7 Simultaneous Decision-Making Models

Decisions about investment programmes often involve simultaneous choices about types and numbers of investment projects. Additionally, models used for simultaneous decision-making might need to accommodate choices within a range of company areas such as financing, production, sales, human resources and tax policy. In this chapter, the finance and production areas—because of their relevance and close connections with investment decisions—are selected to illustrate ways of supporting investment decision-making in a broader sense than has been discussed previously. In the following sections some models are presented in detail, their practical relevance is discussed, and problems with their practical application are analysed.

First DEAN's model is illustrated, which is used to make a simultaneous choice between various investment and finance projects within a single time period. Thus, it is a *static model*, and it is also *single-tiered*, in that alternatives can be realised at only one point in time (normally the beginning of the planning period). Obviously, these characteristics limit the model's utility as a stand-alone decision support tool. Consequently, a model developed by HAX (1964) and WEINGARTNER (1963): a *multi-tier model* for simultaneous investment and finance decisions spanning multiple periods is also analysed. Concluding the chapter, the (multi-tier) extended model of FÖRSTNER and HENN (1970) is presented as an example of simultaneous investment and production decision-making support.

7.1 Static Model for Simultaneous Investment and Financing Decisions (DEAN Model)

Description of the model

A simultaneous analysis of investment and financing alternatives is usually precipitated by interdependencies between them: i.e. the availability and quality of financing choices might determine the feasibility and profitability of investment

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projects, and vice versa. Such interdependencies are taken into account in models of simultaneous investment and finance decision-making. Although the static model developed by DEAN is relatively simplistic and, because only one period is considered, of limited applicability in real life investment decision-making, it is explored here as a good illustration of the basic interdependencies between investment and financing decisions, and as a transparent introduction to simultaneous decision-making.

The simultaneous investment and finance decision-making models of DEAN, HAX and WEINGARTNER and others are based on the assumptions that:

- Certainty exists.
- A limited number of investment and financing alternatives are available.
- The investment and financing projects are not mutually exclusive and can be undertaken independently (although indirect relations might exist, for example, in regard to competition for finance).
- Only monetary effects of the investment and financing alternatives are relevant.
- All relevant effects of the investment and financing projects may be assigned to the separate projects as cash inflows and cash outflows, and to periods of discrete and identical time spans.
- Liquidity is a requirement for all the points in time under consideration.
- Tax payments do not affect the profitability of the alternatives.
- The economic life of the investment projects, or the term of the financing projects, is pre-defined.

In addition to these general assumptions, DEAN's model pre-supposes the following:

- The investment and financing projects involve only one time period, with cash flows at the beginning and at the end.
- All projects are completely divisible and may be undertaken in full, or in part up to a predetermined limit.

The objective considered in the model is to maximise the compound value of the total investment and finance programme (consisting of cash inflows from the investment activities less cash outflows from the financing projects) as at the end of the planning period. It is assumed that investment projects have cash inflows (surpluses) at the end of this period while, due to interest and redemption payments, financing projects have cash outflows (negative net cash flows).

At the beginning of the period, funds necessary to execute the investment projects (i.e. the total initial investment outlays) must be supplied by appropriate financing projects, including internal funds (Such funds can be included without explicit interest claims, i.e. using an interest rate of 0 %, or using an interest rate derived from the appropriate opportunity cost.).

Mathematically, the model can be formulated using the variables and parameters specified below, as follows:

Variables:
x_j = Extent of realising the investment project j (j = 1 . . ., J)
y_i = Extent of realising the financing project i (i = 1 . . ., I)
Parameters:
a_{jt} = Net cash flow per unit of the investment project j for the point in time t (t = 0,1)
d_{it} = Net cash flow per unit of the financing project i for the point in time t (t = 0,1)

Objective function (related to t = 1):

$$\underbrace{\sum_{j=1}^{J} a_{j1} \cdot x_j}_{\textit{Net cash flows of the investment projects}} + \underbrace{\sum_{i=1}^{I} d_{i1} \cdot y_i}_{\textit{Net cash flows of the financing projects}} \Rightarrow \max! \tag{7.1}$$

The sum of the net cash flows resulting from the investment and financing projects is maximised.

The constraints are:
Financing constraint (related to t = 0):

$$\underbrace{\sum_{j=1}^{J} a_{j0} \cdot x_j}_{\textit{Net cash flows of the investment projects}} + \underbrace{\sum_{i=1}^{I} d_{i0} \cdot y_i}_{\textit{Net cash flows of the financing projects}} = 0 \tag{7.2}$$

Project constraints:

$$\begin{aligned} 0 \le x_j \le 1, &\quad \text{for } j = 1, \ldots, J \\ 0 \le y_i \le 1, &\quad \text{for } i = 1, \ldots, I \end{aligned}$$

The financing of cash outflows (initial investment outlay for the investment projects) is required at the beginning of the first (and only) period, when the sum of net cash flows from both investment and financing projects must be zero. Limits set, such as the maximum number of investment projects or loans (maximum number of financing projects), must be considered as well. The investment and financing projects can be undertaken in arbitrary fractions of their maxima ($x_j = 1$ or $y_i = 1$).

One way to find the optimum solution of this model is to use a graphical procedure. For this, capital demand and capital supply functions are illustrated in a diagram. The capital demand function indicates, for all available investment projects, the capital required as a function of the cost of capital. Analogously, the capital supply function shows the available capital as a function of interest rates. The point of intersection of the capital supply and capital demand curves indicates the optimum investment and finance programme. Also, the interest rate that is the hurdle rate for both investment and financing projects can be determined—i.e. the model's endogenous interest rate.

Where investments are *not* completely divisible, the optimum solution cannot be determined graphically. If only a limited number of projects is available the solution may be found using an enumeration procedure, otherwise integer linear optimisation methods must be used. The following example illustrates the optimisation for both completely divisible and discrete investment projects.

Example 7.1
Four completely divisible investment and financing projects are available. They are characterised by the following net cash flows (in €'000) a_{jt} or d_{it}:

Table 7.1 Characterisations of the investment and financing projects

	Data		Intermediate results		
Investment projects	a_{j0}	a_{j1}	Interest rate (in %)	Priority	Accumulated capital demand
IP1	−100.0	113.0	13.0	2	150
IP2	−60.0	66.0	10.0	4	240
IP3	−50.0	58.0	16.0	1	50
IP4	−30.0	33.6	12.0	3	180
Financing projects	d_{i0}	d_{i1}	Interest rate (in %)	Priority	Accumulated capital supply
FP1	25.0	−27.0	8.0	3	105
FP2	60.0	−64.0	6.6	2	80
FP3	100.0	−120.0	20.0	4	205
FP4	20.0	−21.0	5.0	1	20

The optimisation problem is then expressed as:

Objective function:

$$113\ x_1 + 66\ x_2 + 58\ x_3 + 33{,}6\ x_4 - 27\ y_1 - 64\ y_2 - 120\ y_3 - 21\ y_4 \Rightarrow \max!$$

Constraints:

Financing constraint:

$$-100\ x_1 - 60\ x_2 - 50\ x_3 - 30\ x_4 + 25\ y_1 + 60\ y_2 + 100\ y_3 + 20\ y_4 = 0$$

Project constraints:

$$0 \le x_i \le 1, \quad \text{for}\, j = 1, \ldots, 4$$
$$0 \le y_i \le 1, \quad \text{for}\, i = 1, \ldots, 4$$

First, the graphical solution is illustrated. In preparation, the internal rates of return (IRRs) of the investment projects and the effective rates of interest for the financing projects must be calculated, using the following formula:

$$r_j = \left|\frac{a_{j1}}{a_{j0}}\right| - 1 \quad \text{or} \quad r_i = \left|\frac{d_{i1}}{d_{i0}}\right| - 1 \tag{7.3}$$

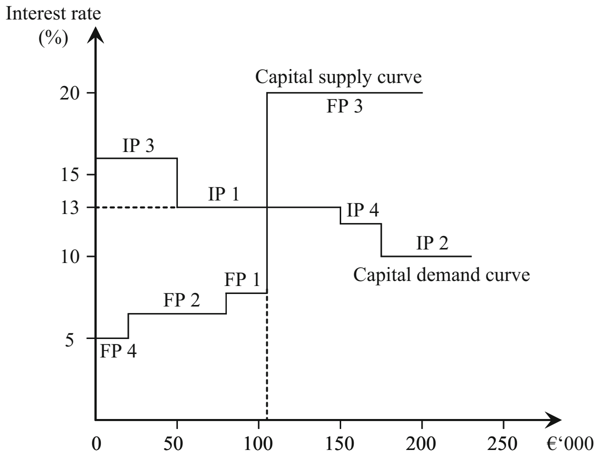
Because all projects are divisible, a ranking according to profitability may be derived from the interest rates calculated. With rising interest rates, the profitability of financing projects decreases and that of investment projects increases: i.e. the most profitable financing project is the one with the lowest interest rate, and the most profitable investment project is the one with the highest rate. The internal rates of return and effective rates, as well as the resultant priority rankings, are shown in Table 7.1 above.

