Stochastic Maximum Principle for Hilbert Space Valued Forward-Backward Doubly SDEs with Poisson Jumps

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Abstract. In this paper we study the stochastic maximum principle for a control problem in infinite dimensions. This problem is governed by a fully coupled forward-backward doubly stochastic differential equation (FBDSDE) driven by two cylindrical Wiener processes on separable Hilbert spaces and a Poisson random measure. We allow the control variable to enter in all coefficients appearing in this system.

Existence and uniqueness of the solutions of FBDSDEs and an extended martingale representation theorem are provided as well.

Keywords: Wiener process · Poisson process · Forward-backward doubly stochastic differential equation · Maximum principle

1 Introduction

Backward stochastic differential equations in infinite dimensions (BSDEs) were studied by Hu and Peng in [6], Tessitore in [17] and Al-Hussein in [3]. Al-Hussein proved in [3] the existence and uniqueness of the solutions to BSDEs in infinite dimensions driven by genuine Q - Wiener processes (and also cylindrical Wiener processes) on separable Hilbert spaces. He gave also a representation of the solution of a system of semi-linear parabolic PDEs and found viscosity solutions to such PDEs. In [4] sufficient conditions of optimality for backward stochastic evolution equations on Hilbert spaces are derived. Several references in these directions are recorded in [4]. These works give a motivational base to study the maximum principle for optimality of forward-backward stochastic differential equations (FBSDEs) in infinite dimensions. In fact, Yin and Wang [19], proved the existence and uniqueness of the solutions of FBSDEs with Poisson jumps in Hilbert space and with bounded random terminal times. Their work relies on

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those in [16] and the method of continuation given in [7]. Developing applications to such FBSDEs as for example in [3] are not yet well studied.

Let us now talk about more general equations. In finite dimensions, a fully coupled forward-backward doubly stochastic differential equation (FBDSDE) was introduced by Peng and Shi in [12]. Such equations are generalizations of stochastic Hamilton systems. Al-Hussein and Gherbal in [5] studied a stochastic control problem governed by a fully coupled multi-dimensional FBDSDE with Poisson jumps.

In the present work, we shall work in infinite dimensions and try to derive the stochastic maximum principle for optimal control of fully coupled FBDSDEs with jumps; see (1) below. Moreover, existence and uniqueness of the solutions to infinite dimensional FBDSDEs along with an extended martingale representation theorem will be provided as well.

Applications of such equations can be gleaned from [5]. Our formulation of these equations as well as cost functionals are given in abstract forms to allow the possibility to work directly in the case of partial information on one hand and on the other hand to cover most of the applications available in the literature. For instance, a linear quadratic case can be given as a concrete and useful example. For more details of this example, we refer the reader to [15] or [18]. In fact, many applications of FBDSDE either in finance or to stochastic PDEs can be developed in parallel to those provided in the literature.

Our results here can be generalized easily to the case of a stochastic relaxed control problem governed by a nonlinear fully coupled FBDSDE with Poisson jumps, which involves relaxed controls. We refer the reader to Ahmed et al. [1], in this respect.

The paper is organized as follows. Notation and an extended martingale representation theorem are recorded in Sect. 2. Section 3 is devoted to stating the stochastic optimal control problem, which is governed by FBDSDE (1). Existence and uniqueness of the solutions of FBDSDEs are included in Sect. 4. Finally, in Sect. 5 we establish the stochastic maximum principle of our control problem.

2 Notation and an Extended Martingale Representation Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let H_1 and H_2 be two separable Hilbert spaces. Assume that $(W_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ are two cylindrical Wiener processes on H_1 and H_2 respectively, where T is a fixed positive number. Let η be a Poisson point process with values in a measurable space $(\Theta, \mathcal{B}(\Theta))$. We denote by $\nu(d\theta)$ to the characteristic measure of η, which is assumed to be a σ-finite measure on $(\Theta, \mathcal{B}(\Theta))$, by $N(d\theta, dt)$ to the Poisson counting measure induced by η with compensator $\nu(d\theta)dt$, and by $\tilde{N}(d\theta, dt) = N(d\theta, dt) - \nu(d\theta)dt$ to the compensation of the jump measure $N(\cdot, \cdot)$ of η. We assume that the three processes B, W and η are mutually independent.

