Morphing Schnyder Drawings of Planar Triangulations

Fidel Barrera-Cruz[1], Penny Haxell[1], and Anna Lubiw[1]

University of Waterloo, Waterloo, Canada
{fbarrera,pehaxell,alubiw}@uwaterloo.ca

Abstract. We consider the problem of morphing between two planar drawings of the same triangulated graph, maintaining straight-line planarity. A paper in SODA 2013 gave a morph that consists of $O(n^2)$ steps where each step is a linear morph that moves each of the n vertices in a straight line at uniform speed [1]. However, their method imitates edge contractions so the grid size of the intermediate drawings is not bounded and the morphs are not good for visualization purposes. Using Schnyder embeddings, we are able to morph in $O(n^2)$ linear morphing steps and improve the grid size to $O(n) \times O(n)$ for a significant class of drawings of triangulations, namely the class of weighted Schnyder drawings. The morphs are visually attractive. Our method involves implementing the basic "flip" operations of Schnyder woods as linear morphs.

Keywords: algorithms, computational geometry, graph theory.

1 Introduction

Given a triangulation on n vertices and two straight-line planar drawings of it, Γ and Γ', that have the same unbounded face, it is possible to morph from Γ to Γ' while preserving straight-line planarity. This was proved by Cairns in 1944 [6]. Cairns's proof is algorithmic but requires exponentially many steps, where each step is a *linear morph* that moves every vertex in a straight line at uniform speed. Floater and Gotsman [13] gave a polynomial time algorithm using Tutte's graph drawing algorithm [19], but their morph is not composed of linear morphs so the trajectories of the vertices are more complicated, and there are no guarantees on how close vertices and edges may become. Recently, Alamdari et al. [1] gave a polynomial time algorithm based on Cairns's approach that uses $O(n^2)$ linear morphs, and this has now been improved to $O(n)$ by Angelini et al. [2]. The main idea is to contract (or almost contract) edges. With this approach, perturbing vertices to prevent coincidence is already challenging, and perturbing to keep them on a nice grid seems impossible.

In this paper we propose a new approach to morphing based on Schnyder drawings. We give a planarity-preserving morph that is composed of $O(n^2)$ linear morphs and for which the vertices of each of the $O(n^2)$ intermediate drawings are on a $6n \times 6n$ grid. Our algorithm works for *weighted Schnyder drawings* which are obtained from a Schnyder wood together with an assignment of positive weights

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to the interior faces. A Schnyder wood is a special type of partition (colouring) and orientation of the edges of a planar triangulation into three rooted directed trees. Schnyder [15,16] used them to obtain straight-line planar drawings of triangulations in an $O(n) \times O(n)$ grid. To do this he defined barycentric coordinates for each vertex in terms of the number of faces in certain regions of the Schnyder wood. Dhandapani [7] noted that assigning any positive weights to the faces still gives straight-line planar drawings. We call these *weighted Schnyder drawings*—they are the drawings on which our morphing algorithm works.

Two weighted Schnyder drawings may differ in weights and in the Schnyder wood. We address these separately: we show that changing weights corresponds to a single planar linear morph; altering the Schnyder wood is more significant.

The set of Schnyder woods of a given planar triangulation forms a distributive lattice [5], [11], [14] possibly of exponential size [12]. The basic operation for traversing this lattice is a "flip" that reverses a cyclically oriented triangle and changes colours appropriately. It is known that the flip distance between two Schnyder woods in the lattice is $O(n^2)$ (see Section 2). Therefore, to morph between two Schnyder drawings in $O(n^2)$ steps, it suffices to show how a flip can be realized via a constant number of planar linear morphs. We show that flipping a facial triangle corresponds to a single planar linear morph, and that a flip of a separating triangle can be realized by three planar linear morphs.

