

Chapter 22

Modelling Interest Rate Dynamics

Abstract In this chapter, we establish the fundamental relationships between interest rates, bond prices and forward rates. We further discuss the modelling of interest rates and analyse typical models for the spot interest rate and the forward rates. As we desire interest rates to be non-negative, we seek stochastic processes with this feature such as the Feller process. Thus we present the motivation of the Feller process and its relevance to the interest rate modelling. We also summarise the main results of Fubini's theorem, that are very useful for modelling forward rates.

22.1 The Relationship Between Interest Rates, Bond Prices and Forward Rates

In this section we clarify the relationship between interest rates, bond prices, yield to maturity and forward rates. Let $P(t, T)$ denote the price at time t of a pure discount bond paying \$1 at time T . The yield to maturity $\rho(t, T)$ is the continuously compounded rate of return causing the bond price to satisfy the maturity condition

$$P(T, T) = 1, \tag{22.1}$$

that is, $\rho(t, T)$ satisfies (see Fig. 22.1)

$$P(t, T)e^{\rho(t, T)(T-t)} = 1. \tag{22.2}$$

The yield may also be expressed as

$$\rho(t, T) = -\frac{\ln P(t, T)}{T - t}. \tag{22.3}$$

The instantaneous spot interest rate, $r(t)$, is the yield on the currently maturing bond, i.e.,

$$r(t) = \rho(t, t). \tag{22.4}$$

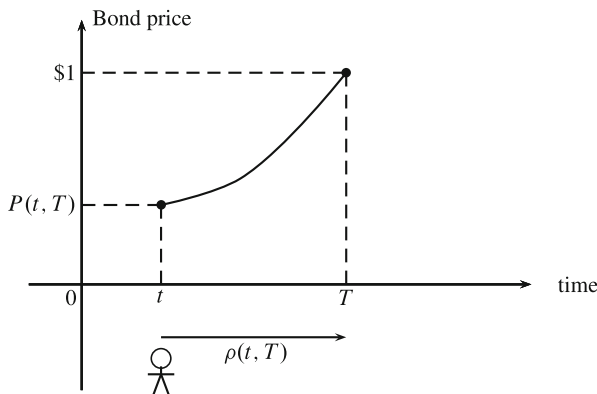


Fig. 22.1 The yield to maturity $\rho(t, T)$; satisfies $P(t, T)e^{\rho(T-t)} = 1$

By allowing $t \rightarrow T$ in (22.3) and applying L'Hôpital's rule we find that¹

$$r(T) = P_t(T, T), \tag{22.5}$$

where $P_t(T, T)$ denotes the partial derivative of $P(t, T)$ with respect to its first argument (running time t), evaluated at the time $t = T$.

The forward rate arises when we consider an investor who holds a bond maturing at T_1 and asking what return he or she would earn between T_1 and $T_2 (> T_1)$, if he or she contracted now at time t . Figure 22.2 displays a time line that is useful when thinking about the relation between forward rates and bond prices. The required rate of return is the forward rate $f(t, T_1, T_2)$ defined by

$$P(t, T_1) = P(t, T_2)e^{f(t, T_1, T_2)(T_2 - T_1)},$$

i.e.

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} \ln \left[\frac{P(t, T_1)}{P(t, T_2)} \right]. \tag{22.6}$$

To see how $f(t, T_1, T_2)$ represents the implicit rate of interest currently available (at time t) on riskless loans from T_1 to T_2 consider the set of transactions illustrated

¹We need to apply L'Hôpital's rule since setting $t = T$ in Eq. (22.3) yields the meaningless ratio $0/0$. Thus by L'Hôpital's rule

$$\lim_{t \rightarrow T} \rho(t, T) = \lim_{t \rightarrow T} -\frac{\frac{\partial}{\partial t} \ln P(t, T)}{\frac{\partial}{\partial t} (T - t)} = \lim_{t \rightarrow T} \frac{\frac{-1}{P(t, T)} P_t(t, T)}{-1} = \lim_{t \rightarrow T} \frac{P_t(t, T)}{P(t, T)} = P_t(T, T).$$

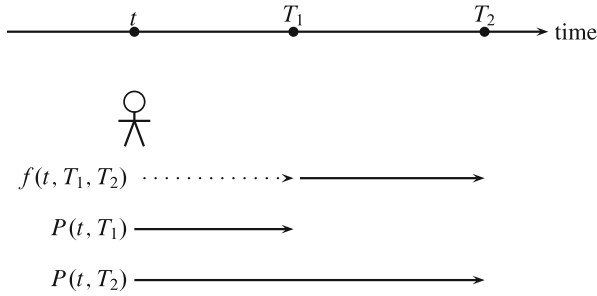


Fig. 22.2 The forward rate $f(t, T_1, T_2)$ and bond prices $P(t, T_1), P(t, T_2)$

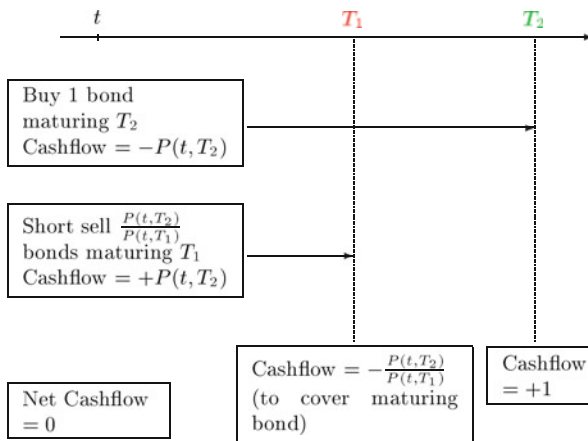


Fig. 22.3 The net zero transaction that relates the forward $f(t, T_1, T_2)$ and bond prices $P(t, T_1)$ and $P(t, T_2)$

in Fig. 22.3. A set of transactions at time t involving zero net cashflow has the net effect of investing $P(t, T_2)/P(t, T_1)$ dollars at time T_1 to yield a dollar for sure at time T_2 . The forward rate defined above is simply the continuously compounded interest rate earned on this investment. Note that calculation of $f(t, T_1, T_2)$ involves only bond prices observable at time t .

The Heath–Jarrow–Morton model to be discussed in a later chapter focuses on the instantaneous forward rate defined by²

$$f(t, T) \equiv f(t, T, T), \tag{22.7}$$

²There is a slight abuse of notation in that we use the symbol f to denote both the three variable function $f(t, T_1, T_2)$ and the two variable function $f(t, T)$.

and is the instantaneous rate of return the bond holder can earn by extending the investment an instant beyond T . Letting $T_2 \rightarrow T_1$, in (22.6) and applying L'Hôpital's rule we see that³

$$f(t, T) = -\frac{P_T(t, T)}{P(t, T)}. \quad (22.8)$$

It follows from (22.8) that the price of the pure discount bond may be written in terms of the forward rate as

$$P(t, T) = e^{-\int_t^T f(t,s)ds}. \quad (22.9)$$

In a world of certainty all securities, in equilibrium, must earn the same instantaneous rate of return so as to exclude the possibility of riskless arbitrage opportunities. The application of this equilibrium condition to discount bonds implies

$$\frac{P_t(t, T)}{P(t, T)} = r(t), \quad (22.10)$$

from which⁴ we obtain the relationship between pure discount bond prices and the instantaneous spot rate in a world of certainty,

$$P(t, T) = e^{-\int_t^T r(s)ds}. \quad (22.11)$$

A comparison of (22.11) and (22.2) shows the relationship between the yield to maturity and the spot rate in a world of certainty,

$$\rho(t, T) = \frac{1}{T-t} \int_t^T r(s)ds. \quad (22.12)$$

³It is best to set $T_1 = T$, $T_2 = T + x$ in Eq. (22.6) and calculate as follows:

$$\begin{aligned} f(t, T) &= \lim_{x \rightarrow 0} \frac{1}{x} [\ln P(t, T) - \ln P(t, T + x)] = \lim_{x \rightarrow 0} -\frac{\frac{\partial}{\partial x} \ln P(t, T + x)}{\frac{\partial}{\partial x} x} \\ &= \lim_{x \rightarrow 0} \frac{-P_T(t, T + x)}{P(t, T + x)} = \frac{-P_T(t, T)}{P(t, T)}. \end{aligned}$$

⁴From Eq. (22.10)

$$\frac{d}{ds} \ln P(s, T) = r(s).$$

Integrating (t, T) we obtain

$$\ln[P(s, T)]_t^T = \ln P(T, T) - \ln P(t, T) = \ln 1 - \ln P(t, T) = -\ln P(t, T) = \int_t^T r(s)ds.$$

Hence (22.11) follows.