For each $t \in [0, T]$, define

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^\eta,$$

where

$$\mathcal{F}_t^W := \sigma\{l(W_s) : 0 \le s \le t, l \in H_1^*\} \vee \mathcal{N},$$
$$\mathcal{F}_{t,T}^B := \sigma\{l(B_r) - l(B_t) : t \le r \le T, l \in H_2^*\},$$
$$\mathcal{F}_t^\eta := \sigma\{\eta_s : 0 \le s \le t\} \vee \mathcal{N},$$

and $\mathcal{N}$ is the collection of all $\mathbb{P}$-null sets of $\mathcal{F}$.

Note that $\{\mathcal{F}_t\}_{t\in[0,T]}$ does not constitute a filtration because it is not increasing nor decreasing.

Let us set the following spaces of solutions.

For a separable Hilbert space E, let $\mathcal{M}^2(0,T;E)$ denote the set of jointly measurable processes $\{\mathcal{Y}_t, t \in [0,T]\}$ taking values in E, and satisfy: $\mathcal{Y}_t$ is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$, and

$$\mathbb{E}\Big[\int_0^T |\mathcal{Y}_t|_E^2 \, dt\Big] < \infty.$$

Let $L^2_\nu(E)$ be the set of $\mathcal{B}(\Theta)$-measurable mapping k with values in K such that

$$|||k||| := \Big[\int_\Theta |k(\theta)|_E^2 \, \nu(d\theta)\Big]^{\frac{1}{2}} < \infty.$$

Denote by $\mathcal{V}_\eta^2(0,T;E)$ to the set of processes $\{\mathfrak{K}_t,\ t \in [0,T]\}$ that take their values in $L^2_\nu(K)$ and satisfy: $\mathfrak{K}_t$ is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$, and

$$\mathbb{E}\Big[\int_0^T \int_\Theta |\mathfrak{K}_t(\theta)|_E^2 \, \nu(d\theta)dt\Big] < \infty.$$

Finally, fixing a fixed separable Hilbert space K, we set

$$\begin{aligned}\mathbb{M}^2 := \mathcal{M}^2(0,T;K) \times \mathcal{M}^2(0,T;K) \ &\times \mathcal{M}^2(0,T;L_2(H_2,K)) \\ &\times \mathcal{M}^2(0,T;L_2(H_1,K)) \times \mathcal{V}_\eta^2(0,T;K).\end{aligned}$$

Here $L_2(E,K)$ denotes the space of all Hilbert-Schmidt operators from E into K, for $E = H_1, H_2$, with inner product dented by $||\cdot||$. Then $\mathbb{M}^2$ is a Hilbert space with respect to the norm $\|\cdot\|_{\mathbb{M}^2}$ given, for $\Lambda. = (x., Y., z., Z., \xi.)$, by

$$\begin{aligned}&\|\Lambda.\|^2_{\mathbb{M}^2} \\ &:= \mathbb{E}\Big[\int_0^T |x_t|^2 \, dt + \int_0^T |Y_t|^2 \, dt + \int_0^T \|z_t\|^2 \, dt + \int_0^T \|Z_t\|^2 \, dt + \int_0^T |||\xi_t|||^2 \, dt\Big].\end{aligned}$$

We close this section by providing an extended martingale representation theorem.

Theorem 1. *Let ρ and g be elements of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$ and $\mathcal{M}^2(0,T; L_2(H_1,K))$, respectively. If M is the martingale*

$$M(t) = \mathbb{E}\left[\rho + \int_0^T g(t) \overleftarrow{dB}_t \,|\, \mathcal{E}_t\right],\ 0 \le t \le T,$$

where $\mathcal{E}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B \vee \mathcal{F}_t^\eta$, *then there exist unique elements* (ϕ, κ) *of* $\mathcal{M}^2(0,T; L_2(H_2,K)) \times \mathcal{V}_\eta^2(0,T;K)$ *such that*

$$M(t) = M(0) + \int_0^t \phi_s \, dW_s + \int_0^t \int_\Theta \kappa_s(\theta) \tilde{N}(d\theta, ds).$$

Here the integral with respect to $\overleftarrow{dB}$ is a backward Itô integral, while the integral with respect to dW is a standard forward Itô integral.

