Chapter 2
Harmonic Oscillator

In scanning probe microscopy, vibrations play a central role in several areas. If, for instance, a scanning tunneling microscope is rests on a table you might wonder what this has to do with vibrations. However, floor vibrations with amplitudes of roughly one tenth of a micrometer (100 nm) have to be compared to an amplitude stability of less than 0.01 nm which is necessary for atomically resolved imaging in STM. Thus the vibrational noise amplitude is about 10,000 times larger than the signal to be measured. This means that knowledge about vibrations and vibration isolation is essential for scanning probe methods. Another area where oscillations are an important topic is atomic force microscopy. In the dynamical mode of atomic force microscopy, a cantilever vibrating close to (or at) its resonance frequency is used as a force detector. The simplest way to study vibrations is to study the harmonic oscillator. In this chapter we will study the mechanical harmonic oscillator.

2.1 Free Harmonic Oscillator

The simplest example of a harmonic oscillator is a mass on a spring (Fig. 2.1). The position to which gravity extends the spring in equilibrium is chosen as the point of zero extension. The displacement relative to this point is called z. The force exerted by the spring on the mass m during the oscillation is given by Hooke's law as

$$F = -kz, \tag{2.1}$$

with k being the spring constant. If the spring deflection has negative values ($z < 0$, longer spring extension), the direction of the force is positive and vice versa. Thus the minus sign in (2.1) appears because the force exerted by the spring has a direction opposite to the deflection z. Newton's second law tells us that the equation of motion for the mass m is

$$ma = m\frac{d^2z}{dt^2} = m\ddot{z} = F = -kz. \tag{2.2}$$

B. Voigtländer, *Scanning Probe Microscopy*,
NanoScience and Technology, DOI 10.1007/978-3-662-45240-0_2

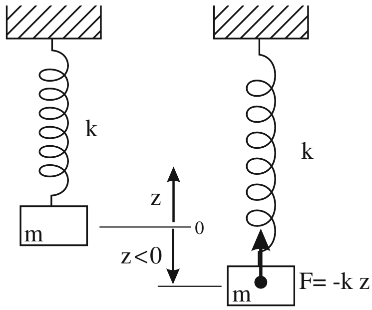

Fig. 2.1 The simplest example of a harmonic oscillator: a mass on a spring

An ansatz for the solution of the equation of motion (2.2) is $z = \cos(\omega_0 t)$ with ω_0 being a parameter which has to be determined.[1] We verify that this is a correct solution by differentiating z two times:

$$\frac{dz}{dt} = -\omega_0 \sin(\omega_0 t); \quad \frac{d^2 z}{dt^2} = -\omega_0^2 \cos(\omega_0 t). \tag{2.3}$$

Formally (2.2) is solved if $\omega_0 = \sqrt{\frac{k}{m}}$. But what is the physical significance of ω_0? We know that the cosine function repeats itself if the argument is larger than 2π. Therefore, the mass makes one complete cycle of oscillation if $\omega_0 t = 2\pi$. This time, we call the period of the oscillation T, and ω_0 is given by

$$\omega_0 = 2\pi / T. \tag{2.4}$$

The angular frequency ω_0 is the number of radians through which the oscillation proceeds per time, while the frequency f_0 is the number of oscillations per time ($\omega_0 = 2\pi f_0$). If the mass is larger it takes a longer time for one oscillation and if the spring constant is stronger the mass will move more quickly. Note that the period of oscillation (and also ω_0) does not depend on how far we stretch the spring at the beginning. Any solution multiplied by a constant factor is still a solution of (2.2).

We have found a solution to the equation of motion. But is this the only one or are there more solutions? Also the sine function provides a valid solution. The most general solution is a linear combination of a sine and a cosine function

$$z = A\cos(\omega_0 t) + B\sin(\omega_0 t). \tag{2.5}$$

There is a more intuitive way to find the general solution. When we used the cosine function as solution, the oscillation started with the maximum extension at time zero. However, alternatively also any other time during the oscillation could be chosen as the start of the oscillation. This shift of the time corresponds to a shift of the phase of the oscillation (the argument of the cosine function is called phase) by a constant

[1] The argument of the cosine is named the phase. The phase increases linearly with time if ω_0 is constant.

phase shift ϕ. Thus all solutions are captured if the solution is shifted by a constant (but arbitrary) phase shift ϕ, and the general solution results as

$$z = a\cos(\omega_0 t + \phi). \tag{2.6}$$

The two solutions given in (2.5) and (2.6) are in fact equivalent. Using the mathematical identity

$$\cos(\alpha + \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta, \tag{2.7}$$

the following relations between A, B in (2.5) and a, ϕ in (2.6) are obtained

$$B = a\cos\phi, \quad A = a\sin\phi. \tag{2.8}$$

Moreover, the solutions given in (2.5) and (2.6) are the general solution to the equation of motion. There are no other solutions.

