the first integral term in Eq. (25.23) can be written

$$\begin{aligned}\int_{0}^{t}\sigma^{*}\left(v,t,\omega(v)\right)dv&=\bar{\sigma}^{2}\int_{0}^{t}g^{2}\left(r(v),f(v,\tau)\right)e^{-\lambda(t-v)}\frac{\left(1-e^{-\lambda(t-v)}\right)}{\lambda}dv\\&=\frac{e^{-\lambda t}\bar{\sigma}^{2}}{\lambda}\int_{0}^{t}g^{2}(r(v),f(v,\tau))e^{\lambda v}dv\\&\quad-\frac{e^{-2\lambda t}\bar{\sigma}^{2}}{\lambda}\int_{0}^{t}g^{2}(r(v),f(v,\tau))e^{2\lambda v}dv\\&\equiv\frac{e^{-\lambda t}}{\lambda}I(t;\lambda)-\frac{e^{-2\lambda t}}{\lambda}I(t;2\lambda).\end{aligned}$$

Next note that the second integral in Eq. (25.23) may be written as

$$\begin{aligned}\int_{0}^{t}\sigma\left(v,t,\omega(v)\right)d\tilde{W}(v)&=\bar{\sigma}\int_{0}^{t}e^{-\lambda(t-v)}g\left(r(v),f(v,\tau)\right)d\tilde{W}(v)\\&=\bar{\sigma}e^{-\lambda t}\int_{0}^{t}g\left(r(v),f(v,\tau)\right)e^{\lambda v}d\tilde{W}(v)\\&\equiv e^{-\lambda t}J(t;\lambda).\end{aligned}$$

Similarly the first integral term in Eq. (25.26) can be written

$$\int_0^t \sigma^* (v, \tau, \omega(v))\, dv = \bar{\sigma}^2 \int_0^t g^2 (r(v), f(v, \tau))\, e^{-\lambda(\tau - v)} \frac{\left(1 - e^{-\lambda(\tau - v)}\right)}{\lambda} dv$$

$$= \frac{e^{-\lambda\tau}}{\lambda} I(t; \lambda) - \frac{e^{-2\lambda\tau}}{\lambda} I(t; 2\lambda).$$

The second integral term in Eq. (25.26) may be similarly treated, so that

$$\int_0^t \sigma (v, \tau, \omega(v)) d\tilde{W}(v) = \bar{\sigma} \int_0^t e^{-\lambda(\tau - v)} g\,(r(v), f(v, \tau)) d\tilde{W}(v)$$

$$= \bar{\sigma} e^{-\lambda\tau} \int_0^t e^{\lambda v} g\,(r(v), f(v, \tau)) d\tilde{W}(v)$$

$$\equiv e^{-\lambda\tau} J(t; \lambda).$$

We may thus write the stochastic integral equations for $r(t)$ and $f(t, \tau)$ in terms of the integrals $I(t; \lambda)$, $I(t; 2\lambda)$ and $J(t; \lambda)$ as

$$r(t) = f(0, t) + \frac{e^{-\lambda t}}{\lambda} I(t; \lambda) - \frac{e^{-2\lambda t}}{\lambda} I(t; 2\lambda) + e^{-\lambda t} J(t; \lambda), \tag{25.124}$$

$$f(t, \tau) = f(0, \tau) + \frac{e^{-\lambda\tau}}{\lambda} I(t; \lambda) - \frac{e^{-2\lambda\tau}}{\lambda} I(t; 2\lambda) + e^{-\lambda\tau} J(t; \lambda). \tag{25.125}$$

We note that Eqs. (25.124) and (25.125) can be re-expressed as

$$r(t) - f(0, t) + \frac{e^{-2\lambda t}}{\lambda} I(t; 2\lambda) = e^{-\lambda t} \left[\frac{I(t; \lambda)}{\lambda} + J(t; \lambda) \right],$$

$$f(t, \tau) - f(0, \tau) + \frac{e^{-2\lambda\tau}}{\lambda} I(t; 2\lambda) = e^{-\lambda\tau} \left[\frac{I(t; \lambda)}{\lambda} + J(t; \lambda) \right].$$

We may combine the above equations to express $I(t; 2\lambda)$ as a function of $r(t)$ and $f(t, \tau)$, i.e.,

$$I(t; 2\lambda) = \frac{\lambda e^{\lambda\tau}}{e^{-\lambda t} - e^{-\lambda\tau}} [f(t, \tau) - f(0, \tau)] - \frac{\lambda e^{\lambda t}}{e^{-\lambda t} - e^{-\lambda\tau}} [r(t) - f(0, t)] \tag{25.126}$$

