Welfare Guarantees for Proportional Allocations⋆

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Abstract. According to the proportional allocation mechanism from the network optimization literature, users compete for a divisible resource – such as bandwidth – by submitting bids. The mechanism allocates to each user a fraction of the resource that is proportional to her bid and collects an amount equal to her bid as payment. Since users act as utility-maximizers, this naturally defines a proportional allocation game. Recently, Syrgkanis and Tardos (STOC 2013) quantified the inefficiency of equilibria in this game with respect to the social welfare and presented a lower bound of 26.8% on the price of anarchy over coarse-correlated and Bayes-Nash equilibria in the full and incomplete information settings, respectively. In this paper, we improve this bound to 50% over both equilibrium concepts. Our analysis is simpler and, furthermore, we argue that it cannot be improved by arguments that do not take the equilibrium structure into account. We also extend it to settings with budget constraints where we show the first constant bound (between 36% and 50%) on the price of anarchy of the corresponding game with respect to an effective welfare benchmark that takes budgets into account.

1 Introduction

The *proportional allocation mechanism*, introduced by Kelly [11], is fundamental in the network optimization literature. According to this mechanism, a divisible resource — such as bandwidth of a communication link — is allocated to users as follows. Each user submits a bid to the mechanism; this corresponds to the user's *willingness-to-pay* for sharing the resource. The mechanism allocates to each user a fraction of the resource that is equal to the ratio of her bid over the total amount of bids. It also receives a payment from each user that is equal to her bid. This naturally defines a *proportional allocation game* among the users who act as players; each player has a (typically concave, non-negative, and non-decreasing) valuation function for the resource share she receives and aims to

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maximize her utility, i.e., her value for the resource share minus her payment to the mechanism. As it is typically the case in games, the *social welfare* (i.e., the total value of the players for the resource share they receive) at *equilibria* is, in general, suboptimal.

We aim to quantify this inefficiency of equilibria by bounding the *price of anarchy* [12] of proportional allocation games. Besides the well-known work of Johari and Tsitsiklis [9] who considered pure Nash equilibria in the full information setting, there has been surprisingly little focus on price of anarchy bounds over more general equilibrium concepts. The only exception we are aware of is the recent work of Syrgkanis and Tardos [22] who studied proportional allocation as part of a broader class of mechanisms. Motivated by their work, we present new bounds on the price of anarchy of proportional allocation under general equilibrium concepts, such as *coarse-correlated* equilibria in the full information setting and *Bayes-Nash* equilibria in the incomplete information setting. In particular, we prove that the social welfare at equilibrium is at least $1/2$ of the optimal social welfare. The bound holds for coarse-correlated and pure Bayes-Nash equilibria in the full information and Bayesian setting, respectively, and improves the bound of 26.8% of [22]. The proof is conceptually simple and is obtained by bounding the utility of every player at equilibrium by the utility this player would have by deviating to a particular deterministic bid.

We also consider the scenario where players have budget constraints representing their *ability-to-pay*. Here, each player has a budget and is never allowed to bid above it. We assess the quality of equilibria in this case in terms of an *effective welfare* benchmark — proposed in previous work but further refined here — that takes budgets into account. We show that the effective welfare at equilibrium is at least a constant fraction of the optimal one. To the best of our knowledge, this is the first constant price of anarchy bound (in particular, between 36% and 50%) with respect to this benchmark[1]. Again, our proofs follow by considering a single deterministic deviation for each player, defined in a slightly different way compared to the deviation we consider in our bound on the social welfare.

Related Work. The proportional allocation mechanism and its variations have received significant attention in the network optimization literature. Proportional allocation games have been considered in [7,13,14] where the existence and uniqueness conditions for pure Nash equilibria are proved. Variations of the mechanism with different definitions for the allocation rule or the payments have been considered in [15,16,17,20] (see also the discussion in [8]).