In the general solution of the equation of motion, we introduced two more constants: A and B, or a and ϕ, respectively. How are these constants determined? They are determined by the initial conditions of the motion. For instance if we start the motion from a static extension z_0, B and ϕ are zero. Now we determine these constants for the most general initial condition: z_0, v_0. The acceleration $a(t)$ cannot be specified as an initial condition. It is given by the spring constant, mass and $z(t)$ according to (2.2). We use the form for the general solution given in (2.5) and its derivative

$$v(t) = -\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t). \tag{2.9}$$

These equations are valid for all times, but we know z and v at time $t = 0$. If we insert $t = 0$ we obtain

$$z_0 = A + B \cdot 0 = A \quad v_0 = -\omega_0 A \cdot 0 + \omega_0 B = \omega_0 B. \tag{2.10}$$

We therefore find that the constants A and B can be determined by the initial conditions as

$$A = z_0 \quad \text{and} \quad B = v_0/\omega_0. \tag{2.11}$$

2.2 Driven Harmonic Oscillator

In dynamic atomic force microscopy, we will consider a cantilever which is exited, driven or moved with a sinusoidal external excitation amplitude. The simplest model for this is a harmonic oscillator in which the upper fixing point of the spring (cf. Fig. 2.1) is oscillated (excited) sinusoidally with $z_{\text{drive}}(t) = A_{\text{drive}}\cos(\omega_{\text{drive}} t)$. The resulting force on the mass m is then $F = -k(z - z_{\text{drive}})$. The equation of motion results as

$$ma = m\ddot{z} = -k(z - z_{\text{drive}}). \tag{2.12}$$

The driving frequency ω_{drive} can be different from the natural frequency of the oscillator ω_0. The question arises at which frequency the driven harmonic oscillator will oscillate. At its natural frequency ω_0, at the driving frequency ω_{drive}, or at some value in between? It turns out that the driven harmonic oscillator will oscillate in the steady-state at the driving frequency ω_{drive}. One special solution for the equation of motion is

$$z(t) = A\cos(\omega_{drive} t). \tag{2.13}$$

Inserting this ansatz into the equation of motion (2.12) results in

$$-m\omega_{drive}^2 A\cos(\omega_{drive} t) = -m\omega_0^2 A\cos(\omega_{drive} t) + kA_{drive}\cos(\omega_{drive} t). \tag{2.14}$$

We find that $z = A\cos(\omega_{drive} t)$ is a solution of the equation of motion if

$$A = \frac{kA_{drive}}{m(\omega_0^2 - \omega_{drive}^2)}. \tag{2.15}$$

The special solution (2.13) means that m oscillates at the driving frequency with an amplitude which depends on the driving frequency and also on the natural frequency of the oscillator. If $\omega_{drive} < \omega_0$ then displacement and driving excitation are in the same direction. If $\omega_{drive} > \omega_0$ then A becomes negative. This is equivalent to a positive amplitude and a phase shift of $-180°$ of the oscillation $z(t)$ relative to the driving excitation. The amplitude and phase for an undamped driven harmonic oscillator are shown in (Fig. 2.2). If $\omega_{drive} \ll \omega_0$ the amplitude A approaches the

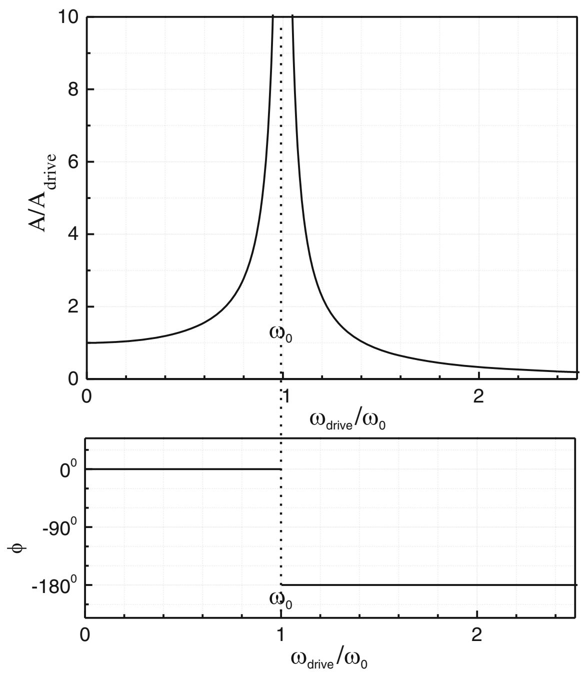

Fig. 2.2 Amplitude and phase of an undamped driven harmonic oscillator as a function of ω_{drive} showing a resonance at ω_0

excitation amplitude A_{drive}. If $\omega_{\text{drive}} \gg \omega_0$ the amplitude approaches zero because the mass can no longer follow the high frequency of the driving excitation.