Next, substituting (22.11) into (22.8) reveals the relationship between the spot rate and the forward rate in a world of certainty viz.

$$f(t, T) = r(T), \quad \text{for all } T \geq t. \quad (22.13)$$

Note that this last equation is a degenerate version of the expectations hypothesis, i.e., the expected instantaneous spot rate for time T is equal to the instantaneous forward rate for time T , calculated at time t .

In subsequent chapters we shall see how the relationships (22.11)–(22.13) generalise quite naturally in a world of uncertainty to be the corresponding relationships under suitable probability measures.

Another important set of interest rates are LIBOR rates. They are akin to the forward rates $f(t, T_1, T_2)$ in that they are rates that one can contract at time t for borrowing over a fixed period in the future. However the time difference between current time and that fixed period in the future remains constant. Whereas with $f(t, T_1, T_2)$ the time difference between t and T_1 decreases as time evolves, in fact it would make no sense to consider t beyond T_1 . We shall discuss LIBOR rates in Chap. 26 when we discuss the Brace–Gatarek–Musiela (BGM) LIBOR market model.

22.2 Modelling the Spot Interest Rate

First we consider models for the dynamics of the spot interest rate. A number of such models have been proposed and most of these are of the general form

$$dr = \mu(r, t)dt + \sigma(r, t)dW. \quad (22.14)$$

Typically the drift term is of the form

$$\mu(r, t) = \kappa(\bar{r} - r(t)), \quad (22.15)$$

where \bar{r} is the long run level of the spot rate of interest. This form implies mean reverting behaviour of the spot interest rate which is confirmed in the empirical studies discussed below. It is less clear what form the volatility function should take. Forms usually used in empirical studies and in development of term structure and interest rate derivative models to be discussed later are of the general form

$$\sigma(r, t) = \sigma r(t)^\gamma, \quad (22.16)$$

where usually $\gamma \geq 0$ is assumed.

With these drift and diffusion terms the interest rate process (22.14) assumes the form

$$dr = \kappa(\bar{r} - r(t))dt + \sigma r(t)^\gamma dW. \quad (22.17)$$

Chan et al. (1992) have estimated a discretised version of this model on US Treasury bill data for the period 1964–1989 and found (in terms of our notation) $\kappa = 0.5921$, $\bar{r} = 0.0689$, $\sigma^2 = 1.6704$ and $\gamma = 1.4999$. The value of σ^2 may seem high but it should be borne in mind that the average volatility is measured by $\sigma \bar{r}^\gamma = 0.0234 (= 2.34\% \text{ p.a.})$ which is a reasonable value for interest rate markets. Noting the interpretation that $1/\kappa$ is the average time for reversion back to the mean we see that the estimated value of $\kappa = 0.5921$ implies that this average time is about 1.69 years. This value also looks reasonable.

In Fig. 22.4 we illustrate simulations of $r(t)$ given by (22.17) for varying values of γ which use the same sequence of random numbers. In these simulations we have used the values $\kappa = 0.6$, $\bar{r} = 0.07$ [which are close to those found by Chan et al. (1992)] and the values of σ displayed in Table 22.1. The values have been chosen so that the overall volatility term at $r = \bar{r}$, viz. $\sigma \bar{r}^\gamma$ remains constant at the value

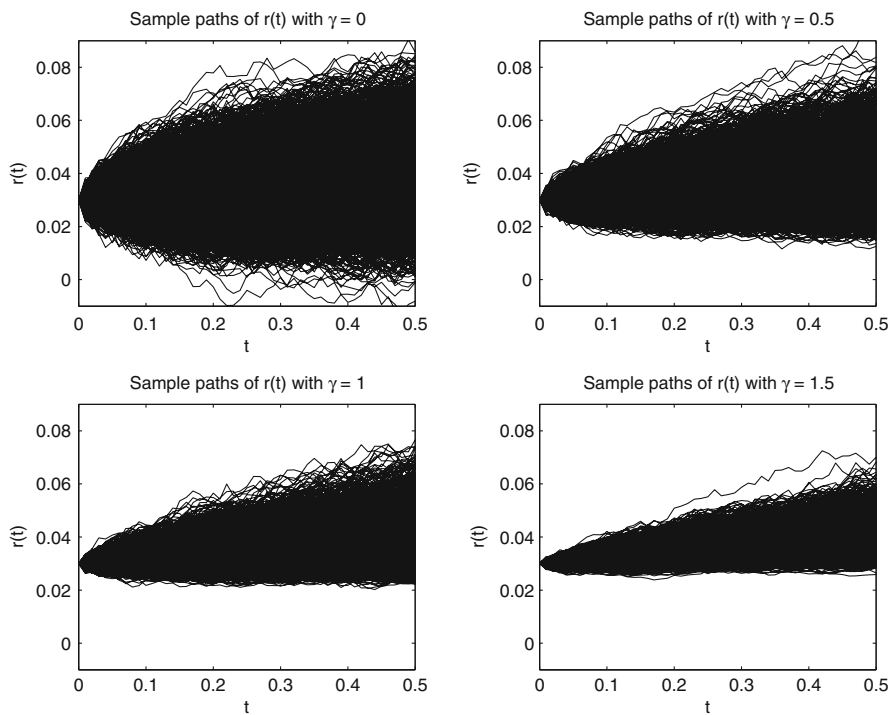


Fig. 22.4 Bundles of simulated paths for the interest rate process (22.17) for various values of γ . The parameters used are calibrated to the estimates of Chan et al. (1992)

Table 22.1 Values of σ and γ used in simulations of the interest rate process; here $\sigma\bar{r}^\gamma = 2.34\%$

γ	0	0.5	1	1.5
σ	0.0234	0.08844	0.33429	1.26348

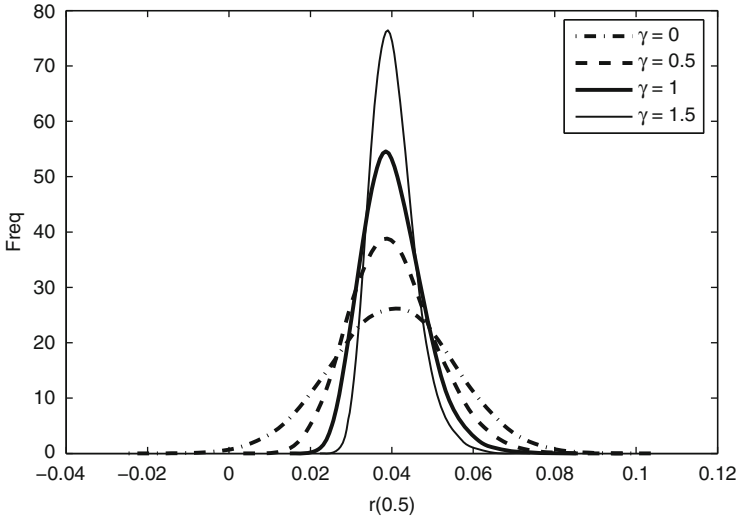


Fig. 22.5 The density functions for r at $t = 0.05$ corresponding to the simulations in Fig. 22.4

0.0234 as γ varies. We have used the initial value, $r_0 = 0.03$. It is of interest to note that when $\gamma = 0$, interest rates can become negative though the probability of this event is very low. As the value of γ increases we can see that the bundle of interest rate paths moves away from zero and the distribution becomes more peaked, as can also be seen from Figs. 22.4 and 22.5.

