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Preface

This book contains the proceedings of the 7th International Symposium on Algorithmic Game Theory (SAGT) held in Haifa, Israel, in October 2014.

The program of SAGT 2014 consisted of five invited lectures and 29 presentations of refereed submissions. The Program Committee selected 29 out of 65 submissions after a careful reviewing process. The PC found that this year's submissions included many papers of high quality, more papers than a regular program can accommodate. The PC therefore decided to accept 26 regular papers, and to invite 3 additional submissions to be presented in the form of a brief announcement at the conference. The PC feels that these brief announcements added inspiration and novelty to the program.

The accepted submissions were invited to these proceedings. They cover various important aspects of algorithmic game theory that were grouped into 8 sessions: Matching Theory, Game Dynamics, Games of Coordination, Networks and Social Choice, Markets and Auctions, Price of Anarchy, Computational aspects of games, and Mechanism Design and Auctions.

I would like to thank all authors who submitted their research work and all Program Committee members and external reviewers for their effort in selecting the program for SAGT 2014.

July 2014 Ron Lavi

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Social Welfare in One-Sided Matchings: Random Priority and Beyond*

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Abstract. We study the problem of approximate social welfare maximization (without money) in one-sided matching problems when agents have unrestricted cardinal preferences over a finite set of items. *Random priority* is a very well-known *truthful-in-expectation* mechanism for the problem. We prove that the approximation ratio of random priority is $\Theta(n^{-1/2})$ while no truthful-in-expectation mechanism can achieve an approximation ratio better than $O(n^{-1/2})$, where n is the number of agents and items. Furthermore, we prove that the approximation ratio of all ordinal (not necessarily truthful-in-expectation) mechanisms is upper bounded by $O(n^{-1/2})$, indicating that random priority is asymptotically the best truthful-in-expectation mechanism and the best ordinal mechanism for the problem.

1 Introduction

We study the problem of approximate social welfare maximization (without money) in one-sided matching problems when agents have unrestricted cardinal preferences over a finite set of items. Specifically, each agent has a valuation function mapping items to real numbers, which can be arbitrary. These valuation functions should be interpreted as von Neumann-Morgenstern utility functions, i.e. they induce orderings on outcomes, which are standardly defined up to positive affine transformations (multiplication by a positive scalar and shift by a scalar).

A (direct revelation) mechanism (without money) is a function J mapping vectors of valuation functions (valuation profiles) to matchings, that is, allocations of items to agents such that each agent is allocated exactly one item. Mechanisms can also be randomized, and then the function J is a random map.

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We will be interested in *truthful* mechanisms, that is, mechanisms that do not give incentives to agents to misreport their valuation functions. For randomized mechanisms, we are interested in mechanisms that are *truthful-in-expectation*, which means that no agent can increase its expected utility by misreporting.

A very intuitive and well-studied truthful-in-expectation mechanism is *random priority* (often also referred to as *random serial dictatorship*), which first fixes a uniformly random ordering of the agents and then serially matches each agent to its most preferred item from the set of still unmatched items, based on that ordering. Random priority is an *ordinal* mechanism, i.e., a mechanism that only depends on the ordering of items induced by the valuation functions and not the actual numerical values.

We will look to maximize the *social welfare*, that is, the sum of agents' valuations for the items they are matched with in the outcome of the mechanism. We measure the performance of a mechanism by its approximation ratio, which is the worst ratio between the (expected) social welfare of the mechanism and the welfare of the optimal allocation, over all valuation profiles. It is easy to see that the mechanism that always outputs the optimal outcome is not truthful.

For a meaningful discussion on social welfare maximization for von Neumann-Morgenstern utilities, one has to fix a canonical representation of the valuation functions [7, 9, 11, 12]. These functions are usually represented in one of two canonical forms, *unit-sum* [7, 12] (the valuations sum up to one) or *unit-range* [4, 11, 19] (all valuations are in the interval $[0, 1]$ with both 0 and 1 in the image of the function). Our main result is given by the following theorem, which holds for both normalizations.

Theorem 1. *The approximation ratio of random priority is $\Theta(1/\sqrt{n})$. Furthermore, random priority is asymptotically the best truthful-in-expectation mechanism and the best ordinal (not necessarily truthful-in-expectation) mechanism for the problem.*

The theorem also holds for an extension to the unit-range representation, when 0 is not required to be in the image of the function; we discuss how in Section 5. An implication of Theorem 1 is that the well-known *Probabilistic Serial* mechanism [6], which is often preferred to random priority, does not give better social welfare guarantees, even if we assume truthful reporting.

1.1 Discussion and Related Work

In the presence of incentives, the one-sided matching problem (often referred to as *the assignment problem* or *house allocation problem*) was originally defined in [13] and has been studied extensively ever since [6, 9, 12, 14, 18, 19]. There are several surveys discussing the problem [1, 17] and we refer the interested reader to those for a more detailed exposition. Random priority is a folklore mechanism that solves the problem fairly (in the sense of anonymity) and satisfies some additional nice properties; it is truthful-in-expectation and *ex-post Pareto efficient*. On the other hand, it is not *ex-ante Pareto efficient*. Most of the previous work in literature [6, 13, 19] has mainly been directed towards designing mechanisms with desired properties that achieve efficiency criteria for different Pareto efficiency notions and less towards whether truthful mechanisms achieve "good levels" of social welfare. This is a really important question to ask, especially since the notion of approximation ratio gives us a systematic way of comparing mechanisms or proving their limitations. The term *approximate mechanism design without money* was used in [16] to describe problems where the goal is to approximately optimize some objective function, given the constraint of truthfulness. This approach has been adopted by a large body of computer science literature [3, 5, 9, 11, 12, 15] and the approximation ratio is now considered to be the predominant measure of efficiency for truthful mechanisms.

Social welfare maximization is arguably a less natural objective when agents are endowed with von Neumann-Morgenstern utilities, because of adding up valuations after normalization. On the other hand, it is quite widespread in *quasi-linear* settings. We strongly believe that considering the social welfare objective for von Neumann-Morgenstern utilities is just as natural and in fact there is a growing amount of literature that embraces the same idea and provides arguments to support it [5, 7, 11, 12].

A different approach, often encountered in literature, is to consider ordinal measures of efficiency. For example, Bhalgat et al [5] calculate the approximation ratio of random priority when the objective is the maximization of *ordinal* social welfare, a notion of welfare that they define based solely on ordinal information. Ordinal measures of efficiency have also been studied in terms of incentives and approximation ratios [8, 15]. However, these measures do not encapsulate the "socially desired" outcome in the way that social welfare does, i.e., they do not necessarily maximize the aggregate happiness of individuals [3]. This is even more evident if one considers that the the standard assumption in social choice and economics theory is that such an underlying cardinal structure exists, even if agents are not asked to report it. In our setting, not reporting the full cardinal information corresponds to using ordinal mechanisms, which by our main theorem, is enough for achieving the (asymptotically) best approximation guarantees. On the other hand, cardinal reports are also often encountered in literature, with the pseudo-market mechanism of [13] being a prominent example. Several cardinal mechanisms were also presented in [11, 12] for social welfare maximization and [10] for information elicitation. Our main theorem does not only prove the capabilities of random-priority but also the limitations of all truthful (including cardinal) mechanisms. In the full version of our paper, we give an example of an non-ordinal mechanism that actually provides better approximation guarantees for concrete sizes of the set of agents.

Independently of our work, Adamczyk et al. [2] study truthful mechanisms for social welfare in one-sided matchings when agents have von Neumann - Morgenstern utilities, normalized in the unit interval, but not necessarily unit-sum or unit-range. Their main result on the approximation ratio of random priority can be combined with some additional arguments to obtain our lower bounds but our upper bounds are more general.

2 Preliminaries

Let $N = \{1, \dots, n\}$ be a finite set of agents and $M = \{1, \dots, n\}$ be a finite set of indivisible items. An *outcome* is a matching of agents to items, that is, an assignment of items to agents where each agent gets assigned exactly one item. We can view an outcome μ as a vector $(\mu_1, \mu_2 \dots, \mu_n)$ where μ_i is the unique item matched with agent i. Let O be the set of all outcomes. Each agent i has a private valuation function mapping outcomes to real numbers that can be arbitrary except for two conditions; agents are indifferent between outcomes that match them to the same item and they are not indifferent between outcomes that match them to different items. The first condition implies that agents only need to specify their valuations for items instead of outcomes and hence the valuation function of an agent i can be instead defined as a map $u_i : M \to \mathbb{R}$ from items to real numbers. The second condition requires that valuation functions are injective, i.e., they induce a total ordering on the items. This is mainly for convenience, to avoid having to specify tie-breaking rules. As we discuss in Section 5, all of our results extend to most natural tie-breaking rules. Valuation functions are standardly considered to be well-defined up to positive affine transformations, that is, for item $j : j \to \alpha u_i(j) + \beta$ is considered to be a different representation of u_i. The two standard ways to fix the canonical representation of u_i in literature are *unit-range*, i.e., $\max_j u_i(j) = 1$ and $\min_j u_i(j) = 0$ and *unit-sum*, that is $\sum_j u_i(j) = 1$.

Let V be the set of all canonically represented valuation functions of an agent. Call $\mathbf{u} = (u_1, u_2, \dots, u_n)$ a *valuation profile* and let V^n be the set of all valuation profiles with n agents. A *direct revelation mechanism* (without money) is a function $J : V^n \to O$ mapping *reported* valuation profiles to matchings. For a randomized mechanism, we define J to be random map $J : V^n \to O$. Let $J(\mathbf{u})_i$ denote the restriction of the outcome of the mechanism to the i'th coordinate, which is the item assigned to agent i by the mechanism.

We will be interested in *truthful mechanisms*, that is, mechanisms that do not incentivize agents to report anything other than their true valuation functions. Formally, a mechanism J is truthful if for each agent i and all $\mathbf{u} = (u_i, u_{-i}) \in V^n$ and $\tilde{u}_i \in V$ it holds that $u_i(J(u_i, u_{-i})_i) \geq u_i(J(\tilde{u}_i, u_{-i})_i)$, where u_{-i} denotes the valuation profile $\mathbf{u}$ without the i'th coordinate. In other words, if u_i is agent i's true valuation function, then it has no incentive to misreport. For randomized mechanisms, a mechanism is *truthful-in-expectation* if for each agent i and all $\mathbf{u} = (u_i, u_{-i}) \in V^n$ and $\tilde{u}_i \in V$ it holds that $\mathbb{E}[u_i(J(u_i, u_{-i})_i)] \geq \mathbb{E}[u_i(J(\tilde{u}_i, u_{-i})_i)]$.

A class of mechanisms that turns out to be important for our purposes is that of neutral and anonymous mechanisms. Formally, a mechanism is *anonymous* if for any valuation profile $(u_1, u_2, \dots, u_n)$, every agent i and any permutation $\pi : N \to N$ it holds that $J(u_1, u_2, \dots, u_n)_i = J(u_{\pi(1)}, u_{\pi(2)}, \dots, u_{\pi(n)})_{\pi(i)}$. By this definition, in an anonymous mechanism, agents with exactly the same valuation functions must have the same probabilities of receiving each item. Similarly, a mechanism is *neutral* if for any valuation profile $(u_1, u_2, \dots, u_n)$, every item j and any permutation $\sigma : M \to M$ it holds that $J(u_1, u_2, \dots, u_n)_i = \sigma^{-1}(J(u_1 \circ \sigma, u_2 \circ \sigma, \dots, u_n \circ \sigma)_i)$, i.e., the mechanism is invariant to the indices of the items.

We will consider both ordinal and cardinal mechanisms. A mechanism J is *ordinal* if for any i, any valuation profile $\mathbf{u} = (u_i, u_{-i})$ and any valuation function u'_i such that for all $j, j' \in M$, $u_i(j) < u_i(j') \Leftrightarrow u'_i(j) < u'_i(j')$, it holds that $J(u_i, u_{-i}) = J(u'_i, u_{-i})$. A mechanism for which the above does not necessarily hold is *cardinal*. Informally, ordinal mechanisms operate solely based on the *ordering* of items induced by the valuation functions and not the actual numerical values themselves.

We measure the performance of a mechanism by its approximation ratio,

$$ar(J) = \inf_{\mathbf{u} \in V^n} \frac{\sum_{i=1}^n u_i(J(\mathbf{u})_i)}{\max_{\mu \in O} \sum_{i=1}^n u_i(\mu_i)}.$$

The quantity $\sum_{i=1}^n u_i(J(\mathbf{u})_i)$ is the *social welfare* of mechanism J on valuation profile $\mathbf{u}$ and $\max_{\mu \in O} \sum_{i=1}^n u_i(\mu_i)$ is the social welfare of the optimal matching. For ease of notation, let $w^*(\mathbf{u}) = \max_{\mu \in O} \sum_{i=1}^n u_i(\mu_i)$. For randomized mechanisms, we are interested in the *expected social welfare* $\mathbb{E}\left[\sum_{i=1}^n u_i(J(\mathbf{u})_i)\right]$ of mechanism J and the approximation ratio is defined accordingly.

Next we will state a lemma that will be useful for our proofs. These kinds of lemmas are standard in literature (e.g. see [11, 12]). The lemma implies that when trying to prove upper bounds on the approximation ratio of mechanisms, it suffices to consider mechanisms that are anonymous.

Lemma 1. *For any mechanism J, there exists an anonymous mechanism J' such that $ar(J') \geq ar(J)$. Furthermore, if J is truthful (for deterministic mechanisms) or truthful-in-expectation (for randomized mechanisms) then it holds that J' is truthful-in-expectation.*

We will particularly be interested in the mechanism *random priority*. Random priority fixes an ordering of the agents uniformly at random and then lets them pick their favorite items from the set of available items based on this ordering. Note that random priority is truthful-in-expectation, ordinal, anonymous and neutral. We conclude the section with the following lemma.

Lemma 2. *For any valuation profile* $\mathbf{u}$*, the optimal allocation on* $\mathbf{u}$ *is a possible outcome of random priority.*

3 Unit-Range Valuation Functions

In this section, we assume that the representation of the valuation functions is unit-range. It will be useful to consider a special class of valuation functions C_ϵ that we will refer to as *quasi-combinatorial valuation functions*, a straightforward adaptation of a similar notion in [11]. Informally, a valuation function is quasi-combinatorial if the valuations of each agent for every item are close to 1 or close to 0 (the proximity depends on ϵ). Formally,

$$C_\epsilon = \{u \in V | u(M) \subset [0, \epsilon) \cup (1 - \epsilon, 1]\},$$

where $u(M)$ is the image of the valuation function u. Let $C_\epsilon^n \subseteq V^n$ be the set of all valuation profiles with n agents whose valuation functions are in C_ϵ. The following lemma implies that when we are trying to prove a lower bound on the approximation ratio of random priority, it suffices to restrict our attention to quasi-combinatorial valuation profiles $C_\epsilon^n \subseteq V^n$ for any value of ϵ.

Lemma 3. *Let J be an ordinal, anonymous and neutral randomized mechanism for unit-range representation, and let $\epsilon > 0$. Then*

$$ar(J) = \inf_{\mathbf{u} \in C_\epsilon^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(J(\mathbf{u})_i)]}{w^*(\mathbf{u})}.$$

The lemma formalizes the intuition that because the mechanism is ordinal, the worst-case approximation ratio is encountered on extreme valuation profiles. In fact, the proof of the lemma inductively "pushes" agents' valuations towards 0 or towards 1, without increasing the approximation ratio each time and ends up with a quasi-combinatorial profile.

For unit-range representation, Theorem 1 is given by the following lemmas.

Lemma 4. *For unit-range representation, $ar(RP) = \Omega\left(n^{-1/2}\right)$.*

Proof. Because of Lemma 3, for computing a lower bound on the approximation ratio of random priority, it is sufficient to only consider quasi-combinatorial valuation profiles. Let $\epsilon \leq 1/n^3$. Then, there exists $k \in \mathbb{N}$ such that

$$|k - w^*(\mathbf{u})| \leq \frac{1}{n^2},$$

where $w^*(\mathbf{u})$ is the social welfare of the maximum weight matching on valuation profile $\mathbf{u}$. Since random priority can trivially achieve an expected welfare of 1 (for any permutation the first agent will be matched to its favorite item), we can assume that $k \geq \sqrt{n}$, otherwise we are done. Note that the maximum weight matching $\mu^* \in O$ assigns k items to agents with $u_i(\mu_i) \in (1-\epsilon, 1]$. Since random priority is anonymous and neutral, without loss of generality we can assume that these agents are $\{1, \ldots, k\}$ and for every agent $j \in N$, it holds that $\mu_j^* = j$. Thus $u_j(j) \in (1-\epsilon, 1]$ for $j = 1, \ldots, k$ and $u_j(j) \in [0, \epsilon)$ for $j = k+1, \ldots, n$.

Consider any run of random priority; one agent is selected in each round. Let $l \in \{0, \ldots, n-1\}$ be any of the n rounds. We will now define the following sets:

$$U_l = \{j \in \{1, \ldots, n\} | \text{ agent } j \text{ has not been selected prior to round } l\}$$
$$G_l = \{j \in U_l | u_j(j) \in (1-\epsilon, 1] \text{ and item } j \text{ is still unmatched}\}$$
$$B_l = \{j \in U_l | u_j(j) \in [0, \epsilon) \text{ or item } j \text{ has already been matched to some agent}\}$$

These three families of sets should be interpreted as three sets that change over the course of the execution of random priority. U_l is the set of agents yet to be matched, which is partitioned into G_l, the set of "good" agents, that guarantee a welfare of almost 1 when picked, and B_l, the set of "bad" agents, that do not guarantee any contribution to the social welfare. For the purpose of calculating a lower bound, we will simply bound the sizes of the sets in these families. Obviously, $G_0 = \{1, \ldots, k\}$ and $B_0 = \{k+1, \ldots, n\}$.

The probability that an agent $i \in G_l$ is picked in round l of random priority is $|G_l|/(|G_l| + |B_l|)$, whereas the probability that an agent $i \in B_l$ is picked is $|B_l|/(|G_l| + |B_l|)$. By the discussion above, we can assume that whenever an agent from G_l is picked its contribution to the social welfare is at least $1 - \epsilon$ whereas the contribution from an agent picked from B_l is less than ϵ. In other words, the expected contribution to the social welfare from round l is at least $|G_l|/(|G_l| + |B_l|) - \epsilon$.

We will now upper bound $|G_l|$ and lower bound $|B_l|$ for each l. Consider round l and sizes $|G_l|$ and $|B_l|$. First suppose that some agent i from G_l is picked and the agent is matched with item j. If $j \neq i$ and agent j is in G_l, then $|G_{l+1}| = |G_l| - 2$ and $|B_{l+1}| = |B_l| + 1$, since agent j no longer has its item from the optimal allocation available and so agent j is in B_{l+1}. On the other hand, if $j = i$ or agent j is in B_l then $|G_{l+1}| = |G_l| - 1$ and $|B_{l+1}| = |B_l|$. In either case, $|G_{l+1}| \geq |G_l| - 2$ and $|B_{l+1}| \leq |B_l| + 1$. Intuitively, the picked agent might take away some item from a good agent and turn it into a bad agent.

