Chapter 13
Option Pricing Under Jump-Diffusion Processes

Abstract This chapter extends the hedging argument of option pricing developed for continuous diffusion processes previously to the situations when the underlying asset price is driven by the jump-diffusion stochastic differential equations. By constructing hedging portfolios and employing the capital asset pricing model, we provide an option pricing integro-partial differential equations and a general solution. We also examine alternative ways to construct the hedging portfolio and to price option when the jump sizes are fixed.

13.1 Introduction

Now let us turn to the problem of developing the hedging argument under the assumption that the underlying asset price x is driven by the jump-diffusion stochastic differential equation (12.11). To develop a hedging argument we need to know the dynamics of the option price. If the option price f is given by

$$f = f(x,t),$$

then application of the results (12.26) implies that

$$\frac{df}{f} = (\mu_f - \lambda k_f)dt + \sigma_f dw + (Y_f - 1)dN,$$

where[1]

$$\begin{aligned} f\mu_f &= \theta + (\mu - \lambda k)x\Delta + \frac{1}{2}\sigma^2 x^2 \Gamma + \lambda f k_f, \\ f\sigma_f &= \sigma x \Delta, \\ fk_f &= \mathbb{E}^{Q_Y}[f(xY,t) - f(x,t)] = \int [f(xY,t) - f(x,t)]G(Y)dY, \end{aligned}$$

[1]We recall the definitions $\theta = \frac{\partial f}{\partial t}$, $\Delta = \frac{\partial f}{\partial x}$, $\Gamma = \frac{\partial^2 f}{\partial x^2}$.

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and

$$Y_f - 1 \equiv (f(xY,t) - f(x,t))/f(x,t)$$

is the random variable percentage change in the option price. If the Poisson event for the asset occurs and the proportional jump size takes on the value Y, then the Poisson event for the option occurs and the proportional jump size in the option value is given by

$$Y_f = \frac{f(xY,t)}{f(x,t)},$$

which is a nonlinear relationship connecting the random variables Y_f and Y.

13.2 Constructing a Hedging Portfolio

Consider a portfolio which contains the asset, the option on the asset and the riskless asset with return r per unit time in the proportions π_x, π_f, and π_r, so that

$$\pi_x + \pi_f + \pi_r = 1.$$

If V is the value of the portfolio then the return dynamics of the portfolio are given by

$$\begin{aligned}\frac{dV}{V} &= \pi_x \frac{dx}{x} + \pi_f \frac{df}{f} + \pi_r dr \\ &= \pi_x[(\mu - \lambda k)dt + \sigma dw + (Y-1)dN] \\ &\quad + \pi_f[(\mu_f - \lambda k_f)dt + \sigma_f dw + (Y_f - 1)dN] + \pi_r r dt.\end{aligned}$$

Collecting terms and using $\pi_r = 1 - \pi_x - \pi_f$ we obtain

$$\frac{dV}{V} = (\mu_V - \lambda k_V)dt + \sigma_V dw + (Y_V - 1)dN, \tag{13.1}$$

where

$$\begin{aligned}\mu_{\mathrm{V}} &= \pi_x(\mu - r) + \pi_f(\mu_f - r) + r, \\ \sigma_{\mathrm{V}} &= \pi_x \sigma + \pi_f \sigma_f, \\ Y_{\mathrm{V}} - 1 &= \pi_x(Y-1) + \pi_f[f(xY,t) - f(x,t)]/f(x,t), \\ k_{\mathrm{V}} &= \mathbb{E}^{Q_Y}[Y_V - 1].\end{aligned} \tag{13.2}$$

Here $(Y_V - 1)$ is the random variable percentage change in the portfolio's value if the Poisson jump event occurs.

When the asset price follows a diffusion process the hedging portfolio is rendered riskless by choosing the portfolio proportions π_x, π_f such that

$$\pi_x \sigma + \pi_f \sigma_f = 0. \tag{13.3}$$

However, this choice of portfolio weights in the case of a jump-diffusion process, while eliminating the σ_V term will not eliminate the jump risk (i.e. the $Y_V - 1$ term). In fact, there is no choice of π_x and π_f which eliminates the jump risk term (i.e. makes $Y_V = 1$).

Let us nevertheless determine the return characteristics of the portfolio when the Black–Scholes hedge is followed. Letting π_x^* and π_f^* denote the values of π_x, π_f satisfying (13.3) and V^* the corresponding portfolio value we have from (13.1)

$$\frac{dV^*}{V^*} = (\mu_V^* - \lambda k_V^*)dt + (Y_V^* - 1)dN. \tag{13.4}$$

The portfolio return has thus been reduced to a pure jump process, and could also be written

$$\frac{dV^*}{V^*} = \begin{cases} (\mu_V^* - \lambda k_V^*)dt, & \text{if the Poisson jump event does not occur,} \\ (\mu_V^* - \lambda k_V^*)dt + (Y_V^* - 1) & \text{if the Poisson jump event occurs.} \end{cases} \tag{13.5}$$

Equation (13.5) tells us that most of the time the portfolio return will be predictable and earn $(\mu_V^* - \lambda k_V^*)$. However every $(1/\lambda)$ units of time, on average, the portfolio return takes an unexpected jump.