This result is known in finite dimensions (i.e. when all Hilbert spaces are Euclidean spaces), as it can be seen easily by combining the well known martingale representation theorem for Brownian motions and Poisson random measure (e.g. see [10, Lemma 4.2] or [8]) and [11, Proposition 1.2]. The proof of this infinite dimensional version can be gleaned by mimicking the ideas of proofs in [11, Proposition 1.2], [2, Theorem 3.1] and [10, Lemma 4.2].

Such a theorem is in fact essential for finding solutions to BDSDEs and decoupled (or coupled) FBDSDEs; see e.g. [9] for using martingale representation theorem to show the existence of solutions to FBSDEs in continuous situations.

3 Statement of the Control Problem

Let $\mathcal{O}$ be a separable Hilbert space and U be a nonempty convex of $\mathcal{O}$. We say that $v_\cdot : [0,T] \times \Omega \to \mathcal{O}$ is *admissible* if $v_\cdot \in \mathcal{M}^2(0,T;\mathcal{O})$ and $v_t \in U$ *a.e.* t, *a.s.* The set of admissible controls will be denoted by $\mathcal{U}_{ad}$. Consider the following controlled K-valued fully coupled FBDSDE with jumps:

$$\begin{cases} dx_t = B(t, x_t, Y_t, z_t, Z_t, \xi_t, v_t)dt + \Sigma(t, x_t, Y_t, z_t, Z_t, \xi_t, v_t)dW_t \\ \qquad\qquad + \int_\Theta \Phi(t, x_t, Y_t, z_t, Z_t, \xi_t, v_t, \theta)\tilde{N}(d\theta, dt) - z_t \overleftarrow{dB}_t, \\ dY_t = -F(t, x_t, Y_t, z_t, Z_t, \xi_t, v_t)dt - G(t, x_t, Y_t, z_t, Z_t, \xi_t, v_t)\overleftarrow{dB}_t \\ \qquad\qquad + Z_t dW_t + \int_\Theta \xi_t(\theta)\tilde{N}(d\theta, dt), \\ x_0 = \pi \in K, Y_T = h(x_T), \ t \in (0,T), \end{cases} \tag{1}$$

where the coefficients

$$\begin{aligned} &B, F : \Omega \times [0,T] \times K \times K \times L_2(H_2;K) \times L_2(H_1;K) \times L_\nu^2(K) \times \mathcal{O} \to K, \\ &\Sigma, G : \Omega \times [0,T] \times K \times K \times L_2(H_2;K) \times L_2(H_1;K) \times L_\nu^2(K) \times \mathcal{O} \to L_2(H_1;K), \\ &\Phi : \Omega \times [0,T] \times K \times K \times L_2(H_2;K) \times L_2(H_1;K) \times L_\nu^2(K) \times \mathcal{O} \times \Theta \to K, \\ &h : \Omega \times K \to K, \end{aligned}$$

are measurable and $v_\cdot \in \mathcal{U}_{ad}$. More conditions will be assumed in Sect. 3. The mapping h is defined, for $(\omega, x) \in \Omega \times K$, by $h(\omega, x) := c\,x + \zeta(\omega)$, where $c \neq 0$ is a constant and ζ is a fixed arbitrary element of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$.

Definition 1. *A solution of (1) is a quintuple* (x, Y, z, Z, ξ) *of stochastic processes such that* (x, Y, z, Z, ξ) *belongs to* $\mathbb{M}^2$ *and satisfies the following two integral equations:*

$$\begin{cases} x_t = \pi + \int_0^t B(s, x_s, Y_s, z_s, Z_s, \xi_s, v_s)ds + \int_0^t \Sigma(s, x_s, Y_s, z_s, Z_s, \xi_s, v_s)dW_s \\ \qquad + \int_0^t \int_\Theta \Phi(s, x_s, Y_s, z_s, Z_s, \xi_s, v_s, \theta)\tilde{N}(d\theta, ds) - \int_0^t z_s \overleftarrow{dB}_s, \\ \\ Y_t = h(x_T) + \int_t^T F(s, x_s, Y_s, z_s, Z_s, \xi_s, v_s)ds \\ \qquad + \int_t^T G(s, x_s, Y_s, z_s, Z_s, \xi_s, v_s)\overleftarrow{dB}_s \\ \qquad - \int_t^T Z_s dW_s - \int_t^T \int_\Theta \xi_s(\theta)\tilde{N}(d\theta, ds), \ t \in [0, T]. \end{cases}$$

In Sect. 4 we shall discuss the existence and uniqueness of (1).