Now suppose that agent i from B_l is picked and the agent is matched with item j. If agent j is in G_l then $|G_{l+1}| = |G_l| - 1$ and $|B_{l+1}| = |B_l|$, since agent j no longer has its item from the optimal allocation available and so agent j is in B_{l+1}. On the other hand, if agent j is in B_l then $|G_{l+1}| = |G_l|$ and $|B_{l+1}| = |B_l| - 1$. In either case, $|G_{l+1}| \geq |G_l| - 2$ and $|B_{l+1}| \leq |B_l| + 1$.

To sum up, in each round l of random priority, we can assume the size of B_l increases by at most 1 and the size of G_l decreases by at most 2. Given this and that $|G_0| = k$ and $|B_0| = n - k$ and that $|G_l| > 0$ for $l \leq \lfloor k/2 \rfloor$, we get

$$\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u})_i)\right] \geq \sum_{l=0}^{n} \left(\frac{|G_l|}{|G_l| + |B_l|} - \epsilon\right) \geq \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k - 2l}{n - l} - n\epsilon$$

and the ratio is

$$\frac{\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u})_i)\right]}{w^*(\mathbf{u})} \geq \frac{\sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k-2l}{n-l} - n\epsilon}{k + \frac{1}{n^2}} \geq \frac{\sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k-2l}{n-l} - n\epsilon}{2k}$$
$$= \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1 - \frac{2l}{k}}{2(n-l)} - \frac{n\epsilon}{2k} > \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1 - \frac{2l}{k}}{2n} - \frac{n\epsilon}{2k} \geq \frac{k - 11}{8n} - \frac{n\epsilon}{2k}.$$

The bound is clearly minimum when k is minimum, that is, $k = \sqrt{n}$. Since this bound holds for any $\mathbf{u} \in C_\epsilon^n$, we get

$$ar(RP) = \inf_{\mathbf{u} \in C_\epsilon^n} \frac{\mathbb{E}[\sum_{i=1}^{n} u_i(J(\mathbf{u})_i)]}{w^*(\mathbf{u})} \geq \frac{\sqrt{n} - 11}{8n} - \frac{n\epsilon}{2\sqrt{n}}.$$

We can choose ϵ so that the approximation ratio is at least $\frac{1}{20\sqrt{n}}$ for $n \geq 400$ and for $n \leq 400$, the bound holds trivially since random priority matches at least one agent with its favorite item. □

Next, we state the following lemma about ordinal mechanisms. We leave the details for the full version.

Lemma 5. *Let J be any ordinal mechanism for unit-range representation. Then $ar(J) = O\left(n^{-1/2}\right)$.*

Our final lemma provides a matching upper bound on the approximation ratio of any truthful-in-expectation mechanism.

Lemma 6. *Let J be a truthful-in-expectation mechanism for unit-range representation. Then $ar(J) = O\left(n^{-1/2}\right)$.*

Proof. By Lemma 1, we can assume that Mechanism J is anonymous. Let $k \geq 1$ be a parameter to be chosen later and let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ be the valuation profile where

$$u_i(j) = \begin{cases} 1, & \text{for } j = i \\ \frac{2}{k} - \frac{j}{n}, & \text{for } 1 \leq j \leq k+1, j \neq i \\ \frac{n-j}{n^2}, & \text{otherwise} \end{cases} \qquad \forall i \in \{1, \ldots, k+1\},$$

$$u_i(j) = \begin{cases} 1, & \text{for } j = 1 \\ \frac{2}{k} - \frac{j}{n}, & \text{for } 2 \leq j \leq k+1 \\ \frac{n-j}{n^2}, & \text{otherwise} \end{cases} \qquad \forall i \in \{k+2, \ldots, n\}.$$

For $i = 2, \ldots, k+1$, let $\mathbf{u^i} = (u'_i, u_{-i})$ be the valuation profile where all agents besides agent i have the same valuations as in $\mathbf{u}$ and $u'_i = u_{k+2}$. Note that when agent i on valuation profile $\mathbf{u^i}$, reports u_i instead of u'_i, the resulting valuation profile is $\mathbf{u}$. Since J is anonymous and $u'_i = u_1 = u_{k+2} = \ldots = u_n$, then agent i receives at most a uniform lottery among these agents on valuation profile $\mathbf{u^i}$ and so it holds that

$$\begin{aligned} \mathbb{E}[u'_i(J(\mathbf{u^i})_i)] &\leq \frac{1}{n-k+1} + \sum_{j=2}^{k+1} \frac{1}{n-k+1}\left(\frac{2}{k} - \frac{j}{n}\right) + \sum_{j=k+2}^{n} \frac{1}{n-k+1} \cdot \frac{n-j}{n^2} \\ &\leq \frac{4}{n-k+1}. \end{aligned}$$

Next observe that since J is truthful-in-expectation, agent i should not increase its expected utility by misreporting u_i instead of u'_i on valuation profile $\mathbf{u^i}$, that is,

$$\mathbb{E}[u'_i(J(\mathbf{u^i})_i)] \geq \mathbb{E}[u'_i(J(\mathbf{u})_i)]. \tag{1}$$

For all $i = 2, \ldots, k+1$, let p_i be the probability that $J(\mathbf{u})_i = i$. Then,

$$\mathbb{E}[u'_i(J(\mathbf{u})_i)] \geq p_i\left(\frac{2}{k} - \frac{i}{n}\right) \geq p_i\left(\frac{2}{k} - \frac{k+1}{n}\right),$$

and by Inequality (1) we get

$$\begin{aligned} p_i\left(\frac{2}{k} - \frac{k+1}{n}\right) &\leq \frac{4}{n-k+1} \\ \Rightarrow p_i &\leq \frac{4}{n-k+1} \cdot \frac{kn}{2n-k(k+1)} \leq \frac{4}{n-k} \cdot \frac{kn}{2n-(k+1)^2}. \end{aligned}$$

Let $p = \frac{4}{n-k} \cdot \frac{kn}{2n-(k+1)^2}$, i.e. for all i, $p_i \leq p$. We will next calculate an upper bound on the expected social welfare achieved by J on valuation profile $\mathbf{u}$.

For item $j = 1$, the contribution to the social welfare is upper bounded by 1. Similarly, for each item $j = k+2, \ldots, n$, its contribution to the social welfare is upper bounded by $1/n$. Overall, the total contribution by item 1 and items $k+2, \ldots, n$ will be upper bounded by 2.

We next consider the contribution to the social welfare from items $j = 2, \ldots, k+1$. Define the random variables

$$X_j = \begin{cases} 1, & \text{if } J(\mathbf{u})_j = j \\ \frac{2}{k} - \frac{j}{n}, & \text{otherwise} \end{cases} .$$

The contribution from items $j = 2, \ldots, k+1$ is then $\sum_{j=2}^{k+1} X_j$ and so we get

$$\mathbb{E}\left[\sum_{j=2}^{k+1} X_j\right] = \sum_{j=2}^{k+1} \mathbb{E}\left[X_j\right] \leq \sum_{j=2}^{k+1} \left(p + \frac{2}{k} - \frac{j}{n}\right) \leq kp + 2.$$

Overall, the expected social welfare of mechanism J is at most $4 + pk$ while the social welfare of the optimal matching is $k + 1 + \sum_{i=k+2}^{n} \frac{n-i}{n^2}$ which is at least k. Since $p = \frac{4}{n-k} \cdot \frac{kn}{2n-(k+1)^2}$, the approximation ratio of J then is

$$ar(J) \leq \frac{4+pk}{k} = \frac{4}{k} + \frac{4}{n-k} \cdot \frac{kn}{2n-(k+1)^2}.$$

Let $k = \lfloor \sqrt{n} \rfloor - 1$ and note that $\sqrt{n} - 2 \leq k \leq \sqrt{n} - 1$. Then,

$$\begin{aligned} ar(J) &\leq \frac{4}{k} + \frac{4}{n-k} \cdot \frac{kn}{2n-(k+1)^2} \leq \frac{4}{\sqrt{n}-2} + \frac{4}{n-\sqrt{n}+1} \cdot \frac{(\sqrt{n}-1)n}{2n-(\sqrt{n})^2} \\ &\leq \frac{4}{\sqrt{n}-2} + \frac{4}{\sqrt{n}} \leq \frac{12}{\sqrt{n}} + \frac{4}{\sqrt{n}} = \frac{16}{\sqrt{n}}. \end{aligned}$$

The last inequality holds for $n \geq 9$ and for $n < 9$ the bound holds trivially. This completes the proof. □

4 Unit-Sum Valuation Functions

In this section, we assume that the representation of the valuation functions is unit-sum. We prove Theorem 1 using the following three lemmas. The first lemma provides a lower bound on the approximation ratio of random priority.

Lemma 7. *For unit-sum representation, $ar(RP) = \Omega\left(n^{-1/2}\right)$.*

Proof. Let $\mathbf{u}$ be any unit-sum valuation profile and let C be the constant in the bound from Lemma 4. Suppose first that the $w^*(\mathbf{u}) < 4\sqrt{n}/C$. We will show that random priority guarantees an expected social welfare of 1, which proves the lower bound for this case. Consider any agent i and notice that in random priority, the probability that the agent is picked by the l'th round is l/n, for any $1 \leq l \leq n$ and hence the probability of the agent getting one of its l most preferred items is at least l/n. Let u_i^l be agent i's valuation for its l'th most preferred item; agent i's expected utility for the first round is then at least u_i^1/n. For the second round, in the worst case, agent i's most preferred item has already been matched to a different agent and so the expected utility of the agent for the first two rounds is at least $u_i^1/n + u_i^2/n$. By the same argument, agent i's expected utility after n rounds is at least $\sum_{i=1}^{n} u_i^l/n = 1/n$. Since this holds for each of the n agents, the expected social welfare is at least 1.

Suppose now $w^*(\mathbf{u}) \geq 4\sqrt{n}/C$. We will transform $\mathbf{u}$ to a unit-range valuation profile $\mathbf{u}''$. By Lemma 2, the optimal allocation can be achieved by a run of random priority, so we know that in the optimal allocation at most 1 agent will be matched with its least preferred item. Now consider the valuation profile $\mathbf{u}'$ where each agent i's valuation for its least preferred item is set to 0 (unless it already is 0) and the rest of the valuations are as in $\mathbf{u}$. Since the ordinal preferences of agents are unchanged, random priority performs worse on this valuation profile, and because of Lemma 2, $w^*(\mathbf{u}') \geq w^*(\mathbf{u}) - 1/n$. Next consider the valuation profile

$$\mathbf{u}'' = \begin{pmatrix} \mathbf{u}' & \mathbf{1} \\ \mathbf{o}^T & 1 \end{pmatrix}.$$

where $\mathbf{o} \in \mathbb{R}^n$ and $\mathbf{o}_j = (j-1)/n^5$. That is, $\mathbf{u}''$ has $n+1$ agents and items, where agents $1, \ldots, n$ have the same valuations for items $1, \ldots, n$ as in $\mathbf{u}'$, every agent has a valuation of 1 for item $n+1$, and agent $n+1$ only has a significant valuation for item $n+1$. Notice that $\mathbf{u}''$ is a unit-range valuation profile, and $w^*(\mathbf{u}'') \geq w^*(\mathbf{u}') + 1$. Furthermore, $\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u}'))\right] \geq \mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u}''))\right] - 2$ and hence

$$\begin{aligned}
\frac{\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u})_i)\right]}{w^*(\mathbf{u})} &\geq \frac{\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u}')_i)\right]}{w^*(\mathbf{u}') + 1/n} \geq \frac{\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u}'')_i)\right] - 2}{w^*(\mathbf{u}'') + 1/n - 1} \\
&\geq \frac{\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP(\mathbf{u}'')_i)\right]}{w^*(\mathbf{u}'')} - \frac{2}{w^*(\mathbf{u}'')} \geq \frac{C}{\sqrt{n}} - \frac{2}{w^*(\mathbf{u})} \\
&\geq \frac{C}{\sqrt{n}} - \frac{2}{4\sqrt{n}/C} = \frac{C}{2\sqrt{n}}.
\end{aligned}$$

This completes the proof. □

Next, we bound the approximation ratio of any ordinal (not necessarily truthful-in-expectation) mechanism. The proof is similar to that of Lemma 5.

Lemma 8. *Let J be an ordinal mechanism for unit-sum representation. Then $ar(J) = O\left(n^{-1/2}\right)$.*

Finally, the upper bound for any truthful-in-expectation mechanism is given by the following lemma.

Lemma 9. *Let J be a truthful-in-expectation mechanism for unit-sum representation. Then $ar(J) = O\left(n^{-1/2}\right)$.*

For the proof, the main observation is that the valuation profile used in the proof of Lemma 6 can easily be modified in a way such that all rows of the matrices of valuations sum up to one, to obtain a unit-sum valuation profile. Then exactly the same steps used in the proof of Lemma 6 also prove Lemma 9.

5 Extensions

Allowing Ties

Our results extend if we allow ties in the image of the valuation function. All of our upper bounds hold trivially. For the approximation guarantee of random priority, first the mechanism clearly must be equipped with some tie-breaking rule to settle cases where indifferences appear. For all natural (fixed before the execution of the mechanism) tie-breaking rules the lower bounds still hold. To see this, consider any valuation profile with ties and a tie-breaking rule for random priority. We can simply add sufficiently small quantities ϵ_{ij} to the valuation profile according to the tie-breaking rule and create a new profile without ties. The assignment probabilities of random priority will be exactly the same as for the version with ties, and random priority achieves an $\Omega(1/\sqrt{n})$ approximation ratio on the new profile. Then since ϵ_{ij} were sufficiently small, the same bound holds for the original valuation profile.

$[\mathbf{0}, \mathbf{1}]$ Valuation Functions

All of our results apply to the extension of the unit-range representation where 0 is not required to be in the image of the function, that is $\max_j u_i(j) = 1$ and for all j, $u_i(j) \in [0, 1]$. This representation captures scenarios where agents can be more or less indifferent between every single item. Since every unit-range valuation profile is also a valid profile for this representation, the upper bounds hold trivially. For the approximation ratio of random priority, we obtain the following corollary.

Corollary 1. *The approximation ratio of random priority, for the setting with $[0, 1]$ valuation functions is $\Omega\left(n^{-1/2}\right)$.*

The intuition for the proof is that, similarly to the proof of Lemma 7, we can "turn" a $[0, 1]$ profile into a unit-range profile with only a constant change in the approximation ratio and then use our bound for unit-range profiles. The details are left for the full version.

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Efficiency of Truthful and Symmetric Mechanisms in One-Sided Matching*

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Abstract. We study the efficiency (in terms of social welfare) of truthful and symmetric mechanisms in one-sided matching problems with *dichotomous preferences* and *normalized von Neumann-Morgenstern preferences*. We are particularly interested in the well-known *Random Serial Dictatorship* mechanism. For dichotomous preferences, we first show that truthful, symmetric and optimal mechanisms exist if intractable mechanisms are allowed. We then provide a connection to online bipartite matching. Using this connection, it is possible to design truthful, symmetric and tractable mechanisms that extract 0.69 of the maximum social welfare, which works under assumption that agents are not adversarial. Without this assumption, we show that Random Serial Dictatorship always returns an assignment in which the expected social welfare is at least a third of the maximum social welfare. For normalized von Neumann-Morgenstern preferences, we show that Random Serial Dictatorship always returns an assignment in which the expected social welfare is at least $\frac{1}{e}\frac{\nu(\mathcal{O})^2}{n}$, where $\nu(\mathcal{O})$ is the maximum social welfare and n is the number of both agents and items. On the hardness side, we show that no truthful mechanism can achieve a social welfare better than $\frac{\nu(\mathcal{O})^2}{n}$.

1 Introduction

We study the efficiency of mechanisms in one-sided matching problems, where the goal is to allocate n *indivisible* items to n unit-demand *rational* agents having *private* preferences over items. Agents are rational, i.e., they would like to be assigned to the best items according to their private preferences. The problem essentially captures variants of practical applications such as allocating houses to residents, assigning professors to courses and so on. In this paper, we mainly focus on *cardinal preferences* in which agents have values for different items. A practical setting would be that residents have values for different houses. A *mechanism* maps preferences that agents report to a matching, which is a one-to-one mapping between agents and items. Throughout the paper, depending on the context, we use sometimes term *matching* and sometimes *assignment*, but they

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always mean essentially the same. One immediate question arises: if there exist mechanisms in which no agent could benefit by deviating from reporting his true preference regardless the preferences reported by other agents? Such mechanisms are often called *truthful* mechanisms. The question was answered in [15], where it was shown that there exists only one *truthful, nonbossy* and *neutral* mechanism. A mechanism is nonbossy if an individual agent cannot change the output of the mechanism without changing his assignment. A mechanism is neutral if the mechanism is independent of the identities of items, e.g., the assignment get permuted accordingly when the items are permuted. The unique mechanism works as follows. First, agents are sorted in a fixed order, and then the first agent chooses his favorite item, the next agent chooses his favorite item among remaining items, etc. When the fixed order is picked uniformly among all possible orderings, the resulted mechanism is called *Random Serial Dictatorship (RSD).*

Besides the truthfulness, an important issue left is to understand the efficiency of mechanisms in one-sided matching problems. The efficiency of a mechanism is defined as the social welfare of the assignment the mechanism returns. Zhou [17] confirmed Gale's conjecture by showing that there is no *symmetric, Pareto optimal* and *truthful* mechanism for general preferences. A mechanism is symmetric if agents are treated equally if they report the same preferences. A mechanism is Pareto optimal if the mechanism never outputs an assignment that the social welfare could be improved without hurting any agent. It is well-known that RSD is truthful and ex post efficient, i.e., it never outputs Pareto dominated outcomes.

We observe that there is few work that study the efficiency of RSD. The main reason is that its average social welfare could be even $O(n)$ far away from the optimal social welfare if the preferences of agents for items are unrestricted. It happens when assigning a particular item to a particular agent contributes most of the optimal social welfare. However, in RSD it is possible that the agent only gets that item with a probability of $1/n$. In this paper, we circumvent this problem by considering smaller but still rich domains of preferences. The first type of preferences we consider is *dichotomous preferences*, where agents have binary preferences over items. We shall call this setting simply *dichotomous.* Dichotomous preferences are fairly natural in assignment problems. For example, professors indicate the courses they like or dislike to teach, or workers choose the working shifts they want. The goal here is to design good mechanisms to assign courses/shifts to professors/workers. One can model these problems with bipartite graphs: workers on one side, shifts on the other, an edge indicates whether a worker wants to participate in a particular shift. Then one can find a maximum matching in the graph to optimize the total value of the assignment. It is shown in [5] that with some careful tie-breaking rule, finding a maximum matching yields a truthful mechanism. However, such mechanisms fail to capture the symmetry. To make this approach symmetric, one could find all maximum matchings and randomly choose one. Note that it implies that Zhou's impossibility result does not pertains to dichotomous preferences. However, since finding all maximum matchings in bipartite graphs is $\#P$-complete, we conjecture that it is computationally infeasible to design truthful and symmetric mechanisms that obtain optimal welfare. Therefore, we turn our attention to investigate how well mechanisms can approximate the maximum social welfare. By the connection to the online bipartite matching problem [11,12], we get the following result:

Result 1 *In dichotomous setting there exists a truthful and symmetric mechanism that is a* 0.69*-approximation to the maximum social welfare.*

Due to the limitation of such mechanisms, next we show that RSD also obtains a constant approximation for dichotomous preferences.