It is possible to say something about the qualitative characteristics of the portfolio return. Note first of all that

$$Y_V^* - 1 = \pi_f^* \frac{f(xY,t) - f(x,t) - f_x(x,t)(xY - x)}{f(x,t)}.$$

Since the option price is a strictly convex function of the asset price it follows that

$$\frac{f(xY,t) - f(x,t)}{xY - x} > f_x(x,t),$$

for $Y > 1$, and

$$\frac{f(xY,t) - f(x,t)}{xY - x} < f_x(x,t),$$

for $Y < 1$. Thus for all values of Y, it follows that

$$f(xY,t) - f(x,t) - f_x(x,t)(xY - x) > 0.$$

Hence

$$\text{sign}(Y_V^* - 1) = \text{sign}(\pi_f^*).$$

Suppose an investor is long the stock and short the option (i.e. $\pi_f^* < 0$) then most of the time he or she would earn more than the expected return on the hedge μ_V^*, since $k_V^* < 0$. The investor will however suffer losses when the asset price jumps from time to time. These losses occur at such a frequency so as to, on average, offset the excess return $-\lambda k_V^*$. If we define as a "quiet" period, that period in between the arrival of Poisson jump events, and if we assume that the jump events are related to asset specific information then the above argument shows that during quiet periods writers of options will tend to make what appear to be positive excess returns. Purchasers of options on the other hand would make negative excess returns and therefore appear as "losers". However, at the arrival (relatively infrequently) of Poisson jump events, the options writers will suffer loss and the buyers appear as "winners". Since the arrival of the Poisson events is random, there is no systematic way of exploiting this understanding of the dynamics. The reverse argument applies when the investor is short the asset and long the option (i.e. $\pi_f^* > 0$).

13.3 Pricing the Option

The clue to pricing the option in the presence of jump-diffusion processes is the alternative approach used by Black–Scholes employing the Capital Asset Pricing model.

We have already stressed that the Poisson jump events are asset specific. It follows that the jump component of the asset's return represents non-systematic risk. It also follows that, since the only uncertainty in the V^* portfolio of the previous section is the Poisson jump component, then its risk is uncorrelated with the market, i.e. it contains only non-systematic risk. From modern portfolio theory we have the result that portfolios containing only non-systematic risk have a beta factor of zero. Furthermore, if the CAPM describes security returns then the return on a zero beta portfolio must equal the riskless rate. It follows that

$$\mu_V^* = r,$$

or, from (13.2) that

$$\pi_x^*(\mu - r) + \pi_f^*(\mu_f - r) = 0,$$

which when combined with

$$\pi_x^* \sigma + \pi_f^* \sigma_f = 0,$$

yields

$$\frac{\mu - r}{\sigma} = \frac{\mu_f - r}{\sigma_f}. \tag{13.6}$$

After applying the definitions of μ_f and σ_f in the last equation, we obtain the following equation for the option price

$$\frac{\partial f}{\partial t} + (r - \lambda k)x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - rf + \lambda \mathbb{E}^{Q_Y}[f(xY,t) - f(x,t)] = 0. \tag{13.7}$$

Because of the expectation operator $\mathbb{E}^{Q_Y}$, Eq. (13.7) is an integro-partial differential equation and solution techniques for it require a degree of complexity beyond those for the Black–Scholes partial differential equation.

We may use (13.6) to obtain a martingale representation of the price. Using an argument familiar from Chaps. 8 and 10, if we use ϕ to denote the market price of risk associated with the risk factor dw then (13.6) may be interpreted as

$$\begin{aligned} \mu &= r + \phi\sigma, \\ \mu_f &= r + \phi\sigma_f. \end{aligned}$$

Thus in the absence of riskless arbitrage opportunities the stochastic differential equations for x and f may be written

$$\begin{aligned} \frac{dx}{x} &= (r - \lambda k + \phi\sigma)dt + \sigma dw + (Y - 1)dN, \\ \frac{df}{f} &= (r - \lambda k_f + \phi\sigma_f)dt + \sigma_f dw + (Y_f - 1)dN. \end{aligned}$$

Or alternatively as

$$\frac{dx}{x} = rdt + \sigma d\tilde{w} + [(Y - 1)dN - \lambda k dt], \tag{13.8}$$

$$\frac{df}{f} = rdt + \sigma_f d\tilde{w} + [(Y_f - 1)dN - \lambda k_f dt], \tag{13.9}$$

where

$$\tilde{w}(t) = w(t) + \int_0^t \phi(s)ds.$$

Under the original measure $\mathbb{P}$, $\tilde{w}$ will not be a standard Wiener process, but application of Girsanov's theorem for processes involving jumps (see Bremaud 1981) allows us to assert that it is possible to obtain an equivalent measure $\tilde{\mathbb{P}}$ under which $\tilde{w}$ is a standard Wiener process and N remains a jump process with jump intensity λ.

We note that (13.9) may be written

$$d(fe^{-rt}) = e^{-rt}\sigma_f f d\tilde{w} + e^{-rt} f[(Y_f - 1)dN - \lambda k_f dt]$$

so that under $\tilde{\mathbb{P}}$ the quantity fe^{-rt}, the option price measured in units of the money market account e^{rt}, is a martingale, i.e.

$$f(x,t) = e^{-r(T-t)}\tilde{\mathbb{E}}_t[f(x_T, T)],$$

where $\tilde{\mathbb{E}}_t$ is the expectation operator under $\tilde{\mathbb{P}}$.

We note that one way to calculate $\tilde{\mathbb{E}}_t$ would be to simulate the jump-diffusion process (13.8) for x. Application of the Feynman–Kac formula for jump-diffusion processes (see Appendix 12.1) would yield the integro-partial differential equation (13.7). Thus we have established the link between the martingale viewpoint and the integro-partial differential equation viewpoint.