Let us now introduce a *cost functional*:

$$J(v.) := \mathbb{E}\big[\int_0^T \ell(t, x_t, Y_t, z_t, Z_t, \xi_t, v_t)dt + \varphi(x_T) + \psi(Y_0)\big], \ v. \in \mathcal{U}_{ad}, \tag{2}$$

with

$$\varphi, \psi : H \to \mathbb{R},$$
$$\ell : \Omega \times [0, T] \times K \times K \times L_2(H_2; K) \times L_2(H_1; K) \times L^2_\nu(K) \times \mathcal{O} \to \mathbb{R},$$

being measurable functions such that (2) is defined. See assumption (A3) below for precise assumptions.

The control problem of system (1) is to minimize J over $\mathcal{U}_{ad}$. Thus an admissible control $u.$ is called an *optimal control* if

$$J(u.) = \inf_{v. \in \mathcal{U}_{ad}} J(v.). \tag{3}$$

In this case we shall say that $(x, Y, z, Z, \xi, u.)$ is an *optimal solution* of the control problem (1)–(3).

Further details on this control problem will be the main purpose of Sect. 5. We discuss next the existence and uniqueness of (1).

4 Forward-Backward Doubly Stochastic Differential Equations

We shall be interested here in the existence and uniqueness of the solution to FBDSDE (1). Keeping the notations in Sect. 3 denote

$$A(t, X, v) = (-F, B, -G, \Sigma, \Phi)(t, X, v)$$

and

$$\langle A, X \rangle = -\langle x, F \rangle_K + \langle Y, B \rangle_K - \langle z, G \rangle_{L_2(H_2;K)} + \langle Z, \Sigma \rangle_{L_2(H_1;K)} + \langle \xi, \Phi \rangle_{L^2_\nu(K)},$$

for $X = (x, Y, z, Z, \xi) \in K \times K \times L_2(H_2; K) \times L_2(H_1; K) \times L^2_\nu(K)$ and $(t, v) \in [0, T] \times \mathcal{O}$. The following three assumptions on the coefficients of system (1) and (2) are our main assumptions.

(A1) $\forall X = (x, Y, z, Z, \xi), \bar{X} = (\bar{x}, \bar{Y}, \bar{z}, \bar{Z}, \bar{\xi}) \in K \times K \times L_2(H_2; K) \times L_2(H_1; K) \times L^2_\nu(K)$, $\forall t \in [0, T]$, $\forall v \in \mathcal{O}$,

$$\left\langle A(t,X,v) - A(t,\bar{X},v), X - \bar{X} \right\rangle \leq -\lambda(|x-\bar{x}|_K^2 + \left|Y - \bar{Y}\right|_K^2 + \|z - \bar{z}\|^2_{L_2(H_2;K)} + \left\|Z - \bar{Z}\right\|^2_{L_2(H_1;K)} + \left\|\xi - \bar{\xi}\right\|^2_{L^2_\nu(K)}),$$

and

$$c > 0,$$

or

(A1)'

$$\left\langle A(t,X,v) - A(t,\bar{X},v), X - \bar{X} \right\rangle \geq \lambda(|x-\bar{x}|_K^2 + \left|Y - \bar{Y}\right|_K^2 + \|z - \bar{z}\|^2_{L_2(H_2;K)} + \left\|Z - \bar{Z}\right\|^2_{L_2(H_1;K)} + \left\|\xi - \bar{\xi}\right\|^2_{L^2_\nu(K)}),$$

and

$$c < 0,$$

for some $\lambda > 0$.

(A2) For each $X \in K \times K \times L_2(H_2; K) \times L_2(H_1; K) \times L^2_\nu(K)$ and $v \in \mathcal{O}$, we have $A(\cdot, X, v) \in \mathbb{M}^2$.