As can be seen in Fig. 2.2 the amplitude A approaches infinity if ω_{drive} approaches ω_0. We will see in the next section that damping of the harmonic oscillator prevents this "resonance catastrophe".

2.3 Driven Harmonic Oscillator with Damping

Including damping to the driven harmonic oscillator is a more realistic case which we consider in the following. An additional friction term has to be included to the equation of motion (2.12). We consider this term as proportional to the speed at which the oscillating mass moves $F_{\text{frict}} = m\gamma\dot{z}$. Also here we assume an external exciting amplitude $z_{\text{drive}}(t) = A_{\text{drive}}\cos(\omega t)$. Here and in the following we replaced $\omega_{\text{drive}} \equiv \omega$, in order to have a simpler notation. The spring force acting on the oscillating mass is again proportional to the difference between the position of the mass z and the excitation amplitude z_{drive} as $F = -k(z - z_{\text{drive}})$. With this the equation of motion reads

$$m\ddot{z} = -m\gamma\dot{z} - k(z - z_{\text{drive}}). \tag{2.16}$$

Replacing $\omega_0^2 = k/m$ results in

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \omega_0^2 z_{\text{drive}}. \tag{2.17}$$

Solving this equation would be quite difficult without the use of complex numbers. The trick here is to consider z and z_{drive} as complex numbers ($\tilde{z}$ and $\tilde{z}_{\text{drive}}$) and find the complex solution for the differential equation. Since the physical quantities are real and the differential equation is linear, at the end only the real part of $\tilde{z}$ is our solution. The amplitudes are regarded as complex numbers as

$$\tilde{z} = Ae^{i(\omega t+\phi)} = Ae^{i\phi}e^{i\omega t} = \hat{z}e^{i\omega t} \quad \text{and} \quad \tilde{z}_{\text{drive}} = A_{\text{drive}}e^{i\omega t}. \tag{2.18}$$

Without loss of generality we set the phase shift of the excitation amplitude z_{drive} to zero, i.e. A_{drive} is real, while $\hat{z}$ is regarded as a complex number with a (real) phase shift ϕ and (real) oscillation amplitude A as, $\hat{z} = Ae^{i\phi}$. The real part of $\tilde{z}$ will later be the real solution for the motion of the mass m. The nice thing about the complex notation is that differentiation of $\tilde{z}$ is now just multiplication with $i\omega$ ($\frac{d\tilde{z}}{dt} = \hat{z}i\omega e^{i\omega t} = i\omega\tilde{z}$). This means differentiation in (2.17) (with $z \rightarrow \tilde{z}$) can be easily executed and this differential equation converts to the simple algebraic equation

$$\left[(i\omega)^2\hat{z} + \gamma(i\omega)\hat{z} + \omega_0^2\hat{z}\right]e^{i\omega t} = \omega_0^2 A_{\text{drive}}e^{i\omega t}. \tag{2.19}$$

After dividing both sides by $e^{i\omega t}$, we obtain the complex solution

$$\hat{z} = \frac{\omega_0^2 A_{\text{drive}}}{\omega_0^2 - \omega^2 + i\gamma\omega}. \tag{2.20}$$

Now the real z is the real part of the complex quantity $\tilde{z}$ as

$$z = Re(\tilde{z}) = Re(\hat{z}e^{i\omega t}) = Re(Ae^{i(\omega t+\phi)}). \tag{2.21}$$

Since A and ϕ are real, the resulting real position z reads

$$z = A\cos(\omega t + \phi), \tag{2.22}$$

with the amplitude A and phase shift ϕ between excitation amplitude and oscillation amplitude.

In order to calculate A we recall that $\hat{z} = Ae^{i\phi}$. Therefore, $\hat{z}\hat{z}^* = A^2$ and A^2 can be written as

$$A^2 = \frac{\omega_0^4 A_{\text{drive}}^2}{\left(\omega_0^2 - \omega^2 + i\gamma\omega\right)\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)} = \frac{\omega_0^4 A_{\text{drive}}^2}{\left(\omega^2 - {\omega_0}^2\right)^2 + \gamma^2\omega^2}. \tag{2.23}$$

Now we introduce as a convenient abbreviation the quality factor $Q = \omega_0/\gamma$. The physical significance of the quality factor will be elucidated later. This replacement results in

$$A^2 = \frac{\omega_0^4 A_{\text{drive}}^2}{\left(\omega^2 - \omega_0^2\right)^2 + \frac{\omega_0^2\omega^2}{Q^2}}. \tag{2.24}$$

Furthermore, the oscillation amplitude A can be written as a function of the normalized frequency ω/ω_0 as

$$A^2 = \frac{A_{\text{drive}}^2}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{1}{Q^2}\frac{\omega^2}{\omega_0^2}}. \tag{2.25}$$