Finally we note that

$$\psi(t) = \int_0^t \sigma^2 (v, t, \omega(v))\, dv = \bar{\sigma}^2 \int_0^t e^{-2\lambda(t - v)} g^2 (r(v), f(v, \tau))\, dv$$

$$= \bar{\sigma}^2 e^{-2\lambda t} \int_0^t e^{2\lambda v} g^2 (r(v), f(v, \tau))\, dv = e^{-2\lambda t} I(t; 2\lambda).$$

Thus we finally have

$$\psi(t) = \lambda\alpha(t,\tau)\,[r(t) - f(0,t)] - \lambda e^{-\lambda(t-\tau)}\alpha(t,\tau)\,[f(t,\tau) - f(0,\tau)]\,, \tag{25.127}$$

where we set

$$\alpha(t,\tau) \equiv \frac{e^{-\lambda t}}{e^{-\lambda\tau} - e^{-\lambda t}}.$$

Appendix 25.2 Proof of Proposition 25.2

It is readily verified that the manipulations that led to Eq. (25.125) of Appendix 25.1 are equally valid for t set to a general maturity T. Thus (25.126) holds for t set to T, i.e.,

$$\begin{aligned} I(t;2\lambda) &= \frac{\lambda e^{\lambda T}}{e^{-\lambda t}-e^{-\lambda T}}\,[f(t,T)-f(0,T)] - \frac{\lambda e^{\lambda t}}{e^{-\lambda t}-e^{-\lambda T}}\,[r(t)-f(0,t)] \\ &= e^{2\lambda t}\psi(t). \end{aligned}$$

Substituting the expression for $\psi(t)$ we find that

$$\begin{aligned} I(t;2\lambda) &= \lambda e^{2\lambda t}\left(\alpha(t,\tau)[r(t) - f(0,t)] - e^{-\lambda(t-\tau)}\alpha(t,\tau)[f(t,\tau) - f(0,\tau)]\right) \\ &= \lambda\left(\frac{e^{\lambda T}}{e^{-\lambda t} - e^{-\lambda T}}[f(t,T) - f(0,T)] - \frac{e^{\lambda t}}{e^{-\lambda t} - e^{-\lambda T}}[r(t) - f(0,t)]\right). \end{aligned}$$

On rearranging

$$\begin{aligned} \frac{e^{\lambda T}}{e^{-\lambda t} - e^{-\lambda T}}[f(t,T) - f(0,T)] = &\frac{e^{\lambda t}}{e^{-\lambda t} - e^{-\lambda T}}[r(t) - f(0,t)] \\ &+ e^{2\lambda t}\alpha(t,\tau)[r(t) - f(0,t)] \\ &- e^{2\lambda t}e^{-\lambda(t-\tau)}\alpha(t,\tau)[f(t,\tau) - f(0,\tau)], \end{aligned}$$

from which

$$\begin{aligned} f(t,T) - f(0,T) = [r(t) - f(0,t)]&\left(\frac{e^{\lambda t}}{e^{\lambda T}} + \frac{e^{2\lambda t}(e^{-\lambda t} - e^{-\lambda T})}{e^{\lambda T}}\alpha(t,\tau)\right) \\ &- \frac{e^{2\lambda t}e^{-\lambda(t-T)}}{e^{\lambda T}}\alpha(t,\tau)(e^{-\lambda t} - e^{-\lambda T})[f(t,\tau) - f(0,\tau)]. \end{aligned} \tag{25.128}$$

Consider the following:

(i)

$$\frac{e^{\lambda t}+e^{2\lambda t}(e^{-\lambda t}-e^{-\lambda T})}{e^{\lambda T}}\alpha(t,\tau)=\frac{e^{\lambda t}}{e^{\lambda T}}+\frac{e^{2\lambda t}(e^{-\lambda t}-e^{-\lambda T})e^{-\lambda t}}{e^{\lambda T}(e^{-\lambda \tau}-e^{-\lambda t})}$$

$$=\frac{e^{\lambda t}(e^{-\lambda \tau}-e^{-\lambda t})+e^{\lambda t}(e^{-\lambda t}-e^{-\lambda T})}{e^{\lambda T}(e^{-\lambda \tau}-e^{-\lambda t})}=\frac{e^{\lambda t}(e^{-\lambda \tau}-e^{-\lambda T})}{e^{\lambda T}(e^{-\lambda \tau}-e^{-\lambda t})}$$

$$=\frac{e^{2\lambda t}}{e^{2\lambda T}}\frac{e^{-\lambda t}}{e^{-\lambda \tau}-e^{-\lambda t}}\frac{e^{-\lambda \tau}-e^{-\lambda T}}{e^{-\lambda T}}=e^{-2\lambda(T-t)}\frac{\alpha(t,\tau)}{\alpha(T,\tau)}$$