(A3) We have

$$\begin{cases} (i)\ F, B, G, \varSigma, \varPhi, \ell \text{ are continuously differentiable with respect to } (x, Y, z, Z, \xi), \\ \quad \text{also } \varphi \text{ and } \psi \text{ are continuously differentiable with respect to } x \text{ and } Y, \\ \quad \text{respectively,} \\ (ii) \text{ the derivatives of } F, B, G, \varSigma, \varPhi \text{ with respect to the above arguments are} \\ \quad \text{bounded,} \\ (iii) \text{ the derivatives of } \ell \text{ are bounded by } C(1 + |x| + |Y| + \|z\| + \|Z\| + |||\xi|||), \\ (iv)\ \varphi_x \text{ and } \psi_Y \text{ are bounded by } C\,(1 + |x|) \text{ and } C\,(1 + |Y|), \text{ respectively,} \end{cases}$$

for some constant $C > 0$.

Remark 1. The condition $c > 0$ in (A1) guarantees the following monotonicity condition of the mapping h:

$$\langle h(x) - h(\bar{x}), x - \bar{x} \rangle_K \geq c\,|x - \bar{x}|_K^2, \ \ \forall\, x, \bar{x} \in K.$$

A similar thing happens also for the case $c < 0$ in (A1)'.

Theorem 2. *If (A1)–(A3) (or (A1)', (A2)–(A3)) hold, then there exists a unique solution* (x, Y, z, Z, ξ) *of the FBDSDE (1).*

By making use of the extended martingale representation theorem (Theorem 1), the proof is standard and can be achieved directly by following the outline of the proofs in [13,14]. So we omit it.

This theorem in its infinite dimensional setting is new. In fact, as far as know, this is the first appearance of such an infinite dimensional result as well as Theorem 1.

5 Stochastic Maximum Principle

To derive the maximum principle we define the *Hamiltonian* $\mathcal{H}$ from $[0,T] \times \Omega \times K \times K \times L_2(H_2;K) \times L_2(H_1;K) \times L^2_\nu(K) \times \mathcal{O} \times K \times K \times L_2(H_2;K) \times L_2(H_1;K) \times L^2_\nu(K)$ to $\mathbb{R}$ by the formula:

$$\begin{aligned}\mathcal{H}(t,x,Y,z,Z,\xi,v,p,P,q,Q,\Upsilon) &:= -\left\langle p, F(t,x,Y,z,Z,\xi,v)\right\rangle_K \\ &+ \left\langle P, B(t,x,Y,z,Z,\xi,v)\right\rangle_K - \left\langle q, G(t,x,Y,z,Z,\xi,v)\right\rangle_{L_2(H_2;K)} \\ &+ \left\langle Q, \Sigma(t,x,Y,z,Z,\xi,v)\right\rangle_{L_2(H_1;K)} + \ell(t,x,Y,z,Z,\xi,v) \\ &+ \int_\Theta \left\langle \Upsilon(\widehat{\theta}), \Phi(t,x,Y,z,Z,\xi,v,\widehat{\theta}) \right\rangle_{L^2_\nu(K)} \nu(d\widehat{\theta}). \end{aligned} \tag{4}$$

Theorem 3. *Let $v.$ be an arbitrary element of $\mathcal{U}_{ad}$. Assume (A1)–(A3). Let $\{(y_t, Y_t, z_t, Z_t, k_t),\ t \in [0,T]\}$ be the corresponding solution of (1). Then there exists a unique solution (p, P, q, Q, Υ) of the following adjoint equations of (1):*

$$\begin{cases} dp_t = -\mathcal{H}_Y(t)dt - \mathcal{H}_Z dW_t - q_t \overleftarrow{dB}_t - \int_\Theta \mathcal{H}_\xi(t)\tilde{N}(d\theta,dt), \\ dP_t = -\mathcal{H}_x(t)dt - \mathcal{H}_z(t)\overleftarrow{dB}_t + Q_t dW_t + \int_\Theta \Upsilon_t(\theta)\tilde{N}(d\theta,dt), \\ p_0 = -\psi_Y(Y_0), P_T = -\,c\,p_T + \varphi_x(x_T), \end{cases} \tag{5}$$

where $\mathcal{H}_x(t)$ is the gradient $\nabla_x \mathcal{H}(t,x,Y_t,z_t,Z_t,\xi_t,v_t,p_t,P_t,q_t,Q_t,\Upsilon_t) \in K, \ldots$ etc.