The phase ϕ of the oscillation relative to the excitation can be obtained as follows. In general the phase φ of a complex number $x = re^{i\varphi}$ can be obtained from the relation $\tan\varphi = \frac{Im(x)}{Re(x)}$. In order to calculate the phase ϕ, we recall that $\hat{z} = Ae^{i\phi}$. However, according to (2.20) the real and imaginary parts of $1/\hat{z}$ are much easier to find. Therefore, we write

$$\frac{1}{\hat{z}} = \frac{1}{Ae^{i\phi}} = \frac{1}{A}e^{-i\phi} = \frac{1}{\omega_0^2 A_{\text{drive}}}\left(\omega_0^2 - \omega^2 + i\gamma\omega\right). \tag{2.26}$$

Using the fact that $\tan(-\phi) = -\tan\phi$, we see that

$$\tan\phi = \frac{-\gamma\omega}{\omega_0^2 - \omega^2} = \frac{-\omega_0\omega}{Q\left(\omega_0^2 - \omega^2\right)}. \tag{2.27}$$

Also the phase ϕ can be written as function of the normalized frequency ω/ω_0 as

$$\tan\phi = \frac{-\frac{\omega}{\omega_0}}{Q\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]}. \tag{2.28}$$

With these results, the amplitude (2.25) and phase (2.28) in the solution (2.22) are calculated as a function of given variables. The resonance curve in Fig. 2.3 shows the amplitude and the phase of a driven damped harmonic oscillator. For small driving frequencies $\omega \ll \omega_0$, the motion of the oscillator mass just follows the outer excitation with a phase approaching zero; i.e. the oscillation is in phase with the excitation. On

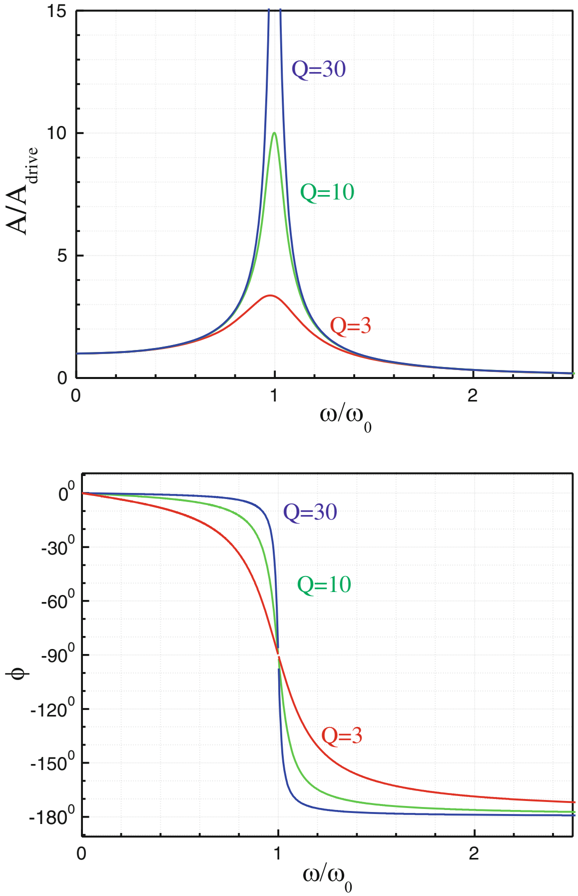

Fig. 2.3 Amplitude and phase of a damped driven harmonic oscillator as a function of $\omega \equiv \omega_{\text{drive}}$, for different values of damping $Q = \omega_0/\gamma$

the other hand for very large frequencies $\omega \gg \omega_0$, the amplitude A approaches zero. In this case the phase approaches $-180°$, i.e. the motion of the oscillator mass is always in opposite to the excitation.

If we take the limit $\omega \gg \omega_0$ in (2.25) we find that the amplitude is proportional to $1/\omega^2$ for small damping, i.e. $\gamma \ll \omega_0$ or $Q \gg 1$. As seen in Fig. 2.3, the smaller the damping, the higher the maximum amplitude is. For small damping the maximum of the resonance curve is very close to the resonance frequency of the free harmonic oscillator ω_0. At any driving frequency the phase is smaller than zero, which means that the oscillator displacement z always lags behind the driving excitation (Fig. 2.3). The phase at resonance ($\omega = \omega_0$) is $-90°$, while it approaches $-180°$ for large driving frequencies.

The amplitude at the resonance frequency $A(\omega_0)$ can be obtained using (2.25) as

$$A(\omega_0) = QA_{\mathrm{drive}}, \tag{2.29}$$

i.e. the amplitude at resonance is Q times higher than the excitation amplitude. For the case of cantilevers in atomic force microscopy this resonance enhancement of the excitation amplitude can be quite high. Due to damping in air, Q-factors of 500 are usual for cantilevers in air. In vacuum, quality factors higher than 10,000 can be reached.