This table also shows total capital demand and supply as a function of interest rates. The priority rankings of the investment projects can be used, together with their maximum initial investment outlay, to determine their total capital demand as a function of interest rates. At any loan interest rate greater than 16 %, no level of capital demand would be considered, because that rate exceeds the interest potentially receivable from the investments. At a rate of 16 %, the decision-maker would be indifferent between investing and not investing in project 3, the most profitable one, because the financing cost equals the rate receivable. With a smaller interest rate, this project would be undertaken. Below an interest rate of 16 %, the cumulative capital demand is currently €50,000, which corresponds to the maximum initial investment outlay of investment project 3, since the other investment projects would be rejected. The second priority investment project (project 1) earns an interest rate of 13 %, so at this interest rate the total capital demand increases by the initial outlay required to undertake this investment project (€100,000). The cumulative capital demand then becomes €150,000. The other investment projects shape capital demand as a function of interest rates in the same way, and the resulting series can be represented in the capital demand curve shown in Fig. 7.1.

By analogy, a curve of capital supply as a function of interest rates can be derived using the maximum loan amounts and effective rates of interest for the financing projects. The capital supply curve obtained in this example is also shown in Fig. 7.1.

The optimum investment and financing programme balances capital demand and capital supply. In order to take the priorities of both investment and financing projects into account, investment projects—beginning with the highest priority project—are included in the optimum programme step by step (ranked by priority)

Fig. 7.1 Graphical solution using the DEAN model



as long as their IRRs exceed the interest rates of the financing projects necessary to finance their initial outlays.

This is the case up to the point where the capital supply and demand curves intersect, so the optimum investment programme and financing programme can be determined from this intersection. All investment and financing projects to the left of the intersection can be realised although, commonly, one project—investment or financing—can be undertaken only partially.

In the example given, financing projects 4, 2 and 1, investment project 3, and part of investment project 1 (55/100 or 11/20) comprise the optimum programme. The compound value (CV) of this programme is obtained from the cash flow surpluses of the optimum investment projects less the interest and redemption of the optimum financing projects (at time $t = 1$) and amounts to (in €'000):

$$CV = \underset{IP\ 3}{58} + \underset{IP\ 1}{11/20 \cdot 113} - \underset{FP\ 4}{21} - \underset{FP\ 2}{64} - \underset{FP\ 1}{27} = 8.15$$

The interest rate at which the capital demand and supply curves meet can be determined from the diagram: in the example it is 13 %. This is the endogenous, or critical, interest rate, which may be used to generate the following rules:

(a) Investment projects (financing projects) are undertaken wholly if their interest rates are higher (lower) than the endogenous rate.
(b) Investment projects (financing projects) are undertaken partially if their interest rates equal the endogenous rate.
(c) Investment projects (financing projects) are not undertaken if their interest rates are lower (higher) than the endogenous rate.

Provided the endogenous rate is known in advance, the optimum programme of financing and investment projects may be derived either from the stated conditions, or by using the net present value (NPV) method. Using the endogenous interest rate as a uniform discount rate, the projects may be assigned to the groups listed above regarding their calculated NPVs. The NPV will be either greater than zero (a), equal to zero (b), or less than zero (c). Incidentally, this example also demonstrates the suitability of the NPV method for decision-making in imperfect capital markets, provided the ‘correct’ interest rate is known. However, the endogenous interest rate is known only after an optimisation procedure and, therefore, can be used only for assessing additional projects once the original programme has been decided upon.

The assumption of complete divisibility will not be realistic for many investment projects. When projects are necessarily discrete and the graphically determined ‘optimum’ programme contains a partial investment project, as is the case in the example (investment project 1), this programme cannot be realised. In that case, neither undertaking project 1 in its entirety nor rejecting it will produce an optimal solution. This is because the rate of return of the investment projects, which was used for priority ordering, is no longer the only relevant criterion for programme optimisation. The size of the investment outlay also matters. It might be more profitable to favour an investment project with a lower capital demand over one with a higher rate of return.

As previously noted, the optimum programme may also be determined using either a complete or a limited enumeration. With a limited enumeration, all possible investment programmes, except for those that are obviously unprofitable, are analysed in the following way. For each combination of investment projects, the optimum financing programme is determined on the basis of the previous rank order, such that the sum of inflows and outflows at $t = 0$ is zero. The total compound value of each combination at $t = 1$ is then calculated. The programme with the maximum total compound value is optimal. This is illustrated in the following example, which is a continuation of the previous one. Obviously unprofitable investment programmes are ignored. Table 7.2 shows the results of the required calculations.

Table 7.2 Compound values of the investment and financing programmes

Investment programme	Capital demand (€’000)	Financing programme	Compound value (€’000)
IP 3	50	FP 4, 0.5 FP 2	5.0
IP 1	100	FP 4, FP 2, 0.8 FP 1	6.4
IP 3, IP 1	150	FP 4, FP 2, FP 1, 0.45 FP 3	5.0
IP 3, IP 4	**80**	**FP 4, FP 2**	**6.6**
IP 3, IP 2	110	FP 4, FP 2, FP 1, 0.05 FP 3	6.0
IP 3, IP 4, IP 1	180	FP 4, FP 2, FP 1, 0.75 FP 3	2.6
IP 4, IP 1	130	FP 4, FP 2, FP 1, 0.25 FP 3	4.6

The optimum in this example, with a compound-value of €6,600, is the combination of investment projects 3 and 4 financed by financing projects 2 and 4.

Assessment of the model

DEAN's model for simultaneous investment and financing decision-making is a relatively simple one and presents no special difficulties for data collection and model solution.

To assess the method's practicability, the reader should refer to comments made on the NPV method (see Chap. 3), with the proviso that the DEAN model does not assume a perfect capital market. However, the fundamental objections to combining 'imperfect capital market' and 'certainty' assumptions expressed in Chap. 4 (concerning the VoFI method) should again be emphasised.

Consumption decisions are largely ignored in this model. If no internal funds are available, or if they are included without interest claims and are, therefore, used to finance investments, the level of consumption is defined at the beginning of the planning period. If, however, interest rates are derived from opportunity costs, the available investment opportunities and alternative financing opportunities determine whether internal funds will be used. Then, assuming the opportunity costs reflect a time preference with regard to consumption, the inclusion of internal funds can be interpreted as a (simplified) integration of the consumption decision into the model.

The assumption that investment and financing projects are independent, and the limitation of a single-period time span, are also problematic. The time span limitation is particularly so, as investments are typically long-term and usually show long-term effects. Differences in the economic life of the investment and financing projects often occur, and misleading rankings can result. Moreover, future investment and financing opportunities are completely ignored where only a single term is considered. A more accurate solution to the simultaneous decision-making problem may be achieved using the following dynamic model.

7.2 Multi-tier Model of Simultaneous Investment and Financing Decisions (HAX and WEINGARTNER Model)

Description of the model

The multi-tier model for simultaneous investment and financing decisions described in this section was developed by both HAX and WEINGARTNER independently, in almost identical form. Most of the assumptions underlying DEAN's model apply to this model also. However, unlike DEAN's model, the HAX and WEINGARTNER model is *multi-tier* in that the investment and financing projects considered may commence at different times.

The objective included in the model is, again, to maximise the compound value of the total investment and financing programme. It is assumed that any investment project surpluses earned before the end of the planning period are reinvested in a 1-year financial investment at a given interest rate. Thus, a uniform discount rate is not required in this model. At the beginning of each time period within the planning period, a liquidity constraint is formulated to ensure that cash inflows and outflows are balanced. In addition, it is assumed that investment and financing projects can be executed repeatedly, but that investment projects are indivisible, or discrete. The cash flow profiles of the investment and financing projects are assumed to be independent of their size, i.e. the interest rate for a loan (financing project) is independent of the total sum borrowed.

The HAX and WEINGARTNER model can be expressed in mathematical form using the variables and parameters specified below. Investment and financing projects are sequentially numbered, but without index references to periods. An exception to this is the short term financial investment which is labelled Jt.