It can be shown that for $\gamma \geq 1/2$ the probability of interest rates becoming negative tends to zero. Hence a commonly used form of the volatility function is (22.16) with $\gamma = 1/2$ and this is the basis of the Cox–Ingersoll–Ross model to be discussed in a later chapter. Stochastic processes having a linear drift term such as (22.15) and a square root volatility have been extensively investigated by Feller (1951). He showed that the conditional transition density function for Eq. (22.14) with

$$\mu(r, t) = \kappa(\bar{r} - r(t)), \quad \sigma(r, t) = \sigma\sqrt{r(t)}, \quad (22.18)$$

where κ , \bar{r} and σ are positive constants, is given by

$$\begin{aligned} p(r, t|r_0, 0) &= g_t \sum_{n=0}^{\infty} \frac{e^{-\lambda_t/2} (\lambda_t/2)^n (1/2)^{n+d/2}}{n! \Gamma(n+d/2)} (g_t r)^{n-1+d/2} e^{-(g_t r)/2} \\ &\equiv g_t P_{\chi^2(d, \lambda_t)}(g_t r), \end{aligned} \quad (22.19)$$

where $P_{\chi^2(d, \alpha)}(x)$ is the density function of a non-central Chi-squared distribution with d degrees of freedom and non-centrality parameter α . Alternatively, the density function may be expressed as (which is really the form given by Feller)

$$p(r, t|r_0, 0) = g_t \frac{e^{-(g_t r + \lambda_t)} (g_t r)^{(d-1)/2} \sqrt{\lambda_t}}{2(\lambda_t g_t r)^{d/4}} I_{d/2-1}(\sqrt{\lambda_t g_t r}), \quad (22.20)$$

where the modified Bessel function of the first kind $I_\alpha(x)$ is defined as

$$I_\alpha(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\alpha}}{n! \Gamma(n+\alpha+1)}.$$

In the above formula the parameters d , g_t and λ_t are defined as

$$\begin{aligned} d &= 4\kappa\bar{r}/\sigma^2, \\ g_t &= \frac{4\kappa}{\sigma^2(1 - e^{-\kappa t})}, \\ \lambda_t &= \frac{4\kappa r_0}{\sigma^2(e^{\kappa t} - 1)}. \end{aligned}$$

We need the condition $2\kappa\bar{r} > \sigma^2$ to ensure that $r(t)$ is positive. A plot of $p(r, t|r_0, 0)$ for a range of values of σ and the values of κ, \bar{r} indicated earlier is displayed in Fig. 22.6. The process having drift and diffusion terms as specified in (22.18) has become known variously as the square root process, the Cox–Ingersoll–Ross process and the Feller process. We shall use the latter term as it is the results obtained by Feller in the cited paper that are used in the quantitative finance literature. In Sect. 22.3 we shall give some motivation for the Feller process that will give some insight into why the underlying distribution is non-central chi squared.

An important criticism of modelling the volatility by use of functional forms such as (22.16) is that volatility of interest rates depend on the level of interest rates. This seems to run against what is often observed as it is possible to identify historical periods when rates were high but relatively stable or rates were low but fairly volatile. It is also often argued that volatility of interest rates should also be a function of the news arrival process (i.e. the dW 's). One way to capture this effect would be to allow σ to itself follow a stochastic process—in this regard see Problem 22.8.

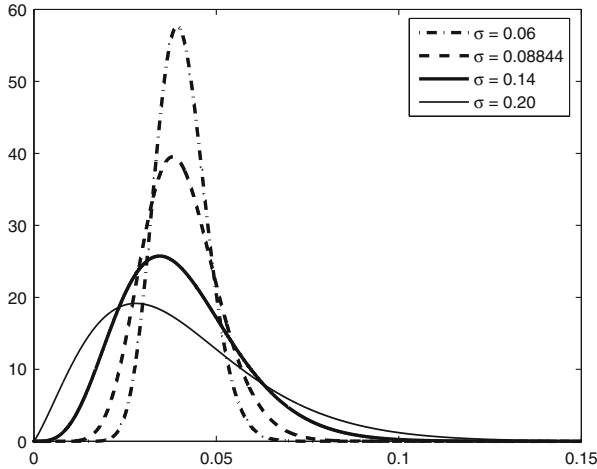


Fig. 22.6 The transition density function of the Feller process for different values of σ . The remaining parameters of the process correspond to those used for the simulations in Fig. 22.4

22.3 Motivating the Feller (or Square Root) Process

A very widely used stochastic process in finance is the Feller or square root process, frequently referred to as the Cox, Ingersoll and Ross (or just CIR) process. As we have stated earlier this process was originally studied by Feller (1951), but introduced into the finance literature by Cox et al. (1985b). Its interest for finance applications lies in the fact that (for certain parameter constellations) the process is guaranteed to generate positive values and its density function is known. This is clearly a desirable feature for a model of interest rate processes, but also in other areas of finance e.g. models of credit spreads or risk-premia. In this section we seek to give some insight into why this process generates positive (or at least non-negative outcomes), and also why its conditional transitional probability density function is a chi-squared distribution.

Consider the process (22.14) when $\mu(r, t) = (\alpha - \beta r(t))$ and $\sigma(r, t) = \sigma(t)$, i.e.

$$dr = (\alpha - \beta r(t))dt + \sigma(t)dW(t). \tag{22.21}$$

This is essentially an Ornstein–Uhlenbeck process (see Sect. 6.3.5) whose solution can be written

$$r(t) = r(0)e^{-\beta t} + \int_0^t \alpha e^{-\beta(t-s)} ds + \int_0^t e^{-\beta(t-s)} \sigma(s) dW(s). \tag{22.22}$$

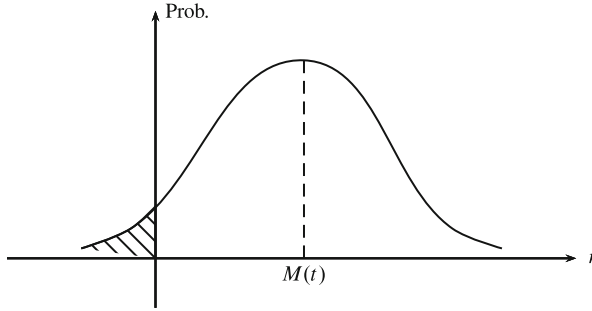


Fig. 22.7 The possibility of negative interest rates from the distribution in Eq. (22.23)

It follows from Eq. (22.22) that

$$r(t) \sim N(M(t), V^2(t)), \tag{22.23}$$

where

$$M(t) = r_0 e^{-\beta t} + \int_0^t \alpha e^{-\beta(t-s)} ds,$$

and

$$\begin{aligned} V^2(t) &= E_0 \left[\left(\int_0^t e^{-\beta(t-s)} \sigma(s) dW(s) \right)^2 \right] \\ &= \int_0^t e^{-2\beta(t-s)} \sigma^2(s) ds. \end{aligned}$$

The fact that $r(t)$ is normally distributed means that there is a positive probability that $r(t)$ can become negative as illustrated in Fig. 22.7. This is not a desirable feature of a process for interest rate dynamics, though in practice this probability may be quite low as can be seen from the simulated distribution in Fig. 22.5 for the value $\gamma = 0$.

We are therefore motivated to find a process that has zero probability of $r(t)$ becoming zero or negative. Consider the set of u (mean reverting to zero) Ornstein–Uhlenbeck processes

$$dX_j = -\frac{1}{2}\beta X_j(t)dt + \frac{1}{2}\sigma dW_j, \quad (j = 1, \dots, u). \tag{22.24}$$

By solving (22.24) we obtain

$$X_j(t) = X_j(0)e^{-\frac{1}{2}\beta t} + \frac{1}{2}\sigma \int_0^t e^{-\frac{\beta}{2}(t-v)} dW_j(v), \tag{22.25}$$

implying that

$$X_j \sim N(M_{X_j}(t), V_X(t)),$$

where

$$M_{X_j}(t) = X_j(0)e^{-\frac{1}{2}\beta t},$$

and

$$V_X(t) = \frac{\sigma^2}{4} \int_0^t e^{-\beta(t-v)} dv.$$

Consider the process

$$r(t) = X_1^2(t) + X_2^2(t) + \dots + X_u^2(t).$$

Ito's lemma for a multi-variable function and some algebraic manipulations imply that

$$dr = \left(\frac{u\sigma^2}{4} - \beta r(t) \right) dt + \sigma \sqrt{r(t)} \left(\sum_{i=1}^u \frac{X_i(t)}{\sqrt{r(t)}} dW_i \right). \tag{22.26}$$

Consider the new process

$$W(t) = \sum_{i=1}^u \int_0^t \frac{X_i(v)}{\sqrt{r(v)}} dW_i(v), \tag{22.27}$$

so that

$$dW(t) = \sum_{i=1}^u \frac{X_i(t)}{\sqrt{r(t)}} dW_i(t).$$

We calculate⁵

$$E[dW(t)] = 0,$$

and

$$var[dW(t)] = \sum_{i=1}^u \frac{X_i^2(t)}{r(t)} E[dW_i^2] = \left(\sum_{i=1}^u X_i^2(t) \right) \frac{dt}{r(t)} = dt.$$

⁵Recall that $E(dW_i dW_j) = 0$ ($i \neq j$) ($j = 1, 2, \dots, d$).