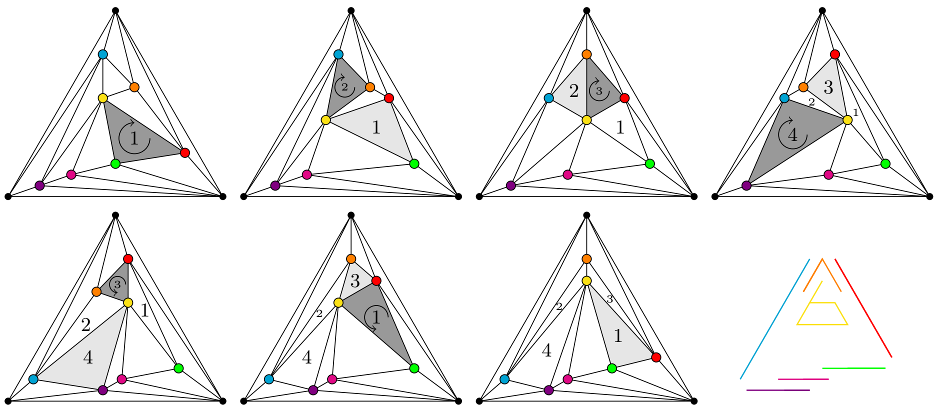

Fig. 1. A sequence of triangle flips, counterclockwise along the top row and clockwise along the bottom row. In each drawing the triangle to be flipped is darkly shaded, and the one most recently flipped is lightly shaded. The linear morph from each drawing to the next one is planar. Vertex trajectories are shown bottom right.

There is hope that our method will give good visualizations for morphing. See Figure 1. The edge-contraction method of Alamdari et al. [1] is not good for visualization purposes—at the end of the recursion, the whole graph has contracted to a triangle. The method of Floater and Gotsman [13] gives good visualizations, based on experiments and heuristic improvements developed by

Shurazhsky and Gotsman [17]. However, their method suffers the same drawbacks as Tutte's graph drawing method, namely that vertices and edges may come very close together. Our intermediate drawings lie on a $6n \times 6n$ grid where vertices are at least distance 1 apart and face areas are at least $\frac{1}{2}$.

Not all straight-line planar triangulations are weighted Schnyder drawings, but we can recognize those that are in polynomial time. The problem of extending our result to all straight-line planar triangulations remains open. There is partial progress in the first author's thesis [3].

This paper is structured as follows. Section 2 contains the relevant background on Schnyder woods. Section 3 contains the precise statement of our main result, and the general outline of the proof. In Section 4 we show that changing face weights corresponds to a linear morph. Flips of facial triangles are handled in Section 5 and flips of separating triangles are handled in Section 6. In Section 7 we explore which drawings are weighted Schnyder drawings.

1.1 Definitions and Notation

Consider two drawings Γ and Γ' of a planar triangulation T. A *morph* between Γ and Γ' is a continuous family of drawings of T, $\{\Gamma^t\}_{t\in[0,1]}$, such that $\Gamma^0 = \Gamma$ and $\Gamma^1 = \Gamma'$. We say a face xyz *collapses* during the morph $\{\Gamma^t\}_{t\in[0,1]}$ if there is $t \in (0,1)$ such that x, y and z are collinear in Γ^t. We call a morph between Γ and Γ' *planar* if Γ^t is a planar drawing of T for all $t \in [0,1]$. Note that a morph is planar if and only if no face collapses during the morph. We call a morph *linear* if each vertex moves from its position in Γ^0 to its position in Γ^1 along a line segment and at constant speed. Note that each vertex may have a different speed. We denote such a linear morph by $\langle \Gamma^0, \Gamma^1 \rangle$.

Throughout the paper we deal with a planar triangulation T with a distinguished exterior face with vertices a_1, a_2, a_3 in clockwise order. The set of interior faces is denoted $\mathcal{F}(T)$. A 3-cycle C whose removal disconnects T is called a *separating triangle*, and in this case we define $T|_C$ to be the triangulation formed by vertices inside C together with C as the exterior face, and we define $T \setminus C$ to be the triangulation obtained from T by deleting the vertices inside C.

2 Schnyder Woods and Their Properties

A *Schnyder wood* of a planar triangulation T with exterior vertices a_1, a_2, a_3 is an assignment of directions and colours 1, 2, and 3 to the interior edges of T such that the following two conditions hold.

(D1) Each interior vertex has three outgoing edges and they have colours 1, 2, 3 in clockwise order. All incoming edges in colour i appear between the two outgoing edges of colours $i-1$ and $i+1$ (index arithmetic modulo 3).

(D2) At the exterior vertex a_i, all the interior edges are incoming and of colour i.

The following basic concepts and properties are due to Schnyder [16]. For any Schnyder wood the edges of colour i form a tree T_i rooted at a_i. The path from internal vertex v to a_i in T_i is denoted $P_i(v)$.

(P1) If T_i^- denotes the tree in colour i with all arcs reversed, then $T_{i-1}^- \cup T_i \cup T_{i+1}^-$ contains no directed cycle. In particular, any two outgoing paths from a vertex v have no vertex in common, except for v, i.e., $P_i(v) \cap P_j(v) = \{v\}$ for $i \neq j$.

