Tolerance Rough Fuzzy Approximation Operators and Their Properties

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Abstract. In the framework of classification, the rough fuzzy set (RFS) deal with the fuzzy decision tables with discrete conditional attributes and fuzzy decision attribute. However, in many applications, the conditional attributes are often real-valued. In order to deal with this problem, this paper extends the RFS model to tolerance RFS, The definitions of the tolerance rough fuzzy set approximation operators are given, and their properties are investigated.

Keywords: Rough set · Fuzzy set · Rough fuzzy set · Tolerance relation · Approximation operator

1 Introduction

Fuzzy set is usually employed to characterize the uncertainty of cognition[1], while rough set is widely used to describe the uncertainty of knowledge[2]. The typical application of fuzzy set is the fuzzy control[3], while the representative application of rough set is feature selection[4].In this paper, we discuss the problem of extension of rough set in the framework of classification, i.e. the target concept is the decision class. In this scenario, the combination of fuzzy set and rough set addresses the following three types of situations:

(1) The conditional attributes and decision attribute are all fuzzy.
In this situation, the corresponding rough set model is called fuzzy rough set (FRS). The fuzzy decision tables dealt with by FRS are called fuzzy I-decision tables in this paper. The table 1 is a small fuzzy I-decision table with 6 samples. Where A_1 and A_2 are two fuzzy conditional attributes. A_1 has two fuzzy language terms A_{11} and A_{12}, A_2 has three fuzzy language terms A_{21}, A_{22} and A_{23}. C is a fuzzy decision attribute, which has two fuzzy language terms C_1 and C_2. The values in the table 1 are fuzzy membership degree of samples belonging to a fuzzy set.

There are 4 FRS models reported in the literature. Based on possibility and necessity theory, in 1990, Dubois and Prade proposed a FRS model[5], which is

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Table 1. Fuzzy I-decision table

Samples	A_1		A_2			C	
	A_{11}	A_{12}	A_{21}	A_{22}	A_{23}	C_{11}	C_{22}
x_1	0.9	0.1	0.7	0.2	0.1	0.3	0.7
x_2	0.6	0.4	0.6	0.2	0.2	0.6	0.4
x_3	0.7	0.3	0.0	0.7	0.3	0.4	0.6
x_4	0.3	0.7	0.2	0.7	0.1	0.9	0.1
x_5	0.7	0.3	0.0	0.1	0.9	1.0	0.0
x_6	0.1	0.9	0.0	0.7	0.3	0.5	0.5

the most popular one. Based on the fuzzy inclusion degree, Kuncheva proposed the second FRS model in 1992, and applies this model to feature selection[6]. Also in 1992, based on lattice, Nanda proposed the third FRS model[7]. Based on α -level sets, Yao proposed the fourth FRS model[8]. A comprehensive survey of FRS can be found in [9].

(2) The decision attribute is crisp, the conditional attributes are fuzzy. By now, there is no model in the literature to characterize this problem, we will investigate this model in the future work, we named the corresponding fuzzy decision tables as fuzzy II-decision tables. The table 2 is a small fuzzy II-decision table with 6 samples.

(3) The decision attribute is fuzzy, the conditional attributes are discrete. The corresponding model is called rough fuzzy set (RFS)[5]. In this paper, we extend the RFS model to tolerance RFS (TRFS) by introducing the tolerance rough fuzzy approximation operators. In addition, we also study the properties of the tolerance rough fuzzy approximation operators. The fuzzy decision tables dealt with by TRFS are called fuzzy III-decision tables. The table 3 is a small fuzzy III-decision table with 6 samples.

This paper is organized as follows. Section 2 provides preliminaries. The definition of the tolerance rough fuzzy approximation operators, their properties and the proof are presented in Section 3. Section 4 concluded this paper.

Table 2. Fuzzy II-decision table

Samples	A_1		A_2			C
	A_{11}	A_{12}	A_{21}	A_{22}	A_{23}	
x_1	0.9	0.1	0.7	0.2	0.1	yes
x_2	0.6	0.4	0.6	0.2	0.2	no
x_3	0.7	0.3	0.0	0.7	0.3	yes
x_4	0.3	0.7	0.2	0.7	0.1	no
x_5	0.7	0.3	0.0	0.1	0.9	no
x_6	0.1	0.9	0.0	0.7	0.3	yes

Table 3. Fuzzy III-decision table

Samples	A_1	A_2	C	
			C_1	C_2
x_1	1.5	2.3	0.3	0.7
x_2	3.6	2.7	0.6	0.4
x_3	4.3	5.0	0.4	0.6
x_4	7.3	5.4	0.9	0.1
x_5	3.7	9.1	1.0	0.0
x_6	1.0	2.5	0.5	0.5

2 Preliminaries

In this section, we briefly review the basic concepts, including rough set, rough fuzzy set, and tolerance rough set.

