Research on a Class of Nonlinear Matrix Equation

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Abstract. In this paper, the nonlinear matrix equation $X^r + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ is discussed. We propose the Newton iteration method for obtaining the Hermite positive definite solution of this equation. And a numerical example is given to identify the efficiency of the results obtained.

Keywords: Nonlinear matrix equation, Positive definite solution, Iteration, Convergence.

1 Introduction

Consider the nonlinear matrix equation

$$X + A^* X^{-1} A = Q. \tag{1}$$

Zhan [5] have studied that the maximal positive definite solution of Eq.(1) with the case $Q = I$. An iterative method for obtaining the maximal positive definite solutions of Eq.(1) is proposed. The convergence of the iterative method is also discussed in his paper. Simultaneously, Zhan and Xie [8] have proved that a necessary and sufficient condition for the existence of solutions of Eq.(1) in which A should satisfy the following demands : $A = P^T \Gamma Q \Sigma P$, where P, Q are orthogonal matrices, and diagonal matrix Γ and Σ satisfy $\Gamma > 0$, $\Sigma \geq 0$, $\Gamma^2 + \Sigma^2 = I$.Meanwhile, the solution of Eq.(1) and some necessary conditions for the existence of the solution of the nonlinear matrix equation are given.When $Q = I$ and A is real matrix, Engwerda [2] have used a simple recursion algorithm to obtain the solution of Eq.(1) and proposed a sufficient and necessary condition for the existence of a positive definite solution.

In this paper, we study Newton iteration method for obtaining Hermite positive definite solutions of equation

$$X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q, \tag{2}$$

where A_i is a nonsingular matrix, Q is a Hermite positive definite matrix, r, m are positive integer and $-1 < \delta_i < 0 (i = 1, 2, \cdots, m)$.

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2 Newton's Method for Obtaining Hermite Positive Definite Solutions of the Matrix Equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$

In this section, we study Newton's method for obtaining Hermite positive definite solutions of matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$, where A_i is a nonsingular complex matrix and Q is a Hermite definite matrix.

2.1 Related Lemmas

Lemma 1. *[6] Let* $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}} A_i Q^{-\frac{1}{2}}\|^2 < \frac{1}{4}$, *and define a matrix set* $T = \{X | \beta_1 Q \leq X \leq \beta_2 Q\}$. *Thus Eq.(2) has a unique solution in* T, *where* β_1 *is a larger root of this equation* $x(1-x) = \lambda_{max}(\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q^{-1} A_i Q^{-\frac{1}{2}})$ *and* β_2 *is a larger root of equation* $x(1-x) = \lambda_{min}(\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q^{-1} A_i Q^{-\frac{1}{2}})$.

Lemma 2. *[1] Let* $B_i \in C^{n\times n}$, *and* $P,\ Q \in {H_+}^{n\times n}$. *If* $\sum_{i=1}^{m} \|Q^{\frac{1}{2}} B_i Q^{-\frac{1}{2}}\|^2 < 1$, *then this matrix equation* $X - \sum_{i=1}^{m} B_i^* X B_i = P$ *has a unique solution* X, *and* $X \geq 0$.

Lemma 3. *[4] Let* $B_i \in C^{n\times n}, P, Q, C \in {H_+}^{n\times n}$. *If* $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}} B_i^* C^{-1} Q^{\frac{1}{2}}\|^2 < 1$, *the matrix equation* $C^* X C - \sum_{i=1}^{m} B_i^* X B_i = P$ *has a unique solution* X, *and* $X \geq 0$.

Lemma 4. *[7] If* C *and* P *are the same order Hermite equations, and* $P > 0$, *then* $CPC + P^{-1} \geq 2C$.

Lemma 5. *[3] If* $F \in R^{n\times m}$, *and* $\|F\|_p < 1$, *then* $I-F$ *is nonsingular. Moreover* $(I-F)^{-1} = \sum_{k=0}^{\infty} F^k$. *Further* $\|(I-F)^{-1}\|_p \leq \frac{1}{1-\|F\|_p}$.