The *descendants* of vertex v in T_i, denoted $D_i(v)$, are the vertices that have paths to v in T_i. For any interior vertex v the three paths $P_i(v), i = 1, 2, 3$ partition the triangulation into three regions $R_i(v), i = 1, 2, 3$, where $R_i(v)$ is bounded by $P_{i+1}(v), P_{i-1}(v)$ and $a_{i+1}a_{i-1}$. Schnyder proved that every triangulation T has a Schnyder wood and that a planar drawing of T can be obtained from coordinates that count faces inside regions:

Theorem 1 (Schnyder [15,16]). *Let T be a planar triangulation on n vertices equipped with a Schnyder wood S. Consider the map $f : V(T) \to \mathbb{R}^3$, where $f(a_i) = (2n-5)e_i$, where e_i denotes the i-th standard basis vector in $\mathbb{R}^3$, and for each interior vertex v, $f(v) = (v_1, v_2, v_3)$, where v_i denotes the number of faces contained inside region $R_i(v)$. Then f defines a straight-line planar drawing.*

Dhandapani [7] noted that the above result generalizes to weighted faces. A *weight distribution* $\mathbf{w}$ is a function that assigns a positive weight to each internal face such that the weights sum to $2n-5$. For any weight distribution, the above result still holds if v_i is defined as:

$$v_i = \sum \{\mathbf{w}(f) : f \in R_i(v)\}. \tag{1}$$

We call the resulting straight-line planar drawing the *weighted Schnyder drawing* obtained from $\mathbf{w}$ and S.

We now describe results of Brehm [5], Ossona De Mendez [14], and Felsner [11] on the flip operation that can be used to convert any Schnyder wood to any other. Let S be a Schnyder wood of planar triangulation T. A flip operates on a cyclically oriented triangle C of T. We use the following properties of such a triangle (proofs in the long version).

(S1) The triangle C has an edge of each colour in S. Furthermore, if C is oriented counterclockwise then the edges along C have colours i, $i-1$, $i+1$.
(S2) If C is a separating triangle, then the restriction of S to the interior edges of $T|_C$ is a Schnyder wood of $T|_C$.

Let $C = xyz$ be oriented counterclockwise with edges xy, yz, zx of colour $1, 3, 2$ respectively. A *clockwise flip* of C alters the colours and orientations of S as follows:

1. Edges on the cycle are reversed and colours change from i to $i-1$. See triangle xyz in Figure 3.

2. Any interior edge of $T|_C$ remains with the same orientation and changes colour from i to $i+1$. See edges incident to b in Figure 4.

Other edges are unchanged. The reverse operation is a *counterclockwise flip*, which Brehm calls a *flop*. Brehm [5, p. 44] proves that a flip yields another Schnyder wood. Consider the graph with a vertex for each Schnyder wood of T and a directed edge (S, S') when S' can be obtained from S by a clockwise flip. This graph forms a distributive lattice [5], [11], [14]. Ignoring edge directions, the distance between two Schnyder woods in this graph is called their *flip distance*.

Lemma 2 (Brehm (see the long version)). *In a planar triangulation on n vertices the flip distance between any two Schnyder woods is $O(n^2)$, and a flip sequence of that length can be found in linear time per flip.*

3 Main Result

Theorem 3. *Let T be a planar triangulation and let S and S' be two Schnyder woods of T . Let Γ and Γ' be weighted Schnyder drawings of T obtained from S and S' together with some weight distributions. There exists a sequence of straight-line planar drawings of T $\Gamma = \Gamma_0, \ldots, \Gamma_{k+1} = \Gamma'$ such that k is $O(n^2)$, the linear morph $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is planar, $0 \le i \le k$, and the vertices of each Γ_i, $1 \le i \le k$, lie in a $(6n-15) \times (6n-15)$ grid. Furthermore, these drawings can be obtained in polynomial time.*