For the case that the oscillation frequency is very close to ω_0, i.e. $\omega \approx \omega_0$, the expression for the resonance curve (2.25) can be approximated as

$$A^2 = \frac{A^2_{\mathrm{drive}}}{\left[\left(1+\frac{\omega}{\omega_0}\right)\left(1-\frac{\omega}{\omega_0}\right)\right]^2 + \frac{1}{Q^2}\frac{\omega^2}{\omega_0^2}} \approx \frac{A^2_{\mathrm{drive}}}{4\left(1-\frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}. \tag{2.30}$$

In order to obtain this we used the approximations $1 + \frac{\omega}{\omega_0} \approx 2$ and $\frac{\omega^2}{\omega_0^2} \approx 1$, which hold if $\omega \approx \omega_0$.

An important quantity is the width of the resonance curve. Therefore, we calculate in the following the frequency $\omega_{1/2}$ at which the amplitude of the oscillation decreases to $1/\sqrt{2}$ of its value[2] at ω_0. This condition for the amplitudes can be written as

$$A_{1/2}(\omega_{1/2}) = \frac{1}{\sqrt{2}}A(\omega_0) = \frac{1}{\sqrt{2}}QA_{\mathrm{drive}}. \tag{2.31}$$

If we insert $\omega = \omega_{1/2}$ in expression (2.30), the following relation results

$$\frac{1}{2}A^2_{1/2}(\omega_{1/2}) \approx \frac{A^2_{\mathrm{drive}}}{4\left(1-\frac{\omega_{1/2}}{\omega_0}\right)^2 + \frac{1}{Q^2}} \approx \frac{1}{2}Q^2A^2_{\mathrm{drive}}. \tag{2.32}$$

---

[2] We use the decrease of the amplitude to $1/\sqrt{2}$ instead of $1/2$, because in this case the energy in the harmonic oscillator, which is proportional to the square of the amplitude, decreases to one half of its maximum value.

Solving this expression for $\omega_{1/2} - \omega_0$ results in $\omega_{1/2} - \omega_0 \approx \frac{1}{2}\frac{\omega_0}{Q}$. Since the full width of the resonance curve is twice this, we obtain

$$\Delta\omega_{1/2} \approx \frac{\omega_0}{Q}. \tag{2.33}$$

This means the larger the Q-factor, the narrower the resonance is.

The maximum of the resonance amplitude, which we determine in the following, lies at a slightly lower frequency than ω_0. The maximum of the resonance curve occurs at the frequency at which the denominator in (2.25) becomes minimal. Differentiating the denominator of (2.25) with respect to ω/ω_0, and equating this derivative to zero results in the following expression for the frequency $\omega_{\max}$ at which the resonance curve has its maximum

$$\omega_{\max}^2 = \omega_0^2\left(1 - \frac{1}{2Q^2}\right). \tag{2.34}$$

The corresponding shift of the resonance curve to lower frequencies results as

$$\delta\omega = \omega_0 - \omega_{\max} = \omega_0\left(1 - \sqrt{1 - \frac{1}{2Q^2}}\right). \tag{2.35}$$

For the case of an AFM cantilever considered as a harmonic oscillator we estimate some values for this frequency shift of the resonance curve due to the damping Q of the cantilever. For a resonance frequency of $\omega_0 = 300$ kHz and quality factors of $Q = 10{,}000$ and $Q = 300$, a frequency shift of 0.8 mHz and 0.8 Hz results, respectively. These are very small values and correspondingly in most cases we will neglect this small shift and consider the maximum of the amplitude to be located at ω_0, unless the quality factor is very low.

2.4 Transients of Oscillations

The solution for the damped driven harmonic oscillator (2.22) is the so called steady-state solution after transients due to the initial conditions have died out. An example for a transient is an oscillation which starts from rest. The amplitude is initially zero, builds up after the excitation starts, and reaches the steady-state amplitude in the limit of large times. The steady-state solution (2.22) does not contain such transients arising from specific initial conditions.

It can be shown that the general solution of the driven damped harmonic oscillator is the specific solution (2.22) plus a solution of the corresponding homogeneous problem. The corresponding homogeneous problem is the damped harmonic oscillator without external driving. Here we do not derive the solution for the damped oscillator without driving but it should be remembered that this is (for small damping) an exponentially decaying oscillation $z_{\text{hom}} = G \exp(-\omega_0/(2Q)t)\cos(\omega_{\text{hom}}t - \phi)$ with the oscillation frequency ω_{hom} being slightly lower than the natural frequency ω_0 of the free harmonic oscillator $\omega_{\text{hom}} = \omega_0\sqrt{1 - 1/(4Q^2)}$ and with G and ϕ as coefficients specified by the initial conditions.