Variables:
x_j = Number of units of investment project type j ($j = 1, \ldots, J-1$)
x_{Jt} = Amount of the short term financial investment (in €) at time t ($t = 0, \ldots, T-1$ or T)
y_i = Extent of financing project type i (in €) for $i = 1, \ldots, I$

Parameters:
a_{jt} = Cash outflow surplus per unit of the investment project j ($j = 1, \ldots, J-1$) at time t ($t = 0, 1, \ldots, T$)
d_{it} = Cash outflow surplus per unit (€) of the financing project i at time t
IF_t = Internal funds at time t
X_j = Maximum number of units of investment project j ($j = 1, \ldots, J-1$)
Y_i = Maximum amount of financing project i ($i = 1, \ldots, I$)
c = Interest rate for the short term financial investment

The objective 'maximisation of the compound value (CV)' may be incorporated into the model in different ways. In the following formula, the cash flows of the last period constitute the objective function explicitly.

$$\begin{aligned} CV = \underbrace{IF_T}_{\substack{\text{Internal}\\ \text{funds}}} &- \underbrace{\sum_{j=1}^{J-1} a_{jT} \cdot x_j}_{\substack{\text{Cash outflow surpluses}\\ \text{of the investment projects}}} - \underbrace{\sum_{i=1}^{I} d_{iT} \cdot y_i}_{\substack{\text{Cash outflow surpluses}\\ \text{of the financing projects}}} \\ &+ \underbrace{(1+c) \cdot x_{JT-1}}_{\substack{\text{Compounded short-term financial}\\ \text{investment at the previous point in time}}} \Rightarrow \max! \end{aligned} \tag{7.4}$$

The compound value represents the surplus at the end of the programme planning period. Cash inflow surpluses in earlier points in time are transformed into a short-term financial investment. Accordingly, the compound value may be interpreted as a hypothetical short-term financial investment at time T.

If a variable x_{JT} and a liquidity constraint at time T are integrated into the model, then the objective function can be formulated as follows:

$$CV = x_{JT} \Rightarrow \max!$$

Constraints:

Liquidity constraints:

For t = 0:

$$\underbrace{\sum_{j=1}^{J-1} a_{j0} \cdot x_j}_{\substack{\textit{Cash outflow}\\ \textit{surpluses of the}\\ \textit{investment projects}}} + \underbrace{\sum_{i=1}^{I} d_{i0} \cdot y_i}_{\substack{\textit{Cash outflow}\\ \textit{surpluses of the}\\ \textit{financing projects}}} + \underbrace{x_{J0}}_{\substack{\textit{Short-term}\\ \textit{financial}\\ \textit{investment}}} = \underbrace{IF_0}_{\substack{\textit{Internal}\\ \textit{funds}}} \tag{7.5}$$

For t = 1, . . . , T:

$$\underbrace{\sum_{j=1}^{J-1} a_{jt} \cdot x_j}_{\substack{\textit{Cash outflow surpluses}\\ \textit{of the investment projects}}} + \underbrace{\sum_{i=1}^{I} d_{it} \cdot y_i}_{\substack{\textit{Cash outflow surpluses}\\ \textit{of the financing projects}}} + \underbrace{x_{Jt}}_{\substack{\textit{Short-term}\\ \textit{financial}\\ \textit{investment}}}$$

$$- \underbrace{(1 + c) \cdot x_{JT-1}}_{\substack{\textit{Compounded short-term}\\ \textit{financial investment}\\ \textit{in the previous period}}} = \underbrace{IF_t}_{\substack{\textit{Internal}\\ \textit{funds}}} \tag{7.6}$$

At t = 0, and throughout the planning period, cash outflow surpluses must at no time exceed the internal funds, i.e. illiquidity must be avoided. This is ensured by the mathematical formulation of the liquidity constraints and, additionally, by the further constraint that the short term financial investments must not be negative ($x_{Jt} \geq 0$). However, the balance of the internal financial funds (parameter IF_t) can become negative if the company managers intend to withdraw funds from the investment and financing programme (in order to make funds available for other parts of the company or the owners).

Project restrictions:

$$
\begin{array}{ll}
x_j \leq X_j, & \text{for } j = 1, \ldots, J-1 \\
y_i \leq Y_i, & \text{for } i = 1, \ldots, I \\
x_j \geq 0 \text{ and integer}, & \text{for } j = 1, \ldots, J-1 \\
x_{Jt} \geq 0, & \text{for } t = 0, \ldots, T-1 \\
y_i \geq 0, & \text{for } i = 1, \ldots, I
\end{array}
$$

The number of units of investment projects j ($j = 1, \ldots, J-1$) and the amounts of financing projects i (in €) may not be negative, nor may they exceed the (given) maximum limits. In addition, all investment projects are discrete, or indivisible.

The optimum solution of the HAX and WEINGARTNER model may be calculated using integer linear programming. Where investment projects are divisible, other useful information may be derived from the optimum solution in the form of endogenous interest rates. This is illustrated in the following example.

Example 7.2

The following table shows the cash flow profiles of investment projects 1–7 and financing projects 1–3. Two investment projects are started at time $t = 1$ (investment projects 6 and 7), i.e. this is a multi-tier example.

Investment projects may be undertaken up to the following maximum numbers: 5 (investment project 1), 4 (investment project 2), 2 (investment project 4), 3 (investment project 5), and 4 (investment project 6). Investment projects 3 and 7 are unrestricted. Maximum loans are €500,000 (financing project 1), €600,000 (financing project 2) and €100,000 (financing project 3), and the short term financial investment used for reinvesting surpluses earns an interest rate of 8 % over the planning period. At time $t = 0$, €50,000 cash are available for investing.

Table 7.3 Net cash flows of the investment and financing projects

	Net cash flows at times			
Investment projects	t = 0	t = 1	t = 2	t = 3
Investment project 1	−90,000	45,000	40,000	40,000
Investment project 2	−45,000	24,000	23,000	24,000
Investment project 3	−80,000	35,000	35,000	40,000
Investment project 4	−170,000	75,000	80,000	85,000
Investment project 5	−100,000	40,000	50,000	50,000
Investment project 6	0	−240,000	160,000	160,000
Investment project 7	0	−160,000	92,000	96,000
Financing projects				
Financing project 1	1	0	0	−1.481544
Financing project 2	1	0	0	−1.404928
Financing project 3	0	1	−0.12	−1.12

The model for this example consists of:

Objective function:

$$x_{83} \Rightarrow \max!$$

Constraints:

Liquidity constraints:

$$\begin{aligned}
t = 0: \quad & 90,000x_1 + 45,000x_2 + 80,000x_3 + 170,000x_4 + 100,000x_5 - y_1 - y_2 + x_{80} = 50,000 \\
t = 1: \quad & -45,000x_1 - 24,000x_2 - 35,000x_3 - 75,000x_4 - 40,000x_5 + 240,000x_6 + 160,000x_7 \\
& -y_3 - 1.08x_{80} + x_{81} = 0 \\
t = 2: \quad & -40,000x_1 - 23,000x_2 - 35,000x_3 - 80,000x_4 - 50,000x_5 - 160,000x_6 - 92,000x_7 \\
& -0,12y_3 - 1.08x_{81} + x_{82} = 0 \\
t = 3: \quad & -40,000x_1 - 24,000x_2 - 40,000x_3 - 85,000x_4 - 50,000x_5 - 160,000x_6 \\
& -96,000x_7 + 1.481544y_1 + 1.404928y_2 - 1.12y_3 - 1.08x_{82} + x_{83} = 0
\end{aligned}$$

Project constraints:

$$\begin{aligned}
& x_1 \leq 5 \\
& x_2 \leq 4 \\
& x_4 \leq 2 \\
& x_5 \leq 3 \\
& x_6 \leq 4 \\
& y_1 \leq 500,000 \\
& y_2 \leq 600,000 \\
& y_3 \leq 100,000 \\
& x_j \geq 0 \text{ and integer,} \quad && \text{for } j = 1, \ldots, 7 \\
& y_j \geq 0, && \text{for } i = 1, 2, 3 \\
& x_{8t} \geq 0, && \text{for } t = 0, 1, 2
\end{aligned}$$

The optimum solution of the model is:

$x_1 = 5$	$x_2 = 4$	$x_3 = 0$	$x_4 = 2$	$x_5 = 0$	$x_6 = 3$	$x_7 = 0$
$x_{80} = 137{,}962.96$		$x_{81} = 0$	$x_{82} = 920{,}000$		$x_{83} = 306{,}150.92$	
$y_1 = 457{,}962.96$		$y_2 = 600{,}000$			$y_3 = 100{,}000$	

This resulting optimum solution is, therefore, to invest in five units of investment project 1, four units of investment project 2, two units of investment project 4 and three units of investment project 6. Loans 1, 2 and 3 should be taken out in the following amounts: €457,962.96, €600,000 and €100,000 (i.e. loans 2 and 3 are used to their maximum value). At times $t = 0$, $t = 2$ and $t = 3$ short-term financial investments are recommended in the amounts of €137,962.96, €920,000 and €306,150.92 respectively. The financial investment at $t = 3$ (x_{83}) is identical to the objective function, i.e. the compound value that is maximised in the optimum programme.