13.4 General Form of the Solution

Recall that in Eq. (13.7), t is the current time. If we switch the time variable to $\tau = T - t =$ time-to-maturity, then Eq. (13.7) becomes

$$-\frac{\partial f}{\partial \tau} + (r - \lambda k)x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - rf + \lambda \mathbb{E}^{Q_Y}[f(xY,\tau) - f(x,\tau)] = 0. \tag{13.10}$$

To fully appreciate the nature of the pricing equation (13.10), recall that $G(Y)$ is the probability density function for the random variable Y then (13.10) may be written

$$\begin{aligned} -\frac{\partial f}{\partial \tau} + (r - \lambda k)x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - rf \\ + \lambda \int_{-\infty}^{\infty} [f(xY,\tau) - f(x,\tau)]G(Y)dY = 0, \end{aligned} \tag{13.11}$$

where

$$k = \int_{-\infty}^{\infty} YG(Y)dY - 1.$$

This type of equation may be classed as a mixed integro-partial differential equation. Whilst the solution of such equations is in general quite difficult, it turns out that the general form of the solution may be expressed in a convenient form even before we specify the density function $G(Y)$.

In the situation when the underlying asset is common stock equation (13.10) must be solved subject to the boundary condition

$$f(0, \tau) = 0, \tag{13.12}$$

and the initial condition

$$f(x, 0) = \max[0, x - E], \tag{13.13}$$

where E is the exercise price of the option. Let $M(x, \tau; E, \sigma^2, r)$ denote the solution to (13.10) in the absence of the jump component, i.e. when $\lambda = 0$. Thus M would be the Black–Scholes solution given by

$$M(x, \tau; E, \sigma^2, r) = x\mathcal{N}(d_1) - Ee^{-r\tau}\mathcal{N}(d_2), \tag{13.14}$$

where

$$d_1 = \frac{\ln(x/E) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}.$$

Define the random variable $X_n = \prod_{i=1}^{n} Y_i$ as one having the same distribution as the product of n independently identically distributed random variables, each identically distributed as the random variable price change Y. It is assumed $X_0 = 1$. Define $\mathbb{E}^n$ to be the expectation operator over the distribution of X_n (Fig. 13.1).

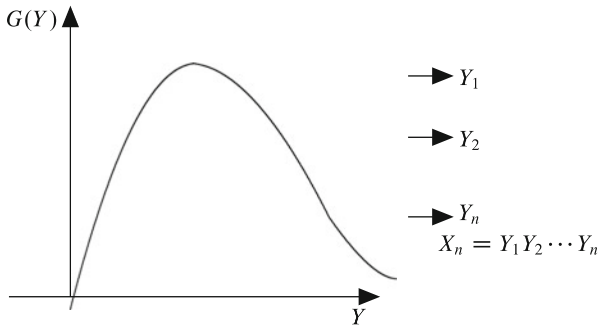

Fig. 13.1 Constructing the random variable X_n

We show in Appendix 13.1 that the solution to (13.10) subject to the boundary and initial conditions (13.12), (13.13) can be written[2]

$$f(x,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathbb{E}^n[M(xX_n e^{-\lambda k\tau}, \tau; E, \sigma^2, r)]. \tag{13.15}$$

To apply the solution (13.15) we need to specify the probability distribution of the random variable Y. Let us consider in particular the case when Y follows a log-normal distribution $\ln Y \sim \phi(\gamma - \delta^2/2, \delta^2)$. It follows that

$$\gamma = \ln(1+k),$$

and that X_n has a log-normal distribution with

$$\mathbb{E}^n[X_n] = e^{n\gamma}, \qquad \mathrm{var}[\ln X_n] = n\delta^2.$$

If we let

$$M_n(x,\tau) = M(x,\tau; E, v_n^2, r_n),$$

where

$$v_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}, \qquad r_n = r - \lambda k + \frac{n\gamma}{\tau},$$

then the solution (13.15) reduces to

$$f(x,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} M_n(x,\tau),$$

where $\lambda' = \lambda(1+k)$. The quantity $M_n(x,\tau)$ is the value of the option, conditional on knowing that exactly n Poisson jumps will occur during the life of the option. The option price is then the expectation of all such values where the expectation is taken over the Poisson distribution (with parameter $\lambda'\tau$) that n jumps will occur during the life of the option.

In Figs. 13.2 and 13.3 we show the effect on the option price and on delta of increasing values of λ. Here we have used the parameter values $T = 1$, $E = 1$, $r = 0.05$, $\sigma = 0.2$, $\gamma = 0$ and $\delta = 0.25$.

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[2]The forms of the solution given here are from the original Merton (1976) paper. He only demonstrates that these solutions indeed satisfy the integro-partial differential equation (13.11) and relevant boundary conditions. Theory on uniqueness of solutions guarantees that this is indeed "the solution". Appendix 13.1 reproduces (modulo some notational changes) Merton's calculations. However this approach gives us no systematic method to solve the integro-partial differential equations encountered in the jump-diffusion case. In Chap. 14 we outline the use of the Fourier transform technique as one such systematic approach.