Result 2 *Random Serial Dictatorship in dichotomous setting returns an assignment in which the expected social welfare is a 3-approximation of the maximum social welfare.*

The second type of preferences we consider is *normalized von Neumann-Morgenstern preferences*, where the value of agent i for item j lies in $[0, 1]$. We shall call this setting simply *normalized.* In this setting our result gives asymptotically tight description of the social welfare achieved by RSD.

Result 3 *In normalized setting with n agents and n items, Random Serial Dictatorship returns a matching which expected social welfare is at least $\frac{1}{e}\frac{\nu(\mathcal{O})^2}{n}$, where $\nu(\mathcal{O})$ is the maximum social welfare.*

This result implies that RSD achieves an $\sqrt{e \cdot n}$-approximation of the optimal social welfare in unit-range preferences, i.e., when $\max_i v_a(i) = 1$, $\min_i v_a(i) = 0$. Recently [6] presented an $O\left(\sqrt{n}\right)$-approximation for RSD in unit-range setting.

Finally, we complement the above result with the following upper-bound.

Result 4 *Given n, for any $k = 1, \ldots, n$ and for any $\epsilon > 0$ there exist an instance of one-sided matching problem with normalized von Neumann-Morgenstern preferences where $\nu\left(\mathcal{O}\right) = k$ and no truthful mechanism can achieve expected social welfare better than $\frac{k^2}{n} + \epsilon$, where k is the optimal social welfare.*

1.1 Related Work

Here we only mention the most relevant work on one-sided matching problems. For more details, we refer the reader to surveys [13,14]. One-sided matching problems modeled in [9] gave a market-like procedure to produce efficient assignments. There, the procedure is Pareto optimal but not truthful. Gale and Shapley [7] considered a similar problem, the marriage problem, but they turned attention to the incentive issues on whether agents would or would not reveal their private preferences. In [8] authors were asking about existence of good mechanisms when preferences are also considered. Zhou [17] answered this question by showing that there is no *symmetric, Pareto optimal* and *truthful* mechanism. Between ex-ante Pareto optimality and ex-post Pareto optimality, Bogomolnaia and Moulin [2] introduced a new concept called *ordinal efficiency.* They gave a *probabilistic serial* mechanism that always returns ordinal efficient assignments. However, the probabilistic serial mechanism is not truthful. Bhalgat et al. [1] studied the efficiency of RSD in a more restricted setting than ours, where agents have values of $\frac{n-j+1}{n}$ for their jth favorite item. Chakrabarty and Swamy [4] introduced the notion of rank approximation to measure the social welfare under ordinal preferences. One-sided matching problems with dichotomous preferences were studied by Bogomolnaia and Moulin [3]. They used the Gallai-Edmonds decomposition of bipartite graphs to characterize the (most) efficient assignments.The most related work to ours is that Filos-Ratsikas et al. [6] independently gave the similar approximation ratio of RSD under unit-range preferences while our results applies to more general settings.

Cardinal preferences enable agents to explicitly express how much they prefer each item, while this can not be done in ordinal preferences. The space of cardinal preferences could be shown to be the same as the space of von Neumann-Morgenstern preferences. In addition, the normalization of preferences is a standard procedure, see [10]. Besides the literature of operational research and decision theory, normalized von Neumann-Morgenstern preferences are widely used to model individual behavior in game-theoretical settings.

2 Preliminaries

The model We model one-sided matching problems as bipartite graphs. In a bipartite graph, its left side is a set A of agents and its right side are a set I of indivisible items. We assume $|A| = |I| = n$ and each agent is matched to exactly one item. For each agent $a \in A$ and each item $i \in I$, there is an edge (a, i) representing a possible allocation of item i to agent a. The preference of agent a for item i is denoted by $v_a(i)$, which is the value that agent a has for item i. We consider two different types of preferences, *dichotomous preferences* and *normalized von Neumann-Morgenstern preferences*. In dichotomous preferences, it holds that $v_a(i) \in \{0, 1\}$, while in normalized von Neumann-Morgenstern preferences, it holds that $v_a(i) \in [0, 1]$. In dichotomous case we shall say shortly that agent a *1-values* item i, if $v_a(i) = 1$, instead of clunky "agent a has value 1 for item i"; the same with value 0.

We say $v_a(\cdot)$ is the preference profile of agent a. Denote by $\mathcal{V}$ the set of all possible preference profiles of a single agent: for dichotomous preferences $\mathcal{V} = \{0, 1\}^I$, for normalized von Neumann-Morgenstern preferences $\mathcal{V} = [0, 1]^I$. Preference profiles of all agents are denoted by $v_A = (v_a)_{a \in A} \in \mathcal{V}^A$; by $v_{-a} = (v_{a'})_{a' \in A \setminus a}$ we denote all profiles except of agent a's. By (v'_a, v_{-a}) we denote agents' preferences with a's preference changed from v_a to v'_a; if (v'_a, v_{-a}) is an argument of a function, then we skip writing double brackets. Consider a set of items $I' \subseteq I$ and suppose that agent a values items $i_1, \ldots, i_k \in I'$ equally and more than any other item in I'. We say that items $i_1, \ldots, i_k$ are *favorite* items of agent a in I'.

We call matrix $p_A = (p_a)_{a \in A}$, where $p_a = \left(p_a^i\right)_{i \in I}$, a feasible *matching* if the following conditions hold: 1) for any $a \in A$ and $i \in I$, $p_a^i \in \{0, 1\}$; 2) for any $a \in A$, $\sum_{i \in I} p_a^i = 1$; 3) for any $i \in I$, $\sum_{a \in A} p_a^i = 1$. Given a feasible matching p_A, we say item i is matched to agent a if $p_a^i = 1$. Thus, the value of agent a for the matching p_A is given by $v_a \cdot p_a = \sum_{i \in I} v_a(i) p_a^i$, where $\cdot$ is an operator of the vector product. The social welfare of the matching p_A is given by $\nu(p_A) = \sum_{a \in A} v_a \cdot p_a$.

From each agent $a \in A$ mechanism $\mathcal{M}$ collects declarations $d_a \in \mathcal{V}$ about his preference profile — we overload notations here a bit, since vector d_a does not always have to be declared completely, i.e., when some of the items are already matched, then the mechanism does not ask a about values for these items. Of course, the connection between true valuations $v_a \in \mathcal{V}$ and declarations $d_a \in \mathcal{V}$, which $\mathcal{M}$ collects, depends heavily on the mechanism $\mathcal{M}$ itself. Mechanism $\mathcal{M}$ maps agents declarations d_A to a feasible matching $\mathcal{M}_A(d_A)$ (i.e., the p_A matrix); $\mathcal{M}_a(v_A)$ denotes the allocation to agent a (i.e., the p_a vector). Mechanism $\mathcal{M}$ might be randomized, and then matching $\mathcal{M}_A(d_A)$ is a random matrix, and allocation $\mathcal{M}_a(v_A)$ is a random vector as well. In this case, $\mathbb{E}[\nu(\mathcal{M}_A(d_A))]$ is the expected social welfare of mechanism $\mathcal{M}_A$, but since all of the mechanisms we analyze are randomized, we shall call it just social welfare.

We measure the performance of the mechanism by comparing the social welfare it produces with the optimal social welfare $\nu(\mathcal{O}(v_A))$, where $\mathcal{O}(v_A)$ denotes a matching that maximizes the social welfare when preferences are given by v_A. Note that $\mathcal{O}(v_A)$ can be seen as a maximum weight matching in the graph $G = (A \cup I, A \times I)$ where weight of edge (a, i) is equal to $v_a(i)$. For simplicity however, throughout the paper we shall just write $\mathcal{O}$, instead of $\mathcal{O}(v_A)$.

A mechanism $\mathcal{M}$ is *truthful*, if for every $a \in A$, every $v_A \in \mathcal{V}^A$ and every $v'_a \in \mathcal{V}$, it holds that (even when the mechanism is randomized)

$$v_a \cdot \mathcal{M}_a(v_A) \geq v_a \cdot \mathcal{M}_a(v'_a, v_{-a}).$$

A mechanism $\mathcal{M}$ is *symmetric* if for every $a_1, a_2 \in A$, every $d_A \in \mathcal{V}^A$ such that $d_{a_1} = d_{a_2}$, it holds that $\mathbb{E}[\mathcal{M}_{a_1}(d_A)] = \mathbb{E}[\mathcal{M}_{a_2}(d_A)]$, i.e., agents with identical declarations have the same (expected) value for the allocation.

RSD and iterative analysis Now let us give the formal description of the Random Serial Dictatorship (RSD) mechanism. RSD first picks an ordering of agents uniformly at random and then asks agents to choose sequentially with respect to the order. We assume that agents are rational, i.e., they will always choose the best items among the unmatched items. Ties are broken randomly, i.e., when agent a is asked in RSD and his favorite items are i_1 and i_2 among unmatched items, agent a will chose items i_1 and i_2 with an equal probability. This is an important assumption for the analysis of RSD with dichotomous preferences. If we would like to analyze RSD when agent would always deterministically choose among the best items, then the competitive ratio guarantees and lower bounds from von Neumann-Morgenstern preferences would apply.

Let us observe a property of RSD that is important for our analysis. Instead of thinking that a random ordering is fixed before any agent is considered sequentially, we can think that RSD chooses an agent randomly from remaining agents in each step. It is easy to see that agents are considered in the same random order in both cases.

RSD is iterative in nature, and so is the analysis. Let us index its time-steps by t, which ranges from 0 to n. $t = 0$ indicates the moment after sorting the agents, but before asking first agent to choose. Let $\mathcal{R}^t$ represent the (partial) matching constructed by RSD after first t steps. Then $\nu(\mathcal{R}^t)$ represents the social welfare obtained after first t steps; in particular $\nu\left(\mathcal{R}^0\right) = 0$. As RSD is being executed, the set of unmatched agents and the set of available items are gradually decreasing. Let A^t and I^t be the set of unmatched agents and the set of available items after step t. For example, $A^0 = A$ and $I^0 = I$. As the sets A^t and I^t are being modified, we also keep track of the way in which $\nu(\mathcal{O})$ is being changed (recall that $\mathcal{O}$ denotes a matching that maximizes the welfare). More precisely, we start with $\nu\left(\mathcal{O}^0\right) = \nu(\mathcal{O})$. Suppose that at step t, RSD asks agent a to choose and then a picks item i, then $\nu(\mathcal{R}^t) = \nu\left(\mathcal{R}^{t-1}\right) + v_a(i)$. We remove a from A^{t-1} and i from I^{t-1}, e.g., $A^t = A^{t-1} - \{a\}$ and $I^t = I^{t-1} - \{i\}$. In addition, we also remove welfare contributed by a and i from $\nu\left(\mathcal{O}^{t-1}\right)$. Certainly, when $t = n$, then $\nu(\mathcal{O}^n) = 0$, while $\nu(\mathcal{R}^n)$ is the social welfare obtained by RSD.

Sequence $\{\nu(\mathcal{R}^t)\}_{t \geq 0}$, which represents the increasing welfare of RSD, is a random process. Moreover, $\mathbb{E}[\nu(\mathcal{R}^n)]$ represents the expected social welfare returned by RSD. The sequence $\{\nu(\mathcal{O}^t)\}_{t \geq 0}$, which represents how the optimal social welfare is affected by the random choices within RSD, is a random process as well. Therefore, we want to describe a relation between $\mathbb{E}[\nu(\mathcal{R}^n)]$ and $\nu\left(\mathcal{O}^0\right)$, and to do so we deploy theory of martingales.

Martingales Below we only introduce notions and properties that we use later in the paper. For a thorough treatment of martingale theory see [16].

Definition 1 Consider a random process $(X^t)_{t=0}^n$. Suppose we observe first k steps of the process, and let $\mathcal{H}^k$ denote the information we have acquired in steps $0, 1, \ldots, k$. Expected value of X^{k+1}, conditioned on the information we have from steps 0 to k, is formally presented as $\mathbb{E}\left[X^{k+1} \middle| \mathcal{H}^k\right]$. If for any $k = 0, \ldots, n-1$, we have $\mathbb{E}\left[X^{k+1} \middle| \mathcal{H}^k\right] = X^k$, then the process is called a *martingale.*

In other words, the process does not change on expectation in one step. We shall also consider a *sub-martingale* $(X^t)_{t=0}^n$ which satisfies $\mathbb{E}\left[X^{k+1} \middle| H^k\right] \geq X^k$ instead of equality in the above definition.

Theorem (Doob's Stopping Theorem) *Let* $(X^t)_{t=0}^n$ *be a martingale, respectively sub-martingale. For any* $k = 0, 1, \ldots, n$ *it holds that* $\mathbb{E}\left[X^k\right] = \mathbb{E}\left[X^0\right]$, *respectively* $\mathbb{E}\left[X^k\right] \geq \mathbb{E}\left[X^0\right]$.

The above is not the Doob's theorem in its full generality, but rather the simplest variant that still holds in our setting.

3 Dichotomous Preferences and Online Bipartite Matching

In this section, we establish a connection between one-sided matching with dichotomous preferences and online bipartite matching. A similar connection was also presented in [1].

Consider a variant of online bipartite matching. We are given a bipartite graph $G = (A \cup B, E)$, where one side A of the graph is given, while vertices from other side B and edges between A and B are unknown. Suppose that vertices from B arrive one by one, and upon the arrival of vertex $b \in B$, all edges adjacent to b are revealed. On vertices of A there is an ordering σ given by a random permutation. Consider RANKING algorithm that upon arrival of vertex $b \in B$ it matches b to the unmatched neighbor in $a \in A$ with the highest ranking $\sigma(a)$. In their seminal paper, Karp et al. [11] have proven that this algorithm constructs a matching of expected size at least $\left(1 - \frac{1}{e}\right) OPT$, where OPT is the offline optimum, and the bound holds even if the vertices of B arrive in an adversarial order. Furthermore, Mahdian and Yan [12] have shown that the performance of RANKING algorithm is even better when the order of vertices in B is also given by a random permutation:

Theorem *Given that the vertices in B arrive uniformly at random and the order of vertices in A is random, RANKING algorithm constructs a matching of expected size at least $0.69 \cdot OPT$, where OPT is the offline optimum.*

Now let us see consider the following mechanisms for one-sided matching with dichotomous preferences. Given the agents and items, mechanism RSD* generates a random ordering on agents and a random ranking on items. RSD* considers agents one by one according to the random ordering. Suppose that agent a is considered at step τ and let $d_a(\cdot)$ be the preference reported by agent a. Denote by I^τ the set of items yet unmatched at step τ. If agent a 1-values any unmatched item, RSD* assigns agent a an item with the highest rank among all remaining items. Otherwise, RSD* assigns nothing to agent a. Finally, RSD* matches any unmatched items to unmatched agents. Truthfulness of RSD* follows from the observation that τ as well as I^τ are independent of a's declaration d_a. More precisely, the moment τ is given only by a random permutation of agents, while set I^τ depends on the permutation of agents and declarations $d_{a'}$ of agents a' that came before a. Therefore, if a declares $d_a(i) = 1$ for item i such that $v_a(i) = 0$, then he can only increase the probability that at moment τ he is matched to a 0-valued item. Analogically, if a declares $d_a(i) = 0$ for item i such that $v_a(i) = 1$, then he can only decrease the probability that at moment τ he is matched to a 1-valued item. Suppose now that agent a has 0 value for all items in I^τ. In this case agent gains nothing regardless of what his declarations are. An agent that reports truthfully in this case, we call *non-adversarial*. Since RSD* is guided by two random permutations, the symmetry of the mechanism is clear.

Theorem 1 *Assuming that agents are non-adversarial, RSD* is a truthful and symmetric mechanism that achieves 0.69-approximation to the maximum social welfare in one-sided matching problems with dichotomous preferences.*

One can imagine that sometimes an agent can be adversarial, and he would not admit that he does not value any of the remaining items. To address this issue, in the next section we present an analysis of RSD mechanism in which every agent can be adversarial.

Algorithm 1. RSD*(A, I)

Let random permutation $\sigma : \{1, ..., n\} \mapsto \{1, ..., n\}$ be the ranking of items;
For each agent $a \in A$ in random order:
    ask agent a about his preference profile $d_a \in \mathcal{V} = \{0,1\}^I$;
    if there is no unmatched item i such that $d_a(i) = 1$, then discard agent a;
    otherwise, assign a to unmatched item i that has the highest rank $\sigma(i)$;
Match any unmatched items to unmatched agents anyhow.

4 Dichotomous Preferences and RSD

Theorem 2 *Random Serial Dictatorship always returns an matching in which the expected welfare is at least* $\frac{1}{3}\nu(\mathcal{O})$ *in one-sided matching problems with dichotomous preferences.*

Proof. Recall, $\mathcal{O}$ is an optimal matching. Let A^t be the set of agents remaining after t steps, let I^t be the set of remaining items, and $\mathcal{O}^t \subset \mathcal{O}$ is what remains from optimal solution after t steps of RSD. Also, $\mathcal{R}^t$ is the partial matching constructed by RSD after t steps, and $\nu(\mathcal{R}^t)$ be its welfare. For an agent a let $\mathcal{O}_a \in I$ be the item to which a is matched in $\mathcal{O}$.

Let Y^t be the set of agents who are matched to an item in $\mathcal{O}^t$ which they value 1, i.e., $\{a \in A^t | \, v_a(\mathcal{O}_a) = 1\}$. Therefore $|Y^t| = \nu(\mathcal{O}^t)$ for every t. It can happen that at time t, an agent does not 1-value any of remaining items I^t, even though he could have 1-valued some of the items in I^0. Thus let $Z^t \subseteq A^t$ be the agents who 0-value all items in I^t. Let us denote $y^t = |Y^t|$ and $z^t = |Z^t|$ for brevity. Consider step $t+1$ of RSD, and assume we have all information available after first t steps, represented by $\mathcal{H}^t$. Let a be the agent who is to make his choice in this step, and let i be the item a chooses. Agent a does not belong to Z^t with probability $1 - \frac{z^t}{n-t}$, and if this happens, then for sure $v_a(i) = 1$, which adds 1 to the welfare of RSD, i.e., $\nu\left(\mathcal{R}^{t+1}\right) = \nu(\mathcal{R}^t) + 1$. Hence $\mathbb{E}\left[\nu\left(\mathcal{R}^{t+1}\right) \middle| \mathcal{H}^t\right] = \nu(\mathcal{R}^t) + 1 - \frac{z^t}{n-t}$.