2.1 Rough Set

Rough set (RS) uses a pair of operators to approximate the target concepts. Let $DT = (U, A \cup C)$ be a decision table, where $U = \{x_1, x_2, \ldots, x_n\}$, which is a set of n objects, U is usually called a universe. $A = \{a_1, a_2, \ldots, a_d\}$ is a set of d attributes used for describing the characteristics of objects. C is class label variable, whose values is in set $d = \{1, 2, \ldots, k\}$). In other words, the objects in U are categorized into k classes: $U_1, U_2, \ldots, U_k$. Let $x \in U$ and R is an equivalence relation induced by a subset of A, the equivalence class containing x is given by:

$$[x]_R = \{y|xRy\} \tag{1}$$

For arbitrary target concept, i.e. a decision class $U_i (1 \leq i \leq k)$, the R-lower approximation operator $\underline{R}$ of U_i is defined as follows,

$$\underline{R}(U_i) = \{[x]_R | [x]_R \subseteq U_i\} \tag{2}$$

The R-upper approximation operator $\overline{R}$ of U_i is defined as follows,

$$\overline{R}(U_i) = \{[x]_R | [x]_R \cap U_i \neq \phi\} \tag{3}$$

The two-tuple $(\underline{R}(U_i), \overline{R}(U_i))$ is called a rough set.

The universe U can be divided into three disjoint regions by R: positive region $POS(U_i)$, negative region $NEG(U_i)$ and boundary region $BND(U_i)$, where

$$POS(U_i) = \underline{R}(U_i) \tag{4}$$

$$NEG(U_i) = U - POS(U_i) \tag{5}$$

$$BND(U_i) = \overline{R}(U_i) - \underline{R}(U_i) \tag{6}$$

2.2 Rough Fuzzy Set

Rough fuzzy set (RFS) is an extended from rough set by replacing the crisp target concept, i.e. the crisp decision class U_i with a fuzzy target concept, i.e. the fuzzy decision class. For the sake of simplicity, we also use U_i , $\underline{R}$ and $\overline{R}$ to describe the fuzzy decision class, the R-lower approximation operator and the upper approximation operator, respectively. We have the following definition[5].

$$\underline{R}(U_i) = \mu_{\underline{R}(U_i)}([x]_R) = inf\{\mu_{U_i}(y)|y \in [x]_R\} \tag{7}$$

and

$$\overline{R}(U_i) = \mu_{\overline{R}(U_i)}([x]_R) = sup\{\mu_{U_i}(y)|y \in [x]_R\} \tag{8}$$

According to the fuzzy extension principle, (7) and (8) can be equivalently written as follows.

$$\underline{R}(U_i) = \mu_{\underline{R}(U_i)}(x) = inf\{max(\mu_{U_i}(y), 1 - \mu_R(x, y))|y \in U\} \tag{9}$$

and

$$\overline{R}(U_i) = \mu_{\overline{R}(U_i)}(x) = sup\{min(\mu_{U_i}(y), \mu_R(x, y))|y \in U\} \tag{10}$$

The two-tuple $(\underline{R}(U_i), \overline{R}(U_i))$ is called a rough fuzzy set.

2.3 Tolerance Rough Set

Tolerance rough set (TRS)[10] is another extension of rough set. TRS extends rough set by replacing a equivalence relation with a similarity relation. The target concept is same as in rough set, which is also a crisp decision class.

Given a decision table $DT = (U, A \cup C)$, R is a similarity relation defined on U, if and only if R satisfies the following conditions:

(1) Reflexivity, i.e. for each $x \in U$, xRx;

(2) Symmetry, i.e. for each x, $y \in U$, xRy, and yRx.