Johari and Tsitsiklis [9] were the first who assessed the quality of proportional allocations in terms of the social welfare. They focused on pure Nash equilibria and proved a lower bound of $3/4$ on their price of anarchy. Their analysis is based on the important observation that a pure Nash equilibrium in a proportional allocation game is also a pure Nash equilibrium in a game where each player

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[1] Previously, Syrgkanis and Tardos [22] had shown that the social welfare at equilibrium is at least $2 - \sqrt{3} \approx 26.8\%$ of the optimal effective welfare. Our techniques can be used to improve this particular guarantee to $1/2$.

has a *linear* valuation function with slope equal to the derivative of the original valuation function at the share value they get at equilibrium. The optimal social welfare in the new game is not smaller than the original one and this allows them to consider the significantly simpler case of linear valuations in their analysis. Then, the price of anarchy bound is obtained by solving a linear program. An alternative proof to the result of [9] without using this argument is presented in [18] (see also [8]).

Unfortunately, this transformation does not apply to more general equilibrium concepts since the resource share each player receives is, in general, a random variable. This is a rather common difficulty that manifests itself in the analysis of games, as we depart from pure Nash equilibria and full information. In particular, Bayes-Nash equilibria have such an extremely rich structure that, typically, the price of anarchy analysis assesses their quality by rather ignoring this structure. Instead, it resorts to bounding the utility of each player by appropriately selected deviations which reveal a relation between the social welfare at equilibrium and the optimal social welfare. This approach has been used in a series of papers that mostly focus on auctions (e.g., see [1,2,3,6,10,19,22]) and is actually the approach we follow in the current paper as well.

Syrgkanis and Tardos [22] present a general analysis framework for the broad class of smooth mechanisms. Among other results, they show a price of anarchy lower bound of 26.8% over coarse-correlated and mixed Bayes-Nash equilibria of proportional allocation games. In their analysis, they bound the utility of each player by the utility she would have by deviating to an appropriately defined *randomized* bid (an approach that has also been used in different contexts in [2,10,21,23]) with a probability distribution that depends only on the optimal allocation and the valuation function of the player. In contrast, the deviating bid we consider depends on the bid strategies at equilibrium (this is in the same spirit as the recent analysis of Feldman et al. [6]) and, more interestingly, it is *deterministic*. In particular, it is defined as the product of the (expected) resource share a bidder receives in the optimal allocation and the expectation of bids of the other players at equilibrium.

Budget constraints are well-motivated in auction settings. In a slightly different context than ours, the effective welfare benchmark is considered by Dobzinski and Paes Leme, who call it *liquid* welfare in [5]. In proportional allocation, Syrgkanis and Tardos [22] prove that the social welfare at equilibrium is a constant fraction of the optimal effective welfare. Note that our guarantee is considerably stronger as we compare directly the effective welfare at equilibrium with its optimal value.

Roadmap. The rest of the paper is structured as follows. We begin with preliminary definitions in Section 2. Our price of anarchy bounds in terms of the social welfare are proved in Section 3. There, we also argue that in order to improve our analysis, radically new ideas are required. The budget-constrained setting is studied in Section 4. We remark that we have not mentioned mixed Bayes-Nash equilibria in the above presentation of our results. Actually, we have observed that such equilibria coincide with pure ones even in the budget-constrained setting.

We discuss related issues as well as additional open problems in Section 5. Due to lack of space, many proofs have been omitted; they will appear in the full version of the paper.

2 Preliminaries

Each player (henceforth called *bidder*) i in a proportional allocation game has a concave[2] non-decreasing valuation function $v_i : [0, 1] \to \mathbb{R}^+$. A strategy for bidder i is simply a non-negative bid. Given a bid vector $\mathbf{b} = (b_1, b_2, ..., b_n)$, with one bid per bidder, the proportional allocation mechanism allocates to each bidder a fraction of the resource that is proportional to the bid submitted by her. Denoting by d_i the resource share that is allocated to bidder i, it is $d_i = \frac{b_i}{\sum_j b_j}$. We often use the notation B_{-i} to denote the sum of bids of all bidders besides i (hence, $d_i = \frac{b_i}{b_i + B_{-i}}$). The utility of bidder i from an allocation is simply the difference of her value for the fraction of the resource she gets minus her bid, i.e., $u_i(\mathbf{b}) = v_i(d_i) - b_i$.