Now let us analyze the expected decrease $\nu(\mathcal{O}^t) - \nu\left(\mathcal{O}^{t+1}\right)$. Suppose that agent a does not belong to Z^t, again with probability $1 - \frac{z^t}{n-t}$. Edge (a, i) is adjacent to at most two 1-value edges in $\mathcal{O}^t$, since $\mathcal{O}^t$ is a feasible matching. Thus when $a \notin Z^t$, then $\nu(\mathcal{O}^t) - \nu\left(\mathcal{O}^{t+1}\right)$ is at most 2 . Now suppose that agent a belongs to Z^t, which happens with probability $\frac{z^t}{n-t}$. Since $v_a(i) = 0$, then a is **not** adjacent to any 1-value edge in $\mathcal{O}^t$, and i may be adjacent to at most one such edge since agent a choose an item randomly from unmatched items. Therefore, when $a \in Z^t$, then $\nu(\mathcal{O}^t) - \nu\left(\mathcal{O}^{t+1}\right)$ is at most 1. Hence, together with noting that $\frac{z^t}{n-t} + \frac{y^t}{n-t} \leq 1$, we can conclude that the expected decrease $\nu(\mathcal{O}^t) - \nu\left(\mathcal{O}^{t+1}\right)$ is:

$$\mathbb{E}\left[\nu\left(\mathcal{O}^t\right) - \nu\left(\mathcal{O}^{t+1}\right) \middle| \mathcal{H}^t\right] \leq 2 \cdot \left(1 - \frac{z^t}{n-t}\right) + \frac{z^t}{n-t} \cdot \frac{y^t}{n-t} \leq 3 \cdot \left(1 - \frac{z^t}{n-t}\right).$$

Since $\mathbb{E}\left[\nu\left(\mathcal{R}^{t+1}\right)\middle|\,\mathcal{H}^t\right] = \nu\left(\mathcal{R}^t\right) + 1 - \frac{z^t}{n-t}$, we get that

$$\mathbb{E}\left[\nu\left(\mathcal{O}^t\right) - \nu\left(\mathcal{O}^{t+1}\right)\middle|\,\mathcal{H}^t\right] \leq 3 \cdot \left(1 - \frac{z^t}{n-t}\right) = 3 \cdot \mathbb{E}\left[\nu\left(\mathcal{R}^{t+1}\right) - \nu\left(\mathcal{R}^t\right)\middle|\,\mathcal{H}^t\right].$$

This means that sequence $(X^t)_{t=0}^n$, defined by $X^0 = 0$ and $X^{t+1} - X^t = 3 \cdot \left(\nu\left(\mathcal{R}^{t+1}\right) - \nu\left(\mathcal{R}^t\right)\right) - \left(\nu\left(\mathcal{O}^t\right) - \nu\left(\mathcal{O}^{t+1}\right)\right)$, satisfies $\mathbb{E}\left[X^{t+1}\middle|\,\mathcal{H}^t\right] \geq X^t$, and therefore is a sub-martingale. From Doobs Stopping Theorem we get that $\mathbb{E}\left[X^n\right] \geq \mathbb{E}\left[X^0\right] = 0$, and hence

$$0 \leq \mathbb{E}\left[X^n\right] = \mathbb{E}\left[\sum_{t=1}^n X^t - X^{t-1}\right] = 3 \cdot \mathbb{E}\left[\sum_{t=1}^n \nu\left(\mathcal{R}^t\right) - \nu\left(\mathcal{R}^{t-1}\right)\right] - $$
$$- \mathbb{E}\left[\sum_{t=1}^n \nu\left(\mathcal{O}^{t-1}\right) - \nu\left(\mathcal{O}^t\right)\right] = 3 \cdot \mathbb{E}\left[\nu\left(\mathcal{R}^n\right)\right] - \mathbb{E}\left[\nu\left(\mathcal{O}^0\right)\right],$$

since $\mathcal{R}^0 = \mathcal{O}^n = \emptyset$. This allows us to conclude that $3 \cdot \mathbb{E}\left[\nu\left(\mathcal{R}^n\right)\right] \geq \nu\left(\mathcal{O}\right)$, which finishes the proof. □

Our analysis is simple, and most likely not tight — approximation ratio should be below 3. On the other hand, it is not very close to 2, as there exist instances with dichotomous preferences in which RSD gives expected outcome close to $\frac{1}{2.28} \cdot \nu\left(\mathcal{O}\right)$. One can see a resemblance between the following instance and the worst case instance for algorithm RANDOM from Karp et al. [11].

Fact 1 Consider the following instance of a problem. We have numbers k, z and $n = z + k$, with k even, and also sets $A = \{1, ..., n\}$, $I = \{1, ..., n\}$. Define the valuations: $v_a\left(i\right) = 1$ if $a = i \in \{1, \ldots, k\}$ or $a \in \left\{1, \ldots, \frac{k}{2}\right\} \wedge i \in \left\{\frac{k}{2}, \ldots, k\right\}$, and 0 otherwise. The optimum solution in this case is obviously k. Simulations indicate that for $k = 10^4$ and $z = 10^7$, the expected performance of RSD is around 4378 giving ratio of $\frac{10^4}{4378} \approx 2.28$. Taking different values of k or z did not significantly changed the outcome of simulations.

5 Normalized Von Neumann-Morgenstern Preferences and RSD

Theorem 3 *Random Serial Dictatorship always returns an assignment in which the expected social welfare is at least* $\frac{1}{e}\frac{\nu(\mathcal{O})^2}{n}$ *in one-sided matching problems with normalized von Neumann- Morgenstern preferences, where* $\nu\left(\mathcal{O}\right)$ *is the maximum social welfare.*

Proof. As before, let $\mathcal{O}$ be the optimal assignment, and $\mathcal{O}^t \subseteq \mathcal{O}$ be the subset of the optimal assignment that remains after t steps of RSD. Consider step $t+1$, and let $\mathcal{H}^t$ be all information available after t steps. We choose agent a uniformly at random from the remaining agents, and then a chooses item i that he prefers the most, i.e., edge (a,i) has the greatest value among edges $\{(a,i)|\, i \in I^t\}$. The number of agents without an assigned item is exactly $n-t$ after t steps, and hence the probability of choosing a particular agent is $\frac{1}{n-t}$.

Let $\mathcal{O}(a)$ denote the item matched to agent a in $\mathcal{O}$. Since agent a has the largest value for item i among remaining items, it has to hold that $v_a(i) \geq v_a(\mathcal{O}(a))$. Therefore, the expected welfare of RSD in step $t+1$ increases at least

$$\sum_{a \in A^t} \frac{v_a(i)}{n-t} \geq \sum_{a \in A^t} \frac{v_a(\mathcal{O}(a))}{n-t} = \frac{\nu(\mathcal{O}^t)}{n-t},$$

and hence $\mathbb{E}\left[\nu\left(\mathcal{R}^{t+1}\right)\middle|\,\mathcal{H}^t\right] \geq \nu(\mathcal{R}^t) + \frac{\nu(\mathcal{O}^t)}{n-t}$. Similar martingale-based reasoning as in Section 4 yields that $\mathbb{E}[\nu(\mathcal{R}^n)] \geq \mathbb{E}\left[\sum_{t=0}^{n-1} \frac{\nu(\mathcal{O}^t)}{n-t}\right]$, so in the remaining part we give a lower bound on this sum.

When we remove agent a and item i in step $t+1$, what is the average decrease $\nu(\mathcal{O}^t) - \nu\left(\mathcal{O}^{t+1}\right)$? Surely, we remove edge $(a, \mathcal{O}^t(a))$ from $\mathcal{O}^t$. However, item i may be assigned a different agent than a in $\mathcal{O}^t$, and the value of this assignment can be arbitrary — let us denote by $L^{t+1} \in [0,1]$ the decrease of $\mathcal{O}^t$ caused by deleting the assignment of i. Therefore, the average decrease at step $t+1$ is $\nu(\mathcal{O}^t) - \mathbb{E}\left[\nu\left(\mathcal{O}^{t+1}\right)\middle|\,\mathcal{H}^t\right] = \mathbb{E}\left[L^{t+1}\middle|\,\mathcal{H}^t\right] + \frac{\nu(\mathcal{O}^t)}{n-t}$, so if we define sequence $(Y_t)_{t=1}^n$, where

$$Y^{t+1} = L^{t+1} + \frac{\nu(\mathcal{O}^t)}{n-t} - \left(\nu\left(\mathcal{O}^t\right) - \nu\left(\mathcal{O}^{t+1}\right)\right), \tag{1}$$

then $\mathbb{E}\left[Y^{t+1}\middle|\,\mathcal{H}^t\right] = 0$ for $t = 0, 1, \ldots, n-1$. We define another sequence $(X^t)_{t=0}^n$ with $X^0 = 0$ and $X^t = \sum_{i=1}^t Y^i$.

Equality $\mathbb{E}\left[Y^{t+1}\middle|\,\mathcal{H}^t\right] = 0$ implies $\mathbb{E}\left[X^{t+1}\middle|\,\mathcal{H}^t\right] = X_t$, which means that $(X^t)_{t=0}^n$ is a martingale, and from Doob's Stopping Theorem, we get that $0 = \mathbb{E}\left[X^0\right] = \mathbb{E}[X^n] = \mathbb{E}[\sum_{t=1}^n Y^t]$. Thus summing equality (1) for t from 1 to $n-1$ and taking expectation yields that

$$\mathbb{E}\left[\sum_{t=0}^{n-1} \frac{\nu(\mathcal{O}^t)}{n-t}\right] = \nu(\mathcal{O}) - \mathbb{E}\left[\sum_{t=1}^{n-1} L^t\right].$$

And since $\mathbb{E}\left[\sum_{t=0}^{n-1} \frac{\nu(\mathcal{O}^t)}{n-t}\right]$ is the outcome of RSD, we just need to upper-bound $\mathbb{E}\left[\sum_{t=1}^{n-1} L^t\right]$ now.

Let us note that equality (1) can be transformed into

$$\frac{Y^{t+1}}{n-t-1} = \frac{L^{t+1}}{n-t-1} - \left(\frac{\nu(\mathcal{O}^t)}{n-t} - \frac{\nu\left(\mathcal{O}^{t+1}\right)}{n-t-1}\right)$$

for $t+1 < n$. Since $\mathbb{E}\left[Y^{t+1}\middle|\,\mathcal{H}^t\right] = 0$, we have $\mathbb{E}\left[\frac{Y^t}{n-t}\middle|\,\mathcal{H}^{t-1}\right] = 0$ as well. Thus sequence $(Z^t)_{t=0}^{n-1}$ with $Z^0 = 0$ and $Z^t = \sum_{i=1}^t \frac{Y^i}{n-i}$ is a martingale, and again from Doob's Stopping Theorem we get that $0 = \mathbb{E}\left[Z^0\right] = \mathbb{E}\left[Z^{n-1}\right] = \mathbb{E}\left[\sum_{t=1}^{n-1} \frac{Y^t}{n-t}\right]$, which gives

$$0 = \mathbb{E}\left[\sum_{t=1}^{n-1} \frac{Y^t}{n-t}\right] = \mathbb{E}\left[\sum_{t=1}^{n-1} \frac{L^t}{n-t}\right] - \mathbb{E}\left[\sum_{t=1}^{n-1} \frac{\nu\left(\mathcal{O}^{t-1}\right)}{n-t+1} - \frac{\nu\left(\mathcal{O}^{t}\right)}{n-t}\right],$$

and since the second sum telescopes we obtain that

$$\mathbb{E}\left[\sum_{t=1}^{n-1} \frac{L^t}{n-t}\right] = \frac{\nu\left(\mathcal{O}^0\right)}{n} - \mathbb{E}\left[\nu\left(\mathcal{O}^{n-1}\right)\right] \leq \frac{\nu\left(\mathcal{O}\right)}{n}. \tag{2}$$

For any $L^t \in [0,1]$ it holds that $\frac{L^t}{n-t} \geq \int_{t-L^t}^{t} \frac{dx}{n-x}$. Moreover all intervals $[t-L^t, t]$ are disjoint, and they are of total length of $\sum_{t=1}^{n-1} L^t$, hence

$$\sum_{t=1}^{n-1} \frac{L^t}{n-t} \geq \sum_{t=1}^{n-1} \int_{t-L^t}^{t} \frac{dx}{n-x} \geq \int_{0}^{\sum_{t=1}^{n-1} L^t} \frac{dx}{n-x} = \ln \frac{n}{n-\sum_{t=1}^{n-1} L^t}.$$

Function $x \mapsto \ln \frac{n}{n-x}$ is convex, so from Jensen's inequality and (2) we get that

$$\frac{\nu\left(\mathcal{O}\right)}{n} \geq \mathbb{E}\left[\sum_{t=1}^{n-1} \frac{L^t}{n-t}\right] \geq \mathbb{E}\left[\ln \frac{n}{n-\sum_{t=1}^{n-1} L^t}\right] \geq \ln \frac{n}{n-\mathbb{E}\left[\sum_{t=1}^{n-1} L^t\right]},$$

which yields $n\left(1-e^{-\frac{\nu(\mathcal{O})}{n}}\right) \geq \mathbb{E}\left[\sum_{t=1}^{n-1} L^t\right]$. We can now finish lowerbounding the outcome of RSD:

$$\nu\left(\mathcal{R}\right) \geq \mathbb{E}\left[\sum_{t=0}^{n-1} \frac{\nu\left(\mathcal{O}^t\right)}{n-t}\right] = \nu\left(\mathcal{O}\right) - \mathbb{E}\left[\sum_{t=1}^{n-1} L^t\right] \geq \nu\left(\mathcal{O}\right) - n + n \cdot e^{-\frac{\nu(\mathcal{O})}{n}} \geq \frac{1}{e}\frac{\nu\left(\mathcal{O}\right)^2}{n},$$

where the last inequality follows from $x - 1 + e^{-x} \geq \frac{1}{e}x^2$ for $x \in [0,1]$. □

On the hardness side, Result 4 says that no truthful mechanism can achieve social welfare greater than $\frac{\nu(\mathcal{O})^2}{n}$. The proof is deferred to the full version of the paper.

6 Open Question

As mentioned in the introduction, we can give the following truthful and symmetric mechanisms that outputs optimal social welfare. The mechanism works as follows. First, collect agents preferences d_a for all $a \in A$. Then consider graph $G = (A, I)$ with edge between every pair $a \in A$, $i \in I$ for which $d_a(i) = 1$.. Next, find the all maximum matchings. Finally, output a maximum matching uniformly at random. Unfortunately, such a mechanism is not feasible when computational efficiency is required. The problem is that it is $\#P$-complete to count all maximum matchings. Therefore, we suspect that any truthful, symmetric and optimal mechanism would be somehow connected with an algorithm for counting all maximum matchings. And because of that, we conjecture that such mechanism should be $\#P$-complete as well.

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Stable Marriage with General Preferences

Extended Abstract

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Abstract. We propose a generalization of the classical stable marriage problem. In our model, the preferences on one side of the partition are given in terms of arbitrary binary relations, which need not be transitive nor acyclic. This generalization is practically well-motivated, and as we show, encompasses the well studied hard variant of stable marriage where preferences are allowed to have ties and to be incomplete. As a result, we prove that deciding the existence of a stable matching in our model is NP-complete. Complementing this negative result we present a polynomial-time algorithm for the above decision problem in a significant class of instances where the preferences are asymmetric. We also present a linear programming formulation whose feasibility fully characterizes the existence of stable matchings in this special case. Finally, we use our model to study a long standing open problem regarding the existence of cyclic 3D stable matchings. In particular, we prove that the problem of deciding whether a fixed 2D perfect matching can be extended to a 3D stable matching is NP- complete, showing this way that a natural attempt to resolve the existence (or not) of 3D stable matchings is bound to fail.

1 Introduction

The Stable Marriage (SM) problem is a classical bipartite matching problem first introduced by Gale and Shapley [8]. An instance of the problem consists of a set n of men, and a set of n women. Each man (woman) has a preference list that is a total order over the entire set of women (men). The goal is to find a stable matching between the men and women, meaning that there is no (man, woman) pair that both prefer each other to their current partners in the matching. Since its introduction, the stable marriage problem has become one of the most popular combinatorial problems with several books being dedicated to its study [10, 15, 21] and more recently [17]. The popularity of this model arises not only from its nice theoretical properties but also from its many applications. In particular, a wide array of allocation problems from many diverse fields can be analyzed within its context. Some well known examples include the labour market for medical interns, auction markets, the college admissions market, the organ donors market, and many more [21].

In their seminal work Gale and Shapley showed that every instance of SM admits a solution and such a solution can be computed efficiently using the so-called

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Gale-Shapley (or man- proposing) algorithm. Among the many new variants of this classical problem, two extensions have received most of the attention: incomplete preference lists and ties in the preferences. Introducing either one of these extensions on its own does not pose any new challenges, meaning that solutions are still guaranteed to exist, all solutions have the same size, and they can be computed using a modification of the original Gale-Shapley algorithm [9, 10]. However, the same cannot be said about the Stable Marriage problem with Ties and Incomplete Lists (SMTI) that incorporates both extensions. In this variant stable matchings no longer need to be of the same size, even though they are still guaranteed to exist. In fact, deciding whether a given instance admits a stable matching of a given size is NP-hard [18], even in the case where ties occur only on one side of the partition. Several papers have studied the approximate variants of this problem (see [17] for a more complete account).

A central assumption in most variants of SM is that agents' preferences are transitive (i.e., if x is preferred to y, and y is preferred to z then x is also preferred to z). However, there are several studies [1, 6, 3, 19] that suggest that non-transitive, and even cyclic preferences arise naturally. Cyclicity, for example, may be introduced in the context of multi-attribute comparisons [7]; e.g., consider the following study from [19] where 62 college students were asked to make binary comparisons between three potential marriage partners x,y and z according to the following three criteria: intelligence, looks and wealth. The candidates had the following attributes: candidate x was very intelligent, plain, and well off; candidate y was intelligent, very good looking, and poor; and candidate z was fairly intelligent, good looking, and rich. From the 62 participants, 17 displayed the following cyclic preference: x was preferred to y, y was preferred to z, and z was preferred to x. In order to better capture such situations there is a need for a model that allows for more general preferences.

Addressing this need we propose the Stable Marriage with General Preferences (SMG) problem. As in SM, in an instance of SMG we are given n men, and n women, and the preferences of men are complete total orders over the set of women. The preferences of women, on the other hand, are given in terms of arbitrary binary relations over the men. Each of these binary relations will be represented by a set of ordered pairs of men. We say that a woman prefers man x at least as much as man y if the ordered pair (x, y) is part of her preference set. A matching is then stable as long as for every unmatched (man, woman) pair at least one member prefers her mate in the matching at least as much as the other member of the pair.