(ii)

$$e^{2\lambda t}e^{-\lambda(t-\tau)}\alpha(t,\tau)\frac{(e^{-\lambda t}-e^{-\lambda T})}{e^{\lambda T}}=e^{2\lambda t-\lambda t+\lambda \tau}\alpha(t,\tau)\frac{(e^{-\lambda t}-e^{-\lambda T})}{e^{2\lambda T}e^{-\lambda T}}$$

$$=\frac{e^{\lambda t}e^{\lambda \tau}e^{-\lambda t}(e^{-\lambda t}-e^{-\lambda T})}{e^{2\lambda T}e^{-\lambda T}(e^{-\lambda \tau}-e^{-\lambda t})}=\frac{e^{2\lambda \tau}}{e^{2\lambda T}}\frac{e^{-\lambda \tau}}{-(e^{-\lambda t}-e^{-\lambda \tau})}\frac{(e^{-\lambda t}-e^{-\lambda T})}{e^{-\lambda T}}$$

$$=-e^{-2\lambda(T-\tau)}\frac{\alpha(\tau,t)}{\alpha(T,t)}.$$

Hence Eq. (25.128) can be rewritten

$$f(t,T)-f(0,T)=e^{-2\lambda(T-t)}\frac{\alpha(t,\tau)}{\alpha(T,\tau)}[r(t)-f(0,t)]$$
$$-e^{-2\lambda(T-\tau)}\frac{\alpha(\tau,t)}{\alpha(T,t)}[f(t,\tau)-f(0,\tau)]$$

where

$$\alpha(\theta_1,\theta_2)\equiv\frac{e^{-\lambda\theta_1}}{e^{-\lambda\theta_2}-e^{-\lambda\theta_1}}.$$

We have thus proved Proposition 25.2.

Appendix 25.3 Details of the Infinitesimal Generator $\mathscr{K}$

We recall the following result from Sect. 5.4 concerning the infinitesimal generator of an n dimensional Ito process. In our application we set

$$X_1\equiv f(t,\tau),$$

$$a_1\equiv\sigma^2(t,\tau,\omega(t))\frac{(e^{\lambda(\tau-t)}-1)}{\lambda},$$

$$\sigma_{11} \equiv \sigma_1 \equiv \sigma(t, \tau, \omega(t)),$$
$$X_2 \equiv r(t),$$
$$a_2 \equiv f_2(0,t) + \lambda f(0,t) + \psi(t) - \lambda r(t),$$
$$\sigma_{21} \equiv \sigma_r \equiv \sigma(t, t, \omega(t)).$$

Thus the matrix S assumes the form

$$\begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_r \\ \sigma_1\sigma_r & \sigma_r^2 \end{bmatrix}.$$

Using the foregoing expression for S the expression for the operator $\mathscr{K}$ in Eq. (25.67) is readily derived.

Appendix 25.4 Proof of Proposition 25.3

Using the relationship

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$$

and Eq. (25.26) for the forward rate $f(t,s)$ we obtain for the bond price the expression

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\left(\int_t^T \int_0^t \sigma^*(v,s,\cdot)dvds + \int_t^T \int_0^t \sigma(v,s,\cdot)d\tilde{W}(v)ds\right)\right],$$

where

$$\sigma(v,T,\cdot) = \bar{\sigma} e^{-\lambda(T-v)} g(r(v), f(v,\tau))$$
$$\sigma^*(v,T,\cdot) = \sigma(v,T,\cdot)\int_v^T \sigma(v,s,\cdot)ds$$
$$= \bar{\sigma}^2 g^2(r(v), f(v,\tau)) e^{-\lambda(T-v)} \int_v^T e^{-\lambda(s-v)}ds.$$