Furthermore we easily calculate that

$$\mathbb{E}[(dW(t))^m] = 0 \text{ for } m \geq 3.$$

So $dW(t)$ has all the statistical properties of a Wiener increment. It thus follows that $W(t)$ is a Wiener process. So we can write (22.26) as

$$dr = \left(\frac{u\sigma^2}{4} - \beta r(t) \right) dt + \sigma \sqrt{r(t)} dW, \quad (22.28)$$

which enjoys the property $r(t) \geq 0$ by construction. The process (22.28) is simply the Feller process already encountered in Sect. 22.2. It is more simply written here as

$$dr = (\alpha - \beta r(t))dt + \sigma \sqrt{r(t)}dW, \quad (22.29)$$

where we choose u as

$$u = \frac{4\alpha}{\sigma^2} > 0. \quad (22.30)$$

Now, for arbitrary α and σ , u is *not* an integer. It can be shown that results for the CIR process with u equal to an integer are also true for u *not* an integer. The value of u for the process affects very much its behaviour. We distinguish two cases:

(i) $u < 2$

We note that $u < 2$ implies $\alpha < \sigma^2/2$. In this case it can be shown that

$$\Pr \left\{ \begin{array}{l} r(t) = 0 \text{ at infinite number} \\ \text{of values of } t \end{array} \right\} = 1.$$

This implies behaviour shown in Fig. 22.8. The excursions at $r(t) = 0$ are also not a desirable feature of a process for interest rate dynamics since it does not accord with what we observed empirically.

(ii) $u > 2$

In this case $\alpha \geq \sigma^2/2$ and it can be shown that

$$\Pr \{ \exists \text{ at least one } t > 0 \text{ s.t. } r(t) = 0 \} = 0$$

which is the desirable feature of the process for r that we seek. Figure 22.9 illustrates what a typical sample path would look like.

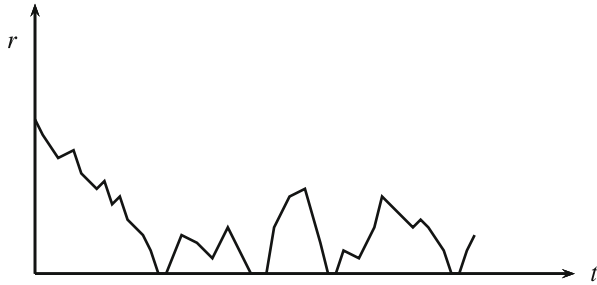


Fig. 22.8 The Feller process for $r(t)$ with $\alpha < \sigma^2/2$

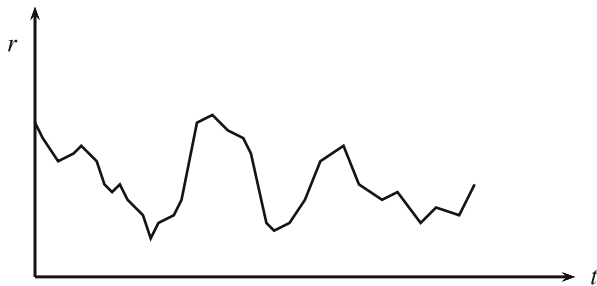


Fig. 22.9 The Feller process for $r(t)$ with $\alpha > \sigma^2/2$

In order to give an intuitive feel for the result that the distribution for $r(t)$ is a non-central Chi-squared distribution, we set

$$\begin{aligned}
 X_1(0) &= X_2(0) = \dots X_{u-1}(0) = 0 \\
 X_u(0) &= \sqrt{r(0)}.
 \end{aligned}$$

Then from (22.25) it follows that

$$X_i(t) \sim N(0, V_X^2(t)) \quad i = 1, \dots, u - 1,$$

and so

$$X_u(t) \sim N(e^{-\frac{1}{2}\beta t} \sqrt{r(0)}, V_X^2(t)).$$

Thus

$$r(t) = V_X^2(t) \sum_{i=1}^{u-1} \left(\frac{X_i(t)}{V_X(t)} \right)^2 + X_u^2(t).$$

Noting that $X_u^2(t)$ is the square of a normally distributed variable with non-zero mean and that each of the $(u - 1)$ terms in the sum on the RHS is the square of a (zero-mean) normally distributed variable, it follows that $r(t)$ has a non-central Chi-squared distribution with $u - 1 = \frac{4\alpha - \sigma^2}{\sigma^2}$ degrees of freedom. The actual density function is derived at Eq. (22.20).

22.4 Fubini's Theorem

In the next section we shall consider models of the forward rate and shall use these to derive corresponding processes for the bond price and spot interest rate. In this analysis we shall frequently encounter stochastic double integrals of the form

$$\int \left(\int g(u, v, \omega) dW(u) \right) dv, \quad (22.31)$$

with a variety of different limits in the integrals. Here ω can be any stochastic variable such as the forward rate, the bond price or the spot interest rate. We will very often need to calculate the stochastic differential of terms like Eq. (22.31) and this is more simply done if we can express it in the form

$$\int \left(\int g(u, v, \omega) dv \right) dW(u). \quad (22.32)$$

In other words we need to change the order of integration in stochastic double integrals. The result that allows us to do this under fairly mild conditions on g is Fubini's theorem. Essentially this result allows us to manipulate stochastic double integrals in the same way that we manipulate ordinary double integrals. Three main results may be derived from Fubini's theorem, these are:

(I)

$$\int_0^\tau \left(\int_0^t g(u, v, \omega) dW(u) \right) dv = \int_0^t \left(\int_0^\tau g(u, v, \omega) dv \right) dW(u), \quad (22.33)$$

for all $0 \leq t \leq \tau$.

(II)

$$\int_t^T \left(\int_0^\tau g(u, v, \omega) dW(u) \right) dv = \int_0^\tau \left(\int_t^T g(u, v, \omega) dv \right) dW(u), \quad (22.34)$$

for all $0 \leq \tau \leq t \leq T$.

In these two results the integration limits do not involve the integration variables u or v . These results state that in this case we may change the order of integration as we do for the same situation with ordinary integrals.

(III)

$$\int_0^t \left(\int_0^v g(u, v, \omega) dW(u) \right) dv = \int_0^t \left(\int_u^t g(u, v, \omega) dv \right) dW(u). \quad (22.35)$$

Now the integration limits depend on the integration variables, however the result again shows that we can handle such stochastic double integrals in the same way that we handle ordinary double integration. Figures 22.10, 22.11 and 22.12 illustrate the relevant regions of integration for cases I, II and III respectively. In each case the region of integration (the shaded area) can be swept out by left to right motion of

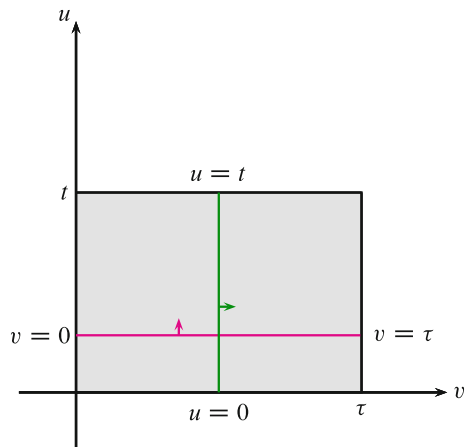


Fig. 22.10 Change of limits for version I of Fubini's theorem

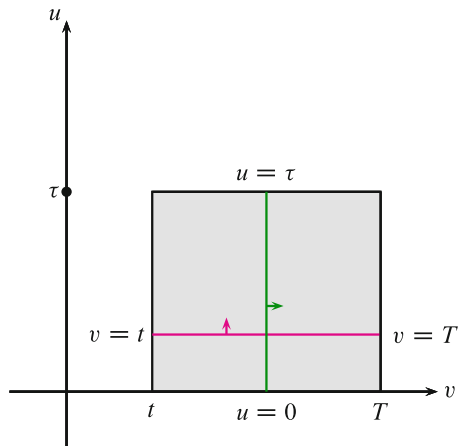
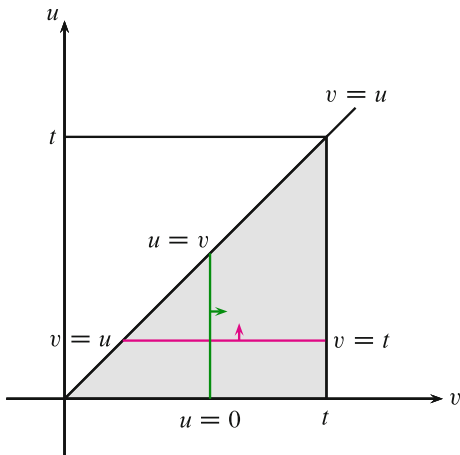


Fig. 22.11 Change of limits for version II of Fubini's theorem

Fig. 22.12 Change of limits for version III of Fubini's theorem



the vertical line (the LHS double integrals) or, from top to bottom motion of the horizontal line (the RHS double integrals).