We can define many similarity relations on U, such as[11, 12]:

$$R_a(x, y) = 1 - \frac{|a(x) - a(y)|}{|a_{max} - a_{min}|} \tag{11}$$

$$R_a(x, y) = exp\left(-\frac{(a(x) - a(y))^2}{2{\sigma_a}^2}\right) \tag{12}$$

$$R_a(x, y) = max\left(min\left(\frac{a(y) - (a(x) - \sigma_a)}{a(x) - (a(x) - \sigma_a)}, \frac{(a(x) + \sigma_a) - a(y)}{(a(x) + \sigma_a) - a(x)}, 0\right)\right) \tag{13}$$

where $a \in A$, $x \in U$, and a_{max} and a_{min} denote the maximum and minimum values of a respectively. The σ_a is variance of attribute a. For $\forall R \subseteq A$, we can define the similarity relations induced by subset R as follows.

$$R_\tau(x, y) = \frac{\sum_{a \in R} R_a(x, y)}{|R|} \geq \tau \tag{14}$$

or

$$R_\tau(x,y) = \prod_{a\in R} R_a(x,y) \geq \tau \tag{15}$$

where τ is a similarity threshold.

For each $x \in U$, the τ tolerance class generated by a given similarity relation R is defined as:

$$[X]_{R_\tau} = \{y|(y \in U) \wedge (xR_\tau y)\} \tag{16}$$

The tolerance lower approximation and upper approximation operators are defined as

$$\underline{R}_\tau(U_i) = \{x|(x \in U) \wedge ([X]_{R_\tau} \subseteq U_i)\} \tag{17}$$

and

$$\overline{R}_\tau(U_i) = \{x|(x \in U) \wedge ([X]_{R_\tau} \cap \ U_i \neq \phi)\} \tag{18}$$

The two-tuple $(\underline{R}_\tau(U_i), \overline{R}_\tau(U_i))$ is called a tolerance rough set.

3 TRFS Approximation Operators and Their Properties

In this section, we present the introduced tolerance rough fuzzy approximation operators and their properties.

3.1 Tolerance Rough Fuzzy Approximation Operators

This paper extends the equivalence relation in rough fuzzy set model to tolerance relation, and the definitions of the tolerance rough fuzzy set approximation operators are given in this section.

Given a fuzzy III-decision table $DT = (U, A \cup C)$, R is a similarity relation defined on U, τ is a similarity threshold, U_i is the ith decision class (i.e. a target concept). The tolerance rough fuzzy lower approximation and tolerance rough fuzzy upper approximation operators are defined as

$$\underline{R}_\tau(U_i) = \mu_{\underline{R_\tau}(U_i)}([x]_{R_\tau}) = inf\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\} \tag{19}$$

and

$$\overline{R}_\tau(U_i) = \mu_{\overline{R_\tau}(U_i)}([x]_{R_\tau}) = sup\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\} \tag{20}$$

Similarly to (9) and (10), we have the following equivalent definitions:

$$\underline{R}_\tau(U_i) = \mu_{\underline{R}_\tau(U_i)}(x) = inf\{max(\mu_{U_i}(y), 1 - \mu_{R_\tau}(x,y))|y \in U\} \tag{21}$$

and

$$\overline{R}_\tau(U_i) = \mu_{\overline{R}_\tau(U_i)}(x) = sup\{min(\mu_{U_i}(y), \mu_{R_\tau}(x,y))|y \in U\} \tag{22}$$

The two-tuple $(\underline{R}_\tau(U_i), \overline{R}_\tau(U_i))$ is called a tolerance rough fuzzy set.

3.2 The Properties of TRFS Approximation Operators

Given a fuzzy III-decision table $DT = (U, A \cup C)$, R is a similarity relation defined on U, τ is a similarity threshold, U_i and U_j are the ith and jth decision class respectively (i.e. a target concept). The proposed TRFS approximation operators satisfy the following properties:

(1) $\underline{R}_\tau(U_i) \subseteq U_i \subseteq \overline{R}_\tau(U_i)$
(2) $\underline{R}_\tau(\phi) = \overline{R}_\tau(\phi) = \phi$, $\underline{R}_\tau(U) = \overline{R}_\tau(U) = U$
(3) $\overline{R}_\tau(U_i \cup U_j) = \overline{R}_\tau(U_i) \cup \overline{R}_\tau(U_j)$
(4) $\underline{R}_\tau(U_i \cap U_j) = \underline{R}_\tau(U_i) \cap \underline{R}_\tau(U_j)$
(5) $U_i \subseteq U_j \Rightarrow \underline{R}_\tau(U_i) \subseteq \underline{R}_\tau(U_j)$
(6) $U_i \subseteq U_j \Rightarrow \overline{R}_\tau(U_i) \subseteq \overline{R}_\tau(U_j)$
(7) $\overline{R}_\tau(U_i \cap U_j) \subseteq \overline{R}_\tau(U_i) \cap \overline{R}_\tau(U_j)$
(8) $\underline{R}_\tau(U_i \cup U_j) \supseteq \underline{R}_\tau(U_i) \cup \overline{R}_\tau(U_j)$