If we call the specific solution z in (2.22) z_s, the general solution for the driven, damped harmonic oscillator is given as $z_{\text{general}} = z_{\text{hom}} + z_s$. It is necessary to include the solution of the damped harmonic oscillator without external driving z_{hom} since it can describe the transients which are not described by z_s. All aspects of z_s are specified in terms of the driving frequency, the driving amplitude, and the phase shift. Yet we still need some way to impose the constraints given by the initial conditions $z(0)$ and $v(0)$ in the general solution. The two coefficients G and ϕ give the freedom to match the general solution to $z(0)$ and $v(0)$.

As an example we consider as initial condition that the oscillation starts from rest. In Fig. 2.4 the general solution for the initial condition: starting from rest, is shown to be composed of the specific solution of the inhomogeneous system (Fig. 2.4a) plus the solution for the homogeneous system (transient) z_{hom} (Fig. 2.4b). In Fig. 2.4c the sum of both is shown for the case that $\omega = \omega_{\text{hom}}$. The specific solution in Fig. 2.4a is approached within the decay time for the homogeneous solution Fig. 2.4b. The fact that the situation is not always simple is shown in Fig. 2.4d. Here the driving frequency deviates from ω_{hom}, which leads to a beating behavior before a steady-state solution is reached.

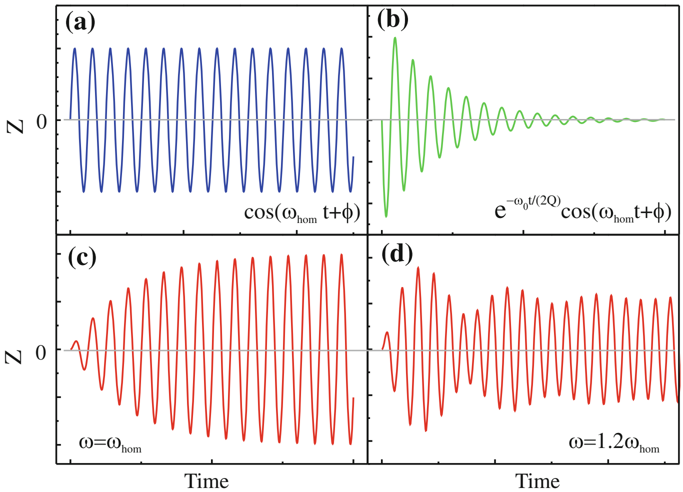

Fig. 2.4 The general solution for a damped driven harmonic oscillator is composed of the specific solution of the inhomogeneous driven system (steady-state solution), shown in (**a**) plus the solution of the homogeneous system without driving (transient), shown in (**b**). The initial conditions are chosen such that the general solution satisfies the given initial conditions (start from rest in this example). **c** and **d** show two examples of general solutions (for two different driving frequencies) starting from rest and approaching the steady-state solution for long times

If the driven damped oscillator is oscillating in steady-state (Fig. 2.4a) and the driving amplitude is stopped suddenly, the problem is converted to a homogeneous one and the oscillator will de-excite as shown in Fig. 2.4b. This is a sinusoidal oscillation with the envelope decreasing as $\exp(-\omega_0/(2Q)t)$. This means that after a time $\tau = 2Q/\omega_0 = TQ/\pi$ the amplitude has decreased by $1/e$. This characteristic time is called ring down time and increases with smaller damping. The same time is needed to build up the steady-state oscillation amplitude after a start from rest.

This time can be expressed in terms of the Q-factor as $\tau = 2Q/\omega_0 = TQ/\pi$. This means that the oscillation builds up (decays) within roughly Q oscillation cycles and Q can be expressed as

$$Q = \frac{1}{2}\tau\omega_0. \tag{2.36}$$

2.5 Dissipation and Quality Factor of a Damped Driven Harmonic Oscillator

When the mass is initially at rest and an external oscillatory excitation is applied, energy is successively stored in the oscillator with the buildup of the oscillation (transient). If the oscillator is finally in a steady-state, the energy stored in the oscillator is constant and all the energy supplied by the external force ends (on average) up in the dissipative term. The instantaneous power dissipated is $F_{\text{frict}}v = \gamma m v^2$ and varies over one period, as v varies. The mean power consumed by the oscillator in steady-state can be written as

$$\langle P\rangle = \langle F_{\text{frict}}v\rangle = \gamma m\langle v^2\rangle. \tag{2.37}$$

The brackets indicate an averaging over one oscillation period. Since $z = A\cos(\omega t + \phi)$, differentiation results in $v^2 = \omega^2 A^2 \sin^2(\omega t + \phi)$. If $\sin^2$ is averaged over one period a factor of one half results. Therefore, the average power results in

$$\langle P\rangle = \gamma m\langle v^2\rangle = \frac{1}{2}\gamma m\omega^2 A^2. \tag{2.38}$$