We now describe how the results in the upcoming sections prove the theorem. Lemma 4 (Section 4) proves that if we perform a linear morph between two weighted Schnyder drawings that differ only in their weight distribution then planarity is preserved. Thus, we may take Γ_1 and Γ_k to be the drawings obtained from the uniform weight distribution on S and S' respectively. By Schnyder's Theorem 1 these drawings lie on a $(2n-5) \times (2n-5)$ grid and we can scale them up to our larger grid. By Lemma 2 (Section 2) there is a sequence of k flips, $k \in O(n^2)$, that converts S to S'. Therefore it suffices to show that each flip in the sequence can be realized via a planar morph composed of a constant number of linear morphs. In Theorem 7 (Section 5) we prove that if we perform a linear morph between two weighted Schnyder drawings that differ only by a flip of a face then planarity is preserved. In Theorem 11 (Section 6) we prove that if two Schnyder drawings with the same uniform weight distribution differ by a flip of a separating triangle then there is a planar morph between them composed of three linear morphs. The intermediate drawings involve altered weight distributions (here Lemma 4 is used again), and lie on a grid of the required size. Putting these results together gives the final sequence $\Gamma_0, \ldots, \Gamma_{k+1}$. All the intermediate drawings lie in a $(6n-15) \times (6n-15)$ grid and each of them can be obtained in $O(n)$ time from the previous one. This completes the proof of Theorem 3 modulo the proofs in the following sections.

4 Morphing to Change Weight Distributions

Lemma 4. *(proof in the long version) Let T be a planar triangulation and let S be a Schnyder wood of T. Consider two weight distributions $\mathbf{w}$ and $\mathbf{w}'$ on the faces of T, and denote by Γ and Γ' the weighted Schnyder drawings of T obtained from $\mathbf{w}$ and $\mathbf{w}'$ respectively. Then the linear morph $\langle \Gamma, \Gamma' \rangle$ is planar.*

5 Morphing to Flip a Facial Triangle

In this section we prove that the linear morph from one Schnyder drawing to another one, obtained by flipping a cyclically oriented face and keeping the same weight distribution, preserves planarity. See Figure 2. We begin by showing how the regions for each vertex change during such a flip and then we use this to show how the coordinates change.

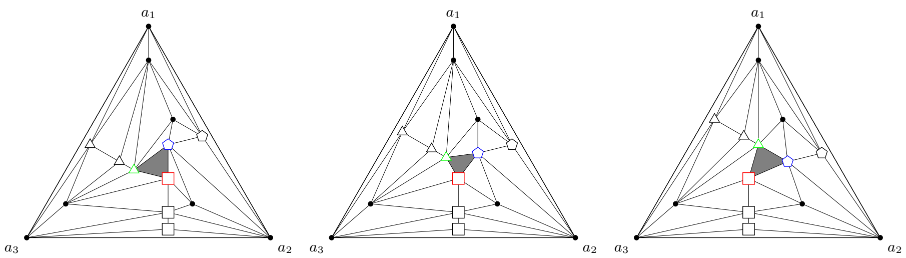

Fig. 2. Snapshots from a linear morph defined by a flip of the shaded face at times $t = 0$, $t = 0.5$ and $t = 1$. The trajectory of rectangular shaped vertices is parallel to a_2a_3. Similar properties hold for triangular and pentagonal shaped vertices.

Let S and S' be Schnyder woods of triangulation T that differ by a flip on face xyz oriented counterclockwise in S with (x, y) of colour 1. Let (v_1, v_2, v_3) and (v'_1, v'_2, v'_3) be the coordinates of vertex v in the weighted Schnyder drawings from S and S' respectively with respect to weight distribution $\mathbf{w}$. For an interior edge pq of T, let $\Delta_i(pq)$ be the set of faces in the region bounded by pq and the paths $P_i(p)$ and $P_i(q)$ in S, and we define $\delta_i(pq)$ to be the weight of that region, i.e., $\delta_i(pq) = \sum_{f \in \Delta_i(pq)} \mathbf{w}(f)$. We use notation $P_i(v)$, $R_i(v)$, and $D_i(v)$ as defined in Section 2 and $\Delta_i(pq)$ as above and add primes to denote the corresponding structures in S'. Let us begin by identifying properties of S and S'. The following two lemmas are proved formally in the long version.

Lemma 5. *The following conditions hold (see Figure 3):*

1. $R_1(x) = R'_1(x)$, $R_3(y) = R'_3(y)$ *and* $R_2(z) = R'_2(z)$.
2. $R'_2(x) = R_2(x) \setminus (\Delta_1(yz) \cup \{f\})$, $R'_3(x) = R_3(x) \cup (\Delta_1(yz) \cup \{f\})$ *and similarly for* y *and* z.