2.2 Newton Iteration Method for Solving the Matrix Equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$

Newton iteration construction process is given firstly. For matrix function

$$F(X) = Q - X - \sum_{i=1}^{m} A_i^* X^{-1} A_i. \tag{3}$$

Let $H = X_{n+1} - X_n$. We have

$$\begin{aligned}
&F(X+H) - F(X)\\
&= \sum_{i=1}^{m} A_i^*[X^{-1} - (X+H)^{-1}]A_i - H\\
&= \sum_{i=1}^{m} A_i^*(X+H)^{-1}HX^{-1}A_i - H\\
&= \sum_{i=1}^{m} A_i^*[(X+H)^{-1} - X^{-1}]HX^{-1}A_i + \sum_{i=1}^{m} A_i^*X^{-1}HX^{-1}A_i - H\\
&= -\sum_{i=1}^{m} A_i^*X^{-1}H(X+H)^{-1}HX^{-1}A_i + \sum_{i=1}^{m} A_i^*X^{-1}HX^{-1}A_i - H.
\end{aligned}$$

For the definition of Frechet derivative, we know that

$$F'(X)H = \sum_{i=1}^{m} A_i^*X^{-1}HX^{-1}A_i - H \tag{4}$$

by Newton's formula

$$F'(X_n)H = -F(X_n). \tag{5}$$

Therefore

$$X_{n+1} - \sum_{i=1}^{m} A_i^*X_n^{-1}X_{n+1}X_n^{-1}A_i = Q - 2\sum_{i=1}^{m} A_i^*X_n^{-1}A_i. \tag{6}$$

Hence we can get the iterative algorithm:

$$\{\, X_0 \in [\frac{1}{2}Q, Q], X_{n+1} - \sum_{i=1}^{m} A_i^*X_n^{-1}X_{n+1}X_n^{-1}A_i = Q - 2\sum_{i=1}^{m} A_i^*X_n^{-1}A_i. \tag{7}$$

Theorem 1. *Let* $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2 < \frac{1}{4}$, $\{X_n\}$ *is matrix sequence which is determined by the iterative Algorithm (7), then* $X_n \in [\frac{1}{2}Q, Q]$, *and*

$$\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_i^*X_n^{-1}Q^{\frac{1}{2}}\|^2 < 1.$$

Proof. For $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2 < \frac{1}{4}$, so

$$\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2 I = \sum_{i=1}^{m} \lambda_{\max}(Q^{-\frac{1}{2}}A_i^*Q^{-1}AQ^{-\frac{1}{2}})I \geq \sum_{i=1}^{m} Q^{-\frac{1}{2}}A_i^*Q^{-1}AQ^{-\frac{1}{2}}.$$

Hence

$$0 \leq \sum_{i=1}^{m} Q^{-\frac{1}{2}}A_i^*Q^{-1}AQ^{-\frac{1}{2}} < \frac{1}{4}I.$$

Similarly, we get

$$\begin{aligned}&\sum_{i=1}^{m} \|Q^{-\frac{1}{2}} A_i^* X_n^{-1} Q^{\frac{1}{2}}\|^2 I\\ \geq &\sum_{i=1}^{m} Q^{\frac{1}{2}} X_n^{-1} A_i Q^{-1} A_i^* X_n^{-1} Q^{\frac{1}{2}}\\ = &\sum_{i=1}^{m} (Q^{\frac{1}{2}} X_n^{-1} Q^{\frac{1}{2}})(Q^{-\frac{1}{2}} A_i Q^{-1} A_i^* Q^{-\frac{1}{2}})(Q^{\frac{1}{2}} X_n^{-1} Q^{\frac{1}{2}})\\ = &(Q^{\frac{1}{2}} X_n^{-1} Q^{\frac{1}{2}})[\sum_{i=1}^{m} (Q^{-\frac{1}{2}} A_i Q^{-1} A_i^* Q^{-\frac{1}{2}})](Q^{\frac{1}{2}} X_n^{-1} Q^{\frac{1}{2}}).\end{aligned}$$

Then by $X_n \in [\frac{1}{2}Q, Q]$, we can obtain $I \leq Q^{\frac{1}{2}} X_n^{-1} Q^{\frac{1}{2}} \leq 2I$. There is

$$\sum_{i=1}^{m} \|Q^{-\frac{1}{2}} A_i^* X_n^{-1} Q^{\frac{1}{2}}\|^2 I \leq I.$$