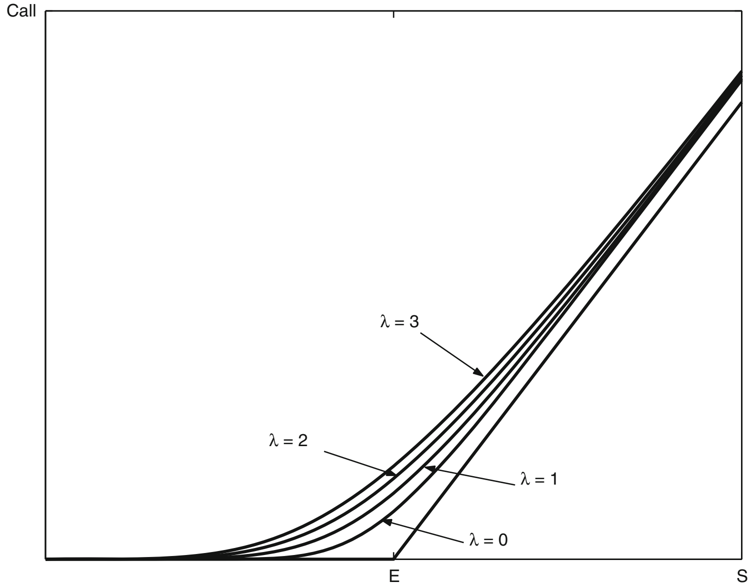

Fig. 13.2 Effect of increasing values of λ on the option price

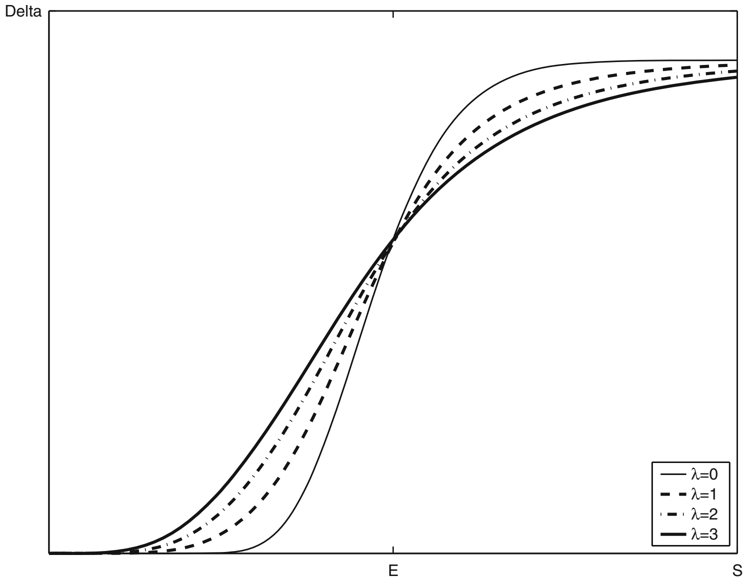

Fig. 13.3 Effect of increasing values of λ on the delta

13.5 Alternative Ways of Completing the Market

As we have seen in the previous sections the incorporation of jumps into the diffusion process governing the dynamics of the asset price introduces an additional source of risk. Namely the risk associated with the Poisson stochastic process governing the jump part of the process followed by the asset price. In order to successfully apply the hedging argument we need some way of hedging this additional risk. The way proposed by Merton in Sect. 13.3 is one way to do this. However other ways are also possible and these usually involve introducing some additional hedging instruments into the hedging portfolio. Such a procedure of introducing a sufficient number of traded instruments to hedge away the number of risk factors is known as "completing the market".

One way of completing the market is to introduce additional options into the hedging portfolio, an approach which was developed by Jones (1984). It is also possible to complete the market by using interest rate market instruments as in Jarrow and Madan (1995).

Here we follow the approach of Jones (1984) and introduce several options into the hedging portfolio (for example, options with different strike prices). For instance we may introduce two options on the stock under consideration. Since we have a finite number of hedging instruments we can only hedge a finite number of "jump risks". Hence in this approach we have to restrict the type of jumps that can occur. In the case of the availability of two options as hedging instruments we allow jumps to have only two amplitudes, as shown in Fig. 13.4.

Hence we write the stock price process as

$$\frac{dx}{x} = \mu dt + \sigma dw + k_1 dN_1 + k_2 dN_2, \tag{13.16}$$

where

$$Pr(dN_i = 1) = \lambda_i dt, \qquad Pr(dN_i = 0) = 1 - \lambda_i dt$$

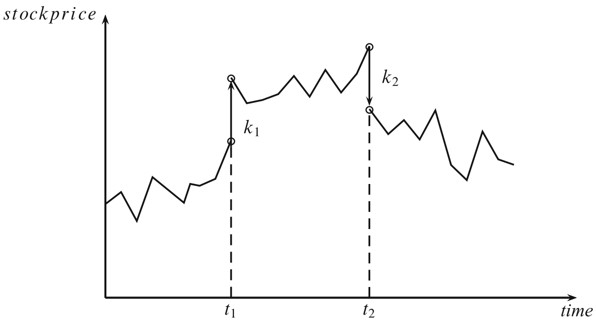

Fig. 13.4 A finite number of fixed jump sizes

for $i = 1, 2$, and k_1, k_2 measure the proportional price jumps in the case of Poisson events.

Let g and h represent the prices of two options written on the stock and assume that the option dynamics contain the same kind of risks as the stock itself. Then the option price dynamics may be written

$$\frac{dg}{g} = \mu_g dt + \sigma_g dw + k_{g_1} dN_1 + k_{g_2} dN_2, \tag{13.17}$$

$$\frac{dh}{h} = \mu_h dt + \sigma_h dw + k_{h_1} dN_1 + k_{h_2} dN_2, \tag{13.18}$$

where the coefficients μ, σ, k represent expected return, volatility and proportional price jumps for each option. All coefficients are assumed to be functions of x, g, h and time t.

We note from (13.16)–(13.18) that the unconditional expected returns are given by

$$\mathbb{E}\left[\frac{dx}{x}\right] = (\mu + k_1\lambda_1 + k_2\lambda_2)dt,$$

$$\mathbb{E}\left[\frac{dg}{g}\right] = (\mu_g + k_{g_1}\lambda_1 + k_{g_2}\lambda_2)dt,$$

$$\mathbb{E}\left[\frac{dh}{h}\right] = (\mu_h + k_{h_1}\lambda_1 + k_{h_2}\lambda_2)dt.$$

Let f be the price of any other option on the stock having an expiry date earlier than that of options g and h. We form a hedging portfolio consisting of the three options, the stock and the risk-free asset. We assume that the price of option f is a function $f(x, g, h, \tau)$ of the stock price, the other two option prices and its time-to-maturity τ in general.