Set

$$I = \int_t^T \int_0^t \sigma^*(v,s,\cdot)dvds + \int_t^T \int_0^t \sigma(v,s,\cdot)d\tilde{W}(v)ds$$
$$\equiv I_1 + I_2$$
$$= \int_0^t \int_t^T \sigma^*(v,s,\cdot)dsdv + \int_0^t \int_t^T \sigma(v,s,\cdot)dsd\tilde{W}(v),$$

where we have interchanged the order of integration to obtain the last equality. Next note that

$$\begin{aligned}\int_t^T \sigma^*(v,s,\cdot)ds &= \sigma\left(r(v), f(v,\tau)\right)\int_t^T e^{-\lambda(s-v)}\int_v^s e^{-\lambda(y-v)}\sigma\left(r(v), f(v,\tau)\right)dyds\\ &= \sigma\left(r(v), f(v,\tau)\right)\int_t^T e^{-\lambda(s-v)}\left\{\int_v^t e^{-\lambda(y-v)}\sigma\left(r(v), f(v,\tau)\right)dy\right.\\ &\quad\left.+\int_t^s e^{-\lambda(y-v)}\sigma\left(r(v), f(v,\tau)\right)dy\right\}ds\\ &= \bar{\sigma}^2 g^2(r(v), f(v,\tau))\int_t^T e^{-\lambda(s-v)}ds\int_v^t e^{-\lambda(y-v)}dy\\ &\quad+\bar{\sigma}^2 g^2(r(v), f(v,\tau))\int_t^T e^{-\lambda(s-v)}\int_t^s e^{-\lambda(y-v)}dyds\\ &= \bar{\sigma}^2 g^2(r(v), f(v,\tau))e^{-\lambda(t-v)}\left(\int_t^T e^{-\lambda(s-t)}ds\right)\int_v^t e^{-\lambda(y-v)}dy\\ &\quad+\bar{\sigma}^2 g^2(r(v), f(v,\tau))e^{-2\lambda(t-v)}\int_t^T e^{-\lambda(s-t)}\int_t^s e^{-\lambda(y-t)}dyds\\ &= \sigma^*(v,t,\cdot)\beta(t,T)+\sigma^2(v,t,\cdot)\alpha(t,T),\end{aligned}$$

where

$$\beta(t,T) = \int_t^T e^{-\lambda(s-t)}ds = \frac{1}{\lambda}\left(1-e^{-\lambda(T-t)}\right),$$
$$\alpha(t,T) = \int_t^T e^{-\lambda(s-t)}\int_t^s e^{-\lambda(y-t)}dyds = \frac{1}{2}\beta^2(t,T),$$

i.e. we have shown that

$$\int_t^T \sigma^*(v,s,\cdot)ds = \beta(t,T)\sigma^*(v,t,\cdot)+\frac{1}{2}\beta^2(t,T)\sigma^2(v,t,\cdot).$$

Next consider

$$\begin{aligned}\int_t^T \sigma(v,s,\cdot)ds &= \int_t^T e^{-\lambda(s-v)}\sigma\left(r(v), f(v,\tau)\right)ds\\ &= \sigma\left(r(v), f(v,\tau)\right)e^{-\lambda(t-v)}\left(\int_t^T e^{-\lambda(s-t)}ds\right),\end{aligned}$$

i.e. we have shown that

$$\int_t^T \sigma(v, s, \cdot)ds = \sigma(v, t, \cdot)\beta(t, T).$$

Returning to the expressions for I_1, I_2 we can now write

$$I_1 = \int_0^t \left[\beta(t, T)\sigma^*(v, t, \cdot) + \frac{1}{2}\beta^2(t, T)\sigma^2(v, t, \cdot)\right]dv,$$

and

$$I_2 = \int_0^t \beta(t, T)\sigma(v, t, \cdot)d\tilde{W}(v),$$

so that

$$\begin{aligned} I &= \frac{1}{2}\beta^2(t, T)\int_0^t \sigma^2(v, t, \cdot)dv \\ &+\beta(t, T)\left[\int_0^t \sigma^*(v, t, \cdot)dv + \int_0^t \sigma(v, t, \cdot)d\tilde{W}(v)\right]. \end{aligned}$$

However we note from Eq. (25.23), for the instantaneous spot rate $r(t)$, that

$$\int_0^t \sigma^*(v, t, \cdot)dv + \int_0^t \sigma(v, t, \cdot)d\tilde{W}(v) = r(t) - f(0, t).$$

Hence

$$I = \frac{1}{2}\beta^2(t, T)\int_0^t \sigma^2(v, t, \cdot)dv + \beta(t, T)\left[r(t) - f(0, t)\right].$$

Recalling the definition of the subsidiary stochastic variable $\psi(t)$ we can finally write

$$I = \frac{1}{2}\beta^2(t, T)\psi(t) + \beta(t, T)\left[r(t) - f(0, t)\right].$$

Hence the expression for the bond price may be written as in Proposition 25.3.