A bid vector $\mathbf{b}$ is a pure equilibrium if the utility of all bidders is maximized, given the bid strategies of the other bidders. So, in a pure equilibrium, no bidder has any incentive to deviate to another strategy. Denoting by $(b_i', \mathbf{b}_{-i})$ the bid vector that is obtained from $\mathbf{b}$ when bidder i unilaterally deviates to bid strategy b_i', we can express this condition as $u_i(\mathbf{b}) \geq u_i(b_i', \mathbf{b}_{-i})$.

The social welfare of an allocation d is the total value of bidders for the resource shares they receive, i.e., $SW(d) = \sum_i v_i(d_i)$. We denote by SW^* the maximum value of the social welfare over all possible allocations. The price of anarchy over pure Nash equilibria is defined as the minimum value of the social welfare among all pure Nash equilibria divided by the optimal social welfare.

The bid strategy of a bidder i can be randomized. In this case, b_i is a random variable and the bidder aims to maximize her expected utility $\mathbb{E}[u_i(\mathbf{b})]$. The bid strategies of different bidders can be independent or correlated. A vector of independent randomized bid strategies is called a mixed Nash equilibrium if it simultaneously maximizes the expected utility of each bidder, given the bid strategies of the other bidders. More generally, coarse-correlated equilibria are solution concepts that capture correlated bid strategies. A vector of (possibly correlated) bid strategies is called a coarse-correlated equilibrium if no bidder has any incentive to unilaterally deviate to any deterministic bid strategy in order to improve her expected utility (again, given the strategies of the other bidders). The notion of the price of anarchy naturally extends to these solution concepts as well. For example, the price of anarchy over correlated equilibria is defined as the minimum value of the expected social welfare among all coarse-correlated equilibria divided by the optimal social welfare.

The above setting is known as the full (or complete) information setting. We consider the incomplete information (or Bayesian) setting as well; in this case,

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[2] Very recently, Correa et al. [4] studied proportional allocation games in the less standard scenario of non-concave valuation functions.

the valuation function $\mathbf{v}_i$ of each bidder i is drawn randomly (and independently from the other bidders) from a probability distribution $\mathbf{F}_i$ over concave, non-decreasing, and non-negative functions in $[0, 1]$. Again, bidder i aims to maximize her expected utility for each possible valuation function v_i drawn from $\mathbf{F}_i$. In the incomplete information setting, each bidder i bases her decision on her exact valuation v_i and on the probability distributions according to which other bidders draw their valuations (and their corresponding bid strategies); these distributions are common knowledge.

So, the bid strategy of bidder i is a (possibly random) bid function $b_i(\mathbf{v}_i)$. A vector with one such strategy per bidder (with independence between bid strategies of different bidders) is called a mixed Bayes-Nash equilibrium if no bidder has any incentive to deviate to some other bid for any valuation function drawn from $\mathbf{F}_i$. In pure Bayes-Nash equilibria, bidders use deterministic bid functions. The price of anarchy over Bayes-Nash equilibria is defined as the minimum value of the expected social welfare among all Bayes-Nash equilibria divided by the expectation of the optimal social welfare. With some abuse in notation, we also use SW^* to denote the expectation of the optimal social welfare in the Bayesian setting.

We also extend the above model by adding budget constraints to the bidders. In this setting, each bidder i has a non-negative budget c_i and she is never allowed to bid above her budget. This restriction can result to equilibria that have extremely low social welfare compared to the optimal one (whose definition does not take budgets into account). Following [22] and [5], we use the effective welfare benchmark in order to assess the quality of equilibria with budget-constrained bidders. The effective welfare of a (deterministic) allocation $d = (d_1, d_2, ..., d_n)$ is defined as $EW(d) = \sum_i \min\{v_i(d_i), c_i\}$. Note that the definition is similar to the definition of the social welfare; the important difference is that the value of each bidder is capped by her budget. We extend this definition to random allocations d as $EW(d) = \sum_i \min\{\mathbb{E}\left[v_i(d_i)\right], c_i\}$. We denote by EW^* the maximum value of the effective welfare over all allocations. The price of anarchy with respect to the effective welfare benchmark (over equilibria in a given class) is the minimum value of the effective welfare (among all allocations induced by equilibria in the class) divided by the optimal effective welfare.