This introduction of non-transitive preferences, even when restricted to just one side of the partition, changes the properties of the model drastically. Like in SM any solution must be a perfect matching. However, solutions are no longer guaranteed to exist. We show that non-transitive preferences generalize both incomplete lists and ties by reducing the SMTI to SMG. In doing so, we prove that the SMG problem is also NP-hard. In addition, we provide results on the structural properties of the SMG problem and give sufficient conditions for the problem to be solvable in polynomial time.

The second half of this paper focuses on three-dimensional stable matching models whose study was initiated by Knuth [15]. We will be particularly interested in the Cyclic 3-Dimensional Stable Matching problem (c3DSM), where we are given a set of n men, a set of n women and in addition a set of n dogs. The preferences of the men are complete total orders over the set of women. Similarly the women have preferences over the dogs, and the dogs have preferences over the men. A 3D matching is said to be stable if there is no (man, woman, dog) triple that is strictly preferred to their current triples in the matching by each of its members. A prominent open question is whether every instance of c3DSM admits a stable matching, and whether it can be computed efficiently.

A natural avenue for attacking c3DSM is to solve the following problem which we refer to as Stable Extension (SE): suppose we fix a perfect matching M on dogs and men, can we efficiently determine whether M is extendible to a 3D stable matching? Recall that women have preferences over dogs only, but note that the given matching M induces preferences over their male owners as well! In essence, this allows us to state the SE problem as a two dimensional bipartite matching problem, and we show in Theorem 4 that SE can be seen as a special case of SMG. We then prove that SE remains NP-complete.

Contributions. In Section 3.1 we show the following result.

Theorem 1. SMG *is* NP*-complete.*

We then identify a significant class of instances that are solvable in polynomial time: those where the preferences are asymmetric, meaning that for every pair of men x, y, each woman prefers at most one to the other. We then prove the following result.

Theorem 2. *For instances of* SMG *with asymmetric preferences, there exists a polynomial time algorithm that finds a solution if and only if one exists.*

We provide two different proofs. The first (given in Section 3.2) employs an adaptation of the classical Gale-Shapley man-proposing algorithm. The second (given in Section 3.3) relies on a polyhedral characterization: we define a polytope that is non-empty if and only if the instance admits a stable matching. We also develop an efficient rounding algorithm for its fractional points. Despite displaying stronger structural properties than SMG, we show that SE remains hard to solve.

Theorem 3. SE *is* NP*-complete.*

The proof of the above theorem is given in Section 4. At a high level, its strategy resembles that of the proof of Theorem 1. The details are however significantly more intricate, mainly due to the fact that SE instances correspond to SMG instances in which preferences are induced by a given 3D matching instance. As an interesting consequence for the c3DSM, Theorem 3 rules out the natural algorithmic strategy of fixing and extending a 2D perfect matching on two of the input sets.

Related work. To the best of our knowledge, the stable marriage problem with non-transitive preferences has not been studied before. However, there is a rich literature about SM and its variants. In particular, there has been significant work concerning the approximation variant of the SMTI problem, where the goal is to find a maximum size stable matching. When ties are allowed on both sides, the problem is NP-hard to approximate within 33/29 [23] and the currently best known ratio is 3/2 [20]. When ties are only allowed on the side of the women the problem is NP-hard to approximate within 21/19 [11] and the currently best known ratio is 25/17 [14].

A related model known as Stable Marriage with Indifference [13, 16], allows for preferences to be given in the form of partial orders. In this case, it is possible for a pair of agents to be incomparable, but transitivity is still assumed to hold. This model allows for several definitions of stability, and depending on which definition is used, solutions might not always exist. However, there is an efficient algorithm for computing each type of stable matchings, whenever they exist.

For c3DSM, it is known that every instance admits a stable matching for $n \leq 4$ [4]. The authors conjectured that this result can be extended to general instances. In [2] it was shown that if we allow unacceptable partners, the existence of a stable matching becomes NP-complete. In the same paper, and also independently in [12], it was shown that the c3DSM problem under a different notion of stability known as strong stability is also NP-complete.

2 Definitions and Notation

Throughout this paper we denote the set of men by B and the set of women by C. In the 3D setting, we have an additional set of n dogs A. A 3D perfect matching $\mathcal{M}$ is a set of n disjoint triples from $A \times B \times C$. For every dog $a \in A$ we denote by $\mathcal{M}(a)$ the man that a is matched to in $\mathcal{M}$. Similarly for every man $b \in B$, $\mathcal{M}(b)$ denotes the woman that b is matched to in $\mathcal{M}$, and for every woman $c \in C$, $\mathcal{M}(c)$ denotes the dog that c is matched to in $\mathcal{M}$. A 3D perfect matching can also be induced by fixing perfect matchings on any two of the following sets $A \times B$, $B \times C$ or $C \times A$. We will use M to denote a perfect matching on $A \times B$ and N to denote a perfect matching on $B \times C$. We then define the 3D matching $M \circ N$ by setting $(a, b, c) \in M \circ N$ if and only if $(a, b) \in M$ and $(b, c) \in N$. For each $q \in A \cup B$ we denote by $M(q)$ the partner of q in M, and similarly for each $q \in B \cup C$ we denote by $N(q)$ the partner of q in N. If the preferences of an agent q are given in terms of an ordering $P(q)$ (with or without ties) over a set A then for all $x, y \in A$ we write $x \succ_q y$ to denote that q strictly prefers x to y and $x \succeq_q y$ to denote that q prefers x at least as much as y.

2.1 Stable Marriage with Ties and Incomplete Lists (SMTI)

An instance $\mathcal{I}$ of SMTI consists of a set B of n men and a set C of n women. Each man $b \in B$ has a preference list $P(b)$ that is an ordering over a subset of C and is allowed to contain ties. Similarly each woman $c \in C$ has a preference list $P(c)$ that is an ordering over a subset of B and is also allowed to contain ties. A pair (b, c) is said to be *acceptable* if b appears in $P(c)$ and c appears in $P(b)$. It is assumed that a woman c is acceptable to a man b if and only if man b is acceptable to woman c. A pair (b, c) is *blocking* with respect to a matching N if (b, c) is an acceptable pair that is not in N , $c \succ_b N(b)$ and $b \succ_c N(c)$. A matching N is *stable* if it uses only acceptable pairs and it has no blocking pairs. In that case, we also say that N is a solution to $\mathcal{I}$. If a solution is of size n we refer to it as a perfect stable matching. While all instances of SMTI admit a stable matching, not all instances admit a perfect stable matching. In this paper we use SMTI to refer to the decision problem of whether a given instance admits a perfect stable matching. This problem is known to be NP-complete [18], even when the ties occur only in the preference lists of the women.

2.2 Stable Marriage with General Preferences (SMG)

An instance $\mathcal{I}$ of SMG consists of a set B of n men and a set C of n women. Each man $b \in B$ has a preference list $P(b)$ that is complete total order over C. Each woman $c \in C$ has a preference relation given in terms of a set of ordered pairs $\mathcal{R}_c \subseteq B \times B$. For a given pair of men $b, b' \in B$ and woman $c \in C$ we interpret $(b, b') \in \mathcal{R}_c$ as woman c preferring man b at least as much as man b'. Note that whether (b, b') is in $\mathcal{R}_c$ is completely independent of whether $(b', b) \in \mathcal{R}_c$. We say that a pair (b, c) is *blocking* with respect to a matching N, if b and c are not matched to each other and neither one prefers its partner in N at least as much as the other. Formally, (b, c) is blocking if $(b, c) \notin N$, $c \succ_b N(b)$ and $(N(c), b) \notin \mathcal{R}_c$. A matching N is *stable* if it has no blocking pairs. It follows from this definition that any stable matching is a perfect matching. In this paper we use SMG to refer to the decision problem of whether a given instance admits a stable matching (which we also call a solution). An instance $\mathcal{I}$ of SMG is said to have *asymmetric preferences* if for every $b_1, b_2 \in B$ and $c \in C$ at most one of the following two conditions holds: $(b_1, b_2) \in \mathcal{R}_c$ or $(b_2, b_1) \in \mathcal{R}_c$.

Note that we could have obtained an alternate definition of stability by saying that a pair (b, c) is blocking if $(b, c) \notin N$, $c \succ_b N(b)$ and $(b, N(c)) \in \mathcal{R}_c$. However, the two models are equivalent via the following correspondence: create a new instance $\mathcal{I}'$ with sets $\mathcal{R}'_c$ where $(b, b') \in \mathcal{R}'_c$ if and only if $(b', b) \notin \mathcal{R}_c$. Then the solutions that are stable for $\mathcal{I}$ under the definition of stability used in this paper, are exactly those that are stable for $\mathcal{I}'$ using the alternate definition of stability. Hence, we can use our definition of stability without loss of generality.

2.3 Stable Extension (SE)

An instance $\mathcal{I}$ of SE consists of a set of n dogs A, a set of n men B, and a set of n women C, together with a fixed perfect matching M on $A \times B$. The preferences are defined cyclically (A over B, B over C, and C over A) and are complete total orders over the corresponding sets. A triple (a, b, c) is *blocking* with respect to a 3D matching $\mathcal{M}$ if $(a, b, c) \notin \mathcal{M}$, $b \succ_a \mathcal{M}(a)$, $c \succ_b \mathcal{M}(b)$ and $a \succ_c \mathcal{M}(c)$. Note that if (a, b, c) is a blocking triple then a, b and c must be part of three disjoint triples in $\mathcal{M}$. A 3D matching $\mathcal{M}$ is stable if it has no blocking pairs. It follows from this definition that any stable 3D matching must be a perfect matching. We say that a perfect matching N on $B \times C$ is a *stable extension*, or a solution to $\mathcal{I}$, if $M \circ N$ is a 3D stable matching, and we use SE to refer to the decision problem of whether a given instance admits a stable extension.

We now demonstrate how an instance $\mathcal{I}$ of SE can be reduced to an SMG instance. First, for each man $b \in B$ we define A_b to be the set of dogs in A that prefer b to the man assigned to them in the fixed perfect matching M. That is $A_b = \{a \in A : b \succ_a M(a)\}$. The set A_b contains exactly those dogs in A with whom man b can potentially be in a blocking triple when extending M to a 3D matching. It follows that if $A_b = \emptyset$ then man b cannot be in a blocking triple in any extension of M to a 3D matching. Now, for each pair (b, c) we define $\alpha(b, c)$ to be the dog in the set A_b that woman c prefers the most. That is $\alpha(b, c) = \max_{\succ_c} A_b$. If $A_b = \emptyset$ then we let $\alpha(b, c)$ be the dog in the last position in woman c's preference list. We now define preferences for each woman $c \in C$

$$\mathcal{R}_c := \{(b, b') \mid b, b' \in B, b \neq b',\ M(b) \succeq_c \alpha(b', c)\} . \tag{1}$$

Note that if $(N(c), b) \in \mathcal{R}_c$ then in the 3D matching $M \circ N$ the woman c will be matched to a dog that she prefers at least as much as any dog in A_b, therefore guaranteeing that the man b and woman c will never be part of the same blocking triple. Hence, in order for $M \circ N$ to be a 3D stable matching it suffices to ensure that for all $(b, c) \notin N$ we either have b matched to someone better than c, meaning $N(b) \succ_b c$, or we have c matched to some $N(c)$ such that $(N(c), b) \in \mathcal{R}_c$. But this is exactly the definition of a stable matching for an instance of SMG. Hence we have the following theorem.

Theorem 4. SE *can be reduced in polynomial time to* SMG.

Note that Theorem 4, together with Theorem 3 imply NP-hardness of SMG, and hence prove Theorem 1 (modulo containment in NP which is straightforward). Nevertheless, we choose to present first the proof of Theorem 1 as a warm-up, as it shares many similarity with that of Theorem 3.

3 Results for SMG

3.1 NP-completeness of SMG (Proof of Theorem 1)

Containment in NP is straightforward, and is based on the observation that deciding whether an edge not in a perfect matching of an instance of SMG is blocking or not can be done in polynomial time in n. The rest of our argument focuses on hardness.

Our proof uses a polynomial time reduction from SMTI where ties occur only in the preference lists of the women. This problem is known to be NP-complete [18]. Let $\mathcal{I}$ be an instance of SMTI where ties occur only on the side of the women. We let $B = \{b_1, \cdots, b_n\}$ denote the set of men and $C = \{c_1, \cdots, c_n\}$

denote the set of women for the instance $\mathcal{I}$. For each person $q \in B \cup C$ we let $P(q)$ denote their preference list.

We now describe how to construct an instance $\mathcal{J}$ of SMG. The set of men for our instance will be given by $B' = B \cup \{b_{n+1}, b_{n+2}\}$ and the set of women by $C' = C \cup \{c_{n+1}, c_{n+2}\}$. The preferences of the men are defined as follows: each original man $b \in B$ ranks the women in $P(b)$ first, in the same order as in $P(b)$, followed by the woman c_{n+1}, and the remaining women of C' ranked arbitrarily; each new man b_{n+i} for $i \in \{1,2\}$ ranks the woman c_{n+i} first, and the remaining women of C' arbitrarily. Now, for each original woman $c \in C$ we define the binary relation $\mathcal{R}_c \subseteq B \times B$ as follows $\mathcal{R}_c := \{(b, b') \mid b, b' \in P(b), b \succeq_c b'\}$. That is, $\mathcal{R}_c$ contains the ordered pair (b, b') when both b and b' are acceptable to c under the instance $\mathcal{I}$ and c prefers b at least as much as b'. Finally, for each extra woman c_{n+i} for $i \in \{1,2\}$ we set $\mathcal{R}_{c_{n+i}} := \emptyset$. This completes the definition of the instance $\mathcal{J}$. The new agents are used to establish the following property.

Lemma 1. *In any solution N to $\mathcal{J}$ every man $b \in B$ is matched to a woman from the set $P(b)$.*

Note that each $b \in B$ ranks c_{n+1} immediately after all acceptable partners and before any unacceptable partners. Now if b is matched to someone outside of $P(b)$, the pair (b, c_{n+1}) will be blocking, since b will prefer c_{n+1} to its current partner, and c_{n+1} cannot be matched to someone she prefers at least as much as b since we defined $\mathcal{R}_{c_{n+1}} = \emptyset$. The full proof can be found in [5]. The following lemma completes the proof of Theorem 1.

Lemma 2. *$\mathcal{I}$ admits a perfect stable matching if and only if $\mathcal{J}$ admits a stable matching.*

Proof. Suppose that N is a perfect stable matching for $\mathcal{I}$. Then complete N to a perfect matching on $B' \cup C'$ by matching b_{n+i} to c_{n+i} for every $i \in \{1,2\}$. To see that this is a stable matching for $\mathcal{J}$, note that both men b_{n+1} and b_{n+2} are matched to their most preferred woman in C', hence they cannot be part of any blocking pairs. It remains to show that no man in B can be part of a blocking pair. Consider a man $b \in B$, and suppose that c is a woman that b strictly prefers to $N(b)$ according to the preferences in $\mathcal{J}$. Then it must be the case that $c \in P(b)$ and b also strictly prefers c to $N(b)$ in $\mathcal{I}$. Since N is a solution to $\mathcal{I}$, woman c prefers $N(c)$ at least as much as b in $\mathcal{I}$. Hence from the way we defined the set $\mathcal{R}_c$ we have $(N(c), b) \in \mathcal{R}_c$, implying that (b, c) is not blocking.

To see the other direction suppose that $\mathcal{J}$ admits a stable matching, and let N be the part of this stable matching obtained by restricting it to the sets $B \cup C$. It follows from Lemma 1 that N is a perfect matching and every man is matched to an acceptable woman. To see that there are no blocking pairs, consider any pair $(b, c) \notin N$ such that (b, c) is an acceptable pair, that is $b \in P(c)$ and $c \in P(b)$. Assume now that in $\mathcal{I}$ man b strictly prefers c to $N(b)$. Since $c \in P(b)$ it follows that b also strictly prefers c to $N(b)$ in $\mathcal{J}$. Hence, since N is a stable matching for $\mathcal{J}$, we must have $(N(c), b) \in \mathcal{R}_c$. From the way we defined $\mathcal{R}_c$ this implies that $N(c)$ is acceptable to c, and c prefers $N(c)$ at least as much as b in $\mathcal{I}$. Therefore N does not have any blocking pairs, and is a stable matching in $\mathcal{I}$.

3.2 Algorithmic Results

In this section we introduce a variant of the Gale-Shapley man- proposing algorithm for instances of SMG that have asymmetric preferences. Let $\mathcal{I}$ be an SMG instance as defined in Section 2.2. Like in the classical algorithm, each man in B is originally declared single and is given a list containing all the women of C in order of preference. In each round, every man b that is still single proposes to its most preferred woman in C that is still in his list. If a woman c accepts a proposal from a man b then they become engaged, and b's status changes from single to engaged. On the other hand, if c rejects b's proposal then b removes woman c from his list and remains single. The difference from the original Gale-Shapley algorithm is in the way that the women decide to accept or reject incoming proposals. A woman c accepts a proposal from a man b if and only if $(b, b') \in \mathcal{R}_c$ for all other men b' that have proposed to c up to that point in the algorithm. This will ensure that whenever a woman c rejects a proposal from a man b, c is guaranteed to be matched at the end of the algorithm to some b' such that $(b', b) \in \mathcal{R}_c$ therefore ensuring that (b, c) will not be a blocking pair. The description of the algorithm is given below.

Algorithm 1. A deferred acceptance algorithm for SMG

1. **while** there a single man in B with a non-empty list **do**
2.   **for all** b single with a non-empty list **do**
3. b proposes to the top c in its list
4.   **for all** b, c such that b proposed to c **do**
5.   **if** $(b, b') \in \mathcal{R}_c$ for all $b' \neq b$ that proposed to c **then**
6. c accepts b ((b, c) become an engaged pair)
7.   **else**
8. c rejects b
9. If the set of engaged pairs forms a perfect matching return this solution, else conclude that $\mathcal{I}$ does not admit a stable matching.

It is easy to see that Algorithm 1 terminates and runs in polynomial time since each man in B proposes to every woman in C at most once. The following lemma is easy to establish and its proof is found in [5].

Lemma 3. *Any solution returned by Algorithm 1 is a stable matching.*

Proof of Theorem 2. Using Lemma 3 it suffices to show that if Algorithm 1 does not find a solution then the given instance does not admit a stable matching. Suppose by contradiction that there exists a stable matching N but Algorithm 1 does not find a solution. First note that since the preferences are asymmetric, no woman $c \in C$ accepts more then one proposal at any point in the algorithm. Hence every woman is engaged to at most one man. Now since Algorithm 1 does not find a solution there is a man b that is rejected by every woman in C. In particular, b is rejected by $N(b)$. Among all pairs $(b, c) \in N$ such that b proposed to c and c rejected b, let (b_0, c_0) be the one that corresponds to the earliest rejection. As observed earlier, no man is rejected because of an arbitrary choice. Therefore, if b_0 was rejected at c_0 then there must be some man b_1 that also proposed to c_0 and $(b_0, b_1) \notin \mathcal{R}_{c_0}$. Let c_1 be the partner of b_1 in N. We must have $c_1 \succ_{b_1} c_0$, since otherwise (b_1, c_0) would be a blocking pair for the stable matching N. Thus b_1 proposed to c_1 before proposing to c_0 and c_1 rejected b_1 before c_0 rejected b_0. But this contradicts our choice of (b_0, c_0). This concludes the proof of the theorem. □

3.3 A Polyhedral Characterization

In this section we provide a polyhedral description for SMG, that is an analogue of the well studied stable marriage polytope, first introduced in [22]. It is well known that the latter polytope is integral, meaning that the optimization version of SM can be solved in polynomial time. For our setting, we show that our polytope can be used to efficiently decide the feasibility of an SMG instance with asymmetric preferences, thus giving an alternative proof of Theorem 2. We remark however that our polytope is not integral for this class of instances. Indeed, one can easily find instances with asymmetric preferences for which our polytope has fractional extreme points.