25.12 Problems

Problem 25.1 Show that the Hull–White model can be obtained within the Heath–Jarrow–Morton framework by setting

$$\sigma(t, T) = \bar{\sigma}e^{-k(T-t)},$$

where $\bar{\sigma}, k$ are constants.

Problem 25.2 The Heath–Jarrow–Morton model takes as its starting point a stochastic differential equation for the instantaneous forward rate of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

and from this determines the stochastic dynamics of the instantaneous spot rate spot rate $r(t)$ and pure discount bond price $P(t,T)$.

Suppose instead we take as the starting point a stochastic differential equation for $P(t,T)$ of the form

$$\frac{dP(t,T)}{P(t,T)} = \beta(t,T)dt + \delta(t,T)dW(t).$$

Determine the corresponding stochastic dynamics for $r(t)$ and $f(t,T)$.

Express in terms of $\beta(t,T)$ and $\delta(t,T)$ the Heath–Jarrow–Morton drift restriction that guarantees no riskless arbitrage opportunities between bonds of different maturities.

Problem 25.3 In Sect. 23.6 we considered the volatility function

$$\sigma(t,T) = \bar{\sigma}e^{-\lambda(T-t)}$$

and showed how this allowed the system dynamics to be Markovianised.

Now consider the volatility function

$$\sigma(t,T) = [\sigma_0 + \sigma_1(T-t)]e^{-\lambda(T-t)}.$$

Show the system dynamics can be Markovianised in this case. In particular obtain the stochastic differential equations for the bond price and the instantaneous spot interest rate.

Hint: You will need to obtain a linked stochastic differential equation system for

$$Z_1(t) = \int_0^t (t-v)e^{-\lambda(t-v)}dW(v),$$

and

$$Z_0(t) = \int_0^t e^{-\lambda(t-v)}dW(v).$$

Problem 25.4 The Ho–Lee model is obtained within the Heath–Jarrow–Morton framework by setting

$$\sigma(t,T) = \bar{\sigma},$$

where $\bar{\sigma}$ is a constant. Show that

$$f(t,T) = r(t) + f(0,T) - f(0,t) + \bar{\sigma}^2 t(T-t).$$

By obtaining the dynamics for $r(t)$ under the risk neutral measure, show also that

$$r(t) = f(0,t) + \frac{1}{2}\sigma^2 t^2 + \bar{\sigma}\tilde{W}(t).$$

Hence, show that for this model the bond price is given by

$$P(t,T) = \exp[-a(t,T) - (T-t)r(t)],$$

where

$$a(t,T) = \ln\frac{P(0,t)}{P(0,T)} - (T-t)f(0,t) + \frac{1}{2}\bar{\sigma}^2 t(T-t)^2.$$

Problem 25.5 Computational Problem—Consider the Heath–Jarrow–Morton model with the volatility function

$$\sigma(t,T) = \sigma_0 e^{-\lambda(T-t)}.$$

We know in this case that the dynamics for the instantaneous spot rate are given by [Eq. (25.54)].

Take $\sigma_0 = 0.02$ and $\lambda = 0.6$. Assume also that the initial forward curve is given by

$$f(0,T) = 0.08 - 0.03e^{-1.5T}.$$

Consider the bond pricing formula [Eq. (25.29)]. Write a program to calculate the bond price by simulating the stochastic differential equation for $r(t)$ from 0 to t and performing the $\tilde{\mathbb{E}}_t$ operation by simulating a large number of paths from t to T. This will give the bond price conditional on the value of $r(t)$ that has been obtained.

You can check the accuracy of your algorithm (and hence choose appropriate Δt and number of paths) by using the fact that when $t = 0$ we have the exact solution

$$P(0,T) = \exp\left(-\int_0^T f(0,s)ds\right).$$

Use this to check the accuracy for $T = 0.5, 1.0, 1.5$ and 2.0.

Then use the simulation procedure to calculate $P(0.5, 1.0)$, $P(0.5, 1.5)$ and $P(0.5, 2.0)$.

Note that the evaluation of $\rho(t, T)$ will be conditional on the interest rate $r(t)$. Obtain $r(t)$ by simulating from 0 to t and be sure to specify the value of $r(t)$ that you are using.

Check the accuracy of these approximations by using the exact bond-pricing formula (here you need to refer to Sect. 23.4.2, but use the $\theta(t)$ that arises in the Heath–Jarrow–Morton model).