Proof

(1)According to the definition (19) and (20), we have

$$\underline{R}_\tau(U_i) = \mu_{\underline{R}_\tau(U_i)}([x]_{R_\tau}) = inf\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\}$$

And

$$\overline{R}_\tau(U_i) = \mu_{\overline{R}_\tau(U_i)}([x]_{R_\tau}) = sup\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\}$$

Because

$$inf\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\} \leq \{\mu_{U_i}(y)|\forall y \in [x]_{R_\tau}\} \leq sup\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\}$$

Hence

$$\underline{R}_\tau(U_i) \subseteq U_i \subseteq \overline{R}_\tau(U_i)$$

(2)Because

$$\mu_{R_\tau(\phi)}([x]_{R_\tau}) = 0$$

Therefore

$$\begin{aligned}\mu_{\underline{R}_\tau(\phi)}([x]_{R_\tau}) &= inf\{\mu_\phi(y)|y \in [x]_{R_\tau}\} = \mu_{\overline{R}_\tau(\phi)}([x]_{R_\tau}) \\ &= sup\{\mu_\phi(y)|y \in [x]_{R_\tau}\} \\ &= 0\end{aligned}$$

Hence

$$\underline{R}_\tau(\phi) = \overline{R}_\tau(\phi) = \phi$$

Because

$$\mu_{R_\tau(U)}([x]_{R_\tau}) = 1$$

Therefore

$$\begin{aligned}\mu_{\underline{R}_\tau(U)}([x]_{R_\tau}) &= inf\{\mu_U(y)|y \in [x]_{R_\tau}\} = \mu_{\overline{R}_\tau(U)}([x]_{R_\tau}) \\ &= sup\{\mu_U(y)|y \in [x]_{R_\tau}\} \\ &= 1\end{aligned}$$

Hence

$$\underline{R}_\tau(U) = \overline{R}_\tau(U) = U$$

(3) Because

$$\begin{aligned}\overline{R}_\tau(U_i \cup U_j) &= \mu_{\overline{R}_\tau(U_i \cup U_j)}([x]_{R_\tau}) = sup\{\mu_{U_i \cup U_j}(y)|y \in [x]_{R_\tau}\} \\ &= max(\mu_{\overline{R}_\tau(U_i)}([x]_{R_\tau}), \mu_{\overline{R}_\tau(U_j)}([x]_{R_\tau})) \\ &= \overline{R}_\tau(U_i) \cup \overline{R}_\tau(U_j)\end{aligned}$$

Hence, the property (3) is hold.

(4) Because

$$\begin{aligned}\underline{R}_\tau(U_i \cap U_j) &= \mu_{\underline{R}_\tau(U_i \cap U_j)}([x]_{R_\tau}) = inf\{\mu_{U_i \cap U_j}(y)|y \in [x]_{R_\tau}\} \\ &= min(\mu_{\underline{R}_\tau(U_i)}([x]_{R_\tau}), \mu_{\underline{R}_\tau(U_j)}([x]_{R_\tau})) \\ &= \underline{R}_\tau(U_i) \cap \underline{R}_\tau(U_j)\end{aligned}$$

Hence, the property (4) is hold.

(5)Because

$$\begin{aligned}U_i \subseteq U_j &\Rightarrow \mu_{R_\tau(U_i)}([x]_{R_\tau}) = \{\mu_{U_i}(y)|y \in [x]_{R_\tau}\} \leq \mu_{R_\tau(U_j)}([x]_{R_\tau}) \\ &= \{\mu_{U_j}(y)|y \in [x]_{R_\tau}\} \\ &\Rightarrow \mu_{\underline{R}\tau(U_i)}([x]_{R_\tau}) = inf\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\} \leq \mu_{\underline{R}\tau(U_j)}([x]_{R_\tau}) \\ &= inf\{\mu_{U_j}(y)|y \in [x]_{R_\tau}\} \\ &\Rightarrow \underline{R}_\tau(U_i) \subseteq \underline{R}_\tau(U_j)\end{aligned}$$

Hence, the property (5) is hold.