By an application of Ito's Lemma in several variables (see Sect. 6.5) and Ito's Lemma for jump processes (Sect. 12.3) the dynamics of the option f are given by

$$\frac{df}{f} = \mu_f dt + \sigma_f dw + k_{f_1} dN_1 + k_{f_2} dN_2,$$

where

$$\mu_f \equiv \frac{1}{f}\left(\mathscr{D}f + \mu x \frac{\partial f}{\partial x} + \mu_g g \frac{\partial f}{\partial g} + \mu_h h \frac{\partial f}{\partial h} - \frac{\partial f}{\partial \tau}\right), \tag{13.19}$$

$$\mathscr{D}f \equiv \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_g^2 g^2 \frac{\partial^2 f}{\partial g^2} + \frac{1}{2}\sigma_h^2 h^2 \frac{\partial^2 f}{\partial h^2}$$

$$+ \sigma x \sigma_g g \frac{\partial^2 f}{\partial x \partial g} + \sigma x \sigma_h h \frac{\partial^2 f}{\partial x \partial h} + \sigma_g g \sigma_h h \frac{\partial^2 f}{\partial g \partial h}, \tag{13.20}$$

$$\sigma_f \equiv \frac{1}{f}\left(\sigma x \frac{\partial f}{\partial x} + \sigma_g g \frac{\partial f}{\partial g} + \sigma_h h \frac{\partial f}{\partial h}\right), \tag{13.21}$$

$$k_{f_i} \equiv \frac{1}{f}[f(xY_i, gY_{g_i}, hY_{h_i}, \tau) - f(x, g, h, \tau)], \qquad (i = 1, 2), \tag{13.22}$$

where,

$$Y_i = (k_i + 1), \qquad Y_{g_i} = k_{g_i} + 1, \qquad Y_{h_i} = k_{h_i} + 1, \qquad (i = 1, 2).$$

We note that all coefficients are functions of the stock price, the first two option prices and time. The dynamics of x, g, h and f each contain the three risk terms dz, dN_1 and dN_2. The stock x and options g, h span the three risk dimensions that they have in common with the option f. Hence by forming a hedge of x, g and h we can cancel any risk due to f. This reflects the redundancy of f since it can be viewed as an instrument which duplicates a return pattern already available via a dynamic portfolio strategy.

Consider the hedging portfolio and suppose that the weights of the risky asset x, options g, h, f and riskless asset r are π, π_g, π_h, π_f, π_r respectively (so that $\pi_r \equiv -(\pi + \pi_h + \pi_g + \pi_f)$ since the weights sum to zero). If V denotes the value of the hedging portfolio then

$$\begin{aligned}
\frac{dV}{V} &= \left[\pi(\mu - r) + \pi_g(\mu_g - r) + \pi_h(\mu_h - r) + \pi_f(\mu_f - r)\right] dt \\
&\quad + [\pi\sigma + \pi_g\sigma_g + \pi_h\sigma_h + \pi_f\sigma_f] dw \\
&\quad + [\pi k_1 + \pi_g k_{g_1} + \pi_h k_{h_1} + \pi_f k_{f_1}] dN_1 \\
&\quad + [\pi k_2 + \pi_g k_{g_2} + \pi_h k_{h_2} + \pi_f k_{f_2}] dN_2 \,.
\end{aligned}$$

The portfolio will be riskless if

$$\pi\sigma + \pi_g\sigma_g + \pi_h\sigma_h + \pi_f\sigma_f = 0, \tag{13.23}$$

$$\pi k_1 + \pi_g k_{g_1} + \pi_h k_{h_1} + \pi_f k_{f_1} = 0, \tag{13.24}$$

$$\pi k_2 + \pi_g k_{g_2} + \pi_h k_{h_2} + \pi_f k_{f_2} = 0. \tag{13.25}$$

The return on the hedging portfolio would then be

$$\frac{dV}{V} = \left[\pi(\mu - r) + \pi_g(\mu_g - r) + \pi_h(\mu_h - r) + \pi_f(\mu_f - r)\right] dt.$$

Following a now standard argument, this return must be zero so that

$$\pi(\mu - r) + \pi_g(\mu_g - r) + \pi_h(\mu_h - r) + \pi_f(\mu_f - r) = 0. \tag{13.26}$$

The four simultaneous Eqs. (13.23)–(13.26) in the weights $(\pi, \pi_g, \pi_h, \pi_f)$ may be written in matrix form as

$$\begin{bmatrix} \mu - r & \mu_g - r & \mu_h - r & \mu_f - r \\ \sigma & \sigma_g & \sigma_h & \sigma_f \\ k_1 & k_{g_1} & k_{h_1} & k_{f_1} \\ k_2 & k_{g_2} & k_{h_2} & k_{f_2} \end{bmatrix} \begin{bmatrix} \pi \\ \pi_g \\ \pi_h \\ \pi_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13.27}$$

Using standard results in linear algebra (13.27) implies that there must exist quantities ξ, γ_1, γ_2 such that

$$\mu - r = \xi\sigma + \gamma_1 k_1 + \gamma_2 k_2, \tag{13.28}$$

$$\mu_g - r = \xi\sigma_g + \gamma_1 k_{g_1} + \gamma_2 k_{g_2}, \tag{13.29}$$

$$\mu_h - r = \xi\sigma_h + \gamma_1 k_{h_1} + \gamma_2 k_{h_2}, \tag{13.30}$$

$$\mu_f - r = \xi\sigma_f + \gamma_1 k_{f_1} + \gamma_2 k_{f_2}. \tag{13.31}$$

Making use of (13.31) and substituting (13.28)–(13.30) and (13.21), we find that the option price f must satisfy

$$\begin{aligned} \mathscr{D} f &+ (r + \gamma_1 k_1 + \gamma_2 k_2) x \frac{\partial f}{\partial x} + (r + \gamma_1 k_{g_1} + \gamma_2 k_{g_2}) g \frac{\partial f}{\partial g} \\ &+ (r + \gamma_1 k_{h_1} + \gamma_2 k_{h_2}) h \frac{\partial f}{\partial h} - (r + \gamma_1 k_{f_1} + \gamma_2 k_{f_2}) f - \frac{\partial f}{\partial \tau} = 0. \end{aligned} \tag{13.32}$$