22.5 Modelling Forward Rates

We recall that $f(t, T)$ denotes the forward rate at time t for instantaneous borrowing at time T . The time line for $f(t, T)$ is illustrated in Fig. 22.13, where we display $f(t, T)$ and $f(t + dt, T)$. The important point to keep in mind is that the maturity date T is fixed, as so the time to maturity decreases as t evolves. This is in contrast to the Brace–Musiela notation that considers fixed period ahead forward rates, as we shall see in Chap. 26 when we come to consider the LIBOR market model.

Heath et al. (1992a) propose to model the forward rate as the driving stochastic process. They write the process for $f(t, T)$ in the form of a stochastic integral equation as

$$f(t, T) = f(0, T) + \int_0^t \alpha(u, T, \omega(u))du + \int_0^t \sigma(u, T, \omega(u))dW(u), \quad (22.36)$$

for $0 \leq t \leq T$.

Figure 22.14 illustrates how the forward curve at time $(t + dt)$ evolves from that at t by the addition of two terms. The first, $\alpha(t, T, \cdot)dt$, represents the average change across the maturities. The second, $\sigma(t, T, \cdot)dW$, represents the impact at different maturities of the shock dW that occurs during $(t, t + dt)$.

More specifically, the quantities $\alpha(u, T, \omega(u))$ and $\sigma(u, T, \omega(u))$ are the drift and volatility of the forward rate process and $f(0, T)$ is the initial forward rate curve which can be obtained from the currently observed yield curve. We allow for possible dependence of α and σ on $\omega(t)$, a vector of path dependent variables such

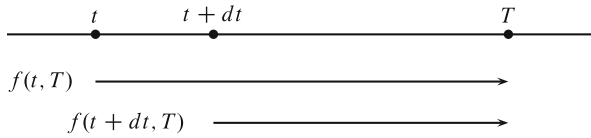


Fig. 22.13 The time line for the fixed terminal data instantaneous forward rate $f(t, T)$

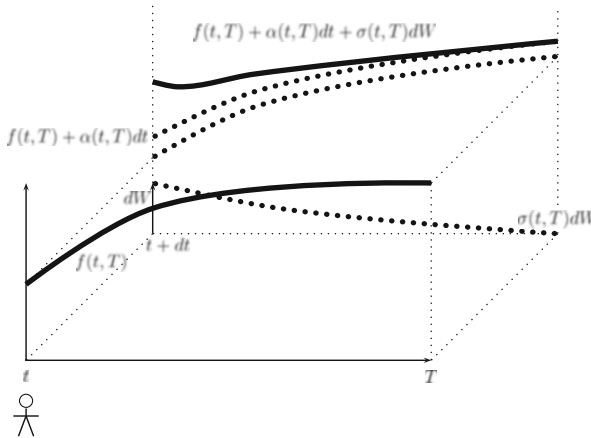


Fig. 22.14 The evolution of the forward rate curve

as the instantaneous spot interest rate $r(t)$, discrete maturity forward rates or even the entire forward curve itself. This would allow the model to capture the effect of the level of interest rates on volatility. This general notation was indeed employed in the original Heath–Jarrow–Morton paper. In the discussion of this section we only allow one shock term dW in order to alleviate the mathematical notation. Heath–Jarrow–Morton allow for multiple shock terms and we consider this case in Chap. 25. In differential form we may write Eq. (22.36) as the stochastic differential equation

$$df(t, T) = \alpha(t, T, \omega(t))dt + \sigma(t, T, \omega(t))dW(t). \tag{22.37}$$

We shall defer for the moment discussion of appropriate functional forms for $\alpha(t, T, \omega(t))$ and $\sigma(t, T, \omega(t))$. The aim of the calculations in the rest of this section is to determine the stochastic processes for the spot rate $r(t)$ and bond price $P(t, T)$ implied by Eq. (22.36).

The reader needs to be wary that the correct path from the forward rate dynamics to the spot rate dynamics has to occur at the level of the stochastic integral equations. Thus we must set $T = t$ in the stochastic integral equation (22.36) for $f(t, T)$ to obtain the stochastic integral equation for $r(t)$. For this latter we can obtain the stochastic differential equation for $r(t)$. It is not correct to obtain the stochastic differential equation for $r(t)$ simply by setting $T = t$ in the stochastic differential

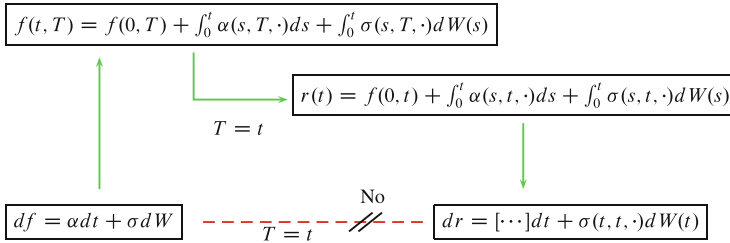


Fig. 22.15 The path from forward rate dynamics to spot rate dynamics

equation (22.37) for $f(t, T)$. This has to do with the fact that the path history matters here and we need to work with the stochastic integral equations, which are properly defined mathematical objects. The stochastic differential equation is merely a (very convenient) short-hand notation as we stressed in Chap. 4. We illustrate the correct path from forward rate dynamics to spot rate dynamics in Fig. 22.15.

Recalling that the spot interest rate and forward rate are related by $r(t) = f(t, t)$ we see from Eq. (22.36) that $r(t)$ satisfies the stochastic integral equation

$$r(t) = f(0, t) + \int_0^t \alpha(u, t, \omega(u)) du + \int_0^t \sigma(u, t, \omega(u)) dW(u). \tag{22.38}$$

This may alternatively be expressed as the stochastic differential equation⁶

$$dr = \left[f_2(0, t) + \alpha(t, t, \omega(t)) + \int_0^t \alpha_2(u, t, \omega(u)) du + \int_0^t \sigma_2(u, t, \omega(u)) dW(u) \right] dt + \sigma(t, t, \omega(t)) dW(t). \tag{22.39}$$

Here f_2, α_2, σ_2 denote partial differentiation of f, α, σ with respect to their second arguments. Close inspection of (22.39) reveals that this process for $r(t)$ is more general than the types of processes considered in Sect. 22.2. The difference stems from the third component of the drift term, viz.,

$$\int_0^t \sigma_2(u, t, \omega(u)) dW(u). \tag{22.40}$$

This term is a weighted sum of the shock terms $dW(u)$ from initial time 0 to current time t , thus it is a path-dependent term. Hence the stochastic differential equation

⁶Here we make use of the result for the differential of a stochastic integral in Sect. 5.5, a result that is used frequently in this and subsequent chapters.

for $r(t)$, Eq. (22.39) is non-Markovian.⁷ Thus the spot interest rate process implied by the forward rate process (22.36) is far more general than those discussed in Sect. 22.2.