Proof. Thanks to assumptions (A1)–(A3) this linear FBDSDE satisfy (A1)', (A2) and (A3). In fact the monotonicity condition follows from the definition of Gâteaux derivatives (as limits) and the fact that the corresponding mappings satisfy originally the monotonicity condition in (A1). Hence the result follows from Theorem 2.

We are now ready to state the stochastic maximum principle for the optimal control problem (1)–(3).

Theorem 4. *Suppose that (A1)–(A3) hold. Given $u. \in \mathcal{U}_{ad}$, let $(x^{u.}, Y^{u.}, z^{u.}, Z^{u.}, \xi^{u.})$ and $(p^{u.}, P^{u.}, q^{u.}, Q^{u.}, \Upsilon^{u.})$ be the corresponding solutions of FBDSDEs (1) and (5), respectively. Assume that the following assumptions hold.*
(i) φ and ψ are convex;
(ii) For all $t \in [0,T]$, $\mathbb{P}$ - a.s., the function $\mathcal{H}(t,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,p^{u.},P^{u.},q^{u.},Q^{u.},\Upsilon^{u.})$ is convex;
(iii) We have

$$\begin{aligned}&\mathcal{H}(t,x^{u.}_t,Y^{u.}_t,z^{u.}_t,Z^{u.}_t,\xi^{u.}_t,u_t,p^{u.}_t,P^{u.}_t,q^{u.}_t,Q^{u.}_t,\Upsilon^{u.}_t) \\ &\qquad = \inf_{v\in U} \mathcal{H}(t,x^{u.}_t,Y^{u.}_t,z^{u.}_t,Z^{u.}_t,\xi^{u.}_t,v,p^{u.}_t,P^{u.}_t,q^{u.}_t,Q^{u.}_t,\Upsilon^{u.}_t), \end{aligned} \tag{6}$$

for a.e. t, $\mathbb{P}$ - a.s.

Then $(x^{u.}, Y^{u.}, z^{u.}, Z^{u.}, \xi^{u.}, u.)$ is an optimal solution of the control problem (1)–(3).

Proof. Let $v.$ be an arbitrary element of $\mathcal{U}_{ad}$. With the help of assumptions (A1)–(A3) let, by using of Theorem 2, $(x^{v.}, Y^{v.}, z^{v.}, Z^{v.}, \xi^{v.})$ be the corresponding solution of FBDSDE (1). Applying (2), the convexity of φ and ψ, the adjoint equations (5) and system (1) it follows that

$$J(v.) - J(u.) \geq \mathbb{E}\big[\, \langle P_T^{u.} + c\, p_T^{u.}, x_T^{v.} - x_T^{u.} \rangle \,\big] - \mathbb{E}\big[\, \langle p_0^{u.}, Y_0^{v.} - Y_0^{u.} \rangle \,\big]$$
$$+\mathbb{E}\big[\int_0^T \big(\ell(t, x_t^{v.}, Y_t^{v.}, z_t^{v.}, Z_t^{v.}, \xi_t^{v.}, v_t) - \ell(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t) \big) dt \big]. \quad (7)$$

Next, by applying a suitable Itô's formula for infinite dimensional SDEs driven by Wiener processes on Hilbert spaces and Poisson measures to compute $\langle p_t^{u.}, Y_t^{v.} - Y_t^{u.} \rangle_K$ and $\langle P_t^{u.}, x_t^{v.} - x_t^{u.} \rangle_K$, we derive with the help of assumptions (A2) and (A3) that