In the following calculation, the liquidity constraint at $t = 0$ is presented with the various outcomes of the optimum solution. The cash flow surplus is invested at this point as a short-term financial investment:

$$\begin{aligned} t = 0: \quad & 90,000 \cdot 5 + 45,000 \cdot 4 + 80,000 \cdot 0 + 170,000 \cdot 2 + 100,000 \cdot 0 \\ & -457,962.96 - 600,000 + x_{80} = 50,000 \\ & \Rightarrow x_{80} = 137,962.96 \end{aligned}$$

At time $t = 1$ there is a particularly high capital demand owing to the initial investment outlays for three investment projects of type $j = 3$. Thus, the short-term financial investment realised is relinquished and an excessive loan is taken out at the beginning of the planning period (identifiable from the positive value of the short-term financial investment at $t = 0$).

Where investment projects are divisible, the optimum solution of a Hax and Weingartner model allows the derivation of endogenous interest rates. In this example, the following optimum solution is obtained:

$x_1 = 5$	$x_2 = 4$	$x_3 = 0$	$x_4 = 2$	$x_5 = 1.8$	$x_6 = 2.68$	$x_7 = 0$
$x_{80} = 0$		$x_{81} = 0$	$x_{82} = 958{,}666.70$		$x_{83} = 324{,}297.87$	
$y_1 = 500{,}000$		$y_2 = 600{,}000$			$y_3 = 100{,}000$	

In decision problems involving divisible investment projects, useful information about scarce resources may be gained from the optimum solution. Opportunity costs or shadow prices can be identified that indicate changes in the objective function caused by easing the constraints. With the Hax and Weingartner model, the shadow prices of liquidity constraints are particularly interesting.

Key Concept
The shadow price of the liquidity constraint at time t indicates the increase in the value of the objective function (i.e. the compound value) that would result from an additional unit of financing (from internal funds) becoming available.

This value may be interpreted as an endogenous compounding factor indicating how an additional monetary unit, made available at time t, yields interest up to time T.

The value of the model endogenous compounding factor depends on the alternatives considered in the model and their effects. In the example, the model endogenous compounding factors q_t^* are:

$$q_0^* = 1.5947, \quad q_1^* = 1.3867, \quad q_2^* = 1.08, \quad \text{and} \quad q_3^* = 1$$

From these model endogenous compounding factors, model endogenous interest rates may be derived, which indicate the endogenous rates of interest for each period. The relationships between the model endogenous compounding factors q_t^*

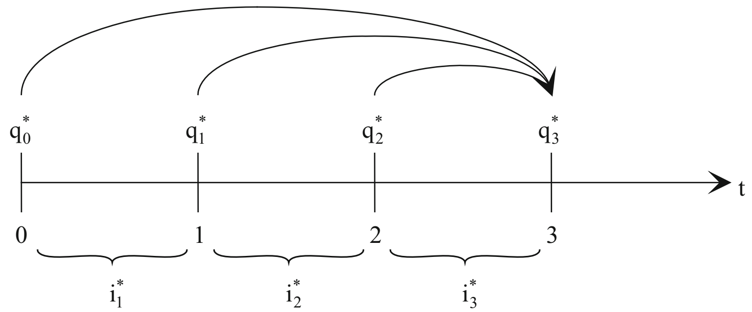


Fig. 7.2 Relationships between the model endogenous compounding factors and the model endogenous interest rates

and the model endogenous interest rates i_t^* for the current example are illustrated as follows (Fig. 7.2):

The model endogenous compounding factor at time t is the product of all model endogenous compounding factors related to the individual periods from time t to the end of the planning period. The compounding factor relevant to a period is the sum of 1 plus the model endogenous interest rate for that period. Therefore, for the model endogenous compounding factor q_t^*, the following applies:

$$q_t^* = \prod_{\tau=t+1}^{T} \left(1 + i_\tau^*\right) \tag{7.7}$$

Model endogenous compounding factors are derived from optimum solutions of linear optimisation problems. From these factors, the model endogenous interest rates may be calculated by changing the equation above. This is demonstrated for the example given in the following:

$$q_2^* = 1 + i_3^* \qquad \Rightarrow \quad i_3^* = q_2^* - 1 = 0.08$$

$$q_1^* = \left(1 + i_2^*\right) \cdot \left(1 + i_3^*\right) = \left(1 + i_2^*\right) \cdot q_2^* \qquad \Rightarrow \quad i_2^* = \frac{q_1^*}{q_2^*} - 1 = 0.2840$$

$$q_0^* = \left(1 + i_1^*\right) \cdot \left(1 + i_2^*\right) \cdot \left(1 + i_3^*\right) = \left(1 + i_1^*\right) \cdot q_1^* \quad \Rightarrow \quad i_1^* = \frac{q_0^*}{q_1^*} - 1 = 0.15$$

The interest rate in the second period (28.4 %) is particularly high: this is the result of high demands from the investment projects at time t = 1, as discussed above for the model stipulating investment project indivisibility. The endogenous interest rate for the third period (8 %) equals the interest rate of the short-term financial investment, because at t = 2 no other investment opportunities exist.

Model endogenous interest rates may be used to assess single investment and financing projects separately. If these interest rates are used as uniform discount rates for calculating the NPVs of the separate projects (as in the Dean model), the following relationships may be stated:

(a) Investment or financing projects with an NPV greater than zero are undertaken to their maxima in an optimum programme.
(b) Investment or financing projects with an NPV of zero are usually undertaken only partially in an optimum programme, i.e. not to their maxima.
(c) Investment or financing projects with an NPV of less than zero are not included in an optimum programme.

If the endogenous interest rates were known, no optimisation of a simultaneous model would be needed. However, they are derived only as the result of an optimisation and, therefore, the application of endogenous interest rates (as uniform discount rates) is useful only for assessing additional projects considered once the optimum programme has already been determined.

In the example, it is now assumed that an additional investment project 9 becomes available. It has the following cash flow profile:

Table 7.4 Cash flow profile for investment project 9

Times	t = 0	t = 1	t = 2	t = 3
Net cash flows	−10,000	4,000	4,500	5,000

Utilising the endogenous interest rates determined above as uniform discount rates, the NPV of this additional investment project can be calculated:

$$c_9 = -€10,000 + \frac{€4{,}000}{1.15} + \frac{€4{,}500}{1.15 \cdot 1.284} + \frac{€5{,}000}{1.15 \cdot 1.284 \cdot 1.08}$$
$$c_9 = -€338.87$$

Because of its negative NPV, investment project 9 should not be included in the programme; the calculated optimum would be unaffected by this additional investment opportunity.

Additionally, the guidelines given in Sect. 3.6, for determining upper and lower bounds for the interest rates of investment and financing opportunities, can also be applied to assessing investment and financing projects separately within a simultaneous investment and financing decision process. It should be possible to define an interval within which the endogenous interest (or discount) rate falls, so that a range of possible NPV results for investment and financing projects can be calculated. Using this approach, it is easy to identify investment and financing projects which are definitely profitable (positive NPV at the upper limit for an investment and at the lower limit for a financing project) or definitely unprofitable (negative NPV at the opposite limits). Only the remaining projects would then require a model for simultaneous decision-making.

Model assessment and model extensions

The Hax and Weingartner model requires the collection of data on forecasted project cash flow profiles, maximum numbers of projects, and internal funds.

Determining the optimum solution may—depending on the number of variables and periods under consideration—be difficult, particularly where the investment projects are indivisible. This has motivated the development of heuristic solution procedures for simultaneous investment and financing decisions; these also rely in part on knowing endogenous interest rates. While heuristic procedures might not always determine the optimum result, they will usually find acceptable solutions with relatively little computational effort. Nevertheless, improvements in computer resources have greatly improved the potential for solving integer linear optimisation problems.

A fundamental criticism of the HAX and WEINGARTNER model that should be emphasised is the combination of 'imperfect capital market' and 'certainty' assumptions.

Because of the model's multi-tier structure, interdependencies between investment and financing opportunities in different periods may be included in the analysis. Therefore, the optimum investment timing may be determined using the model.

Short-term financial investments that yield different rates of interest in different periods (or an interest rate of 0 %, i.e. keeping surpluses as cash) may also be included. Alternative short- or long-term investments with divergent interest rates, and alternative short-term loans, can be accommodated as well.

Withdrawals may be interpreted as payments the company receives from the investment and financing programme. They can be included either as nominated amounts of (negative) internal funds, or as periodic withdrawals from the investment and financing programme that must be maximised. This approach requires a pre-set level for both the compound value and the desired cash withdrawal pattern. The objective then consists of the one variable to be maximised—the withdrawal level. The desired cash withdrawal pattern is taken into account by multiplying time-specific factors (which express the demand for cash at a specific point in time) by the withdrawal level, and integrating the products into the liquidity constraints. Thus, consumption decisions can be integrated into the model (in a simplified form) either by nominating withdrawal amounts, or by maximising the withdrawal flow.