22.5.1 From Forward Rate to Bond Price Dynamics

In this section we determine the bond price dynamics implied by the forward rate dynamics (22.36). Before proceeding we compute a quantity that will be useful in simplifying the expression for the bond price that we shall calculate below. We note from Eq. (22.38) that

$$\int_0^t r(s)ds = \int_0^t f(0, s)ds + \int_0^t \int_0^s \alpha(u, s, \omega(u))duds + \int_0^t \int_0^s \sigma(u, s, \omega(u))dW(u)ds,$$

which by use of result (III) of Fubini's theorem in Sect. 22.4 can be written

$$\begin{aligned} \int_0^t r(s)ds &= \int_0^t f(0, s)ds + \int_0^t \left(\int_u^t \alpha(u, s, \omega(u))ds \right) du \\ &\quad + \int_0^t \left(\int_u^t \sigma(u, s, \omega(u))ds \right) dW(u). \end{aligned} \tag{22.41}$$

In order to determine the stochastic differential equation for the bond price that is implied by the forward rate process (22.36), we recall first the definitional relationship between bond prices and forward rates, viz.,

$$P(t, T) = \exp \left(- \int_t^T f(t, s)ds \right),$$

or

$$\ln P(t, T) = - \int_t^T f(t, s)ds. \tag{22.42}$$

Substituting Eq. (22.36) into this last equation we have that

$$\begin{aligned} \ln P(t, T) &= - \int_t^T f(0, s)ds - \int_t^T \int_0^t \alpha(u, s, \omega(u))duds - \int_t^T \int_0^t \sigma(u, s, \omega(u))dW(u)ds. \end{aligned}$$

⁷In fact even without the term Eq. (22.40) we have non-Markovian dynamics because of the term $\int_0^t \alpha_2(u, t, \omega(u))du$, which depends on the path followed by the path dependent variable ω between 0 and t .

Applying result (II) of Fubini' theorem (see Sect.22.4) we can re-express $\ln P(t, T)$ as

$$\begin{aligned} \ln P(t, T) = & - \int_t^T f(0, s) ds - \int_0^t \left(\int_t^T \alpha(u, s, \omega(u)) ds \right) du \\ & - \int_0^t \left(\int_t^T \sigma(u, s, \omega(u)) ds \right) dW(u), \end{aligned}$$

which may further be re-arranged to⁸

$$\begin{aligned} \ln P(t, T) = & - \int_0^T f(0, s) ds - \int_0^t \left(\int_u^T \alpha(u, s, \omega(u)) ds \right) du \\ & - \int_0^t \left(\int_u^T \sigma(u, s, \omega(u)) ds \right) dW(u) + \int_0^t f(0, s) ds \quad (22.43) \\ & + \int_0^t \left(\int_u^t \alpha(u, s, \omega(u)) ds \right) du + \int_0^t \left(\int_u^t \sigma(u, s, \omega(u)) ds \right) dW(u). \end{aligned}$$

From Eq. (22.41) we see that we can represent the last three terms of (22.43) as $\int_0^t r(s) ds$, and from Eq. (22.42) we see that

$$- \int_0^T f(0, s) ds = \ln P(0, T).$$

Hence Eq. (22.43) simplifies to

$$\begin{aligned} \ln P(t, T) = & \ln P(0, T) + \int_0^t r(s) ds - \int_0^t \left(\int_u^T \alpha(u, s, \omega(u)) ds \right) du \\ & - \int_0^t \left(\int_u^T \sigma(u, s, \omega(u)) ds \right) dW(u). \quad (22.44) \end{aligned}$$

Taking differentials it follows immediately that the log bond price $B(t, T) \equiv \ln P(t, T)$ satisfies the stochastic differential equation

$$dB(t, T) = [r(t) - \alpha_B(t, T)] dt + \sigma_B(t, T) dW(t), \quad (22.45)$$

where we define

$$\alpha_B(t, T) = \int_t^T \alpha(t, s, \omega(t)) ds \quad \text{and} \quad \sigma_B(t, T) = - \int_t^T \sigma(t, s, \omega(t)) ds. \quad (22.46)$$

⁸Since $\int_t^T f(0, s) ds = \int_0^T f(0, s) ds - \int_0^t f(0, s) ds$ and for any function g we have $\int_0^t \left(\int_t^T g(u, s) ds \right) du = \int_0^t \left(\int_t^u g(u, s) ds + \int_u^T g(u, s) ds \right) du = \int_0^t \left(- \int_u^t g(u, s) ds + \int_u^T g(u, s) ds \right) du$.

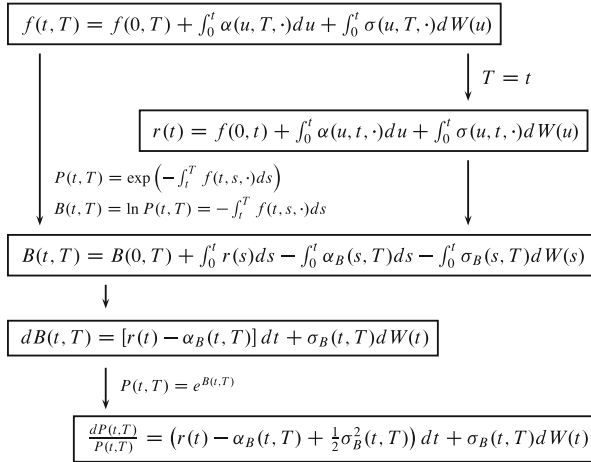


Fig. 22.16 The steps from forward rate to bond price dynamics

By a straightforward application of Ito’s Lemma (since $P(t, T) = e^{B(t, T)}$) we find the corresponding stochastic differential equation for the bond price to be

$$dP(t, T) = \left[r(t) - \alpha_B(t, T) + \frac{1}{2} \sigma_B^2(t, T) \right] P(t, T) dt + \sigma_B(t, T) P(t, T) dW(t). \tag{22.47}$$

The stochastic differential equations (22.39) and (22.47) form a linked system for the instantaneous spot rate and the bond price. This system will typically be non-Markovian and many applications in the Heath–Jarrow–Morton framework seek simplifications which reduce this system to Markovian form. We give one example of how this may be done in the next section.

Since the steps from the forward price dynamics to the bond price dynamics are rather round-about we feel it may be useful to summarise these in Fig. 22.16.

22.5.2 A Specific Example

For the forward rate volatility in Eq. (22.37) we assume that the path dependent variable ω is associated to the instantaneous spot interest rate $r(t)$ in the functional form

$$\sigma(u, t, \omega(u)) = \sigma(u, t, r(u)) = \sigma e^{-\lambda(t-u)} r(u)^\gamma. \tag{22.48}$$

Note that in this case

$$\sigma_2(u, t, r(u)) = -\lambda \sigma e^{-\lambda(t-u)} r(u)^\gamma = -\lambda \sigma(u, t, r(u)), \quad (22.49)$$

so that the non-Markovian term in Eq. (22.39) may be written

$$\begin{aligned} \int_0^t \sigma_2(u, t, r(u)) dW(u) &= -\lambda \int_0^t \sigma(u, t, r(u)) dW(u) \\ &= -\lambda \int_0^t \sigma e^{-\lambda(t-u)} r(u)^\gamma dW(u). \end{aligned} \quad (22.50)$$

Define the subsidiary stochastic variable

$$\phi(t) = \int_0^t \sigma e^{-\lambda(t-u)} r(u)^\gamma dW(u), \quad (22.51)$$

and note that it satisfies the stochastic differential equation

$$d\phi = -\lambda \phi(t) dt + \sigma r(t)^\gamma dW(t). \quad (22.52)$$

Using $\phi(t)$ to substitute the non-Markovian term in Eq. (22.39) we see that $r(t)$ and $\phi(t)$ form a linked *Markovian* system⁹ viz.

$$dr = \left[f_2(0, t) + \alpha(t, t, r(t)) + \int_0^t \alpha_2(u, t, r(u)) du - \lambda \phi(t) \right] dt + \sigma r(t)^\gamma dW(t), \quad (22.53)$$

$$d\phi = -\lambda \phi(t) dt + \sigma r(t)^\gamma dW(t). \quad (22.54)$$

In this system the past levels of volatility and the news arrival process affect the dynamics of $r(t)$ via the variable $\phi(t)$. This captures the type of processes estimated by Brenner et al. (1996) though in a different way to their specification. Here the drift rather than the volatility is affected directly by the $\phi(t)$ process.

In Fig. 22.17 we illustrate some simulations of $r(t)$ and $\phi(t)$ in the system (22.53)–(22.54) for $\gamma = 0.5$. For the drift term we have taken

$$\alpha(t, T, r(t)) = \sigma(t, T, r(t)) \int_t^T \sigma(t, u, r(t)) du, \quad (22.55)$$

⁹We are assuming here that the term $\int_0^t \alpha_2(u, t, r(u)) du$ can be expressed in terms of the state variable, $r(t)$ and $\phi(t)$. This will be the case if we assume the form Eq. (22.55) below that ensures arbitrage free dynamics. Otherwise it may be necessary to introduce further state variables to obtain a Markovian system.