With this the energy dissipated per cycle is

$$\text{Energy dissipated per cycle} = \langle P\rangle T = \langle P\rangle 2\pi/\omega = \pi\gamma m\omega A^2. \tag{2.39}$$

Another important quantity is the total energy stored in the oscillator. If we consider driving frequencies close to ω_0, the energy stored in the driven oscillator is approximately the energy of the free oscillator with the same amplitude A

$$\langle E\rangle \approx \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 A^2. \tag{2.40}$$

The efficiency of an oscillator is defined by how much energy is stored, compared with how much work is supplied (dissipated) by the external force per oscillation cycle. This is called the quality factor of the oscillator and is defined by 2π times the mean energy stored, divided by the energy dissipated per cycle

$$Q = 2\pi \times \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated per cycle}}. \tag{2.41}$$

Close to the resonance frequency ($\omega \approx \omega_0$), Q can be written using (2.39) and (2.40) as

$$Q \approx \frac{\omega_0}{\gamma}. \tag{2.42}$$

This is consistent with the abbreviation for Q introduced in the previous section.

2.6 Effective Mass of a Harmonic Oscillator

In this chapter, we always considered an idealized system consisting of a mass-less spring and a mass m at its end. However, in some cases of practical relevance this approximation is not fulfilled. For instance, in the case of a cantilever-type spring, often used in atomic force microscopy, the mass (of the cantilever) is distributed throughout the whole cantilever (Fig. 2.5b). We introduce the concept of the effective mass for the example of a coil spring (with mass m_{spring}) and assume that the mass

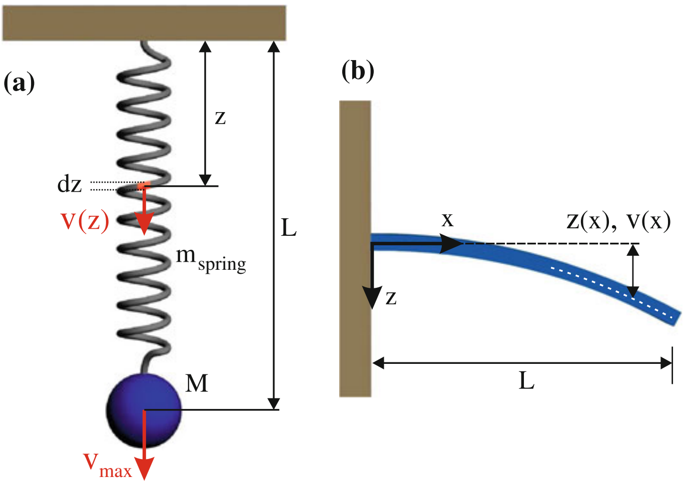

Fig. 2.5 **a** For a spring with mass m_{spring}, the velocity of a volume element depends on the position, i.e. $v = v(z)$. The effective mass turns out to be $1/3$ of the spring mass. **b** For a cantilever beam the deflection and the velocity are non-linear as a function of x

is distributed homogeneously along its length. In the following, we calculate the maximum kinetic energy (which corresponds to the total energy) of the spring with a mass and we do not consider a mass M at the end of the spring.

When calculating the (maximum) kinetic energy of the spring, we regard $v(z)$ as the maximum velocity during one oscillation cycle. The (maximum) kinetic energy of a length element dz of the spring is given by

$$dE_{\text{kin}} = \frac{1}{2}\frac{m_{\text{spring}}}{L}v^2(z)dz. \tag{2.43}$$

According to Fig. 2.5a, the velocity distribution along the spring is linear with z and can be written as $v(z) = v_{\max}z/L$, with $v_{\max}$ being the maximum velocity at the end of the spring, i.e. $v(L)$. Integrating the (maximum) kinetic energy along the spring results in

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2}\int_0^L \frac{m_{\text{spring}}}{L}v^2(z)dz = \frac{1}{2}\frac{m_{\text{spring}}}{L}\int_0^L v_{\max}^2\frac{z^2}{L^2}dz \\ &= \frac{1}{2}\left(\frac{1}{3}m_{\text{spring}}\right)v_{\max}^2 = \frac{1}{2}m_{\text{eff}}v_{\max}^2. \end{aligned} \tag{2.44}$$

Thus a mass-containing spring is equivalent to a massless spring with an effective mass $m_{\text{eff}} = 1/3m_{\text{spring}}$ fixed to the end of the spring. If an additional mass M at the end of a spring is also considered, the effective mass becomes $m_{\text{eff}} = M + 1/3m_{\text{spring}}$.

While we only considered the expression of the kinetic energy here, the same effective mass also enters into the equations of motion, and thus also into all following results. For instance, when calculating the natural frequency of a harmonic oscillator in which the spring contains mass, the effective mass has to be used instead of the mass M at the end of a massless spring.