To accommodate more realistic scenarios, the project conditions might need to be modified, e.g. where multiple iterations of a project are not possible and/or investment projects (particularly financial investment projects) are divisible.

The analyses in this chapter have assumed that the last relevant cash flows occur at the end of the planning period. However, defining the planning period itself represents an additional decision problem for the analyst. This problem is exacerbated if a project's cash flows occur in the relatively distant future, in which case the planning period must be extended to incorporate the last cash flow. An alternative approach would be to choose a shorter planning period and discount the cash flows that arise beyond it. In that case, the following objective function is maximised:

$$x_{JT} - \sum_{t=T+1}^{\hat{T}} \left(\sum_{j=1}^{J-1} a_{jt} \cdot x_j + \sum_{i=1}^{I} d_{it} \cdot y_i \right) \cdot q^{-t+T} \Rightarrow \max! \tag{7.8}$$

With:

q^{-t+T} = Discounting factor for time t
$\hat{T}$ = The time at which the last cash flow occurs

As with all models of simultaneous investment and financing decisions, the assumptions indicated above (certainty of the model data, independence of the projects, exclusion of non-monetary effects, the ability to allocate the effects to specific projects and periods, irrelevance of tax payments, nominated production programme and economic life etc.) may not apply in reality. Also questionable is the assumption that the cash flow profiles of investment and financing projects are independent of the number of projects undertaken. In addition, since cash flows are allocated at the beginning or end of each period, liquidity can only be assured for those points in time, and not for in-between periods. Therefore some financing decisions, despite their connections to decisions illustrated here, must be made outside the model. In practice, it is advisable to check the extent to which such divergences between reality and the model's assumptions might impact on the profitability of projects.

Some of the assumptions of the model can be avoided by modifying its formulation. This would make it possible, for example:

- To allow project interdependence.
- To accommodate balance sheet structures.
- To integrate tax payments into the model.
- To include different economic lives for investment projects and/or terms for financing projects within the model.

At this point, it should be noted that the model assumes centralised decision-making about investment and financing projects. However, the complexity resulting from centralised decision-making, together with possible problems with information transfer and the motivation of managers in decentralised company units, may create the need to decentralise decision-making processes. With decentralisation, the use of mathematical decomposition procedures, transfer prices and investment budgeting may become necessary in order to coordinate investment and financing activities. Also, managers in decentralised units might not, owing to goal conflicts or to asymmetric distribution of information, make decisions that are in the best interests of the company. To deal with this problem, incentive systems are often used.

Up to this point, production decisions have been assumed to be a given. In the following section this premise is discarded in order to consider decisions about production alongside investment programme decisions.

7.3 Multi-tier Model of Simultaneous Investment and Production Decisions (Extended Förstner and Henn Model)

Description of the model

Models for simultaneous investment and production decisions analyse the following types of interdependency:

- The profitability of investment projects as a function of the production programme (i.e. the types and numbers of products produced).
- Investment in increased production capacity as an essential condition for a production programme decision.To consider these interdependencies, product variables that indicate how many units of a product type will be produced are now introduced. Cash flows are also allocated to these variables, and the capacity used by the variables (or the products they represent) is incorporated into capacity constraint formulae.

The extended Förstner and Henn model described in this section is a linear optimisation model. Similar to the models for simultaneous investment and financing decisions, the following assumptions apply:

- There is no uncertainty concerning the model data.
- A limited number of suitable investment and production alternatives is available.
- The investment and production alternatives are not mutually exclusive and each may be undertaken independently (although indirect relationships might exist—for example, investment projects might compete with each other for scarce funds, or they might be designed to increase production capacity).
- Only the monetary effects of the investment and production alternatives are relevant.
- All effects relevant to the investment and production alternatives can be assigned to the relevant projects as cash inflows and outflows, and to the relevant periods (which are discrete and of identical length). All relevant effects from other areas of the company are recorded in these cash inflows.
- All relationships between variables and their effects are linear (for example, cash inflows are proportionate to the levels of production).
- A production process with more than one production step is assumed, and Capacity demands per unit can be allocated to products at every production step.
- The order in which products are manufactured has no influence on cash outflows and capacity demands.
- No storage is necessary, i.e. production volumes correspond to sales volumes.
- Solvency must be maintained for all periods under consideration.
- The financing programme is pre-set.

The Förstner and Henn model for making simultaneous investment and production decisions is derived from the basic model for a production programme decision. This is described briefly next. The specific production situation is

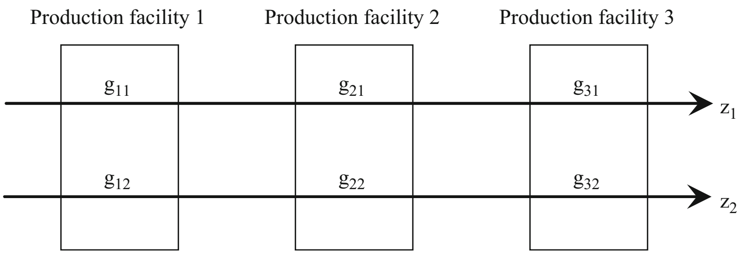


Fig. 7.3 Production structure in the basic model for a production programme decision

illustrated in Fig. 7.3 assuming the existence of two product types and three production facilities (i.e. machines).

Products pass through three production facilities j ($j = 1, 2, 3$), and for every unit of a product k ($k = 1, 2$) there is a specific and constant capacity requirement g_{jk}. This unit-related capacity requirement, the so-called production coefficient, is known. Also known are: the available capacity (in units) for each machine; the per unit variable costs for each product; and product prices (which are independent of sales volumes). The production volumes z_k are identical to the sales volumes—i.e. products are not stored.

The basic model is a static one, and the ultimate objective is to maximise profits. Restrictions result from machine capacity limits and, for obvious reasons, production volumes cannot be negative.

The Förstner and Henn model extends this basic model by removing one of its crucial assumptions: the fixed capacity of the production machines or facilities. Investment variables are introduced to indicate the extent to which the capacity can be raised.

In this Sect. 7.3, an extended version of the Förstner and Henn model is described. In contrast to the original model, it incorporates cash flows from product sales within the liquidity constraints. Moreover, cash outflows resulting from the investment projects are included.

The objective is to maximise compound value. Surpluses from a period may be reinvested in unlimited amounts as short-term, single-period financial investments, as in the Hax and Weingartner model. Thus, a uniform discount rate is unnecessary. It is assumed that the economic lives of the investment projects purchased (here, production machines) are fixed. Liquidation values are taken into account at the end of the economic life and/or planning period.

In a multi-tier model, decisions (about investment and production) and the consequences resulting from them (cash inflows and outflows, creation and use of capacities) must be assigned to specific points in time. The following model assumes that:

- Investment projects' initial outlays, resultant cash flows and capacity increases occur at time t exactly (i.e. the beginning of period $t + 1$).

- Production and sales volumes for the period $t+1$ are assigned at time t. The associated machine capacity demand occurs at time t, but production and sales do not result in product-related cash inflows and outflows until time $t+1$.
- The liquidation values of investment projects become payable at either the end of the economic life, or the end of the planning period (if the end of the economic life is not reached within the planning period).

In formulating the model, the following variables and parameters are used:

Variables:
x_{jt} = Number of production machines of type j ($j = 1, \ldots, J-1$), purchased at time t ($t = 0, \ldots, T-1$)
z_{kt} = Production volume of product k ($k = 1, \ldots, K$), assigned at time t ($t = 0, \ldots, T-1$)
x_{Jt} = Short term financial investment at time t ($t = 0, \ldots, T-1$)

Parameters:
p_{kt} = Price of a unit of product k, produced at time t
a_{vkt} = Variable cash outflow per unit of product k, produced at time t
$I_{fjt\tau}$ = Fixed cash outflow at time t for production machine of type j, purchased at the point in time τ ($\tau = -T^*, -T^*+1, \ldots, 0, \ldots, T$) (If a machine of this type exists at the beginning of the planning period, $-T^*$ is the time at which the oldest machine was purchased.)
$x_{j\tau}$ = Number of machines of type j purchased at time τ, for $\tau < 0$ (this data is known with certainty when formulating the model)
X_{jt} = Maximum number of machines of type j that can be purchased at time t
I_{0jt} = Initial investment outlay for a machine of type j bought at time t
$\hat{L}_{jt}$ = Liquidation value per machine of type j bought at time t, which is received at the end of the planning period
$L_{j\tau}$ = Liquidation value for one machine of type j purchased at time τ at the end of its economic life
$u_{jt\tau}$ = Parameter indicating whether a machine of type j purchased at time τ has reached the end of its economic life at time t. If so, the parameter has the value of one; otherwise its value is zero
c = Interest rate for the short term financial investment
g_{jkt} = Capacity demand of machine j per unit of product k whose production is assigned at time t
$G_{jt\tau}$ = Capacity of a machine of type j purchased at time τ related to time t
Z_{kt} = Maximum sales volume of the product k related to time t
IF_t = Available internal funds at time t