Hence

$$\sum_{i=1}^{m} \|Q^{-\frac{1}{2}} A_i^* X_n^{-1} Q^{\frac{1}{2}}\|^2 < 1.$$

We will prove $X_{n+1} \in [\frac{1}{2}Q, Q]$ using the inductive method as follow.
If $k = 0$, according to known, we can obtain $x_0 \in [\frac{1}{2}Q, Q]$.
Assume that $k = n$, $X_n \in [\frac{1}{2}Q, Q]$, $\forall n \in \mathbb{N}$. We know $k = n+1$, $X_{n+1} \in [\frac{1}{2}Q, Q]$.
The Eq.(6) can become

$$Q - X_{n+1} - \sum_{i=1}^{m} A_i^* X_n^{-1}(Q - X_{n+1})X_n^{-1} A_i = \sum_{i=1}^{m} A_i^* X_n^{-1} (2X_n - Q) X_n^{-1} A_i. \tag{8}$$

According to $X_n \in [\frac{1}{2}Q, Q]$, that is $2X_n - Q \geq 0$. We can deduce

$$A_i^* X_n^{-1}(2X_n - Q)X_n^{-1} A_i \geq 0, \ \ \forall i = 1, 2, \cdots, m.$$

That is

$$\sum_{i=1}^{m} A_i^* X_n^{-1} (2X_n - Q) X_n^{-1} A_i \geq 0.$$

Thus from Lemma 2, we get $Q - X_{n+1} \geq 0$. That is $X_{n+1} \leq Q$.
Similarly, the matrix Eq.(6) can also become

$$\begin{aligned}&X_{n+1} - \frac{1}{2}Q - \sum_{i=1}^{m} A_i^* X_n^{-1}(X_{n+1} - \frac{1}{2}Q)X_n^{-1} A_i\\ = &\frac{1}{2}Q - \sum_{i=1}^{m} A_i^*(2X_n^{-1} - \frac{1}{2}X_n^{-1} Q X_n^{-1})A_i.\end{aligned} \tag{9}$$

From $X_n \in [\frac{1}{2}Q, Q]$, we can obtain $Q^{-\frac{1}{2}} X_n Q^{-\frac{1}{2}} \in [\frac{1}{2}I, I]$, and

$$2(Q^{-\frac{1}{2}} X_n Q^{-\frac{1}{2}})^{-1} - \frac{1}{2}(Q^{-\frac{1}{2}} X_n Q^{-\frac{1}{2}})^{-2} \in [\frac{2}{3}I, 2I].$$

Hence

$$\begin{aligned}&\sum_{i=1}^{m} A_i^*(2X_n^{-1} - \tfrac{1}{2}X_n^{-1}QX_n^{-1})A_i\\ &= \sum_{i=1}^{m} Q^{\frac{1}{2}}(Q^{-\frac{1}{2}}A_i^*Q^{-\frac{1}{2}})[2(Q^{-\frac{1}{2}}X_nQ^{-\frac{1}{2}})^{-1} - \tfrac{1}{2}(Q^{-\frac{1}{2}}X_nQ^{-\frac{1}{2}})^{-2}](Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}})Q^{\frac{1}{2}}\\ &\leq 2\sum_{i=1}^{m} A_i^*Q^{-1}A_i\\ &\leq \tfrac{1}{2}Q.\end{aligned}$$

So the right side of Eq.(9) is positive semi-definite. Hence,

$$X_{n+1} \geq \tfrac{1}{2}Q.$$

In conclusion, $\forall n = 0, 1, \cdots$, we have $X_n \in [\frac{1}{2}Q, Q]$, and

$$\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_i^*X_n^{-1}Q^{\frac{1}{2}}\|^2 < 1.$$

□

Theorem 2. *For iterative Algorithm(7), if* $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2 < \frac{1}{4}$, *then we have*

$$X_1 \geq X_2 \geq \cdots \geq X_n \geq \cdots. \tag{10}$$

Proof. According to Newton iterative Eq.(5), for any $n \geq 1$, we have

$$X_n - X_{n+1} - \sum_{i=1}^{m} A_i^*X_n^{-1}(X_n - X_{n+1})X_n^{-1}A_i = X_n + \sum_{i=1}^{m} A_i^*X_n^{-1}A_i - Q.$$