3. $D_1(x) = D'_1(x)$, $D_2(z) = D'_2(z)$ *and* $D_3(y) = D'_3(y)$.
4. *The interiors of* $R_1(x)$, $R_2(z)$ *and* $R_3(y)$ *are pairwise disjoint.*
5. $D_1(x) \setminus \{x\}$ *is contained in the interior of* $R_1(x)$ *and similarly for* y *and* z. *Consequently* $D_1(x)$, $D_2(z)$ *and* $D_3(y)$ *are pairwise disjoint.*

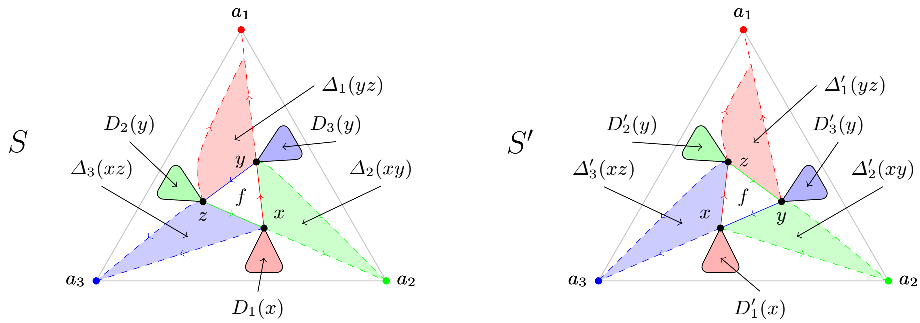

Fig. 3. A flip of a counterclockwise oriented face triangle xyz showing changes to the regions. Observe that $\Delta_1(yz) \cup \{f\}$ leaves $R_2(x)$ and joins $R'_3(x)$.

Next we study the difference between the coordinates of the weighted Schnyder drawings corresponding to S and S'.

Lemma 6. *For each* $v \in V(T)$,

$$(v'_1, v'_2, v'_3) = \begin{cases} (v_1, v_2, v_3) & \text{if } v \notin D_1(x) \cup D_2(z) \cup D_3(y) \\ (v_1, v_2 - (\delta_1(yz) + \mathbf{w}(f)), v_3 + \delta_1(yz) + \mathbf{w}(f)) & \text{if } v \in D_1(x) \\ (v_1 + \delta_2(xy) + \mathbf{w}(f), v_2, v_3 - (\delta_2(xy) + \mathbf{w}(f))) & \text{if } v \in D_2(z) \\ (v_1 - (\delta_3(xz) + \mathbf{w}(f)), v_2 + \delta_3(xz) + \mathbf{w}(f), v_3) & \text{if } v \in D_3(y). \end{cases}$$

We are ready to prove the main result of this section. We express it in terms of a general weight distribution since we will need that in the next section.

Theorem 7. *Let* S *be a Schnyder wood of a planar triangulation* T *that contains a face* f *bounded by a counterclockwise directed triangle* xyz, *and let* S' *be the Schnyder wood obtained from* S *by flipping* f. *Denote by* Γ *and* Γ' *the weighted Schnyder drawings obtained from* S *and* S' *respectively with weight distribution* $\mathbf{w}$. *Then* $\langle \Gamma, \Gamma' \rangle$ *is a planar morph.*

Proof. If a triangle collapses during the morph, then it must be incident to at least one vertex that moves, i.e., one of $D_1(x)$, $D_2(y)$ or $D_3(z)$. By Lemma 5, apart from x, y, z these vertex sets lie in the interiors of regions $R_1(x)$, $R_2(y)$, $R_3(z)$ respectively. Thus it suffices to show that no triangle in one of these regions collapses, and that no triangle incident to x, y or z collapses.

Let t be a triangle such that $t \in R_1(x)$. (The argument for triangles in other regions is similar.) Any vertex of $R_1(x)$ that moves is in $D_1(x)$ and by Lemma 6 these vertices are all translated by the same amount. We argue that if triangle b, c, e in clockwise order collapses as we translate a subset of its vertices then the end result is triangle b, c, e in counterclockwise order. This contradicts the fact that Γ and Γ' have the same faces. A rigorous proof is in the long version. The same argument applies to a triangle in $\Delta_3(xz) \cup \Delta_2(xy)$ that is incident to x but not incident to either y or z.