Note that Eqs. (13.28)–(13.31) extend the familiar interpretation of the no-riskless arbitrage condition. First we interpret ξ as the market price of risk associated with the uncertainty due to the continuous diffusion part of the asset price process and γ_i as the market price of risk associated with the ith jump component. Then Eqs. (13.28)–(13.31) assert that in equilibrium the expected return on each risky asset equals the risk free rate plus the sum of the market price of each risk component times the amount of associated risk.

A considerable simplification of the option pricing equation (13.32) is possible if we assume that all parameters are functions of the stock price and time alone i.e. $f(x, g, h, \tau) = f(x, \tau)$. Then Eq. (13.32) reduces to

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + (r + \gamma_1 k_1 + \gamma_2 k_2) x \frac{\partial f}{\partial x} - (r + \gamma_1 k_{f_1} + \gamma_2 k_{f_2}) f - \frac{\partial f}{\partial \tau} = 0, \tag{13.33}$$

where we recall that

$$k_{f_i} = \frac{1}{f}[f((k_i + 1)x, \tau) - f(x, \tau)], \quad (i = 1, 2).$$

If we assume that all parameters are constant then Eq. (13.33) may be solved in a way similar to that used to solve Merton's equation (13.11) and the solution turns out to be

$$f(x, \tau) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[e^{-(\gamma+\delta)\tau} \frac{(\gamma_1 \tau)^m}{m!} \frac{(\gamma_2 \tau)^n}{n!} \right] \times \\ M[xY_1^m Y_2^n e^{-(\gamma_1 k_1 + \gamma_2 k_2)\tau}, \tau; E, \sigma^2, r], \tag{13.34}$$

where $Y_i = k_i + 1$ for $i = 1, 2$. Suppose we maintain our assumption that all parameters are functions of stock price and time only. Then in the argument leading up to Eq. (13.34) the roles of f, g and h can be interchanged. It follows that g and h must also satisfy an equation like (13.34).

If we assume knowledge of σ, k_1, k_2 is already available, then we have two unknown parameters γ_1, γ_2. Using market values of g, h we may solve $g(x, \tau; \gamma_1, \gamma_2) = g_{market}$ and $h(x, \tau; \gamma_1, \gamma_2) = h_{market}$ to obtain $\widehat{\gamma_1}, \widehat{\gamma_2}$, which may then be used to price the option f.

13.6 Large Jumps

In this section we restrict our attention to binomial jumps. That is we assume $Y_1 = k_1 + 1, Y_2 = k_2 + 1$ satisfy $Y_1 Y_2 = 1$. In this case, if we define

$$k_2 = 1/Y_1 - 1,$$

$$k_{f_2} = \frac{1}{f}(f(x/Y_1, \tau) - f(x, \tau)),$$

then Jones (1984) shows that the option pricing formula (13.34) specialises to

$$f(x, \tau) = \sum_{n=-\infty}^{\infty} (\gamma_1/\gamma_2)^{n/2} e^{-\upsilon\tau} I_n(2\tau\sqrt{\gamma_1\gamma_2}) M(xY_1^n, e^{-\upsilon\psi\tau}, \tau; E, \sigma^2, r),$$

where

$$I_n(z) = \sum_{j=1}^{\infty} \frac{z^{n+2}}{j!(n+j)!}$$

is a modified Bessel function of the first kind of integer order n,

$$v \equiv \gamma_1 + \gamma_2 = \text{the probability of a jump},$$

and

$$\psi \equiv (\gamma_1 k_1 + \gamma_2 k_2)/v = \text{the expected jump amplitude}.$$

These last two results are derived in Feller (1966).

We wish to consider the limiting case in which the jump amplitude becomes large, but at the same time the expected jump amplitude remains constant. In such a case the expected returns on the stock remains finite. If we define $\chi \equiv \ln Y_1 = -\ln Y_2$, then we can define the conditional probabilities for upward versus downward jumps as

$$\gamma_1/v = (\psi + 1 - e^{-\chi})/2\sinh\chi, \qquad \gamma_2/v = (e^{\chi} - \psi - 1)/2\sinh\chi.$$

Note that

$$\lim_{\chi\to\infty} \frac{\gamma_1}{v} = 0,$$

whilst

$$\lim_{\chi\to\infty} \frac{\gamma_2}{v} = 1.$$

These results indicate that large positive jumps are “rare” compared to large negative jumps.