Given an instance $\mathcal{I}$ of SMG we associated with each pair $(b, c) \in B \times C$ a variable x_{bc}, with the intended meaning that $x_{bc} = 1$ if b and c are matched to each other and $x_{bc} = 0$ otherwise. We then consider the following relaxation of the problem and let $P(\mathcal{I})$ denote the set of all vectors satisfying the constraints below

$$\sum_c x_{bc} = 1 \quad \forall b \in B \tag{2}$$

$$\sum_b x_{bc} = 1 \quad \forall c \in C \tag{3}$$

$$x_{bc} + \sum_{c' \succ_b c} x_{bc'} + \sum_{(b',b) \in \mathcal{R}_c} x_{b'c} \geq 1 \quad \forall b \in B, c \in C \tag{4}$$

$$x_{bc} \geq 0 \quad \forall b \in B, c \in C \tag{5}$$

It is easy to check that x is the incidence vector of a stable matching for $\mathcal{I}$ if and only if x is an integer vector in $P(\mathcal{I})$. Our main result is the following.

Theorem 5. *Let $\mathcal{I}$ be an instance of* SMG *with asymmetric preferences. Then $P(\mathcal{I}) \neq \emptyset$ if and only if $\mathcal{I}$ admits a stable matching. Furthermore any fractional point $x \in P(\mathcal{I})$ can be efficiently rounded to a stable matching solution for $\mathcal{I}$.*

Proof. The first direction is trivial since if $\mathcal{I}$ admits a stable matching then the incidence vector corresponding to this stable matching is clearly in P. Now assume $P \neq \emptyset$ and let x be any point in P. We will show how to efficiently round x to a stable matching, thus completing the proof of the theorem. For each $b \in B$ let $f(b)$ be b's most preferred woman in the set $\{c \in C : x_{bc} > 0\}$. Define $N = \{(b, f(b)) : b \in B\}$. We first show that N is a perfect matching. Since each man selects exactly one woman it suffices to show that no two men select the same woman. Suppose by contradiction that $f(b_1) = f(b_2) = c$ for some $b_1 \neq b_2$. Note that

$$\begin{aligned} f(b) = c \quad &\Rightarrow \sum_{c' \succ_b c} x_{bc'} = 0 && \text{from the definition of } f(b) \\ &\Rightarrow x_{bc} + \sum_{(b',b) \in \mathcal{R}_c} x_{b'c} \geq 1 && \text{from the stability constraint for } (b,c) \\ &\Rightarrow x_{bc} + \sum_{(b',b) \in \mathcal{R}_c} x_{b'c} = 1 && \text{from the matching constraint for } c \\ &\Rightarrow \sum_{(b',b) \notin \mathcal{R}_c} x_{b'c} = 0. \end{aligned}$$

Therefore $f(b_1) = f(b_2) = c$ implies that $(b_1, b_2) \in \mathcal{R}_c$ and $(b_2, b_1) \in \mathcal{R}_c$. But this contradicts the assumption that the preferences are asymmetric. Hence N must be a perfect matching. To see that N satisfies the stability constraints consider any pair (b, c). If $f(b) = c$ or $f(b) \succ_b c$ then (b, c) cannot be blocking, since b will be matched in N to someone he prefers at least as much as c. Hence it suffices to consider the case where $c \succ_b f(b)$. But then we must have $x_{bc} + \sum_{c' \succ_b c} x_{bc'} = 0$ and since $x \in P(\mathcal{I})$ this implies that $\sum_{(b',b) \in \mathcal{R}_c} x_{b'c} = 1$. Now, since N uses only edges in the support of x, it follows that $(N(c), b) \in \mathcal{R}_c$ and hence (b, c) is not blocking. Therefore N is a stable matching for $\mathcal{I}$. □

4 NP-completeness of SE (Proof Outline of Theorem 3)

In this section we outline the major ideas behind the proof of Theorem 3. The full version of the proof can be found in [5].

At a high level, our proof adopts a similar strategy as that of Theorem 1. In particular, we will provide a polynomial-time reduction for SMTI to SE. However, due to the additional structural properties of SE instances this reduction becomes significantly more intricate. As in the proof of Theorem 1 we start with an instance $\mathcal{I}$ of SMTI where ties occur only on the side of the women. We then create an instance $\mathcal{J}$ of SE that will consist of three sets A', B' and C'. We will have $B \subset B'$ and $C \subset C'$. We will also define some additional agents and fix a perfect matching M on $A' \times B'$. Our goal will be to show that $\mathcal{I}$ admits a perfect stable matching if and only if $\mathcal{J}$ admits a stable extension. Using Theorem 4, we can show that $\mathcal{J}$ admits a stable extension by arguing instead that the corresponding instance of SMG defined from $\mathcal{J}$ admits a stable matching.

The major difference from the proof of Theorem 1 is that there we had the freedom of defining the sets $\mathcal{R}_c$ directly, but here the sets $\mathcal{R}_c$ are implicitly defined through the structure of the SE instance $\mathcal{J}$. This implicit definition entails a significant amount of inter-dependence between the $\mathcal{R}_c$ sets. In particular it now becomes more difficult to establish the following property of the sets $\mathcal{R}_c$:

for every acceptable pair (b_j, c_i) of the instance $\mathcal{I}$ we have $(b, b_j) \in \mathcal{R}_{c_i}$ if and only if b is a man that c_i prefers at least as much as b_j in $\mathcal{I}$. This property was crucial for establishing a correspondence between the pairs that are blocking in $\mathcal{I}$ and those that are blocking in $\mathcal{J}$. In order to establish this property here we will need to introduce new agents in $\mathcal{J}$ corresponding to each position of woman c_i's preference list $P(c_i)$ from $\mathcal{I}$. In particular if $P(c_i)$ consists of t_i positions, then there will be t_i new dogs $a_{i,1}, \cdots, a_{i,t_i}$. We will then define the preference of the dogs as well as the fixed perfect matching M in such a way as to ensure that if man b_j appears in position k in $P(c_i)$ then the only dog from the set $\{a_{i,1}, \cdots, a_{i,t_i}\}$ that prefers man b_j over his partner in M will be $a_{i,k}$. Finally we will define the preference of woman c_i over the set of dogs in $\mathcal{J}$ as follows: c_i will rank the dogs that are matched in M to men from the first position in $P(c_i)$ at the top of its list, in any arbitrary order among them, followed by the dog $a_{i,1}$; then the dogs that are matched in M to men from the second position in $P(c_i)$ followed by the dog $a_{i,2}$, and so on until the dogs that are matched in M to men from the last position in $P(c_i)$ followed by the dog a_{i,t_i}. The remainder of c_i's preference list will be completed arbitrarily. We will then show how this will imply the desired structure for the set $\mathcal{R}_{c_i}$.

The other major difference from the proof of Theorem 1 lies in ensuring that the new agents will be matched between themselves in any solution to $\mathcal{J}$. Before, if we wanted to ensure that a given pair b_{n+1}, c_{n+1} are always matched to each other, it was sufficient to set c_{n+1} as the most preferred woman of b_{n+1} and $\mathcal{R}_{c_{n+1}} = \emptyset$. However, since our instance of SMG arises from the SE instance $\mathcal{J}$ the sets $\mathcal{R}_c$ are never empty. Indeed it follows from their definition that $(M(\alpha(b, c), b) \in \mathcal{R}_c$ for every pair $b \in B$ and $c \in C$ we have. Thus, in order to ensure that a given pair b_{n+1}, c_{n+1} are always matched to each other we now need to introduce an extra gadget consisting of two new men b_{n+2}, b_{n+3} and two new women c_{n+2}, c_{n+3}. We will have b_{n+i} rank c_{n+i} first in its preference list for all $i \in \{1, 2, 3\}$. It is then possible to ensure that $(b, b_{n+1}) \in \mathcal{R}_{c_{n+1}}$ if and only if $b = b_{n+2}$, $(b, b_{n+3}) \in \mathcal{R}_{c_{n+3}}$ if and only if $b = b_{n+2}$, and $(b, b_{n+2}) \in \mathcal{R}_{c_{n+2}}$ if and only if $b = b_{n+3}$. This will then guarantee that in any solution to $\mathcal{J}$ the men b_{n+2} and b_{n+3} are matched to the women c_{n+2} and c_{n+3} (in any of the two possible ways) and that man b_{n+1} is always matched to woman c_{n+1} as desired. The full details of the proof can be found in [5].

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The Convergence Time for Selfish Bin Packing

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Abstract. In classic bin packing, the objective is to partition a set of n items with positive rational sizes in $(0, 1]$ into a minimum number of subsets called bins, such that the total size of the items of each bin at most 1. We study a bin packing game where the cost of each bin is 1, and given a valid packing of the items, each item has a cost associated with it, such that the items that are packed into a bin share its cost equally. We find tight bounds on the exact worst-case number of steps in processes of convergence to pure Nash equilibria. Those are processes that are given an arbitrary packing. As long as there exists an item that can reduce its cost by moving from its bin to another bin, in each step, a controller selects such an item and instructs it to perform such a beneficial move. The process terminates when no further beneficial moves exist. The function of n that we find is $\Theta(n^{3/2})$, improving the previous bound of Han et al., who showed an upper bound of $O(n^2)$.

1 Introduction

We study a class of bin packing games, that are based on the well-known standard bin packing problem [15,3,5,4], a basic combinatorial optimization problem. In this problem, a set of n items $I = \{1, 2, \ldots, n\}$ is given, where the size of item t, denoted by s_t, satisfies $0 < s_t \le 1$. The goal is to partition (or pack) the items into a minimum number of subsets or blocks. Each such block is *packed* into a unit capacity bin, and the load of a bin is defined to be the total size of items packed into it (and can never exceed 1). Here, we study bin packing from the point of view of algorithmic game theory.

We now define the game theoretical concepts required for the definition of the bin packing game. In a *strategic game*, there is a finite set of players, and a finite and non-empty set of strategies (or actions) that players can perform. Each player has to choose a strategy (possibly independently from other players). Each player has a cost for each one of the possible situations or outcomes, where an outcome is a possible set of strategies of all players, containing one strategy for each player. A classic form of a stable solution is a *Nash equilibrium* (NE) [19]. This is a kind of solution concept of a game with at least two players, where no player can decrease its cost by changing only its own strategy unilaterally. That is, if each player has chosen a (pure or mixed) strategy and no player can benefit by changing its strategy while the other players keep theirs unchanged,

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then the current set of strategy choices and the corresponding costs result in an outcome or solution that is a Nash equilibrium (NE). We are interested in pure Nash equilibria, where the actions of each player are chosen in a deterministic way, and will discuss only this kind of NE.

Given an input for bin packing, the set of players are the items. The pure strategy of a player is the bin in which it is packed (the number of possible bins is n, as this number of bins is always sufficient). We say that a bin $B \subseteq I$ is a valid bin if $\sum_{t \in B} s_t \leq 1$, that is, if its load does not exceed 1. Changing the strategy of an item means that it moves to be packed in a different (non-empty or empty) bin. For $0 \leq k \leq n$, we define a k-bin to be a bin that has exactly k items, and a k^+-bin is a bin that has at least k items. The cost of an item packed into a valid k-bin (for $k \geq 1$) is $\frac{1}{k}$. We let the cost of an item that is not packed into a valid bin be infinite. The deviation of an item t packed in a k_1-bin B_1 (where t is included in the number of items of B_1) to a k_2-bin B_2 (where t is not included in the number of items of B_2) is beneficial if $s(B_2) + s_t \leq 1$ (since otherwise the cost of the item in the alternative bin is infinite) and $k_2 \geq k_1$ (as otherwise its cost is not reduced by moving). The standard bin packing problem can be therefore seen as a class of games, where every input corresponds to a game.

In this paper, we are interested in convergence processes. Such a process receives a set of items and a packing. The packing obviously corresponds to an outcome of the game whose players are those items. The process stops when it reaches a solution that is an NE. As long as it is not an NE, a step is performed. In each step, a controller selects an item that can benefit (reduce its cost) by moving to another bin, and instructs it to move from its current bin to a specified bin (where its cost will be smaller). In each step a single item moves and decreases its cost, while other items may be affected (those that were packed with the moving item will have larger costs, and those that were packed into the bin where it moved will have smaller costs). It is shown by Han et al. [12] (who were the first to study the variant with equal sharing of the bin costs) that such a process always converges in $O(n^2)$ steps. This kind of games are in fact singleton congestion games [13,14], but the number of resources has an exponential size in the number of players, and it is not given explicitly (these are all possible subsets of items that can be packed into a bin), so the convergence and existence of NE can be deduced from previous work on congestion games, but the polynomial time convergence cannot be deduced from those.

Bin packing in general, and more specifically bin packing games, have a number of applications [2,7,8]. Equal sharing is the simplest form of sharing and does not require prior information given by the players (who may or may not be truthful). Bin packing games where items share the cost of the bin proportionally (according to sizes) rather than equally was introduced by Bilò [2], who was the first to study the bin packing problem from this kind of game-theoretic perspective. He proved that every game in this class has an NE. He also proved that any such bin packing game converges to an NE after a finite (but possibly exponentially long) sequence of steps, starting from any initial configuration of the items. The time of convergence for this type of cost sharing was also studied in [17,18]. Multiple papers studied the quality of NE and other types of equilibria [2,7,8,6,1]. Polynomial time algorithms that compute an NE for games with proportional cost sharing equal cost sharing are given in [20,12,6]. Note that the term "bin packing games" is used in the literature for a completely different type of games [10,11,16], and there is recent interest in those games as well.

Our result We show the the worst-case number of steps for convergence is $\Theta(n^{1.5})$. The exact function expressing the worst-case number of steps is

$$\frac{i(i+1)(i-1)}{3} + j - ij \ ,$$

where

$$n = \frac{i(i+1)}{2} - j \quad \text{for} \quad 0 \le j \le i-1 \ .$$

We prove the lower bound by defining a sequence of steps, while the upper bound is proved using two potential functions, one of which is used in [12] and the other one is completely different. Interestingly, combining the two potential functions allows us to find a tight bound for any $n \ge 1$.

2 The Exact Convergence Time

In [12], processes of the following kind were studied. The process starts with an arbitrary packing, and in each step one item that can reduce its cost by moving to another bin is selected by a controller and is moved to another bin such that its cost becomes smaller. The number of steps for convergence was shown to be $O(n^2)$ [12]. In this section we find the exact worst-case number of steps, which turns out to be $\Theta(n^{3/2})$. Note that [17] showed using methods from [9] (where convergence for scheduling problems is studied) that for the case of proportional cost sharing, the number of steps can be exponential.

Given an integer $n \ge 1$, we let

$$i = \min\{h | h(h+1)/2 \ge n\} \quad \text{and} \quad j = i(i+1)/2 - n \ .$$

Thus, $n = i(i+1)/2 - j$, where $i \ge 1$, and $0 \le j \le i-1$ (since $n > i(i-1)/2 = i(i+1)/2 - i$). Additionally, since $n \le i(i+1)/2 < (i+1)^2$ and $n > i(i-1)/2 > (i-1)^2/4$, we have $i > \sqrt{n} - 1$ and $i < 2\sqrt{n} + 1$, and thus $i = \Theta(\sqrt{n})$. We show that the maximum number of steps that can be performed for any set of items and initial configuration is exactly

$$\nabla_{i,j} = \frac{i(i+1)(i-1)}{3} + j - ij \ .$$

Note, that in case $i \ge 12$, all next inequalities are valid: $i - 1 \ge i/2$, $i/6 - 1 \ge i/12$ and $i + 1 \le 2i$. Thus we can easily check that

$$\nabla_{i,j} \le \frac{i(i+1)(i-1)}{3} < i^3$$

and

$$\nabla_{i,j} > \frac{i^3}{6} - i^2 = i^2(i/6-1) \geq i^3/12 \ .$$

Thus, $\nabla_{i,j} = \Theta(n^{\frac{3}{2}})$.

We start with the lower bound.

Lemma 1. *There exists an input of n items, for which there is an initial packing, and a sequence of*

$$\nabla_{i,j} = \frac{i(i+1)(i-1)}{3} + j - ij$$

steps that are performed until no additional steps can be done.

Proof. Consider a set of n items, each of size $\frac{1}{n}$, and an initial packing where each one of the items is packed in its own bin. Let a *staircase packing* be a packing where for every $1 \leq \eta \leq i$, $\eta \neq j$, there is exactly one bin with η items.