For the situation of a cantilever beam the situation is more complicated, because the deflection z (in reaction to a force applied at the end of the cantilever) is not linear along the cantilever beam as shown in Fig. 2.5b. According to [1], the bending has the form $z(x) \propto -x^3 + 3x^2L$. Since a harmonic oscillation is considered throughout the beam, the velocity distribution along the beam is proportional to the deflection $v(x) = cz(x)$. The constant of proportionality is determined by the condition $v(L) = v_{\max}$ as $c = v_{\max}/(2L^3)$. Thus the maximum velocity at position x along the beam results as

$$v(x) = \frac{v_{\max}}{2L^3}\left(-x^3 + 3x^2L\right). \tag{2.45}$$

Using this expression for the velocity distribution along the beam, the (maximum) kinetic energy can be obtained by integration along the beam as

$$E_{\text{kin}} = \frac{1}{2}\int_0^L \frac{m_{\text{cant}}}{L}\frac{v_{\max}^2}{4L^6}\left(-x^3+3x^2L\right)^2 dx = \frac{1}{2}\left(\frac{33}{140}m_{\text{spring}}\right)v_{\max}^2$$
$$= \frac{1}{2}m_{\text{eff}}v_{\max}^2. \tag{2.46}$$

Thus the effective mass for a cantilever beam turns out to be ~ 0.2357, instead of $1/3$ for a coil spring with a linear extension.

In the case of a cantilever spring, an effective mass has to be used in the equation of motion and all subsequently derived expressions such as $\omega_0 = \sqrt{k/m_{\text{eff}}}$. Throughout this text we use the concept of the harmonic oscillator and denote the mass as m in order to keep the notation simple. It has to be kept in mind that in fact the appropriate effective mass haves to be used.

2.7 Linear Differential Equations

At the end of this chapter, we consider some general properties of linear differential equations with constant coefficients. A homogeneous linear differential equation up to the second order can be written as

$$0 = a_1x + a_2\dot{x} + a_3\ddot{x}. \tag{2.47}$$

The following propositions hold for the homogeneous equation.

- Homogeneity: If x is a solution of the linear differential equation, Cx is also a solution.
- Superposition: If x_1 and x_2 are solutions of the linear differential equation, $x_1 + x_2$ is also a solution.
- Combining the two, we see that all linear combinations of two solutions are also solutions.

The corresponding inhomogeneous equations including an external driving force $F(t)$ can be written as

$$F(t) = a_1x + a_2\dot{x} + a_3\ddot{x}. \tag{2.48}$$

If we have a (special) solution of the inhomogeneous equation x_1, we can add any solution x_2 of the homogenous (free) equation $F(t) = 0$ and the sum $x = x_1 + x_2$ will be also a solution of the inhomogeneous system as we see if we add the inhomogeneous equation and the homogeneous equation as

$$F(t) = a_1(x_1 + x_2) + a_2(\dot{x}_1 + \dot{x}_2) + a_3(\ddot{x}_1 + \ddot{x}_2) = a_1x + a_2\dot{x} + a_3\ddot{x}. \tag{2.49}$$

Finally, we come to another important property of linear differential equations. If we have a solution x_1 for an external force $F_1(t)$ and a second solution x_2 for another external force $F_2(t)$, then a solution for the problem with the force $F_1(t) + F_2(t)$ is $x_1 + x_2$. This superposition principle is remarkable and is the basis for decomposing a complicated (arbitrary) force into Fourier components and composing the solution of the problem with a complicated force as a superposition of the solutions obtained for simple harmonic forces. This is also a late justification for why we only considered an external excitation (force) of simple harmonic form for the harmonic oscillator.

2.8 Summary

- The free harmonic oscillator has the natural frequency of $\omega_0 = \sqrt{\frac{k}{m}}$.
- The driven harmonic oscillator oscillates at the driving frequency ω with an amplitude depending on ω and ω_0.
- If $\omega = \omega_0$ the amplitude becomes very large (resonance).
- For the damped driven oscillator the amplitude at resonance is damped with increasing damping force $F_{\text{frict}} = m\gamma\dot{z}$.
- The phase between driving excitation and oscillation is zero if $\omega \ll \omega_0$, it is $-90°$ if $\omega = \omega_0$, and $-180°$ if $\omega \gg \omega_0$.
- The quality factor of the oscillation Q is the ratio between the energy stored in the oscillator to the energy dissipated per cycle. $Q \approx \frac{\omega_0}{\gamma} \approx \frac{\omega_0}{\Delta\omega} \approx A(\omega_0)/A_{\text{drive}}$, with $\Delta\omega$ being the width of the resonance curve and A_{drive} the excitation amplitude.
- The build up or the decay of the steady-state amplitude takes about Q oscillations, i.e. the corresponding time constant is $\tau = 2Q/\omega_0$.
- If a spring has a non-negligible mass, the effective mass has to be used in the equations of the harmonic oscillator.