$$\mathbb{E}\big[\, \langle P_T^{u.}, x^{v.} - x_T^{u.} \rangle \,\big] - \mathbb{E}\big[\, \langle p_0^{u.}, Y_0^{v.} - Y_0^{u.} \rangle \,\big] = -\mathbb{E}\big[\, \langle p_T^{u.}, Y_T^{v.} - Y_T^{u.} \rangle \,\big]$$
$$- \mathbb{E}\big[\int_0^T \langle p_t^{u.}, F(t, x_t^{v.}, Y_t^{v.}, z_t^{v.}, Z_t^{v.}, \xi_t^{v.}, v_t) - F(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t) \rangle \, dt \big]$$
$$+ \mathbb{E}\,\big[\int_0^T \langle \mathcal{H}_Y(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.}, \Upsilon_t^{u.}), Y_t^{v.} - Y_t^{u.} \rangle \, dt \big]$$
$$- \mathbb{E}\,\big[\int_0^T \langle q_t^{u.}, G(t, x_t^{v.}, Y_t^{v.}, z_t^{v.}, Z_t^{v.}, \xi_t^{v.}, v_t) - G(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t) \rangle \, dt \big]$$
$$+ \mathbb{E}\,\big[\int_0^T \langle \mathcal{H}_Z(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.}, \Upsilon_t^{u.}), Z_t^{v.} - Z_t^{u.} \rangle \, dt \big]$$
$$+ \mathbb{E}\,\big[\int_0^T \int_\Theta \langle \mathcal{H}_\xi(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.}, \Upsilon_t^{u.}),$$
$$\xi_t^{v.}(\theta) - \xi_t^{u.}(\theta) \rangle \nu(d\theta) dt \big]$$
$$+ \mathbb{E}\big[\int_0^T \langle P_t^{u.}, B(t, x_t^{v.}, Y_t^{v.}, z_t^{v.}, Z_t^{v.}, \xi_t^{v.}, v_t)$$
$$- B(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t) \rangle dt \big]$$
$$+ \mathbb{E}\big[\int_0^T \langle \mathcal{H}_x(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.}, \Upsilon_t^{u.}), x_t^{v.} - x_t^{u.} \rangle \, dt \big]$$
$$+ \mathbb{E}\big[\int_0^T \langle \mathcal{H}_z(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.}, \Upsilon_t^{u.}), z_t^{v.} - z_t^{u.} \rangle \, dt \big]$$
$$+ \mathbb{E}\big[\int_0^T \langle Q_t^{u.}, \Sigma(t, x_t^{v.}, Y_t^{v.}, z_t^{v.}, Z_t^{v.}, \xi_t^{v.}, v_t) - \Sigma(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t) \rangle \, dt \big]$$
$$+ \mathbb{E}\big[\int_0^T \int_\Theta \langle \Upsilon_t^{u.}(\theta), \Phi(t, x_t^{v.}, Y_t^{v.}, z_t^{v.}, Z_t^{v.}, \xi_t^{v.}, v_t, \theta)$$
$$- \Phi(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, \theta) \rangle \nu(d\theta) dt \big]. \quad (8)$$

On the hand, from the formula $h(\omega, x) := c\, x + \xi(\omega), x \in K$, one gets easily the cancelation:

$$\mathbb{E}\big[\, \langle c\, p_T^{u.}, x_T^{v.} - x_T^{u.} \rangle \,\big] - \mathbb{E}\big[\, \langle p_T^{u.}, Y_T^{v.} - Y_T^{u.} \rangle \,\big] = 0. \tag{9}$$

Therefore, by applying (8) and (9) in (7), using the formula of $\mathcal{H}$ in (4) and then the convexity of $\mathcal{H}$ in condition (ii) we obtain

$$\begin{aligned} &J(v.) - J(u.) \\ &\geq \mathbb{E}\big[\int_0^T \langle \mathcal{H}_v(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.}, \Upsilon_t^{u.}), v_t - u_t \rangle_{\mathcal{O}}\, dt \big]. \end{aligned} \tag{10}$$

But the minimum condition (iii) yields

$$\langle \mathcal{H}_v(t, x_t^{u.}, Y_t^{u.}, z_t^{u.}, Z_t^{u.}, \xi_t^{u.}, u_t, p_t^{u.}, P_t^{u.}, q_t^{u.}, Q_t^{u.}, \Upsilon_t^{u.}), v_t - u_t \rangle_{\mathcal{O}} \geq 0.$$

Consequently (10) becomes

$$J(v.) - J(u.) \geq 0.$$

Since $v.$ is an arbitrary element of $\mathcal{U}_{ad}$, then $u.$ is an optimal control, and so the proof is complete.

Remark 2. Condition (A1) assumed in Theorem 4 is only needed to guarantee the existence and uniqueness of the solutions of (1) and (5), and so if one can get such solutions without assuming (A1) there will not any necessity to assume it in advance in this theorem.

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