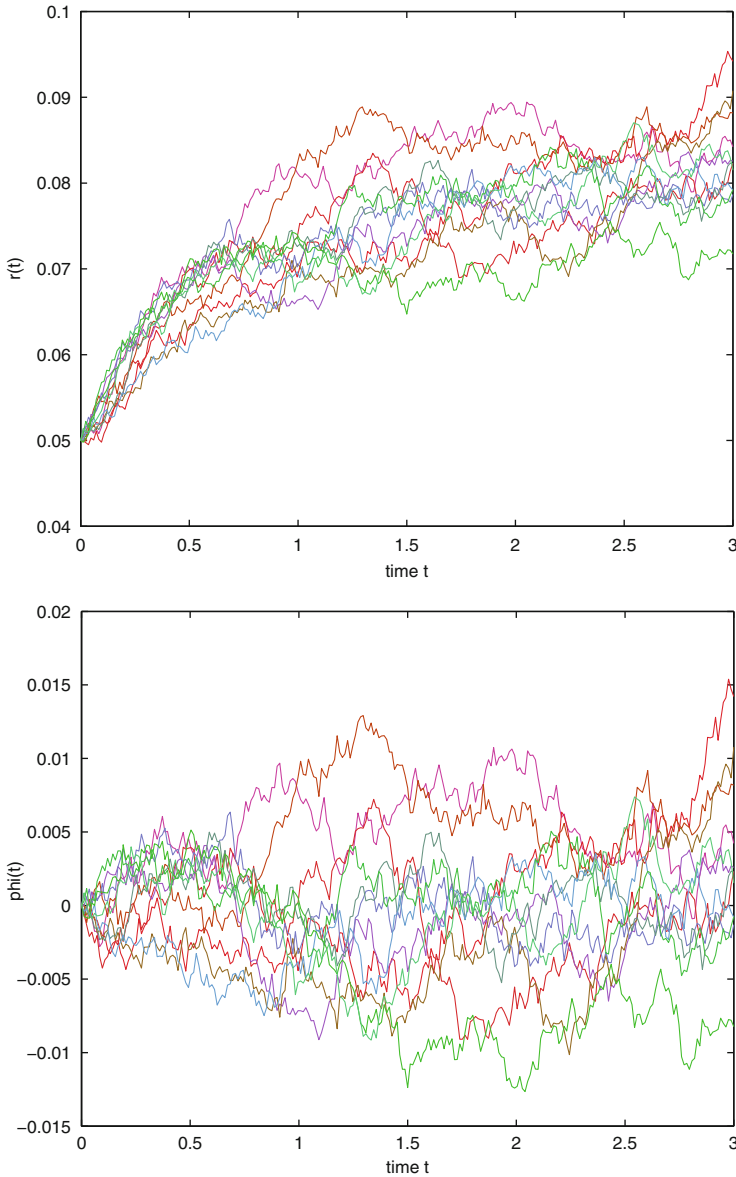


Fig. 22.17 Simulations of $r(t)$ and $\phi(t)$ for (22.53)–(22.54) with drift specified by (22.55)

which ensures that the economy is arbitrage free as we shall show in Chap. 25. The values of σ , λ were taken from Bhar and Chiarella (1997b). The initial forward rate curve $f(0, t)$ is relevant to 90-day bank bill futures data traded on the SFE in 1991, and was calculated using the polynomial fitting procedure described in Bhar and Hunt (1993).

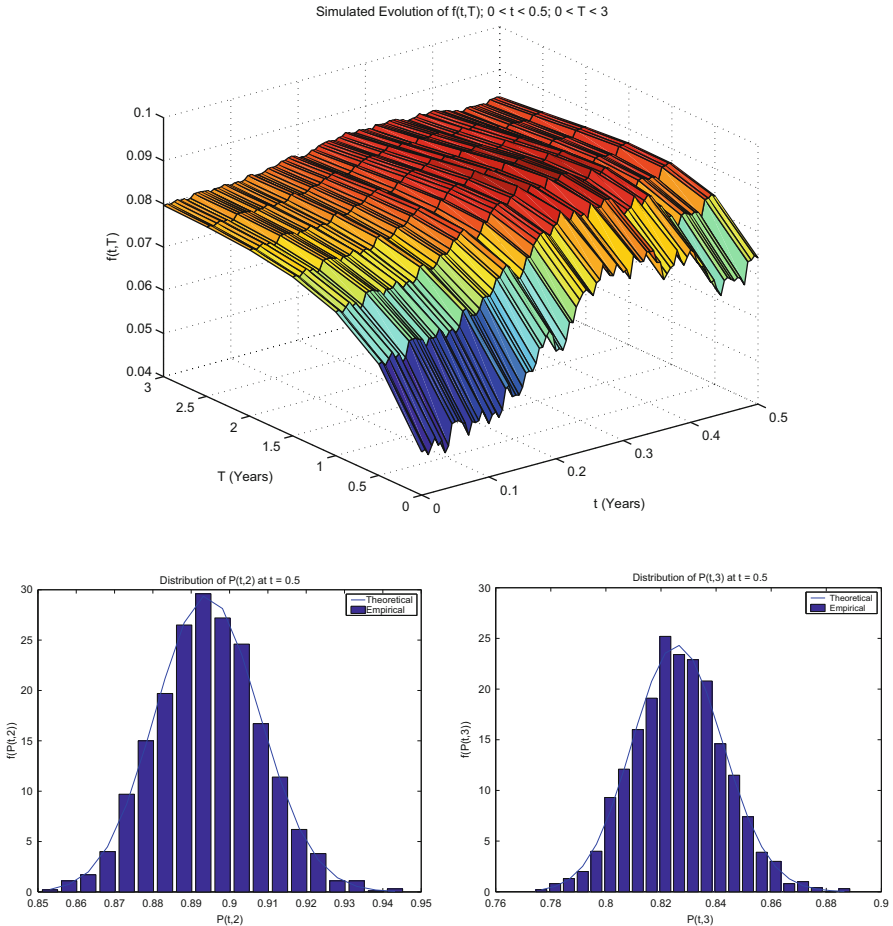


Fig. 22.18 Simulations of the process for forward rates and simulated distribution density of bond prices

In order to give some intuitive feel for the type of dynamics implied by the forward rate process, we also display in Fig. 22.18 simulations of the forward rate process and the simulated distribution density of the corresponding bond prices $P(t, 2)$ and $P(t, 3)$ when $t = 0.5$, for the same values of γ .

22.6 Appendix

Appendix 22.1 Calculation of Covariance

Consider the interest rate process equation (22.38) in the case of a deterministic drift and volatility so that

$$r(t) = f(0, t) + \int_0^t \alpha(u, t) du + \int_0^t \sigma(u, t) dW(u).$$

We can readily calculate that

$$m_r(t) = \mathbb{E}[r(t)] = f(0, t) + \int_0^t \alpha(u, t) du.$$

We can then calculate the covariance between interest rates at two maturities s and t as

$$\begin{aligned} \text{cov}(r(s), r(t)) &= E_0[\{r(s) - m_r(s)\}\{r(t) - m_r(t)\}] \\ &= E_0 \left[\int_0^s \sigma(u, s) dW(u) \int_0^t \sigma(u, t) dW(u) \right]. \end{aligned}$$

For definiteness suppose $s < t$

$$\begin{aligned} &E_0 \left[\int_0^s \sigma(u, s) dW(u) \int_0^t \sigma(u, t) dW(u) \right] \\ &= E_0 \left[\int_0^s \sigma(u, s) dW(u) \int_0^s \sigma(u, t) dW(u) + \underbrace{\int_0^s \sigma(u, s) dW(u) \int_s^t \sigma(u, t) dW(u)}_{\text{these } dW(u) \text{ are non overlapping } \therefore E_0 = 0} \right] \\ &= E_0 \left[\int_0^s \sigma(u, s) dW(u) \int_0^s \sigma(u, t) dW(u) \right] \\ &= \int_0^s \sigma^2(u, t) du. \end{aligned}$$

In general (i.e. $s < t$ or $s > t$) we have

$$\text{cov}(r(s), r(t)) = \int_0^{\min(s,t)} \sigma^2(u, t) du = \int_0^{s \wedge t} \sigma^2(u, t) du$$

where $s \wedge t \equiv \min(s, t)$.