Since $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_i^*X_n^{-1}Q^{\frac{1}{2}}\|^2 < 1$, if we can prove $X_n + \sum_{i=1}^{m} A_i^*X_n^{-1}A_i - Q \geq 0$, then we can get $X_n - X_{n+1} \geq 0$ directly.
Consider the Eq.(6)

$$\sum_{i=1}^{m} A_i^*X_{n-1}^{-1}X_nX_{n-1}^{-1}A_i - 2\sum_{i=1}^{m} A_i^*X_{n-1}^{-1}A_i - X_n + Q = 0.$$

Then

$$\begin{aligned}&X_n + \sum_{i=1}^{m} A_i^*X_n^{-1}A_i - Q\\ &= \sum_{i=1}^{m} A_i^*X_{n-1}^{-1}X_nX_{n-1}^{-1}A_i + \sum_{i=1}^{m} A_i^*X_n^{-1}A_i - 2\sum_{i=1}^{m} A_i^*X_{n-1}^{-1}A_i\\ &= \sum_{i=1}^{m} A_i^*X_{n-1}^{-1}(X_n - X_{n-1})X_{n-1}^{-1}A_i + \sum_{i=1}^{m} A_i^*X_{n-1}^{-1}(X_{n-1} - X_n)X_n^{-1}A_i\\ &= \sum_{i=1}^{m} A_i^*X_{n-1}^{-1}(X_{n-1} - X_n)X_n^{-1}(X_{n-1} - X_n)X_{n-1}^{-1}A_i\\ &\geq 0.\end{aligned}$$

So $X_n - X_{n+1} \geq 0, \ \ \forall n = 1, 2, \ldots$. □

Theorem 3. *Consider the iterative Algorithm(7). If* $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2 < \frac{1}{4}$, *then the iterative sequence* $\{X_n\}$ *converges to* X_L.

Proof. According to Theorem 1 and Theorem 2, in Newton iterative method, for $\forall n \in N$, we get $X_n \in [\frac{1}{2}Q, Q]$, and $\{X_n\}$ is bounded monotonic sequence. Hence, for enough n, $\{X_n\}$ converges to X, and X is a solution of the Eq.(2). Let $X_n \to X'$, X' is a solution of the Eq.(2), then we have $X' \leq X_L$. On the other hand, notice that X' is obtained by sequence $\{X_n\}$, for the any solution X of the Eq.(2), we will prove that $X_n \geq X, \forall n \geq 1$ as follow.
Eq.(6) then becomes

$$\begin{aligned} & X_{n+1} - X - \sum_{i=1}^{m} A_i^* X_n^{-1}(X_{n+1} - X)X_n^{-1}A_i \\ = & Q - 2\sum_{i=1}^{m} A_i^* X_n^{-1}A_i - X + \sum_{i=1}^{m} A_i^* X_n^{-1} X X_n^{-1} A_i. \end{aligned}$$

According to X is a solution of Eq.(2), then X satisfies $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q$. Therefore

$$\begin{aligned} & X_{n+1} - X - \sum_{i=1}^{m} A_i^* X_n^{-1}(X_{n+1} - X)X_n^{-1}A_i \\ = & \sum_{i=1}^{m} A_i^* X^{-1} A_i + \sum_{i=1}^{m} A_i^* X_n^{-1} X X_n^{-1} A_i - 2\sum_{i=1}^{m} A_i^* X_n^{-1} A_i \\ = & \sum_{i=1}^{m} A_i^* X^{-1}(X_n - X)X_n^{-1}A_i - \sum_{i=1}^{m} A_i^* X_n^{-1}(X_n - X)X_n^{-1}A_i \\ = & \sum_{i=1}^{m} A_i^* X_n^{-1}(X_n - X)X^{-1}(X_n - X)X_n^{-1}A_i \geq 0. \end{aligned}$$

According to Lemma 2, whether or not $X_n - X$ is positive definite, we can obtain $X_{n+1} - X \geq 0$. In fact, by $X_n \geq X, n = 1, 2, \cdots$, we can obtain $X_n \geq X_L$, $n = 1, 2, \cdots$, and $X' \geq X_L$.