We show using induction on i that there exists a sequence of exactly $\Delta_{i,j} = i(i+1)(i-1)/6 - j(j-1)/2$ steps which results in a staircase packing. Note that for $i = 1$, $j = 0$ follows, and thus $\Delta_{1,0} = 0$. Otherwise $i \geq 2$, and in this case $j \geq 0$, $0 \leq j(j-1) \leq (i-1)(i-2)$ are valid. Thus we get the next chain of inequalities:

$$i(i+1)(i-1) - 3j(j-1) \geq i(i+1)(i-1) - 3(i-1)(i-2) = (i-1)(i^2-2i+6) > 0$$

where we also used that $i^2-2i+6 \geq 6$. First, we show the claim for the case $j = 0$ (where $\Delta_{i,0} = i(i+1)(i-1)/6$) by induction on i. For $i = 1$, in every packing there is exactly one bin with one item, and this packing is a staircase packing. For a given value of i, $n = i(i+1)/2$. We consider a subset of $n' = n - i = i(i-1)/2$ items. By the induction hypothesis it is possible to obtain a packing such that for any $1 \leq \eta \leq i-1$ there is a bin with η items. Considering the complete set of n items, we get that for any $2 \leq \eta \leq i-1$ there is a bin with η items, and additionally there are $i+1$ bins, each with a single item. By the induction hypothesis, this packing is obtained in $i(i-1)(i-2)/6$ steps. Let B_η denote a specific bin with η items for $1 \leq \eta \leq i-1$, where the bin B_1 is chosen arbitrarily. The i other items packed in dedicated bins are called *free items*. For $k = 1, \ldots, i-1$, the k-th free item is moved from its bin, to the bins B_1, B_2, ..., B_{i-k}, in this order. B_{i-k} will now contain $i-k+1$ items and will not be used again in this process. After all these steps, B_η (for $1 \leq \eta \leq i-1$) will contain $\eta+1$ items. The i-th free item remains packed in its own bin, so as a result, for any $1 \leq \eta \leq i$ there is a bin with η items. The number of additional steps for the free items (the steps that are applied after the bins B_η are created using the induction hypothesis) is

$$\sum_{k=1}^{i-1}(i-k) = i(i-1)/2 \ ,$$

as the number of steps for the kth free item is $i-k$. The total number of steps is therefore

$$\frac{i(i-1)(i-2)}{6}+\frac{i(i-1)}{2}=\frac{i(i+1)(i-1)}{6}\ .$$

To show the claim for the case for $j \neq 0$ (and $i \geq 2$), we use the claim that was proved for $j=0$. Assume that $n=i(i+1)/2-j$ where $0<j<i$. In this case, first we create a staircase packing of a subset of $n'=i(i-1)/2$ items, leaving $i-j$ free items. For $k=1,\ldots,i-j$, the k-th free item is moved from its bin, to the bins B_1, B_2, …, B_{i-k}, in this order. The bin B_{i-k} will contain $i-k+1$ items as a result and will not be used for later steps. After this is done for $i-j$ items, B_η will contain $\eta+1$ items for $j \leq \eta \leq i-1$, and for $1 \leq \eta \leq j-1$, B_η still contains η items. Thus, for every $1 \leq \eta \leq i$, $\eta \neq j$, there is exactly one bin with η items and this is exactly a staircase packing as required. The number of additional steps (after the bins B_η are created using the claim for $j=0$) is

$$\sum_{k=1}^{i-j}(i-k)=i(i-1)/2-j(j-1)/2\ .$$

The total number of steps is

$$\frac{i(i+1)(i-1)}{6}-\frac{j(j-1)}{2}\ .$$

Once a staircase packing is achieved, we show that it is possible to reach a packing where all items are packed in one bin together using exactly $i(i+1)(i-1)/6-ij+j(j+1)/2$ steps. We define a phase as follows. In the beginning of a phase there are bins with different numbers of items. Let

$$J=\{j_1<j_2<\cdots<j_{|J|}\}$$

be the set of numbers of items before some phase, and let the bin B_η for $\eta \in J$ be the bin with η items. If $|J|>1$, we repeatedly take an item from B_{j_1}, and move it to B_{j_2} then to B_{j_3} and so forth until it reaches $B_{j_{|J|}}$. A phase ends when all items of B_{j_1} were moved. If $j=0$, then initially $J=\{1,\ldots,i\}$, there are $i-1$ phases, and the number of steps in all phases is

$$\sum_{k=1}^{i-1}k(i-k)=\frac{i^2(i-1)}{2}-\frac{(i-1)i(2i-1)}{6}=\frac{i(i-1)(i+1)}{6}\ .$$

Otherwise, initially $J=\{1,\ldots,i\}-\{j\}$, there are $i-2$ phases, and the number of steps is

$$\sum_{k=1}^{j-1}k(i-1-k)+\sum_{k=j+1}^{i-1}k(i-k)=\sum_{k=1}^{i-1}k(i-k)-\sum_{k=1}^{j-1}k-j(i-j)$$

$$=\frac{i^2(i-1)}{2}-\frac{i(i-1)(2i-1)}{6}-\frac{j(j-1)}{2}-j(i-j)=\frac{i(i+1)(i-1)}{6}-ij+\frac{j}{2}+\frac{j^2}{2}\ .$$

The total number of steps is therefore

$$\frac{i(i+1)(i-1)}{6}-\frac{j(j-1)}{2}+\frac{i(i-1)(i+1)}{6}+\frac{j}{2}-ij+\frac{j^2}{2}=\frac{i(i+1)(i-1)}{3}+j-ij\ .$$

□

Next, we prove the main result of this paper.

Theorem 1. *The maximum number of steps until convergence is at most*

$$\frac{i(i+1)(i-1)}{3}+j-ij=\Theta(n^{\frac{3}{2}})\ ,$$

and there exists an input of n items where this bound can be achieved.

Proof. The lower bound was proved in the previous lemma. For the upper bound, consider an input I of $n = i(i + 1)/2 - j$ items for $0 \le j \le i - 1$, an initial configuration and a sequence of moves. Let $p_{\min}$ denote the smallest item size in I. Let $\varepsilon = \min\{p_{\min}, 1/n\}$, and let I' be the input where $s_t = \varepsilon$ for $1 \le t \le n$. For the input I' there cannot be invalid moves, since all items can be packed into one bin.

Lemma 2. *The initial configuration and the sequence of moves of I are valid for I' as well.*

Proof. Since no item size was increased, all configurations of I are valid for I'. Since the cost of an item in a packing depends only on numbers of items in its bin and not on their sizes, modifying the sizes may only increase sets of beneficial deviations, that is, every move which was beneficial and possible for I remains such for I' and the sequence of moves is still valid. □

In what follows, we will consider only sequences of moves for I'. In particular, we consider only sequences with a maximum number of moves. Such a sequence must exist since from the results of [12] every sequence of moves has a finite length.

Lemma 3. *Every sequence with a maximum number of moves starts with the configuration where every item is packed in a separate bin, and ends with the configuration that all items are packed in one bin.*

Proof. Consider a sequence of ℓ moves. Assume that there is a bin B with $k \ge 2$ items in the initial configuration, and let $\phi \in B$. Modify the configuration such that instead of B the starting configuration has the two bins $B \setminus \{\phi\}$ and $\{\phi\}$ (other bins remain unchanged). Next, add a step in the beginning of the sequence of moves where ϕ moves to join the items of $B \setminus \{\phi\}$. This is an improving step since ϕ reduces its cost from 1 to $\frac{1}{k}$. This results in a sequence of $\ell + 1$ steps, which contradicts maximality.

Next, assume that after the sequence of moves there are at least two non-empty bins, containing k_1 and k_2 items respectively, where $k_1 \le k_2$. Let ψ be an item packed in the first bin. Add a move of ψ to the second bin in the end of the sequence. This is an improving step since ψ reduces its cost from $\frac{1}{k_1}$ to $\frac{1}{k_2+1} \leq \frac{1}{k_1+1} < \frac{1}{k_1}$. This results in a sequence of $\ell + 1$ steps, which contradicts maximality. □

Let $k > 0$ be an integer. We define a *level k small step* to be a move where an item moves from a k-bin to another k-bin. A step is called a *small step* if there is an integer k such that the step is a level k small step. Given the set of sequences of steps of maximum length we focus on sequences where the prefix of small steps has maximum length.

Lemma 4. *Assume that after a prefix of the sequence of steps is applied there are at least two k-bins. Then the first step in the remainder of the sequence of steps involving a k-bin is a level k small step.*

Proof. Assume by contradiction that there is no level k small step in the remaining part of the sequence. Since the sequence of steps terminates only when all items are packed in one bin, there is at least one item in the union of the k-bins that will perform a move (in fact, all the items of all the k-bins except for possibly one such bin will do that). Consider the first step after the current configuration was reached that involves a k-bin (either an item moving to the bin or moving out of it).

There are two possible moves. If an item ψ moves from a k-bin into a bin with $k' > k$ items, we modify the sequence as follows. First ψ moves to another k-bin, and then it moves to the bin with k' items. The second step is still beneficial for ψ since in the second step it moves from a $(k+1)$-bin to a bin with $k' \geq k+1$ items. This modification augments the length of the sequence by 1, which contradicts maximality.

If an item ϕ moves from a bin with $\tilde{k} < k$ items to one of the k-bins, we modify the sequence as follows. First choose an arbitrary item from one of the k-bins and move it to another k-bin. Then, move ϕ to the bin out of which the item was just moved (which now has $k - 1$ items). This last move is beneficial since $\tilde{k} \leq k - 1$. This modification augments the length of the sequence by 1, which contradicts maximality. □

Lemma 5. *Consider the prefix of small steps. After this prefix is performed, every bin has a different number of items.*

Proof. Assume by contradiction that at this time there are two k-bins. Using Lemma 4, there will be a level k small step later in the sequence, which will be the first move which involves k-bins. Since all items are identical, it is possible to perform such a step immediately instead of at a later time. This does not change the number of steps in the sequence, and it increases the length of the prefix of small steps, which contradicts maximality of the prefix. □

Lemma 6. *Consider the prefix of small steps. After this prefix is performed, there is one bin of each number of items in $\{1, 2, \ldots, i\} \setminus \{j\}$, that is, a staircase packing is created.*

Proof. We prove an invariant which is kept as long as only small steps are done. Let b_k be the number of bins with k items, and recall that initially $b_1 = n$ and $b_\ell = 0$ for $0 < \ell \leq n$. Assume that at a given time, k_m is the maximum integer such that $b_{k_m} > 0$. We say that a number $1 \leq k \leq k_m - 1$ is bad if $b_k = 0$, and otherwise it is good. That is, a number k is bad if there are no k-bins, but there exists at least one $(k+1)^+$-bin. If $b_k \geq 2$ then we say that k is *very good.* Two bad numbers are called *consecutive* bad numbers if all numbers between them are good, that is, if k_1 and k_2 such that $k_1 < k_2 < k_m$ are both bad ($b_{k_1} = 0$ and $b_{k_2} = 0$), and for all k' such that $k_1 < k' < k_2$, $b_{k'} > 0$.

The invariant is as follows. For every pair of consecutive bad numbers k_1, k_2, where $1 \leq k_1 < k_2 < k_m$, there exists a number $\tilde{k}$, where $k_1 < \tilde{k} < k_2$, such that $\tilde{k}$ is very good.

Initially, $k_m = 1$, thus there are no bad numbers, and the invariant holds trivially. Recall that we only analyze small steps and consider the change resulting from a single level k small step. Every level k small step implies that before this step there are at least two k-bins and so k is very good.

Note that k is the only number that can become bad as a result of a level k small step. Moreover, if $k = k_m$, then the value k_m increases by 1. Assume first that k remains very good. No bad numbers are created, and since no number stops beings very good then the invariant holds (even if some number stops being bad). If k remains good, but not very good, then still no new bad numbers are created and we only need to consider the case that k was the only very good number between two consecutive bad numbers. Let these two numbers be $k_1 < k < k_2$. If $k_2 > k+1$ and $k_1 < k-1$, then the numbers of k_1-bins and k_2-bins are unchanged (that is, these numbers remain zero) and the numbers k_1, k_2 remain consecutive bad numbers between which we need to show that a very good number exists after the step. Since $k+1$ was good, as a result of the move $b_{k+1} \geq 2$, and since $k_1 < k+1 < k_2$, there is a very good number between k_1 and k_2, as required. If $k_1 = k-1$ but $k_2 > k+1$ then k_1 becomes good. If k_1 was the minimum bad number then we are done. Otherwise, let $k_3 < k_1$ be a bad number such that k_3 and k_1 were consecutive bad numbers. We now have that k_3 and k_2 are consecutive bad numbers and $b_{k+1} \geq 2$ so $k+1$ is a very good number between them. If $k_1 < k-1$ but $k_2 = k+1$ then k_2 becomes good. If k_2 was the maximum bad number then we are done. Otherwise, let $k_4 > k_2$ be a bad number such that k_2 and k_4 were consecutive bad numbers. We now have that k_3 and k_4 are consecutive bad numbers and $b_{k-1} \geq 2$ so $k-1$ is a very good number between them. Finally, if both $k_1 = k-1$ and $k_2 = k+1$ hold, then the only case of interest is when k_1 was not the minimum bad number and k_2 was not the maximum bad number. We let $k_3 < k_1$ be a bad number such that k_3 and k_1 were consecutive bad numbers, and let $k_4 > k_2$ be a bad number such that k_4 and k_2 were consecutive bad numbers. Now k_3 and k_4 are consecutive bad numbers. There is a very good number in (k_3, k_1) which is now a very good number between k_3 and k_4.

Finally, we consider the case where k becomes bad. If there previously was a bad number k_2 such that $k_2 > k$, we distinguish two cases. If $k_2 > k+1$, then

k and k_2 becomes a consecutive bad pair of numbers, and $k+1$ becomes a very good number between them. Otherwise, $k_2 = k+1$ becomes good. If k_2 was the maximum bad number then we are done, and otherwise, let $k_4 > k_2$ be such that k_2 and k_4 were consecutive bad numbers. Instead, k and k_4 are now consecutive bad numbers, and the very good number between them is the same one which was very good between k_2 and k_4. The proof is symmetric for the case that there previously was a bad number k_1 such that $k_1 < k$.

To complete the proof, consider the configuration after the prefix of small steps. Since every bin has a different number of items, there are no very good numbers, and hence, by the invariant, there is at most one bad number. If there exists a bin with at least $i+1$ items, and there is just one bad number, then there are at least $(i+1)(i+2)/2 - i = i(i+1)/2 + 1 > n$ items. If there is no bin with at least i items, then there are at most $i(i-1)/2 < n$ items. Thus, there is a bin with i items, and since there is at most one bad number, the bad number must be j if $j \neq 0$, and otherwise there is no bad number. Therefore, the packing at this time is a staircase packing. □

Lemma 7. *The number of steps in the prefix of small steps is at most*

$$\frac{i(i+1)(i-1)}{6} - \frac{j(j-1)}{2} \ .$$

Proof. We use the potential function as in [12] which is the sum of squares of number of items in the bins. In the beginning every item is in a dedicated bin, so the potential is equal to $n = i(i+1)/2 - j$. Consider a level k small step. The potential function increases by exactly 2 in this step, since the only change is that instead of two k-bins, there is a $(k-1)$-bin a $(k+1)$-bin, and the increase in the potential is exactly

$$(k+1)^2 + (k-1)^2 - 2k^2 = 2 \ .$$

Since a staircase packing is achieved in the prefix of small steps, the value of the potential after this prefix is

$$\sum_{k=1}^{i} k^2 - j^2 = \frac{i(i+1)(2i+1)}{6} - j^2 \ .$$

Thus, the number of steps cannot exceed half the difference between the final potential and the initial potential, which is

$$\left(\frac{i(i+1)(2i+1)}{6} - j^2 - \left(\frac{i(i+1)}{2} - j\right)\right)/2 = \frac{i(i+1)(i-1)}{6} - \frac{j(j-1)}{2} \ .$$

□

Lemma 8. *The number of steps in the remainder of the sequence after the prefix of small steps is at most*

$$\frac{i(i+1)(i-1)}{6} - ij + \frac{j(j+1)}{2} \ .$$

Proof. In this case we define a different potential function. Sort the bins in non-increasing order according to numbers of items. Let the index of an item be the index of the bin into which it is packed. The potential of a packing is sum of indices of items.

The potential is clearly positive at all times. The final potential is n, since all items are packed in one bin. Consider a step in which an item moves from a k_1-bin B_v to a k_2-bin B_u (where $k_2 \geq k_1$). Since all items are identical, we assume that B_v is the k_1-bin of maximum index, and B_u is the k_2 bin of minimum index. This holds even if $k_1 = k_2$, since in this case there are at least two bins with this number of items. Since the bins are sorted by non-increasing order according to numbers of items we have $v > u$. As a result of the move, B_v now has $k_1 - 1$ items, and B_u now has $k_2 + 1$ items. By definition, if $u > 1$ then B_{u-1} has at least $k_2 + 1$ items. Similarly, if B_{v+1} exists then it has at most $k_1 - 1$ items, so the sorted order is still valid. The change in the potential in this step is the change in the index of the bin of the moving item, which is $v - u \geq 1$.

If $j = 0$, then the potential before the remainder of the sequence of moves is performed is

$$\sum_{k=1}^{i} k(i-k+1) = \frac{i(i+1)^2}{2} - \frac{i(i+1)(2i+1)}{6} = \frac{i(i+1)(i+2)}{6}$$

while $n = \frac{i(i+1)}{2}$, so the number of steps is at most

$$\frac{i(i+1)(i+2)}{6} - \frac{i(i+1)}{2} = \frac{i(i+1)(i-1)}{6} .$$

If $j > 0$, then the potential before the remainder of the sequence of moves is performed is

$$\sum_{k=1}^{i-j} k(i-k+1) + \sum_{k=i-j+1}^{i-1} k(i-k) = i\sum_{k=1}^{i-1} k + \sum_{k=1}^{i-j} k - \sum_{k=1}^{i-1} k^2$$
$$= \frac{i^2(i-1)}{2} + \frac{(i-j)(i-j+1)}{2} - \frac{i(i-1)(2i-1)}{6}$$
$$= \frac{i(i-1)(i+1)}{6} + \frac{i^2}{2} + \frac{j^2}{2} - ij + \frac{i}{2} - \frac{j}{2} .$$

In each step the function decreases by at least 1, so the number of steps is at most

$$\frac{i(i-1)(i+1)}{6} + \frac{i^2}{2} + \frac{j^2}{2} - ij + \frac{i}{2} - \frac{j}{2} - \frac{i(i+1)}{2} + j = \frac{i(i+1)(i-1)}{6} - ij + \frac{j}{2} + \frac{j^2}{2} .$$

□

Summing up the maximum number of steps in the prefix and in the remainder we get

$$\frac{i(i+1)(i-1)}{6} - \frac{j(j-1)}{2} + \frac{i(i+1)(i-1)}{6} - ij + \frac{j(j+1)}{2} = \frac{i(i+1)(i-1)}{3} + j - ij .$$

□

3 Further Research - Open Questions

In this paper the completely adversarial/arbitrary strategy is considered only. What about adversarial order of players, but choosing the move that best improves the individual cost? This question is very natural, and is of sense.

Another issue is the following: We determined the worst-case number of steps for convergence, when in the initial configuration there is only one item in any bin. This setting can be seen in a very natural manner, from the reverse side: Consider a game, when there are balls in one bin, and there are many further bins. Any ball (i.e. any item) wants to be in a more safe place, i.e. it moves to another bin where it will share the bin with a smaller number of another balls. In this game the final configuration will be just where any ball has an own bin. Thus the number of steps is the same. Returning to our original bin packing game, it remains open, that how many steps are possible in the worst case, starting with an arbitrary initial configuration. It seems that the investigation can be handled in a similar way.

Both questions are open now, and remain for further research.

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Short Sequences of Improvement Moves Lead to Approximate Equilibria in Constraint Satisfaction Games*

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Abstract. We present an algorithm that computes approximate pure Nash equilibria in a broad class of constraint satisfaction games that generalize the well-known cut and party affiliation games. Our results improve previous ones by Bhalgat et al. (EC 10) in terms of the obtained approximation guarantee. More importantly, our algorithm identifies a polynomially-long sequence of improvement moves from any initial state to an approximate equilibrium in these games. The existence of such short sequences is an interesting structural property which, to the best of our knowledge, was not known before. Our techniques adapt and extend our previous work for congestion games (FOCS 11) but the current analysis is considerably simpler.