The model can be formulated as follows:

Objective function (related to the point in time T):

$$\underbrace{x_{JT-1} \cdot (1+c)}_{\substack{\textit{Compounded short-term}\\ \textit{financial investment of}\\ \textit{the previous period}}} + \underbrace{\sum_{k=1}^{K} z_{kT-1} \cdot \left(p_{kT-1}\text{-}a_{vkT-1}\right)}_{\substack{\textit{Cash inflow surpluses due to}\\ \textit{product-related payments}}}$$

$$+ \underbrace{\sum_{j=1}^{J-1} \sum_{\tau=-T*}^{T-1} \hat{L}_{j\tau} \cdot x_{j\tau}}_{\substack{\textit{Cash inflows due to the}\\ \textit{liquidation of equipment at}\\ \textit{the end of the planning period}}} \quad \Rightarrow \max! \tag{7.9}$$

Liquidity constraints:

$$\underbrace{\sum_{j=1}^{J-1} x_{jt} \cdot I_{0jt}}_{\substack{\textit{Initial investment}\\ \textit{outlays}}} + \underbrace{\sum_{j=1}^{J-1} \sum_{\tau=-T*}^{t} I_{fjt\tau} \cdot x_{j\tau}}_{\substack{\textit{Equipment-dependent}\\ \textit{cash outflows}}} - \underbrace{\sum_{k=1}^{K} z_{kt-1} \cdot \left(p_{kt-1}\text{-}a_{vkt-1}\right)}_{\substack{\textit{Cash inflow surpluses due to}\\ \textit{product-related payments}}}$$

$$- \underbrace{\sum_{j=1}^{J\text{-}1} \sum_{\tau=\text{-}T*}^{t\text{-}1} L_{j\tau} \cdot x_{j\tau} \cdot u_{jt\tau}}_{\substack{\textit{Cash inflows due to the liquidation}\\ \textit{of equipment, that has reached}\\ \textit{the end of its economic life}}} - \underbrace{x_{Jt-1} \cdot (1+c)}_{\substack{\textit{Compounded short-term}\\ \textit{financial investment of}\\ \textit{the previous period}}} + \underbrace{x_{Jt}}_{\substack{\textit{Short-term}\\ \textit{financial}\\ \textit{investment}}}$$

$$= \underbrace{IF_t}_{\substack{\textit{Available}\\ \textit{internal}\\ \textit{funds}}} \tag{7.10}$$

At all times t (t = 0, ..., T − 1) the company must remain solvent.

Capacity constraints:

$$\underbrace{\sum_{k=1}^{K} g_{jkt} \cdot z_{kt}}_{\textit{Use of capacity}} \leq \underbrace{\sum_{\tau=\text{-}T*}^{t} G_{jt\tau} \cdot x_{j\tau}}_{\textit{Available capacity}} \tag{7.11}$$

Capacity demands on all investment projects j (j = 1, ..., J − 1) and at all times t (t = 0, ..., T − 1) must not exceed their available capacities.

Sales constraints:

$$\underset{\textit{Volume of sales}}{z_{kt}} \leq \underset{\textit{Maximum volume of sales}}{Z_{kt}} \tag{7.12}$$

At all times t ($t = 0, \ldots, T-1$) and for all products k ($k = 1, \ldots, K$) the maximum sales volumes must not be exceeded.

Project constraints:

$$\begin{array}{ll} x_{jt} \leq X_{jt}, & \text{for } j = 1, \ldots, J-1; t = 0, \ldots, T-1 \\ x_{jt} \geq 0 \text{ and integer}, & \text{for } j = 1, \ldots, J-1; t = 0, \ldots, T-1 \\ x_{Jt} \geq 0, & \text{for } t = 0, \ldots, T-1 \\ z_{kt} \geq 0, & \text{for } k = 1, \ldots, K; t = 0, \ldots, T-1 \end{array}$$

Example 7.3
In the following example, a simultaneous investment and production programme decision is required to cover three periods. The company produces three product types k ($k = 1, 2, 3$). For these product types, the following differences between prices and variable cash outflows per unit have been estimated (these are assumed to remain constant throughout the 3 years):

$$\begin{array}{lcl} p_1 - a_{v1} & = & €1.40 \text{ per unit} \\ p_2 - a_{v2} & = & €1.35 \text{ per unit} \\ p_3 - a_{v3} & = & €1.00 \text{ per unit} \end{array}$$

In each period, the maximum market demand for products of type k is indicated by the following parameters Z_k:

$$Z_1 = 8{,}000 \text{ units} \quad Z_2 = 6{,}000 \text{ units} \quad Z_3 = 5{,}000 \text{ units}$$

Three machines j ($j = 1, 2, 3$) are required to produce each of the three products. The following matrix shows the production coefficients, e.g. the requirement of capacities of the three machines j per unit of the products k (in time units). These capacities are also assumed to be constant throughout the three periods:

Table 7.5 Production coefficients for the three machines j and the products k

	Machine		
Product	1	2	3
1	3	3	3
2	4	3	2
3	5	2	4

Initially, two machines of types 1 and 2 and four of type 3 are already in use. All have a remaining economic life of one period. Their capacities, relevant cash outflows and liquidation values are the same as for newly purchased machines, described next.

New machines can be purchased at the beginning of each period, without limit. The economic lives of the projects are three periods each and, if the machines are acquired at the beginning of period 1 (at time $t = 0$), their initial investment outlays (in €), capacities (in time units), and cash outflows (in €/machine) are:

Table 7.6 Data for the machines

Machine	Initial investment outlay	Capacity	Cash operating outflows
1	1,000	5,000	195
2	960	4,000	185
3	880	3,500	225

By acquiring the machines at $t = 1$ or $t = 2$, the initial investment outlays, the outflows and the production coefficients remain unchanged, but the capacities increase by 10 % each period. The liquidation value at the end of the economic life is 10 % of the initial investment outlay. The decline in investment project liquidation value occurs continuously over all periods of the economic life, starting from the initial investment outlay.

The following internal funds are available:

$$t = 0: \text{ €}25,000 \quad \text{and} \quad t = 1: \text{ €}5,000.$$

The interest rate on the short-term financial investment is 6 %. The objective is to maximise the compound value.

For this decision problem, a simultaneous model must be formulated. The temporal structure of the liquidity, capacity and sales restrictions (R) as well as the objective function (OF) are shown in Fig. 7.4.

Objective function (related to $t = 3$):

$$\begin{aligned} &x_{42} \cdot 1.06 + 0.1 \cdot (1,000x_{10} + 960x_{20} + 880x_{30}) + 0.4 \cdot (1,000x_{11} + 960x_{21} + 880x_{31}) \\ &+ 0.7 \cdot (1,000x_{12} + 960x_{22} + 880x_{32}) + 1.4z_{12} + 1.35z_{22} + z_{32} \Rightarrow \max! \end{aligned}$$

The objective function refers to the end of the last period. At this time, the short-term financial investment initiated at the beginning of that period is recouped (including interest), and the liquidation values of the investments made at different points in time, as well as the cash flow surpluses of the products produced at time $t = 2$, are included. The liquidation values amount to 10 %, 40 % or 70 % of the respective initial investment outlays, according to the age of the investment projects.

Liquidity constraints:

$$\begin{aligned} &t = 0: \\ &\quad 1,000x_{10} + 960x_{20} + 880x_{30} + 195(2 + x_{10}) + 185(2 + x_{20}) + 225(4 + x_{30}) \\ &\quad + x_{40} = 25,000 \end{aligned}$$

The liquidity constraint for $t = 0$ includes the initial investment outlays for the machines purchased at the beginning of the first period and the cash outflows for both new and existing machines. The short-term financial investment is also included. In keeping with the assumptions of the model, all cash outflows must be financed using available funds.