In conclusion, we have $X' = X_L$, it is equivalent to $\lim_{n\to\infty} X_n = X_L$. □

Theorem 4. *If* $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2 < \frac{1}{4}$, *then the matrix sequence* $\{X_n\}$ *of Algorithm (7) satisfies*

$$\|X_{n+1} - X_L\| \leq \frac{4\sum_{i=1}^{m} \lambda max(Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}})\|Q^{-\frac{1}{2}}\|^2 \cdot \|Q^{\frac{1}{2}}\|^2}{\beta_1(1-4\sum_{i=1}^{m} \lambda max(Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}}))} \|X_n - X_L\|^2, \quad (11)$$

where $\beta_1 = \frac{1+\sqrt{1-4\lambda max(\sum_{i=1}^{m} Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}})}}{2}$.

Proof. Obviously, from Lemma 1 and Theorem 3, we know that the maximal solution X_L of Eq.(2) satisfies $X_L \in (\beta_1 Q, \beta_2 Q)$, and $Q = X_L + \sum_{i=1}^{m} A_i^* X_L^{-1} A_i$, then

$$\|Q^{\frac{1}{2}}X_L^{-1}Q^{\frac{1}{2}}\| \le \frac{1}{\beta_1},$$

where β_1 is a large root of equation $x(1-x) = \lambda\max(\sum_{i=1}^{m} Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}})$. Therefore, we can obtain

$$\beta_1 = \frac{1+\sqrt{1-4\lambda\max(\sum_{i=1}^{m} Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}})}}{2}.$$

In Eq.(6), we use $X_L + \sum_{i=1}^{m} A_i^*X_L^{-1}A_i$ to replace Q, then

$$\sum_{i=1}^{m} A_i^*X_n^{-1}X_{n+1}X_n^{-1}A_i - 2\sum_{i=1}^{m} A_i^*X_n^{-1}A_i - X_{n+1} + X_L + \sum_{i=1}^{m} A_i^*X_L^{-1}A_i = 0.$$

So

$$\begin{aligned}
&X_{n+1} - X_L \\
&= \sum_{i=1}^{m} A_i^*X_n^{-1}(X_{n+1}-X_n)X_n^{-1}A_i + \sum_{i=1}^{m} A_i^*X_n^{-1}(X_n - X_L)X_L^{-1}A_i \\
&= \sum_{i=1}^{m} A_i^*X_n^{-1}(X_{n+1}-X_L)X_L^{-1}A_i - \sum_{i=1}^{m} A_i^*X_n^{-1}(X_n - X_L)X_n^{-1}A_i \\
&\quad + \sum_{i=1}^{m} A_i^*X_n^{-1}(X_n - X_L)X_L^{-1}A_i \\
&= \sum_{i=1}^{m} A_i^*X_n^{-1}(X_{n+1}-X_L)X_n^{-1}A_i \\
&\quad + \sum_{i=1}^{m} A_i^*X_n^{-1}(X_n - X_L)X_L^{-1}(X_n - X_L)X_n^{-1}A_i.
\end{aligned}$$

Since $X_n \in [\frac{1}{2}Q, Q]$, then

$$\|Q^{\frac{1}{2}}X_n^{-1}Q^{\frac{1}{2}}\| \le 2.$$

So

$$\begin{aligned}
&\|Q^{-\frac{1}{2}}(X_{n+1}-X_L)Q^{-\frac{1}{2}}\| \\
&\le (\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2)\|Q^{\frac{1}{2}}X_n^{-1}Q^{\frac{1}{2}}\|^2\|Q^{-\frac{1}{2}}(X_{n+1}-X_L)Q^{-\frac{1}{2}}\| \\
&+ (\sum_{i=1}^{m} \|Q^{-\frac{1}{2}}A_iQ^{-\frac{1}{2}}\|^2)\|Q^{\frac{1}{2}}X_n^{-1}Q^{\frac{1}{2}}\|^2\|Q^{\frac{1}{2}}X_L^{-1}Q^{\frac{1}{2}}\|\|Q^{-\frac{1}{2}}(X_{n+1}-X_L)Q^{-\frac{1}{2}}\|^2 \\
&\le 4\sum_{i=1}^{m} \lambda\max(Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}})\|Q^{-\frac{1}{2}}(X_{n+1}-X_L)Q^{-\frac{1}{2}}\| \\
&+ \frac{4\sum_{i=1}^{m} \lambda\max(Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}})}{\beta_1}\|Q^{-\frac{1}{2}}\|^2\|X_n - X_L\|^2.
\end{aligned}$$