In the Bayesian setting, both the budget $\mathbf{c}_i$ of bidder i and her valuation $\mathbf{v}_i$ are drawn randomly according to the probability distribution $\mathbf{F}_i$. We refine the effective welfare benchmark in this case as

$$EW(d) = \sum_i \mathbb{E}_{(\mathbf{v}_i, \mathbf{c}_i) \sim \mathbf{F}_i} \left[\min\{\mathbb{E}_{(\mathbf{v}_{-i}, \mathbf{c}_{-i}) \sim \mathbf{F}_{-i}} \left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\} \right],$$

where the inner expectation is taken over the valuation-budget value pairs of the other bidders once the pair for bidder i has been fixed (and over the corresponding bid strategies). In order to simplify notation in the proofs below, we will not explicitly use the subscripts in the expectations.

3 Bounding the Social Welfare of Equilibria

In this section, we prove the price of anarchy bounds with respect to the social welfare. We consider both coarse-correlated equilibria in the full information setting as well as pure Bayes-Nash equilibria in the Bayesian setting. Our proofs use the following lemma (its proof is omitted) which bounds the utility of a bidder at a deterministic deviation. We also use this lemma later in Section 4 where we study budget-constrained bidders.

Lemma 1. *Consider a bidder with a concave and non-decreasing valuation function $v : [0, 1] \to \mathbb{R}^+$ and let Γ be the random variable denoting the sum of bids of the other bidders. Then, for every $z \in [0, 1]$ and for every $\mu > 0$, the expected utility the bidder would have by deviating to the deterministic bid $\mu z \mathbb{E}[\Gamma]$ is at least $\frac{3\mu - 1}{4\mu} v(z) - \mu z \mathbb{E}[\Gamma]$.*

We are ready to prove our price of anarchy bounds. We begin with the case of coarse-correlated equilibria in the full information setting which is much simpler.

Theorem 1. *The price of anarchy of proportional allocation games over coarse-correlated equilibria is at least $1/2$.*

Proof. Consider a full information proportional allocation game with n bidders in which bidder i has valuation function v_i and denote by x_i the resource fraction bidder i gets in the optimal allocation. Let $\mathbf{b}$ be a coarse-correlated equilibrium that induces a random allocation $d = (d_1, ..., d_n)$ and let $B = \sum_i b_i$ be the random variable denoting the sum of bids of all bidders, with B_{-i} being the sum of bids of all bidders besides bidder i. Since $\mathbf{b}$ is a coarse-correlated equilibrium, bidder i has no incentive to deviate to any deterministic bid (including the deviating bid $x_i \mathbb{E}[B_{-i}]$). By applying Lemma 1 for bidder i with $z = x_i$, $\mu = 1$ and $\Gamma = B_{-i}$, we obtain that

$$\mathbb{E}[u_i(\mathbf{b})] \geq \mathbb{E}[u_i(x_i \mathbb{E}[B_{-i}], \mathbf{b}_{-i})] \geq \frac{1}{2} v_i(x_i) - x_i \mathbb{E}[B_{-i}].$$

Summing over all bidders and using the fact that $B_{-i} \leq B$ for every bidder i, we have

$$\sum_i \mathbb{E}[u_i(\mathbf{b})] \geq \frac{1}{2} \sum_i v_i(x_i) - \sum_i x_i \mathbb{E}[B_{-i}] \tag{1}$$

$$\geq \frac{1}{2} \sum_i v_i(x_i) - \sum_i x_i \mathbb{E}[B]$$

$$= \frac{1}{2} SW^* - \mathbb{E}[B].$$

The theorem follows by this inequality since the social welfare equals the sum of bidders' utilities plus their bids, i.e., $\mathbb{E}[SW(d)] = \sum_i \mathbb{E}[u_i(\mathbf{b})] + \mathbb{E}[B]$. $\square$