(6)Because

$$\begin{aligned}U_i \subseteq U_j &\Rightarrow \mu_{R_\tau(U_i)}([x]_{R_\tau}) = \{\mu_{U_i}(y)|y \in [x]_{R_\tau}\} \leq \mu_{R_\tau(U_j)}([x]_{R_\tau}) \\ &= \{\mu_{U_j}(y)|y \in [x]_{R_\tau}\} \\ &\Rightarrow \mu_{\overline{R}\tau(U_i)}([x]_{R_\tau}) = sup\{\mu_{U_i}(y)|y \in [x]_{R_\tau}\} \leq \mu_{\overline{R}\tau(U_j)}([x]_{R_\tau}) \\ &= sup\{\mu_{U_j}(y)|y \in [x]_{R_\tau}\} \\ &\Rightarrow \overline{R}_\tau(U_i) \subseteq \overline{R}_\tau(U_j)\end{aligned}$$

Hence, the property (6) is hold.

(7) Because

$$\begin{aligned}\overline{R}_\tau(U_i \cap U_j) &= \mu_{\overline{R}_\tau(U_i \cap U_j)}([x]_{R_\tau}) = sup\{\mu_{U_i \cap U_j}(y)|y \in [x]_{R_\tau}\} \\ &\leq min(\mu_{\overline{R}_\tau(U_i)}([x]_{R_\tau}), \mu_{\overline{R}_\tau(U_j)}([x]_{R_\tau})) \\ &= \overline{R}_\tau(U_i) \cap \overline{R}_\tau(U_j)\end{aligned}$$

Hence, we have $\overline{R}_\tau(U_i \cap U_j) \subseteq \overline{R}_\tau(U_i) \cap \overline{R}_\tau(U_j)$.

(8) Because

$$\underline{R}_\tau(U_i \cup U_j) = \mu_{\underline{R}_\tau(U_i \cup U_j)}([x]_{R_\tau}) = inf\{\mu_{U_i \cup U_j}(y) | y \in [x]_{R_\tau}\}$$
$$\geq max(\mu_{\underline{R}_\tau(U_i)}([x]_{R_\tau}), \mu_{\underline{R}_\tau(U_j)}([x]_{R_\tau}))$$
$$= \underline{R}_\tau(U_i) \cup \underline{R}_\tau(U_j)$$

Hence, we have $\underline{R}_\tau(U_i \cup U_j) \supseteq \underline{R}_\tau(U_i) \cup \overline{R}_\tau(U_j)$.

4 Conclusions

This paper combines the tolerance rough set and rough fuzzy set. Two TRFS approximation operators are introduced, their properties are investigated, and the proofs of these properties are given. The proposed TRFS approximation operators extended the range of application of the classical rough approximation operators, which can directly deal with the decision table with real value conditional attributes and fuzzy value decision attribute. In the future works, we will study the approximation operators for decision table with fuzzy value conditional attributes and discrete value decision attribute.

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References

1. Zadeh, L.A.: Fuzzy sets. Inform. Control **8**, 338–353 (1965)
2. Pawlak, Z.: Rough sets. International Journal of Information and Computer Sciences **11**(5), 341–356 (1982)
3. Zimmermann, H.J.: Fuzzy set theory. Journal of Advanced Review **2**, 317–332 (2010)
4. Shen, Q., Jensen, R.: Rough Sets, their Extensions and Applications. International Journal of Automation and Computing **4**(1), 1–12 (2007)
5. Dubois, D., Prade, H.: Rough fuzzy sets and fuzzy rough sets. International Journal of General Systems **17**, 191–208 (1990)
6. Kuncheva, L.I.: Fuzzy rough sets: application to feature selection. Fuzzy Sets and Systems **51**(2), 147–153 (1992)
7. Nanda, S.: Fuzzy rough sets. Fuzzy Sets and Systems **45**, 157–160 (1992)
8. Yao, Y.Y.: Combination of rough and fuzzy sets based on α-level sets. In: Lin, T.Y., Cercone, N. (eds.) Rough Sets and Data Mining: Analysis for Imprecise Data, pp. 301–321. Kluwer Academic Publishers, Boston (1997)

9. Yeung, D.S., Chen, D.G., Tsang, E.C.C., et al.: On the generalization of fuzzy rough sets. IEEE Transactions on Fuzzy Systems **13**(3), 343–361 (1992)
10. Skowron, A.: Tolerance approximation spaces. Fundamenta Informaticae **27**(2–3), 245–253 (1996)
11. Parthaláin, N.M., Shen, Q.: Exploring the boundary region of tolerance rough sets for feature selection. Pattern Recognition **42**, 655–667 (2009)
12. Parthaláin, N.M., Shen, Q., Jensen, R.: A distance measure approach to exploring the rough set boundary region for attribute reduction. IEEE Transactions on Knowledge and Data Engineering **22**, 306–317 (2010)