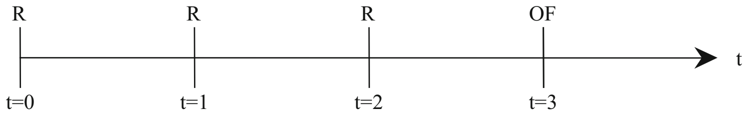


Fig. 7.4 Temporal structure of the liquidity, capacity and sales restrictions and the objective function

$$\begin{aligned}
&t = 1: \\
&-1.4z_{10} - 1.35z_{20} - z_{30} + 1,000x_{11} + 960x_{21} + 880x_{31} + 195(x_{10} + x_{11}) \\
&\quad + 185(x_{20} + x_{21}) + 225(x_{30} + x_{31}) - 2 \cdot 0.1 \cdot 1,000 - 2 \cdot 0.1 \cdot 960 \\
&\quad - 4 \cdot 0.1 \cdot 880 - 1.06 \cdot x_{40} + x_{41} = 5,000
\end{aligned}$$

The liquidity restriction for $t = 1$ includes the initial investment outlays of the machines acquired at the beginning of the second period, the operating cash outflows for the machines purchased at $t = 0$ and $t = 1$, and the short-term financial investment. Cash inflows result from liquidation values, from the balance of the relevant cash inflows and outflows for products produced in the first period (assigned at $t = 0$), and from the compounded short-term financial investment undertaken in the time $t = 0$.

$$\begin{aligned}
&t = 2: \\
&-1.4z_{11} - 1.35z_{21} - z_{31} + 1,000x_{12} + 960x_{22} + 880x_{32} + 195(x_{10} + x_{11} + x_{12}) \\
&\quad + 185(x_{20} + x_{21} + x_{22}) + 225(x_{30} + x_{31} + x_{32}) - 1.06 \cdot x_{41} + x_{42} = 0
\end{aligned}$$

Capacity constraints:

$$\begin{aligned}
&t = 0: \\
&\quad 3z_{10} + 4z_{20} + 5z_{30} \leq 10,000 + 5,000x_{10} \\
&\quad 3z_{10} + 3z_{20} + 2z_{30} \leq 8,000 + 4,000x_{20} \\
&\quad 3z_{10} + 2z_{20} + 4z_{30} \leq 14,000 + 3,500x_{30}
\end{aligned}$$

$$\begin{aligned}
&t = 1: \\
&\quad 3z_{11} + 4z_{21} + 5z_{31} \leq 5,000 + 5,500x_{11} \\
&\quad 3z_{11} + 3z_{21} + 2z_{31} \leq 4,000 + 4,400x_{21} \\
&\quad 3z_{11} + 2z_{21} + 4z_{31} \leq 3,500 + 3,850x_{31}
\end{aligned}$$

$$\begin{aligned}
&t = 2: \\
&\quad 3z_{12} + 4z_{22} + 5z_{32} \leq 5,000x_{10} + 5,500x_{11} + 6,050x_{12} \\
&\quad 3z_{12} + 3z_{22} + 2z_{32} \leq 4,000x_{20} + 4,400x_{21} + 4,840x_{22} \\
&\quad 3z_{12} + 2z_{22} + 4z_{32} \leq 3,500x_{30} + 3,850x_{31} + 4,235x_{32}
\end{aligned}$$

Sales constraints:

$$\begin{aligned}
z_{1t} &\leq 8,000, \quad \text{for } t = 0, 1, 2 \\
z_{2t} &\leq 6,000, \quad \text{for } t = 0, 1, 2 \\
z_{3t} &\leq 5,000, \quad \text{for } t = 0, 1, 2
\end{aligned}$$

Project constraints:

$$
\begin{aligned}
&x_{it} \geq 0 \,\text{and}\, \text{integer}, &&\text{for j} = 1, 2, 3;\ t = 0, 1, 2\\
&z_{4t} \geq 0, &&\text{for t} = 0, 1, 2\\
&z_{kt} \geq 0, &&\text{for k} = 1, 2, 3;\ t = 0, 1, 2
\end{aligned}
$$

The optimum solution of the model is:

$x_{10}=6$	$x_{20}=7$	$x_{30}=5$	$x_{11}=3$	$x_{21}=3$	$x_{31}=5$	$x_{12}=0$	$x_{22}=0$	$x_{32}=0$
$x_{40}=2{,}630.00$			$x_{41}=8{,}867.63$			$x_{42}=22{,}338.44$		
$z_{10}=7{,}666.67$			$z_{11}=8{,}000$			$z_{12}=8{,}000$		
$z_{20}=4{,}250$			$z_{21}=5{,}625$			$z_{22}=5{,}625$		
$z_{30}=0$			$z_{31}=0$			$z_{32}=0$		

This optimum solution recommends that six units of machine 1, seven units of machine 2 and five units of machine 3 should be purchased at the beginning of the planning period; as well as three units of machines 1 and 2 and five units of machine 3 at time $t=1$. The production and sales volumes of the products are: in the first period (i.e. at $t=0$): 7,666.67 units of product 1 and 4,250 units of product 2. Short-term financial investments should be made at the beginning of the planning period (amount = €2,630.00) and at times $t=2$ and $t=3$ (amounts = €8,867.63 and €22,338.44 respectively). The objective function value (i.e. maximum compound value) is €48,296.50.

Assessment of the model

The model presented here captures the interdependencies between investment and production decisions relatively well by including product variables, investment variables, and their linkage via the capacity constraints. Thus, it also circumvents the assumption that a cash inflow must be allocated to a specific investment project—a potentially problematic assumption that is common to all other models discussed so far.

Difficulties may arise from the optimum determination process (particularly if the projects must be discrete) and from the processes of data collection. Moreover, deviations between the real environment and the 'model-world' may apply to all the assumptions mentioned. The assumption about available internal funds is one example; financing decisions remain outside the model, apart from short-term financial investments (although financing can be integrated into the model by introducing financing variables). Decisions about *how* to produce products are not part of the model, and the economic lives of the investment projects are assumed to be known. These and other weaknesses of the model can, however, be largely eliminated by extending the model further. Yet, extensions inevitably increase the complexities of data collection and calculation.

Although this and other models for simultaneous investment and production programme decision-making represent planning problems relatively well, they are rarely applied in company practice, for various reasons. One problem is that such generalised theoretical models must be adapted to the specific company situation.

The most crucial barriers to applying this sort of model are the challenging planning requirements and the effort involved in data acquisition and model solution. Difficulties are primarily due to the high complexity of these models including the requirement for projects to be discrete. This requirement may result in problems within the optimum solution calculation process, despite recent progress in computer technology. The model may also lead to data procurement problems, since a huge amount of data might need to be collected from across a company. In addition, the data relates almost exclusively to future periods, so must be forecasted. It is, therefore, highly uncertain, thus reducing the reliability of the model. The considerable influence that investment model data uncertainties have on the profitability of investment objects is considered in the following chapters.

Assessment Material

Exercise 7.1 (Dean Model for Simultaneous Financing and Investment Decisions)

The choice is between the investment and finance projects below, each with their given cash flows, a_{jt} or d_{it} (in €'000):

Table 7.7 Cash flows for the investment projects

Investment projects j	1	2	3	4	5
a_{j0} (€'000)	−120	−160	−70	−60	−30
a_{j1} (€'000)	+144	+170	+77	+78	+36

Table 7.8 Cash flows of for financing projects

Financing projects i	A	B	C	D
d_{i0} (€'000)	+50	+70	+160	+80
d_{i1} (€'000)	−54	−78	−200	−84

(a) For each project, calculate the internal rate of return (IRR) or the effective rate of interest. From this, deduce the capital supply and capital demand curves and draw these on a graph. Determine the optimum investment and financing programme as well as the endogenous rate of interest. What is the maximum compound value?

(b) Take another look at the choice of investment and financing projects in part a) of the exercise. Assume all the investment projects must be realised in full (i.e. they are indivisible). Ascertain the optimum investment and financing programme and calculate the maximum compound value.

(c) State the assumptions made by the Dean model.

Exercise 7.2 (Multi-tier Model for Simultaneous Financing and Investment Decisions)

A company faces the task of planning its investment and financing programme. It must choose between three investment projects (x_1, x_2, x_3). At any point in time, excess funds may be invested in the short term (x_{4t}). Interest on such short-term investments is 5 %. The investment projects are characterised by the following net cash flows (€'000):

Table 7.9 Net cash flows for the investment projects

Time t	x_1	x_2	x_3
0	−100	0	−120
1	50	−80	60
2	50	55	40
3	50	55	40

There are also two financing projects available to the company (y_1, y_2) with the following net cash flows (€'000):

Table 7.10 Net cash flows for the financing projects

Time t	y_1	y_2
0	100	0
1	−10	100
2	−10	0
3	−115	−118

Each loan can be drawn down for up to €600,000 and divided up at will. Each investment project may be undertaken up to five times, but must be realised in full each time (i.e. the projects are indivisible).

The company invests internal funds as follows: €200,000 at the beginning of the first period and €100,000 each at the beginning of the second and third periods.

Formulate a multi-tier model for the simultaneous planning of an investment and financing programme appropriate to the problem described above.

Exercise 7.3 (Multi-tier Model for Simultaneous Financing and Investment Decisions)

A company is faced with two investment projects (x_1, x_2) and two forms of long-term financial investment (x_3, x_4) plus one short-term financial investment (x_{5t}) in each period. The company may take up two loans (y_1, y_2) of up to €1,000,000 each.