The last step of the proof above begins with inequality (1). Essentially, this inequality has the form

$$\sum_i \mathbb{E}\left[u_i(\mathbf{b})\right] \geq \lambda SW^* - \mu \sum_i x_i \mathbb{E}\left[B_{-i}\right].$$

The price of anarchy bound of [22] follows after first proving an inequality of this type and then concluding to a price of anarchy bound of $\frac{\lambda}{\max\{1,\mu\}}$. The smoothness arguments of [22] lead to a version of this inequality with $\lambda = 2 - \sqrt{3}$ and $\mu = 1$. Here, we have been able to improve the parameters to $\lambda = 1/2$ and $\mu = 1$. The next lemma demonstrates that these parameters cannot be improved further; the proof is omitted.

Lemma 2. *For every $\epsilon > 0$, there exists a proportional allocation game such that for every λ, μ satisfying*

$$\sum_i u_i(\mathbf{b}) \geq \lambda SW^* - \mu \sum_i x_i B_{-i} \tag{2}$$

where x_i is the resource fraction of bidder i in the optimal allocation and B_{-i} is the sum of bids of all bidders besides bidder i at a (pure Nash) equilibrium, it holds that $\frac{\lambda}{\max\{1,\mu\}} \leq \frac{1}{2} + \epsilon$.

The proof for Bayes-Nash equilibria (omitted) follows the same general approach with that of Theorem 1.

Theorem 2. *The price of anarchy of proportional allocation games over pure Bayes-Nash equilibria is at least $1/2$.*

4 Budget-Constrained Bidders

In this section, we consider budget-constrained bidders and prove a lower bound of approximately 36% and an upper bound of 50% on the price of anarchy in terms of the effective welfare benchmark. Here, we prove Theorem 3 for Bayes-Nash equilibria only; the (simpler) proof for coarse-correlated equilibria is omitted. Our upper bound (Theorem 4) applies even to pure Nash equilibria.

Before proceeding to the presentation of our bounds for budget-constrained bidders, we remark that minor modifications of the proofs in the previous section can show that the social welfare over equilibria with budget-constrained bidders is at least $1/2$ of the optimal effective welfare, improving a corresponding bound of 26.8% from [22]. The necessary modifications are as follows. First, we need to define the deviating bids in terms of the resource shares in the allocation that maximizes the effective welfare. Then, there is a subtle case where Lemma 1 cannot be used, namely when the deviating bid for a bidder exceeds her budget. Fortunately, the inequality provided by Lemma 1 follows trivially in this case (actually, we use this argument in the proof below). By repeating the analysis in the proofs of Theorems 1 and 2, we can conclude that the social welfare at

equilibrium is at least $1/2$ of the social welfare of the allocation that maximizes the effective welfare. The bound then follows by observing that the effective welfare of this allocation is upper-bounded by its social welfare.

Theorem 3. *The price of anarchy of proportional allocation games with budget-constrained bidders over coarse-correlated or Bayes-Nash equilibria is at least 0.3596.*

Proof. Let $\mu \in (1/3, 1]$ be a parameter whose exact value will be defined later. Consider an incomplete information proportional allocation game with n bidders in which the valuation function $\mathbf{v}_i$ and the budget $\mathbf{c}_i$ of bidder i are drawn from the probability distribution $\mathbf{F}_i$, independently for each bidder. Let x_i be the random variable denoting the resource fraction bidder i gets in the allocation that maximizes the effective welfare. Let $\mathbf{b}$ be a pure Bayes-Nash equilibrium that induces a random allocation $d = (d_1, ..., d_n)$ and B be the random variable denoting the sum of bids of all bidders; again, B_{-i} denotes the sum of bids of all bidders besides bidder i. We denote by A_i the set that contains all pairs of a valuation function and a corresponding budget value (v_i, c_i) that are drawn from the probability distribution $\mathbf{F}_i$ and satisfy $\mathbb{E}\left[\mathbf{v}_i(d_i)|v_i\right] \leq c_i$. Consider a bidder i with valuation-budget pair $(v_i, c_i) \notin A_i$. By the definition of A_i, we have