22.7 Problems

Problem 22.1 Consider the spot rate process

$$dr = (\mu_0 + \mu_1 r(t))dt + \sigma_r dW,$$

where μ_0 , μ_1 and σ_r are constants. Calculate the expected average value of the spot rate i.e.

$$\mathbb{E}_t \left[\frac{1}{T-t} \int_t^T r(u) du \right].$$

Problem 22.2 Consider the mean-reverting process

$$dr = \kappa(\theta - r(t))dt + \sigma dW,$$

where κ , θ and σ are constants. Show that for $s \geq t$

$$r(s) = e^{-\kappa(s-t)}r(t) + \theta(1 - e^{-\kappa(s-t)}) + \sigma \int_t^s e^{-\kappa(s-u)} dW(u).$$

Then by using the appropriate version of Fubini's theorem, show that

$$\int_t^T r(s) ds = (r(t) - \theta)B(t, T) + \theta(T - t) + \sigma \int_t^T B(u, T) dW(u),$$

where

$$B(t, T) \equiv \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}).$$

Problem 22.3 In Sect. 22.5 we have discussed how, starting with the dynamics for the forward rate, we can obtain the dynamics for the spot rate and bond price.

Suppose that instead we start with the dynamics for the bond price, i.e. we assume

$$\frac{dP(t, T)}{P(t, T)} = \alpha_P(t, T)dt + \sigma_P(t, T)dz(t).$$

What process does this imply for $f(t, T)$ and $r(t)$?

Suppose we take

$$\sigma_P(t, T) = \bar{\sigma} e^{-\lambda(T-t)}.$$

What will be the corresponding volatility for the forward rate process?

For this volatility function write out the stochastic differential equation for the spot rate $r(t)$.

Is it still possible to write it in Markovian form in this case?

Hint: Determine the stochastic integral equation satisfied by $B(t, T) = \ln P(t, T)$. Then use this to determine the stochastic dynamics for $G(t, T) = \frac{\partial}{\partial T} B(t, T)$.

Problem 22.4 Following on from Problem 22.3, we could also take as our starting point the dynamics of the yield to maturity. That is we assume

$$d\rho(t, T) = \alpha_\rho(t, T)dt + \sigma_\rho(t, T)dz(t).$$

Now determine the stochastic integral equations and stochastic differential equations satisfied by $P(t, T)$, $f(t, T)$ and $r(t)$.

Suppose that we take

$$\sigma_\rho(t, T) = \bar{\sigma} e^{-\lambda(T-t)}$$

what will be the corresponding volatilities for the bond price $P(t, T)$ and forward rate $f(t, T)$?

Write out the stochastic differential equation satisfied by $r(t)$ in this case. Can you express it in Markovian form?

Hint: Recall the definition of the yield to maturity

$$\rho(t, T) = -\frac{\ln P(t, T)}{T - t}.$$

Use Ito's Lemma to find the stochastic differential equation satisfied by $B(t, T) = \ln P(t, T)$. A further application of Ito's Lemma gives the stochastic differential equation for $P(t, T)$.

Problem 22.5 In Sect. 22.5.1 consider the case in which σ and α depend on the forward rate itself, i.e. $\sigma(t, T, \omega(t)) = \sigma(t, T, f(t, T))$. Show that then the forward rate, the derivative of the forward rate with respect to maturity time, the spot interest rate and the bond price form the linked stochastic differential system

$$df(t, T) = \alpha(t, T, f(t, T))dt + \sigma(t, T, f(t, T))dW(t), \tag{22.56}$$

$$df_2(t, T) = [\alpha_2(t, T, f(t, T)) + \alpha_3(t, T, f(t, T))f_2(t, T)]dt + [\sigma_2(t, T, f(t, T)) + \sigma_3(t, T, f(t, T))f_2(t, T)]dW(t), \tag{22.57}$$

$$dr = \left\{ f_2(0, t) + \alpha(t, t, r(t)) + \int_0^t [\alpha_2(u, t, f(u, t)) + \alpha_3(u, t, f(u, t))f_2(u, t)]du \right\} dt + \int_0^t [\sigma_2(u, t, f(u, t)) + \sigma_3(u, t, f(u, t))f_2(u, t)]dW(u)dt + \sigma(t, t, r(t))dW(t), \tag{22.58}$$

$$dP(t, T) = \left[r(t) - \alpha_B(t, T, \cdot) + \frac{1}{2} \sigma_B^2(t, T, \cdot) \right] P(t, T) dt + \sigma_B(t, T, \cdot) P(t, T) dz(t), \quad (22.59)$$

where

$$\alpha_B(t, T, \cdot) = \int_t^T \alpha(t, s, f(t, s)) ds \quad \text{and} \quad \sigma_B(t, T, \cdot) \equiv - \int_t^T \sigma(t, s, f(t, s)) ds. \quad (22.60)$$

Comment on whether this system is Markovian or non-Markovian.

Problem 22.6 Work through the calculations leading to the stochastic differential equation system (22.56)–(22.59). Obtain the form taken by this stochastic differential equation system when it is assumed that

$$\sigma(t, T, f(t, T)) = \sigma_0 e^{-\lambda(T-t)} f(t, T)^\gamma.$$

Problem 22.7 Computational Problem—Simulate the interest rate process

$$dr = \kappa(\bar{r} - r(t))dt + \sigma r^\gamma dW.$$

Use the values that were used to generate Fig. 22.4.

Reproduce Figs. 22.4 and 22.5.

Problem 22.8 Computational Problem—Consider again the interest rate process of Problem 22.7, but now suppose the volatility is stochastic. That is we have

$$\begin{aligned} dr &= \kappa_r(\bar{r} - r(t))dt + \sqrt{v(t)} r(t)^\gamma dZ_r, \\ dv &= \kappa_v(\bar{v} - v(t))dt + \sigma_v \sqrt{v(t)} dZ_v, \end{aligned}$$

where

$$\mathbb{E}[dZ_r dZ_v] = \rho dt.$$

For κ_r and \bar{r} use the same values as in Problem 22.7. For \bar{v} use the values of σ^2 in Table 22.1. Set the initial value of the v process to \bar{v} .

Simulate this model to determine the impact of stochastic volatility on the distributions in Fig. 22.5. In particular see how the skewness and kurtosis of the distribution is affected. Experiment with values of κ_v between 1 and 3, and values of σ_v equal to $\sqrt{\kappa_v \bar{v}}/2$. Consider $\rho = 0.5, 0.0$ and -0.5 .

Problem 22.9 Computational Problem—Consider the forward rate process

$$f(t, T) = f(0, T) + \int_0^t \bar{\sigma}(s, T) ds + \int_0^t \sigma(s, T) dW(s),$$

where

$$\bar{\sigma}(t, T) = \sigma(t, T) \int_t^T \sigma(t, v) dv,$$

so that the dynamics are arbitrage free.

Take the volatility function

$$\sigma(t, T) = \beta_0 [1 + \alpha_0 r(t) + \alpha_1 f(t, T_1)]^\gamma e^{-\lambda(T-t)},$$

and the initial forward curve having the functional form

$$f(0, T) = 0.08 - 0.03 e^{-1.5T}.$$

Using simulation obtain and graph the distribution for $r(t)$, $f(t, T)$ and $P(t, T)$ when $T = 3$, $T_1 = 5$, at $t = 0.5$ and $t = 1.0$ and for the following sets of parameter specifications

$$(a) \quad \gamma = 0, \quad \beta_0 = 0.02, \quad \lambda = 0.6;$$

$$(b) \quad \gamma = \frac{1}{2}, \quad \beta_0 = 0.015, \quad \alpha_0 = 4.41, \quad \alpha_1 = 6.97, \quad \lambda = 0.6.$$

Problem 22.10 Computational Problem—Consider the stochastic differential equation system (22.56)–(22.59) when it is assumed that $\sigma(t, T, f(t, T)) = \sigma_0 e^{-\lambda(T-t)} f(t, T)^\gamma$ (see Problems 22.6 and 22.5). Using simulation obtain and graph the distribution for $r(t)$, $f(t, T)$ and $P(t, T)$ when $T = 3$ at $t = 0.5$ and $t = 1.0$ for the parameter specification

$$(a) \quad \gamma = 0, \sigma_0 = 0.02, \lambda = 0.6;$$

$$(b) \quad \gamma = \frac{1}{2}, \sigma_0 = 0.02236, \lambda = 0.6.$$

Take $f(0, T)$ as in Problem 22.9 and $\alpha(t, T)$ is given by Eq. (22.55) with r replaced by $f(t, T)$.