For the available investment projects and loans, the following monetary consequences are expected (€'000):

Table 7.11 Net cash outflows per unit of the variables (projects)

Time t	x_1	x_2	x_3	x_4	x_{50}	x_{51}	x_{52}	x_{53}	y_1	y_2
0	100	80	50	100	100	0	0	0	−100	−100
1	−60	−50	0	−10	−105	100	0	0	0	0
2	−60	−50	0	−10	0	−105	100	0	0	0
3	−50	−40	−90	−120	0	0	−105	100	140	130

There are no internal funds available.

(a) Formulate a multi-tier model for maximising the compound value of the investment and financing programme.
(b) The following programmes are proposed:
 (i) $x_1 = 1.5$; $x_2 = 1$; $x_3 = 1$; $y_1 = 1$; $y_2 = 1$
 (ii) $x_1 = 1$; $x_2 = 1$; $y_1 = 1$; $y_2 = 1$
 (The values of the variables x_{5t} are not given here but may be deduced from the other variables.)
 Are the programmes feasible and, if so, optimal? Briefly outline the reasons for this.
(c) How does the model change if additional cash inflows in the amount of €10,000 are expected for each unit of investment project 1 at each of the times $t = 4$ and $t = 5$, and 10 % is the rate of interest for calculation purposes?
(d) In optimising a Hax and Weingartner model, the following endogenous compounding factors q_t^* were determined for the times t:

$$q_0^* = 1.93908;\ q_1^* = 1.4916;\ q_2^* = 1.243;\ q_3^* = 1.1;\ q_4^* = 1$$

 Determine the endogenous rates of interest for periods 1–4, and assess the profitability of an additional project with the following cash flow profile:

Table 7.12 Cash flow profile of the additional project

Time t	0	1	2	3	4
Cash flows (€'000)	−300	120	120	120	110

Exercise 7.4 (Static and Multi-tier Models for Simultaneous Financing and Investment Decisions)

(a) A choice must be made between the investment and finance projects below, each with their forecasted cash flows, a_{jt} or d_{it} (in €'000).

Table 7.13 Cash flows of the investment projects

Investment project	A	B	C	D
a_{j0}	−60	−70	−40	−100
a_{j1}	78	87.5	45	124

Table 7.14 Cash flows of the financing projects

Financing project	1	2	3
d_{i0}	100	100	100
d_{i1}	−110	−120	−132

(a1) Determine the optimum investment and financing programme when the investment and finance projects may all be divided at will.
What is the maximum compound value?

(a2) Ascertain the optimum investment and finance programme assuming the investment projects cannot be divided.
What is the maximum compound value?

(a3) Which programme is optimal if neither the finance projects nor the investment projects can be divided?
What is the maximum compound value?

(b) A company wishes to plan its investment and financing programme simultaneously. There are four investment projects to choose from, A-D (investment variables $x_A - x_D$), with the following net cash flows (€'000):

Table 7.15 Net cash flows for the four investment projects

Time t	A	B	C	D
0	−100	−150	−80	−50
1	40	40	25	15
2	40	50	25	20
3	40	55	25	15
4	40	55	25	10

Investment projects A and C may be realised a maximum of three times. Internal funds available at $t = 1$ amount to €80,000. The investment projects A and B may also be realised at $t = 1$ (investment variables x_E, x_F), and an upper limit of 3 applies to the realisation of investment project A also at this time.

The following information on the financing projects 1–3 (financial variables $y_1 - y_3$) is available:

- If the first financing project is realised, 60 % of the cash inflows will be received at $t = 0$ and 40 % at $t = 1$. At each time, interest at a rate of 10 % is payable on the capital borrowed, which is to be repaid at $t = 4$.

- A payment of the full nominal amount of the second financing project will be received at $t = 0$ if this project is realised. 50 % of the capital is to be repaid at $t = 3$, and the remaining 50 % at $t = 4$. At each time, interest at a rate of 9 % is also payable on capital previously received and not yet repaid.
- The third financing project generates only one positive payment at $t = 0$. Payments of interest and compound interest, as well as capital repayments, are due at times $t = 1$ to $t = 4$. The total amounts payable stay the same and the applicable rate of interest is 6 %.
- For each of the financing projects the maximum amount is €200,000.

At each time, a short-term, single-period financial investment may be made, yielding interest at 3 % (investment variables x_{Gt}, $t = 0, 1, 2, 3, 4$).

Also at each time (except $t = 4$), a short-term, single-period loan may be accessed bearing interest at 7 % (financial variables y_{4t}, $t = 0, 1, 2, 3$), while the maximum amount available at each time is, as for the other financing projects, €200,000.

Formulate a multi-tier model for this problem. Relate the objective function to $t = 4$ and assume a discount rate of 5 % for period 5.

(c) The models formulated in (a), and (b) aim to decide simultaneously on an investment and financing programme. Work out the differences between the models and, in so doing, state the differing assumptions involved.

Exercise 7.5 (Extended Förstner and Henn Model)

The head of a company's planning department wishes to decide about production and investments simultaneously. The following data are available: The company produces two kinds of products, k ($k = 1, 2$). For each unit of product, it achieves a price p_k and has to pay variable cash outflows of a_{vk} resulting from the production process. It can sell maximum amounts of Z_k.

Table 7.16 Data for the two kinds of products

k	p_k (€ per unit)	Z_k (unit)	a_{vk} (€ per unit)
1	12.00	1,000	8.00
2	18.00	16,000	10.00

Both products are produced on three machines, j ($j = 1, 2, 3$). The utilisation of these machines j for each unit of the product k is given below (in units of capacity).

Table 7.17 Utilisation of the machines

	Machine j		
Product k	1	2	3
1	3	4	6
2	2	5	7

At the beginning of the planning period there is an initial stock of machinery with the capacity given below in time units:

Table 7.18 Existing capacity of the machines

Machine j	Capacity
1	300
2	400
3	800

Identical machines may be acquired at the beginning of each period. For each type of machine j, I_{0j} represents the initial investment outlays (in €), and G_j the relevant expansion in capacity (in time units).

Table 7.19 Data for the machines

Machine j	I_{0j}	G_j
1	1,700	60
2	1,400	80
3	3,200	100

The liquidation value at the end of the economic life is 20 % of the initial investment outlay for each machine. The decrease in liquidation value occurs evenly throughout all periods of the economic life.

In each case, the total economic life of the existing machine is 2 years and all existing machines have a remaining economic life of 1 year. The cash outflows to acquire these existing machines were equal to those for the machines available for purchase at $t = 0$.

(a) Formulate a two-period model with the objective 'maximising the compound value'. In so doing, assume that the data given here—with the exception of the cash outflows for the aggregates acquired at $t = 1$ (which rise by 10 % compared with the figures given)—are also valid for the second period. Note that the company must remain liquid at all times. Interest on the short-term financial investment is 10 %. €10,000 of internal funds are available at $t = 0$ and again at $t = 1$.
(b) What problems might be expected in setting up and solving such a model in a real business environment?

Exercise 7.6 (Extended Förstner and Henn Model)

Prepare a simultaneous investment and production decision using the following underlying data.

A company produces two kinds of product, k ($k = 1, 2$). It has a monopoly position in the market and achieves prices p_k according to the following formulae.

The maximum volumes it can sell, Z_k, and the variable cash outflows per unit, cof_{vk}, are also given below (with z_k = production amount and sales volume).

Table 7.20 Data for the two products

k	p_k (€ per unit)	Z_k (unit)	cof_{vk} (€ per unit)
1	$120 - 0.2 \cdot z_1$	600	50
2	$180 - 0.1 \cdot z_2$	1,800	100

Both products are manufactured on the machines j (j = 1, 2) and take up the following time units per unit of product on these machines.

Table 7.21 Data for the machines

	Machine j	
Product k	1	2
1	4	6
2	5	5

At the beginning of the planning period, machine 1 has a capacity of 360 time units and a remaining economic life of one period. Its further characteristics are equal to those given below for new type 1 machines.

New type 1 and type 2 machines can be acquired at the beginning of each period. Their economic life is four periods and the liquidation value at the end of the economic life amounts to 20 % of the initial investment outlay. The decrease in their liquidation value occurs linearly throughout all periods of their economic life.

Regardless of the date of acquisition, the cash outflows are €2,000 for the acquisition of machine 1, and €2,500 for machine 2. Each new machine purchased expands capacity by 90 time units (machine 1) and 100 time units (machine 2).

The rate of interest for short-term financial investments is 10 %; there is €40,000 of internal funds available at t = 0.

Given the above, formulate a dynamic two-period model for determining an optimum investment and production programme with the objective of maximising the compound value. In so doing, assume that the data given—with the exception of the variable cash outflows per unit, which rises by 10 %—are valid for both periods. Bear in mind that the company must remain liquid throughout both periods.

Further Reading – Part IV

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