The jump magnitude becoming large is captured by considering $\chi \to \infty$. In this case Jones (1984) shows that the conditional expected upward jump in the option price satisfies

$$\lim_{\chi\to\infty} \left[\frac{\gamma_1}{v}(f(xe^{\chi}, \tau) - f(x, \tau))\right] = (\psi + 1)x,$$

and that the conditional expected downward jump satisfies

$$\lim_{\chi\to\infty} \left[\frac{\gamma_2}{v}(f(xe^{\chi}, \tau) - f(x, \tau))\right] = f(0, \tau) - f(x, \tau) = -f.$$

Note that in the present notation the partial differential equation (13.33) for f may be written

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + (r - v\chi)x\frac{\partial f}{\partial x} - \frac{\partial f}{\partial \tau}$$
$$+ \gamma_1[f(xe^{\chi}, \tau) - f(x, \tau)] + \gamma_2[f(x, e^{-\chi}, \tau) - f(x, \tau)] - rf = 0.$$

Taking the limit as $\chi \to \infty$ we obtain the partial differential equation

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + (\eta - \theta)x\frac{\partial f}{\partial x} - \eta f + \frac{\partial f}{\partial \tau} + \theta x = 0, \tag{13.35}$$

where

$$\eta \equiv r + v \quad \text{and} \quad \theta \equiv v(\psi + 1).$$

The solution to (13.35) turns out to be

$$f(x, \tau) = x[1 - e^{-\theta\tau} N(-b_1)] - Ed^{-\eta\tau} N(b_2),$$

where

$$b_1 \equiv \frac{\ln(x/E) + (\eta - \theta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad b_2 \equiv b_1 - \sigma\sqrt{\tau}.$$

13.7 Appendix

Appendix 13.1 The Solution of the Integro-Partial Differential Equation

To simplify the notation put

$$P_n(\tau) = \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}, \qquad V_n = xX_n e^{-\lambda k\tau}.$$

We note the derivatives

$$\frac{dP_n(\tau)}{d\tau} = -\lambda\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} + \lambda\frac{e^{-\lambda\tau}(\lambda\tau)^{n-1}}{(n-1)!}$$
$$= \begin{cases} -\lambda P_n(\tau) + \lambda P_{n-1}(\tau), & (n > 0), \\ -\lambda P_n(\tau), & (n = 0), \end{cases}$$

and

$$\frac{\partial V_n}{\partial \tau} = -\lambda k x X_n e^{-\lambda k \tau} = -\lambda k V_n.$$

Using the above notation the proposed solution (13.15) may be written

$$f(x,\tau) = \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{M(V_n,\tau;E,\sigma^2,r)\}. \tag{13.36}$$

We shall simply show that (13.36) satisfies the integro-partial differential equation (13.10) and the associated boundary and initial conditions (13.12) and (13.13). Observe that

$$\begin{aligned}\frac{\partial f}{\partial x} &= \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\left\{\frac{\partial}{\partial x}M(V_n,\tau;E,\sigma^2,r)\right\}\\ &= \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\left\{\frac{\partial V_n}{\partial x}M^{(1)}(V_n,\tau;E,\sigma^2,r)\right\}\\ &= \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{X_n e^{-\lambda k\tau}M^{(1)}(V_n,\tau;E,\sigma^2,r)\}.\end{aligned} \tag{13.37}$$

Here $M^{(1)}$ indicates the first partial derivative of M with respect to its first argument. Upon multiplying through by x the last equation reads

$$x\frac{\partial f}{\partial x} = \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{V_n M^{(1)}(V_n,\tau;E,\sigma^2,r)\}.$$

Differentiating (13.37) again with respect to x we obtain

$$\frac{\partial^2 f}{\partial x^2} = \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\left\{(X_n e^{-\lambda k\tau})^2 M^{(11)}(V_n,\tau;E,\sigma^2,r)\right\},$$

which after multiplication by x^2 becomes

$$x^2\frac{\partial^2 f}{\partial x^2} = \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{V_n^2 M^{(11)}(V_n,\tau;E,\sigma^2,r)\},$$

where $M^{(11)}$ indicates the second partial derivative of M with respect to its first argument. Finally

$$\frac{\partial f}{\partial \tau} = \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\left\{\frac{d}{d\tau}M(V_n,\tau;E,\sigma^2,r)\right\} + \sum_{n=0}^{\infty}\frac{dP_n(\tau)}{d\tau}\mathbb{E}^n\{M(v_n,\tau;E,\sigma^2,r)\}. \tag{13.38}$$

Since

$$\frac{d}{d\tau}M(V_n,\tau;E,\sigma^2,r) = \frac{dV_n}{d\tau}M^{(1)}(V_n,\tau;E,\sigma^2,r) + M^{(2)}(V_n,\tau;E,\sigma^2,r),$$

Eq. (13.38) becomes

$$\begin{aligned}\frac{\partial f}{\partial \tau} &= \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{-\lambda k V_n M^{(1)} + M^{(2)}\} + \sum_{n=0}^{\infty}(-\lambda)P_n(\tau)\mathbb{E}^n\{M\} \\ &\quad + \sum_{n=1}^{\infty}\lambda P_{n-1}(\tau)\mathbb{E}^n\{M\},\end{aligned}$$

where M, $M^{(1)}$ and $M^{(2)}$ are all evaluated at $(V_n,\tau;E,\sigma^2,r)$. Upon rearranging, the last expression can be written as

$$\begin{aligned}\frac{\partial f}{\partial \tau} &= -\lambda f - \lambda k \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{V_n M^{(1)}\} + \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{M^{(2)}\} \\ &\quad + \lambda \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{M(V_{n+1},\tau;E,\sigma^2,r)\}.\end{aligned}$$

Now

$$\begin{aligned}&\frac{-\partial f}{\partial \tau} + (r-\lambda k)x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2} - rf \\ &= \lambda f + \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{\lambda k V_n M^{(1)} + M^{(2)}\} + (r-\lambda k)\sum_{n-0}^{\infty} P_n(\tau)\mathbb{E}^n\{V_n M^{(1)}\} \\ &+ \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\left\{\frac{1}{2}\sigma^2 V_n^2 M^{(11)}\right\} - r\sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{M\} \\ &- \lambda\sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^{n+1}\{M(V_{n+1},\tau;E,\sigma^2,r)\}\end{aligned}$$

$$= \lambda f - \lambda \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^{n+1}\{M(V_{n+1}, \tau; E, \sigma^2, r)\}$$

$$+ \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n \left\{ M^{(2)} + rV_n M^{(1)} + \frac{1}{2}\sigma^2 V_n^2 M^{(11)} - rM \right\}. \qquad (13.39)$$