$$\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)|v_i\right], c_i\} \geq \min\{\mathbb{E}\left[\mathbf{v}_i(x_i)|v_i\right], c_i\}.$$

By considering all valuation-budget pairs not belonging to A_i, we obtain

$$\mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], c_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \geq \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], c_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right],$$

and summing over all bidders, we have

$$\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right]$$

$$\geq \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right]. \tag{3}$$

Now consider a valuation-budget pair $(v_i, c_i) \in A_i$ for bidder i that is drawn from $\mathbf{F}_i$. If $\mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right] \leq c_i$, we can bound the expected utility $\mathbb{E}\left[u_i(\mathbf{b})|v_i\right]$ by considering the deviation of bidder i to bid $\mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right]$ (which is within bidder i's budget c_i). By Lemma 1, we have

$$\mathbb{E}\left[u_i(\mathbf{b})|v_i\right] \geq \frac{3\mu - 1}{4\mu}v_i(\mathbb{E}\left[x_i|v_i\right]) - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right]$$

$$\geq \frac{3\mu - 1}{4\mu}\mathbb{E}\left[v_i(x_i)|v_i\right] - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right]$$

$$\geq \frac{3\mu - 1}{4\mu}\min\{\mathbb{E}\left[v_i(x_i)|v_i\right], c_i\} - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right].$$

The second inequality follows by Jensen's inequality and by the fact $\mathbb{E}\left[B_{-i}|v_i\right] = \mathbb{E}\left[B_{-i}\right]$. Otherwise, if $\mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right] > c_i$, the same inequality follows easily

since

$$\mathbb{E}\left[u_i(\mathbf{b})|v_i\right] \geq 0$$
$$> c_i - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right]$$
$$\geq \frac{3\mu - 1}{4\mu}\min\{\mathbb{E}\left[v_i(x_i)|v_i\right], c_i\} - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right].$$

Hence, when $(v_i, c_i) \in A_i$, we have

$$\mathbb{E}\left[u_i(\mathbf{b})|v_i\right] + \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right] \geq \frac{3\mu - 1}{4\mu}\min\{\mathbb{E}\left[v_i(x_i)|v_i\right], c_i\}.$$

By considering all valuation-budget values belonging to A_i, we have

$$\mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]\mathbb{E}\left[B\right]$$
$$\geq \frac{3\mu - 1}{4\mu}\mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right].$$

Using the obvious fact that $\mathbb{E}\left[x_i\right] \geq \mathbb{E}\left[x_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]$ and the above inequality, we obtain that

$$\mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\right]\mathbb{E}\left[B\right]$$
$$\geq \mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]\mathbb{E}\left[B\right]$$
$$\geq \frac{3\mu - 1}{4\mu}\mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], c_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]. \tag{4}$$

Now, we have

$$\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]$$
$$+\mu\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right]$$
$$\geq \sum_i \mathbb{E}\left[(u_i(\mathbf{b}) + b_i)\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\sum_i \mathbb{E}\left[b_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right]$$
$$\geq \sum_i \mathbb{E}\left[(u_i(\mathbf{b}) + \mu b_i)\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\sum_i \mathbb{E}\left[b_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right]$$
$$= \sum_i \mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[B\right]$$
$$= \sum_i \left(\mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\right]\mathbb{E}\left[B\right]\right)$$
$$\geq \frac{3\mu - 1}{4\mu}\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]. \tag{5}$$