It remains to prove that no triangle t incident to at least two vertices of x, y and z collapses. Here we only consider the case where $t = xyz$, the other case can be handled similarly. We will show that x never lies on the line segment yz during the morph. (The other two cases are similar.) Since (x, y) has colour 1 in S, it follows that $x \in R_1(y)$. Similarly, since (z, x) has colour 2 in S, we have that $x \in R_1(z)$. Therefore $x_1 < y_1, z_1$. Using a similar argument on S' we obtain that $x'_1 < y'_1, z'_1$. Finally, note that $x_1 = x'_1$. This implies that x never lies on the line segment yz during the morph. □

6 Morphing to Flip a Separating Triangle

In this section we prove that there is a planar morph between any two weighted Schnyder drawings that differ by a separating triangle flip. Our morph will be composed of three linear morphs. Throughout this section we let S and S' be Schnyder woods of a planar triangulation T such that S' is obtained from S after flipping a counterclockwise oriented separating triangle $C = xyz$, with (x, y) coloured 1 in S. Let Γ and Γ' be two weighted Schnyder drawings obtained from S and S' respectively with weight distribution $\mathbf{w}$. For the main result of the section, it suffices to consider a uniform weight distribution because we can get to it via a single planar linear morph, as shown in Section 4. However, for the intermediate results of the section we need more general weight distributions.

We now give an outline of the strategy we follow. Morphing linearly from Γ to Γ' may cause faces inside C to collapse. An example is provided in the long version. However, we can show that there is a "nice" weight distribution that prevents this from happening. Our plan, therefore, is to morph linearly from Γ to a drawing $\overline{\Gamma}$ with a nice weight distribution, then morph linearly to drawing $\overline{\Gamma}'$ to effect the separating triangle flip. A final change of weights back to the uniform distribution gives a linear morph from $\overline{\Gamma}'$ to Γ'.

This section is structured as follows. First we study how the coordinates change between Γ and Γ'. Next we show that faces strictly interior to $T|_C$ do not collapse during a linear morph between $\overline{\Gamma}$ and $\overline{\Gamma}'$. We then give a similar result for faces of $T|_C$ that share a vertex or edge with C *provided that* the weight distribution satisfies certain properties. Finally we prove the main result by showing that there is a weight distribution with the required properties.

Let us begin by examining the coordinates of vertices. For vertex $b \in V(T)$ let (b_1, b_2, b_3) and (b'_1, b'_2, b'_3) denote its coordinates in Γ and Γ' respectively. For b an interior vertex of $T|_C$ let β_i be the i-th coordinate of b in $T|_C$ when considering the restriction of S to $T|_C$ with weight distribution $\mathbf{w}$. By analyzing

Figure 4, we can see that the coordinates for b in Γ are

$$\begin{aligned}(b_1, b_2, b_3) &= (x_1 + \delta_3(xz) + \beta_1, z_2 + \delta_1(yz) + \beta_2, y_3 + \delta_2(xy) + \beta_3) \\ &= (x_1, z_2, y_3) + (\delta_3(xz), \delta_1(yz), \delta_2(xy)) + (\beta_1, \beta_2, \beta_3).\end{aligned} \tag{2}$$

We now analyze how the coordinates of vertices change from Γ to Γ'. We use $\mathbf{w}_C$ to denote the weight of faces inside C, i.e., $\mathbf{w}_C = \sum_{f\in\mathcal{F}(T|_C)} \mathbf{w}(f)$.

Lemma 8. *(proof in the long version) For each $b \in V(T)$,*

$$(b'_1, b'_2, b'_3) = \begin{cases} (b_1, b_2 - (\delta_1(yz) + \mathbf{w}_C), b_3 + \delta_1(yz) + \mathbf{w}_C) & \text{if } b \in D_1(x) \\ (b_1 + \delta_2(xy) + \mathbf{w}_C, b_2, b_3 - (\delta_2(xy) + \mathbf{w}_C)) & \text{if } b \in D_2(z) \\ (b_1 - (\delta_3(xz) + \mathbf{w}_C), b_2 + \delta_3(xz) + \mathbf{w}_C, b_3) & \text{if } b \in D_3(y) \\ (x_1, z_2, y_3) + (\delta_2(xy), \delta_3(xz), \delta_1(yz)) + (\beta_3, \beta_1, \beta_2) & \text{if } b \in \mathcal{I} \\ (b_1, b_2, b_3) & \text{otherwise} \end{cases}$$

where $\mathcal{I}$ is the set of interior vertices of $T|_C$.