And then we obtain

$$\begin{aligned}
\|X_{n+1} - X_L\| &\le \|Q^{\frac{1}{2}}\|^2 \cdot \|Q^{-\frac{1}{2}}(X_{n+1}-X_L)Q^{-\frac{1}{2}}\| \\
&\le \frac{4\sum_{i=1}^{m} \lambda\max(Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}})\|Q^{-\frac{1}{2}}\|^2\cdot\|Q^{\frac{1}{2}}\|^2}{\beta_1(1-4\sum_{i=1}^{m} \lambda\max(Q^{-\frac{1}{2}}A_i^*Q^{-1}A_iQ^{-\frac{1}{2}}))}\|X_n - X_L\|^2,
\end{aligned}$$

where $\beta_1 = \frac{1+\sqrt{1-4\lambda \max(\sum_{i=1}^{m} Q^{-\frac{1}{2}} A_i^* Q^{-1} A_i Q^{-\frac{1}{2}})}}{2}$. □

By this theorem, we know, if $\sum_{i=1}^{m} \|Q^{-\frac{1}{2}} A_i Q^{-\frac{1}{2}}\|^2 < \frac{1}{4}$ and the condition of this select initial original $X_0 \in [\frac{1}{2}Q, Q]$, the constructed Newton iterative convergence is quadratic.

3 Numerical Examples

In this section, we will give a numerical example to identify the two algorithm of achieve extremal positive definite solutions of the equation. For different matrices A_i and different α, r, δ_i, $i = 1, 2, \ldots, m$. We compute the solutions of the equation

$$X + \sum_{i=1}^{m} A_i^* X^{\delta i} A_i = Q.$$

We will operate all programs by using MATLAB 7.0. We note that

$$\varepsilon(X_k) = \|X_k^r + \sum_{i=1}^{m} A_i^* X_k^{\delta_i} A_i - Q\|.$$

Example 1. We consider the equation by using the two matrices A_1 and A_2 as follows:

$$A_1 = 0.5 * \begin{pmatrix} -0.1 & 0.2 & -0.06 \\ 0.2 & -0.3 & 0.16 \\ -0.1 & 0 & 0.02 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.01 & 0.02 & 0.03 \\ 0.01 & 0.225 & 0.12 \\ 0 & 0.09 & 0.07 \end{pmatrix},$$

where $Q = I$, $r = 5$, $\delta_1 = -\frac{1}{3}$, $\delta_2 = -\frac{1}{4}$.

By Algorithm 7, the following table records the different values of parameter $\alpha(\alpha > 1)$, and the values satisfy the number of iterations which is needed by a stop condition.

Table 1.

α	k		
	$tol = 10^{-4}$	$tol = 10^{-6}$	$tol = 10^{-8}$
1.3	3	4	5
1.5	3	4	5
1.7	3	4	5
2.1	3	4	5

,

$$X_L = \begin{pmatrix} 0.9969 & 0.0037 & -0.0021 \\ 0.0037 & 0.9808 & -0.0040 \\ -0.0021 & -0.0040 & 0.9943 \end{pmatrix}, \quad X_L - D = \begin{pmatrix} 0.9969 & 0.0037 & -0.0021 \\ 0.0037 & 0.9808 & -0.0040 \\ -0.0021 & -0.0040 & 0.9943 \end{pmatrix},$$

where $D = \frac{1}{2} \sum_{i=1}^{2} (A_i Q^{-1} A_i^*)^{-\frac{1}{\delta_i}}$.

The eigenvalue of $X_L - D$ is $(0.9791, 0.9995, 0.9933)$.

The eigenvalue of $Q^{\frac{1}{5}} - X_L$ is $(0.0005, 0.0067, 0.0208)$.

It implies that $X_L \in \Omega$.

From the Table 1, we conclude that the number of iterations increased with increasing of α within some extent errors.

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