The first inequality follows since the quantity $\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}$ equals $\mathbb{E}\left[\mathbf{v}_i(d_i)\right]$ when $(\mathbf{v}_i, \mathbf{c}_i) \in A_i$ and $\mathbf{c}_i$ otherwise; in the latter case, the budget is clearly not smaller than the bid of bidder i. The second inequality follows since $\mu \leq 1$, the

two equalities are obvious, and the last inequality follows by (4). Now, using (3) and (5), we have

$$
\begin{aligned}
EW(d) &= \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right],\mathbf{c}_i\}\right] \\
&= \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right],\mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i,\mathbf{c}_i) \in A_i\right] \\
&\quad +\mu \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right],\mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i,\mathbf{c}_i) \notin A_i\right] \\
&\quad +(1-\mu)\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right],\mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i,\mathbf{c}_i) \notin A_i\right] \\
&\geq \frac{3\mu-1}{4\mu}\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right],\mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i,\mathbf{c}_i) \in A_i\right] \\
&\quad +(1-\mu)\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right],\mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i,\mathbf{c}_i) \notin A_i\right] \\
&\geq \min\left\{\frac{3\mu-1}{4\mu}, 1-\mu\right\}\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right],\mathbf{c}_i\}\right] \\
&= \min\left\{\frac{3\mu-1}{4\mu}, 1-\mu\right\} EW^*.
\end{aligned}
$$

Hence, the price of anarchy with respect to the effective welfare benchmark is bounded by the quantity $\min\left\{\frac{3\mu-1}{4\mu}, 1-\mu\right\}$ which is maximized to $\frac{7-\sqrt{17}}{8} \approx 0.3596$ for $\mu = \frac{1+\sqrt{17}}{8}$. $\qquad\square$

We conclude this section by presenting our upper bound on the price of anarchy; note that it holds even for pure Nash equilibria.

Theorem 4. *For every $\epsilon > 0$, there exists a proportional allocation game among budget-constrained bidders with price of anarchy at most $1/2 + \epsilon$ over pure Nash equilibria, with respect to the effective welfare benchmark.*

5   Discussion and Open Problems

Our work leaves the obvious open problem of computing the tight bound on the price of anarchy over coarse-correlated and Bayes-Nash equilibria. So far, the only upper bound that is known is the counter-example of 3/4 from [9] for pure Nash equilibria. Is 3/4 the tight bound for all equilibrium concepts considered in the current paper? Actually, we have not been able to identify any coarse-correlated equilibrium in the full information model that is non-pure. Do such equilibria really exist? Interestingly, we can show that mixed Nash equilibria coincide with pure ones (see the statement in Lemma 3). More generally, this statement applies to mixed Bayes-Nash equilibria in the budget-constrained setting. Does it extend to coarse-correlated ones? We believe that this is an interesting open problem.

Lemma 3. *The set of mixed Bayes-Nash equilibria in any proportional allocation game (possibly with budget-constrained bidders) coincides with that of pure Bayes-Nash equilibria.*

In the Bayesian setting, we have not considered more general equilibrium concepts such as coarse-correlated Bayesian equilibria. The main reason is that our analysis requires that the expectation of the sum of bids of the other bidders is the same for any possible valuation bidder i can draw from her distribution; this property is not satisfied by more general equilibrium concepts. What is the price of anarchy in this case? Interestingly, the answer cannot be 3/4 as our next counter-example indicates; the proof is omitted due to lack of space.

Lemma 4. *There exists a proportional allocation game that has price of anarchy at most* 0.7154 *over coarse-correlated Bayesian equilibria.*

Also, recall that we have assumed that bidders have independent valuations. This is a typical assumption in the Bayes-Nash price of anarchy literature [1,3,6,10,19,21,22] with [2] being the only exception we are aware of. Unfortunately, our proof of the pure Bayes-Nash price of anarchy bound does not carry over to the case of correlated valuations either (for the same reason mentioned above). Still, we have not been able to find any counter-example with non-constant price of anarchy in this setting. Again, what is the price of anarchy in this case? These questions are interesting in the budget-constrained setting as well.

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