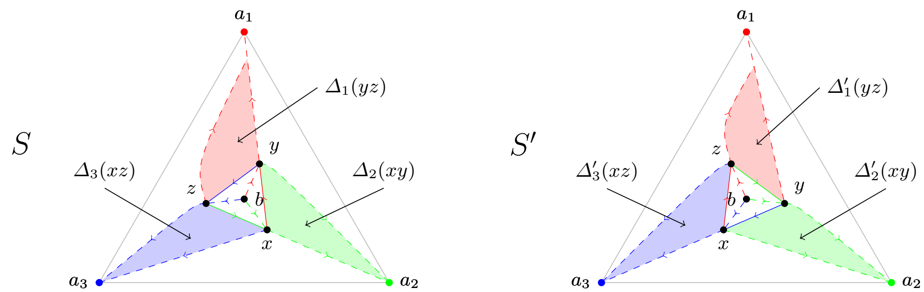

Fig. 4. A flip of a counter-clockwise oriented separating triangle xyz

We now examine what happens during a linear morph from Γ to Γ'. We first deal with faces strictly interior to C. The following two lemmas are proved formally in the long version.

Lemma 9. *For an arbitrary weight distribution no face formed by interior vertices of $T|_C$ collapses in the morph $\langle\Gamma, \Gamma'\rangle$.*

Proof sketch. Consider a face inside C formed by internal vertices b, c, e whose coordinates with respect to $T|_C$ are $\beta, \gamma, \varepsilon$, respectively. Examining (2) and Lemma 8 we see that the coordinates of b, c, e in Γ and Γ' depend in exactly the same way on the parameters from $T \setminus C$ and differ only in the parameters $\beta, \gamma, \varepsilon$. Therefore triangle bce collapses during the morph if and only if it collapses during the linear transformation on $\beta, \gamma, \varepsilon$ where we perform a cyclic shift of coordinates, viz., $(\beta_1, \beta_2, \beta_3)$ becomes $(\beta_3, \beta_1, \beta_2)$, etc. No triangle collapses during this transformation because it corresponds to moving each of the three outer vertices x, y, z in a straight line to its clockwise neighbour. □

Next we consider the faces interior to C that share an edge or vertex with C. We show that no such face collapses, provided that the weight distribution $\mathbf{w}$ satisfies $\delta_1 = \delta_2 = \delta_3$ where we use δ_1, δ_2 and δ_3 to denote $\delta_1(yz)$, $\delta_2(xy)$ and $\delta_3(xz)$ respectively.

Lemma 10. *Let* $\mathbf{w}$ *be a weight distribution for the interior faces of* T *such that* $\delta_1 = \delta_2 = \delta_3$. *No interior face of* $T|_C$ *incident to an exterior vertex of* $T|_C$ *collapses during* $\langle \Gamma, \Gamma' \rangle$.

Proof sketch. We examine separately the case where the interior face is incident to the edge xy of C and the case where the interior face is only incident to the vertex x of C. The other cases follow by analogous arguments.

Consider the case of an interior face bxy incident to edge xy. Suppose by contradiction that at time $r \in [0,1]$ during the morph the face collapses with b^r lying on segment $x^r y^r$, say $b^r = (1-s)x^r + sy^r$ for some $s \in [0,1]$. We use formula (2) and Lemma 8 to re-write this equation. Some further algebraic manipulations (details in the long version) show that there is no solution for r. The case of a face involving vertex x and two interior vertices is similar. □

We are now ready to prove the main result of this section.

Theorem 11. *Let* T *be a planar triangulation and let* S *and* S' *be two Schnyder woods of* T *such that* S' *is obtained from* S *by flipping a counterclockwise cyclically oriented separating triangle* $C = xyz$ *in* S. *Let* Γ *and* Γ' *be weighted Schnyder drawings obtained from* S *and* S', *respectively, with uniform weight distribution. Then there exist weighted Schnyder drawings* $\overline{\Gamma}$ *and* $\overline{\Gamma}'$ *on a* $(6n-15) \times (6n-15)$ *integer grid such that each of the following linear morphs is planar:* $\langle \Gamma, \overline{\Gamma} \rangle, \langle \overline{\Gamma}, \overline{\Gamma}' \rangle$, *and* $\langle \overline{\Gamma}', \Gamma' \rangle$.

Proof. Our aim is to define the planar drawings $\overline{\Gamma}$ and $\overline{\Gamma}'$. Each one will be realized in a grid that is three times finer than the $(2n-5) \times (2n-5)$ grid, i.e., in a $(6n-15) \times (6n-15)$ grid with weight distributions that sum to $6n-15$. Under this setup, the initial uniform weight distribution $\mathbf{u}$ takes a value of 3 in each interior face.

Drawings $\overline{\Gamma}$ and $\overline{\Gamma}'$ will be the weighted Schnyder drawings obtained from S and S' respectively with a new weight distribution $\overline{\mathbf{w}}$. We use Δ_1, Δ_2 and Δ_3 to denote the regions $\Delta_1(yz)$, $\Delta_2(xy)$ and $\Delta_3(xz)$ respectively, in S. We use δ_i and $\overline{\delta}_i$ to denote the weight of $\Delta_i, i = 1,2,3$ with respect to the uniform weight distribution and the new weight distribution $\overline{\mathbf{w}}$, respectively.