The expression in the curly bracket in the third term of (13.39) is zero since $M(V_n, \tau; E, \sigma^2, r)$ is the solution of

$$M^{(2)} + rV_n M^{(1)} + \frac{1}{2}\sigma^2 V_n M^{(11)} - rM = 0.$$

Thus (13.39) reduces to

$$\frac{-\partial f}{\partial \tau} + (r - \lambda k)x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - rf$$
$$= \lambda f - \lambda \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^{n+1}\{M(V_{n+1}, \tau; E, \sigma^2, r)\}. \qquad (13.40)$$

The final step in the proof is to show that the term on the right-hand side of (13.40) equals

$$\lambda \mathbb{E}^{Q_Y}[f(x, \tau) - f(xY, \tau)],$$

where we write Q_Y to indicate clearly that expectations are being taken with respect to the distribution of the random variable Y. Replacing x by xY in (13.36) and applying the operator $\mathbb{E}^{Q_Y}$ we have

$$\mathbb{E}^{Q_Y}\{f(xY, \tau)\} = \mathbb{E}^{Q_Y}\left[\sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{M(YV_n, \tau; E, \sigma^2, r)\}\right]. \qquad (13.41)$$

Given the definition of X_n as the product of n independent drawings from the distribution of Y and $\mathbb{E}^n$ as the expectation operator over the distribution of X_n it should be clear that

$$\mathbb{E}^{Q_Y}\mathbb{E}^n M(YV_n, \ldots.) = \mathbb{E}^{n+1} M(V_{n+1}, \ldots.).$$

Thus (13.41) becomes

$$\mathbb{E}^{Q_Y}\{f(xY, \tau)\} = \sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^{n+1}\{M(V_{n+1}, \tau; E, \sigma^2, r)\}.$$

The summation on the right-hand side above is the same as the summation in the second term on the right-hand side of (13.41), so that this last equation may be written

$$\frac{-\partial f}{\partial \tau}+(r-\lambda k)x\frac{\partial f}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}-rf = \lambda f(x,\tau)-\lambda\mathbb{E}^{Q_Y}\{f(xY,\tau)\}, \qquad (13.42)$$

which may be rearranged to

$$\frac{-\partial f}{\partial \tau}+(r-\lambda k)x\frac{\partial f}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}-rf+\lambda\mathbb{E}^{Q_Y}[f(xY,\tau)-f(x,\tau)] = 0,$$

which is Eq. (13.10).

We have thus shown that Eq. (13.15) is the general form of the solution. It remains only to show that this form of the solution also satisfies the boundary and initial conditions. Since $x = 0$, implies $V_n = 0$ and given that

$$M(0,\tau;E,\sigma^2,r) = 0,$$

it follows that

$$f(0,\tau) = 0,$$

indicating that the boundary condition (13.13) is satisfied by the solution (13.15). To show that the initial condition (13.13) is satisfied requires a little more analysis. Note first of all that

$$M(V_n,0;E,\sigma^2,r) = \max[0, V_n - E],$$

and so

$$\begin{aligned}\mathbb{E}^n\{M(V_n,0;E,\sigma^2,r)\} &= \mathbb{E}^n\{\max[0,V_n-E]\}\\ &\le \mathbb{E}^n\{V_n\} = \mathbb{E}^n\{xX_n\} = x\mathbb{E}^n\{X_n\} = x(1+k)^n.\end{aligned}$$

The last equality follows from the definition of k as $k = \mathbb{E}^{Q_Y}(Y-1)$ and the fact that $\mathbb{E}^n$ is the expectation over the distribution of n independent drawings from the distribution of Y. Now

$$\begin{aligned}f(x,0) &= \lim_{\tau\to 0}\sum_{n=0}^{\infty} P_n(\tau)\mathbb{E}^n\{M(V_n,\tau;E,\sigma^2,r)\}\\ &= P_0(\tau)\mathbb{E}^0\{M(V_0,0:E,\sigma^2,r)\} + \lim_{\tau\to 0}\sum_{n=1}^{\infty} P_n(\tau)\mathbb{E}^n\{M(V_n,\tau;E,\sigma^2,r)\}.\end{aligned}$$

Since $P_0(\tau) = 1$ and

$$\mathbb{E}^0\{M(V_0, 0; E, \sigma^2, r)\} = \mathbb{E}^0\{M(x, 0; E, \sigma^2, r)\} = \max[0, x - E],$$

we have

$$f(x, 0) = \max[0, x - E] + \lim_{\tau \to 0} \sum_{n=1}^{\infty} P_n(\tau)\mathbb{E}^n\{M(V_n, \tau; E, \sigma^2, r)\}.$$

Thus we need to show that the summation term on the right-hand side is zero. To show this proceed as follows:

$$\begin{aligned}
&\lim_{\tau \to 0} \sum_{n=1}^{\infty} P_n(\tau)\mathbb{E}^n\{M(V_n, \tau; E, \sigma^2, r)\} \\
&\leq \lim_{\tau \to 0} x e^{-\lambda\tau} \sum_{n=1}^{\infty} \frac{[(1+k)\lambda\tau]^n}{n!} \qquad \text{(using (13.43))} \\
&= \lim_{\tau \to 0} x e^{-\lambda\tau} [e^{(1+k)\lambda\tau} - 1] \\
&= 0.
\end{aligned}$$

Thus we have shown that $f(x, 0) = \max[0, x - E]$ which is the final step in the demonstration that Eq. (13.15) is the general form of the solution.