We will define $\overline{\mathbf{w}}$ so that $\overline{\delta}_1, \overline{\delta}_2$, and $\overline{\delta}_3$ all take on the average value $\delta := (\delta_1 + \delta_2 + \delta_3)/3$. The idea is to remove weight from faces in a region of above-average weight, and add weight to faces in a region of below-average weight. The new face weights must be positive integers. Note first that δ is an integer. Note secondly that $\delta > \delta_i/3$ for any i since the other δ_j's are positive. Thus $\delta_i - \delta < \frac{2}{3}\delta_i$. This means that we can reduce δ_i to the average δ without removing more than 2 weight units from any face (of initial weight 3) in any region. There is more than one solution for $\overline{\mathbf{w}}$, but the morph might look best if $\overline{\mathbf{w}}$ is as uniform as possible. To be more specific, we can define new face weights $\overline{\mathbf{w}}$ via the following algorithm: Initialize $\overline{\mathbf{w}} = \mathbf{w}$. While some $\overline{\delta}_i$ is greater than the average δ, remove 1 from a maximum weight face of Δ_i and add 1 to a minimum weight face in a region Δ_j whose weight is less than the average.

This completes the description of $\overline{\Gamma}$ and $\overline{\Gamma}'$. It remains to show that the three linear morphs are planar. The morphs $\langle \Gamma, \overline{\Gamma} \rangle$ and $\langle \overline{\Gamma}', \Gamma' \rangle$ only involve changes to the weight distribution so they are planar by Lemma 4. Consider the linear morph $\langle \overline{\Gamma}, \overline{\Gamma}' \rangle$. The two drawings differ by a flip of a separating triangle. They have the same weight distribution $\overline{\mathbf{w}}$ which satisfies $\overline{\delta}_1 = \overline{\delta}_2 = \overline{\delta}_3$. By Lemmas 9, and 10 no interior face of $T|_C$ collapses during the morph. By Theorem 7 no face of $T \setminus C$ collapses during the morph. Thus $\langle \Gamma, \Gamma' \rangle$ defines a planar morph. □

7 Identifying Weighted Schnyder Drawings

In this section we give a polynomial time algorithm to test if a given straight-line planar drawing Γ of triangulation T is a weighted Schnyder drawing. The first step is to identify the Schnyder wood. A recent result of Bonichon et al. [4] shows that, given a point set P with triangular convex hull, a Schnyder drawing on P is exactly the "half-Θ_6-graph" of P, which can be computed efficiently. Thus, given drawing Γ, we first ignore the edges and compute the half-Θ_6 graph of the points. If this differs from Γ, we do not have a weighted Schnyder drawing. Otherwise, the half-Θ_6 graph determines the Schnyder wood S. We next find the face weights. We claim that there exists a unique assignment of (not necessarily positive) weights $\mathbf{w}$ on the faces of T such that Γ is precisely the drawing obtained from S and $\mathbf{w}$ as described in (1). Furthermore, $\mathbf{w}$ can be found in polynomial time by solving a system of linear equations in the $2n-5$ variables $\mathbf{w}(f), f \in \mathcal{F}(T)$. The equations are those from (1). The rows of the coefficient matrix are the characteristic vectors of $R_i(v), i \in \{1, 2, 3\}, v$ an interior vertex of T, and the system of equations has a solution because the matrix has rank $2n-5$. This was proved by Felsner and Zickfeld [10, Theorem 9]. (Note that their theorem is about coplanar orthogonal surfaces; however, their proof considers the exact same set of equations and Claims 1 and 2 give the needed result.)

8 Conclusions and Open Problems

We have made a first step towards morphing straight-line planar graph drawings with a polynomial number of linear morphs and on a well-behaved grid. Our method applies to weighted Schnyder drawings. There is hope of extending to all straight-line planar triangulations. The first author's thesis [3] gives partial progress: an algorithm to morph from any straight-line planar triangulation to a weighted Schnyder drawing in $O(n)$ steps—but not on a nice grid.

It might be possible to extend our results to general (non-triangulated) planar graphs using Felsner's extension [8,9] of Schnyder's results. The problem of efficiently morphing planar graph drawings to preserve convexity of faces is wide open—nothing is known besides Thomassen's existence result [18].

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