1 Introduction

Constraint satisfaction games are generalizations of the well-known cut games and party affiliation games. In a constraint satisfaction game, there is a set of boolean variables and a set of weighted constraints; each constraint depends on some of these variables. Each player controls the value of a distinct variable and has two possible strategies: setting the value of the variable to either 0 (false) or 1 (true). The payoff (or utility) of a player is the total weight in satisfied constraints where her variable appears. Constraint satisfaction games are potential games. The total weight of satisfied constraints serves as an exact potential function in the sense that the difference in the potential between two states that differ in the strategy of a single player equals the change in the utility of that player. Hence, pure Nash equilibria (i.e., states in which no player has an incentive to unilaterally move in order to improve her utility) can be computed by solving

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the local search problem (see [14] for a theoretical treatment of local search) of computing a local maximum of the potential function. Unfortunately, this is a computationally-hard problem [19]. In this paper, we resort to the question of whether relaxed solution concepts — namely, approximate (pure Nash) equilibria — can be computed efficiently.

In particular, we consider constraint satisfaction games where each constraint depends on the value of at most k variables and has the property that its value can change from false to true by a unilateral change in any of its variables. In general, we refer to such games as P_k–Flip games following the terminology of Bhalgat et al. [3]. Particular examples of this type of constraints include "parity" and "not–all–equal" constraints. An odd (respectively, even) parity constraint requires that the number of its true variables is odd (respectively, even). A not-all-equal constraint consists of literals (i.e., variables or their negations) and requires that at least two of its literals have different values. We refer to P_k–Flip games consisting of parity constraints as Parity–k–Flip games; P_k–Flip games with not–all–equal constraints with at least $\bar{k}$ literals are called nae–$(\bar{k}, k)$–Flip games. Party affiliation games are Parity–2–Flip games and, in particular, cut games are Parity–2–Flip games with odd constraints or nae–$(2, 2)$–Flip games whose constraints have no negative literals.

By adapting and extending our techniques in [4] for congestion games, we present a polynomial-time algorithm that computes approximate equilibria in P_k–Flip games. The approximation guarantee is related to the stretch θ of the potential function of games in a given class, defined as the maximum over all games in the class of the maximum ratio between the potential values in two equilibria. As we show, P_k–Flip games have a stretch of $k+1$; hence, for general P_k–Flip games, the approximation guarantee $\theta + \varepsilon$ of our algorithm improves a previous one of $2k - 1 + \varepsilon$ by Bhalgat et al. [3] for $k \geq 3$. By bounding the stretch of nae–$(\bar{k}, k)$–Flip and Parity–k–Flip games, we are able to show further improvements. For nae–$(\bar{k}, k)$–Flip games, the approximation guarantee becomes $3 + \varepsilon$ for $\bar{k} = 2$ and $2 + \varepsilon$ for $\bar{k} \geq 3$; these results improve a bound of $\frac{2\bar{k}}{\bar{k}-1} + \varepsilon$ from [3]. For Parity–k–Flip games with odd k, the approximation guarantee is $k+\varepsilon$. The running time of the algorithm is bounded by a polynomial of the number of players, k, and $1/\varepsilon$. Our analysis follows the same general structure of [4] but uses different technical arguments and is considerably simpler due to the simplicity in the definition of P_k–Flip games.

More importantly, for every initial state of the game, our algorithm identifies a polynomially-long sequence of improvement moves of the players that lead to an approximate equilibrium. The existence of such short sequence suggests an interesting structural property of P_k–Flip games which, to the best of our knowledge, was not known before. Actually, Bhalgat et al. [3] argue about the limitations of (uncoordinated) improvement move sequences by presenting a particular cut game in which any sequence of ρ-moves (i.e., moves that improve the utility of the moving player by a factor of at least ρ) from some states to any ρ–approximate equilibrium has exponential length for any $\rho \in [1, 21/20)$. This negative result complements nicely with the structural property we prove.

Our algorithm is simple. Players are classified into blocks so that the players within the same block have polynomially-related maximum utility (i.e., total weight of the constraints a player can affect). Then, a set of phases is executed. In each phase the players in two consecutive blocks are allowed to move. The players in the block of higher maximum utility are allowed to make p-moves and the players of the other block are allowed to make q-moves. Then, the strategies of the players that were allowed to perform p-moves within a phase are irrevocably decided at its end. Clearly, this defines a sequence of improvement moves by the players. We show that by setting the parameters q and p appropriately, the algorithm terminates in polynomial time and, furthermore, the players whose strategies are irrevocably decided at the end of a phase will not be affected significantly by later moves. In order to do so, we select a value for parameter p that is slightly higher than the stretch of the class of games to which the input game belongs and a value for parameter q that is very close to 1.

Related Work. Schäffer and Yannakakis [19] proved that the problem of computing a pure Nash equilibrium in constraint satisfaction games is complete for the class PLS — standing for polynomial local search — that has been introduced by Johnson et al. [11]. The negative result of [19] covers all games considered in the current work and have been strengthened in [12,13] to capture instances in which each player participates in a constant number of constraints. Among the few rare non-trivial positive results is an algorithm by Poljac [17] who shows that a local maximum of the potential function in cut games can be computed in polynomial time when each player participates in at most three constraints.

The algorithm of [3] for approximate equilibria in P_k–FLIP games has the following structure. Players are partitioned into layers in a similar way to the block partitioning that we use in the current paper. Then, a rearrangement phase moves players across blocks in order to guarantee that the total weight of constraints, in which a player i participates together only with players in the same block or ones having lower maximum utility, is at least $1/k$ of player i's maximum utility. This can be done in such a way that, eventually, each layer contains players with polynomially-related maximum utility. Then, a top-down layer dynamics phase takes place, where players within each layer play $(1+\varepsilon/k)$-moves in a restricted game among them until they reach an $(1+\varepsilon/k)$–approximate equilibrium in this restricted game. The authors of [3] show that the state computed in this way is a $(2k-1+\varepsilon)$–approximate equilibrium for the original game. They also present a variation of their algorithm for NAE–$(\bar{k},k)$–FLIP games that computes $(\frac{2\bar{k}}{k-1}+\varepsilon)$–approximate equilibria. As the authors of [3] emphasize, in general, the moves during the top-down layer dynamics phase are not improvement moves in the original game. In contrast, our algorithm consists only of improvement moves.

Another class of potential games where the problem of computing an (approximate) equilibrium has received a lot of attention is that of congestion games. A classical potential function for these games has been defined by Rosenthal [18]. Fabrikant et al. [8] prove that computing a local minimum of this function (corresponding to a pure Nash equilibrium) is PLS-hard as well. Even worse, for sufficiently general congestion games, Skopalik and Voecking [20] show that computing a ρ–approximate equilibrium is PLS-hard for every reasonable (i.e., polynomially-computable) value of ρ. In our previous work [4], we have presented an algorithm to compute $O(1)$-approximate equilibria for congestion games under mild assumptions for the structure of the game. The current paper adapts and extends the main algorithmic techniques in that paper, which have also been applied to (non-potential) weighted variants of congestion games in [5]. Exact or almost exact equilibria can be computed in several special cases, e.g., see [6,8].

We remark that, even though it is hard to compute exactly, a local optimum of a potential function can be approximated with extremely low precision under very mild assumptions [15]. This does not imply that equilibria can be approximated with a similar precision, as the negative results of [20] show. Also, uncoordinated move sequences have been shown to reach states of high social value quickly [1,2,7], i.e., to states with low potential in the case of P_k–FLIP games. Unfortunately, these states are not approximate equilibria either, since some player typically has a high incentive to move.

Roadmap. The rest of the paper is structured as follows. We begin with preliminary definitions in Section 2. Section 3 is devoted to our upper bounds on the stretch of P_k–FLIP games. The algorithm and the statement of our main result are presented in Section 4 and the analysis follows in Section 5. We conclude with open problems in Section 6.

2 Preliminaries

A constraint satisfaction game consists of a set N of n players, a set of at least n boolean variables $V = \{s_1, s_2, ..., s_{|V|}\}$, and a set C of constraints (henceforth called clauses) over the variables in V. Each clause $c \in C$ has a non-negative weight w_c. Player $j \in N$ controls the value of a distinct variable s_j from V and has two possible strategies: setting the value of s_j to either 0 (false), or 1 (true). The variables of V that are not controlled by any player (if any) are frozen to certain boolean values. A state S of the game is simply a snapshot of variable values (or a snapshot of players' strategies complemented with the fixed values of the frozen variables), i.e., $S = (s_1, s_2, ..., s_{|V|})$. Given a state S of the game, we denote by $SAT(S)$ the set of satisfied clauses. For a subset of players $R \subseteq N$, we denote by $SAT_R(S)$ the subset of $SAT(S)$ that consists of clauses in which the variable of some player from R appears. With some abuse of notation, we simplify $SAT_{\{j\}}(S)$ to $SAT_j(S)$. The utility of a player j is the total weight of the true clauses in which her variable appears, i.e., $u_j(S) = \sum_{c \in SAT_j(S)} w_c$. We also denote by C_R the set of clauses in which at least one player of R participates and simplify $C_{\{j\}}$ to C_j. We use U_j to denote the maximum possible utility that player j might have, i.e., $U_j = \sum_{c \in C_j} w_c$.

Given a state $S = (s_1, s_2, ..., s_{|V|})$ and a player j, we denote by (S_{-j}, s'_j) the state obtained from S when player j unilaterally changes her strategy from s_j to its complement s'_j. This is an improvement move (or simply, a move) for player j if her utility increases, i.e., $u_j(S_{-j}, s'_j) > u_j(S)$. We call it a ρ–move when the utility increases by more than a factor of ρ, i.e., $u_j(S_{-j}, s'_j) > \rho \cdot u_j(S)$. A state S is a pure Nash equilibrium (or simply, an equilibrium) if no player has a move to make. Similarly, S is a ρ–approximate (pure Nash) equilibrium if no player has a ρ-move.

We specifically consider clauses with the following property: any false clause can become true by changing the value of any of its variables. We will refer to games with clauses satisfying this property and with at most k variables per clause as P_k–FLIP games. This class is broad enough and contains (generalizations of) several well-studied games such as cut games and party affiliation games. We are particularly interested in two subclasses of P_k–FLIP games. A NAE-clause contains literals (i.e., variables or their negations) and equals 1 if and only if there are two literals with different values. We will refer to games consisting of NAE-clauses with at least $\bar{k} \geq 2$ and most k literals as NAE–$(\bar{k}, k)$–FLIP games. Observe that these games are P_k–FLIP games since changing the value of any variable that appears in a clause can change the value of the clause from 0 to 1. In PARITY–k–FLIP games, each clause is characterized as odd or even; an odd (respectively, even) clause is true if the number of its variables which are 1 is odd (respectively, even). An important property of P_k–FLIP games is that for any state S and any player j, it holds that $U_j \leq u_j(S) + u_j(S_{-j}, s'_j)$.

Given a state S of a P_k–FLIP game, we denote by $\Phi(S)$ the total weight of all true clauses, i.e., $\Phi(S) = \sum_{c \in SAT_N(S)} w_c$. The function Φ is a potential function for this game. In particular, it has the remarkable property that for every two states S and (S_{-j}, s'_j) differing only in the strategy of player j, the difference of the potential is equal to the difference of the utility of player j, i.e., $\Phi(S) - \Phi(S_{-j}, s'_j) = u_j(S) - u_j(S_{-j}, s'_j)$.

In the following, we will be often considering sequences of moves in which only players in a certain subset $R \subseteq N$ are allowed to move. We can view such moves as moves in a subgame among the players in R, with the set of clauses C_R (each clause in C_R has the same weight as in the original game), and with fixed values for the variables that are not controlled by players in R. Observe that any subgame of a P_k–FLIP game is a P_k–FLIP game as well. Similarly, any subgame of a NAE–$(\bar{k}, k)$–FLIP (respectively, PARITY–k–FLIP) game is a NAE–$(\bar{k}, k)$–FLIP (respectively, PARITY–k–FLIP) game as well. The function $\Phi_R(S) = \sum_{c \in SAT_R(S)} w_c$ is an exact potential function for the subgame among the players in R. The next lemma follows easily by the definitions.

Lemma 1. *For every state S of a P_k–FLIP game and any set of players $R \subseteq N$, it holds that $\Phi_R(S) \leq \sum_{j \in R} u_j(S) \leq k\Phi_R(S)$. Furthermore, for every set of players $R' \subseteq R$, it holds that $\Phi_{R'}(S) \leq \Phi_R(S)$.*

3 The Stretch of P_k–FLIP Games

The approximation guarantee of our algorithm depends on a quantity related to the potential function of P_k–FLIP games that we call the stretch.

Definition 1. *Given $\eta \geq 0$, the $(1+\eta)$-stretch of a P_k–*FLIP *game is the ratio between the maximum and the minimum value of the potential function taken over all $(1+\eta)$-approximate pure Nash equilibria of the game.*

We use the term stretch as a synonym of 1-stretch; observe that it is simply the ratio between the maximum and minimum potentials of (exact) equilibria. In Theorem 1, we present upper bounds on the $(1+\eta)$–stretch of P_k–FLIP games. Note that these bounds may be of independent interest; bounds on the stretch of congestion games from our previous work [4] have been used by Piliouras et al. [16] in order to quantify the price of anarchy of congestion games in settings with uncertainty where players have particular risk attitudes.

Theorem 1. *For any $\eta > 0$, the $(1+\eta)$–stretch of P_k–*FLIP *games,* NAE–$(3,k)$–FLIP *games,* NAE–$(2,k)$–FLIP *games, and* PARITY–k–FLIP *games with odd k is at most $k+1+k\eta$, $2+k\eta$, $3+k\eta$, and $k+k\eta$, respectively.*

Due to lack of space, the proof (as well as counterexamples showing that the bounds are tight) is omitted. In the following, we use the notation $\theta(1+\eta)$ to denote our upper bound on the $(1+\eta)$-stretch of P_k–FLIP games (and clarify when we refer to the stretch of particular subclasses of P_k–FLIP games). We use simply θ to denote the upper bound on the 1-stretch.

4 The Algorithm

The pseudocode of our algorithm appears below as Algorithm 1. We supplement this formal description with a detailed line–by–line explanation. The algorithm takes as input a P_k–FLIP game $\mathcal{G}$ with n players, an initial state S_{in}, and an accuracy parameter $\varepsilon \in (0,1]$. Starting from state S_{in}, it identifies a sequence of moves that lead to a state S_{out}; this is the output of the algorithm. As we will prove later, S_{out} is an approximate equilibrium. The algorithm starts (lines 1 and 2) by setting the values of parameters q and p. Parameter q has a value very close to 1 (namely, $q = 1 + \frac{\varepsilon}{3k}$) and parameter p has a value slightly higher than the q-stretch of the class to which the input game belongs (namely, $p = \theta(q) + \varepsilon/3$). In particular, using our upper bounds on $\theta(q)$ from Theorem 1, p is set to be $k+1+2\varepsilon/3$ in general, $2+2\varepsilon/3$ if $\mathcal{G}$ is a NAE–$(3,k)$–FLIP game, $3+2\varepsilon/3$ if it is a NAE–$(2,k)$–FLIP games, and $k+2\varepsilon/3$ if it is a PARITY–k game and k is odd. The algorithm also sets the value of parameter Δ to be a polynomial depending on n, k, p, and $1/\varepsilon$ (line 3). Then (lines 4-5), it implicitly partitions the players into blocks B_1, B_2, ..., B_m according to their maximum utility. Denoting by $U_{\max}$ the maximum values among all players' maximum utilities, block B_i consists of the players j with maximum utility $U_j \in (U_{\max}\Delta^{-i}, U_{\max}\Delta^{1-i}]$. By the definition of Δ, the players in the same block have polynomially related maximum utilities.

The sequence of moves from state S_{in} to state S_{out} is computed by the code in the lines 6-15. The subsequence of moves described in lines 7-9 constitutes phase 0. During phase 0, the players in block B_1 make q-moves. After that, each phase i for $i \geq 1$ consists of p-moves of players in block B_i and q-moves of players in block B_{i+1}. Strategies of players in block B_i are irrevocably decided at the end of phase i.

We are ready to state our main result which we will prove in the next section.

Theorem 2. *On input a P_k–FLIP game $\mathcal{G}$ with n players, an initial state S_{in}, and $\varepsilon \in (0, 1]$, Algorithm 1 computes a sequence of at most poly$(n, k, 1/e)$ moves that starts from S_{in} and converges to a $(k + 1 + \varepsilon)$–approximate pure Nash equilibrium S_{out}. The approximation guarantee is at most $2 + \varepsilon$ when $\mathcal{G}$ is a NAE–$(3, k)$–FLIP game, at most $3 + \varepsilon$ when it is a NAE–$(2, k)$–FLIP games, and at most $k + \varepsilon$ when it is a PARITY–k–FLIP game and k is odd.*

Input : A P_k–FLIP game $\mathcal{G}$ with a set N of n players, an arbitrary initial state S_{in}, and $\varepsilon \in (0, 1]$
Output: A state S_{out} of $\mathcal{G}$

$q \leftarrow 1 + \frac{\varepsilon}{3k}$;
$p \leftarrow \theta(q) + \varepsilon/3$;
$\Delta = 200p^3nk/\varepsilon^2$;
Set $U_{\min} \leftarrow \min_{j\in N} U_j$, $U_{\max} \leftarrow \max_{j\in N} U_j$, and $m \leftarrow 1 + \lfloor \log_\Delta (U_{\max}/U_{\min}) \rfloor$;
(Implicitly) partition players into *blocks* $B_1, B_2, \ldots, B_m$, such that $j \in B_i$ implies that $U_j \in \left(U_{\max}\Delta^{-i}, U_{\max}\Delta^{1-i}\right]$;
$S \leftarrow S_{\text{in}}$;
while *there exists a player $j \in B_1$ such that $u_j(S_{-j}, s'_j) > q \cdot u_j(S)$* **do**
  $S \leftarrow (S_{-j}, s'_j)$;
end
for *phase $i \leftarrow 1$* **to** *$m - 1$ such that $B_i \neq \emptyset$* **do**
  while *there exists a player j that either belongs to B_i and satisfies $u_j(S_{-j}, s'_j) > p \cdot u_j(S)$ or belongs to B_{i+1} and satisfies $u_j(S_{-j}, s'_j) > q \cdot u_j(S)$* **do**
    $S \leftarrow (S_{-j}, s'_j)$;
  end
end
$S_{\text{out}} \leftarrow S$;

Algorithm 1. Computing approximate equilibria in P_k–FLIP games.

5 Proof of Theorem 2

Before presenting the proof of Theorem 2, we give some intuition behind our analysis. The analysis uses two properties that are formally stated in Lemma 2. What this lemma essentially says is that, during each phase, the total utility of the moving players as well as an increase in the potential of the subgame among these players are small. The first property is used in Lemma 4 to prove that, once the strategy of a player is irrevocably decided, later phases may have only a negligible effect on her. And since no player has a p-move to make at the end of the phase when her strategy is decided, she cannot improve her utility by a factor of (almost) p until the end of the algorithm. Together with the fact that each player's move increases her utility by some non-negligible amount, the second property is used in Lemma 5 to bound the total number of moves.

In our analysis, we denote by S^i the state reached at the end of phase $i \geq 0$, i.e., $S_{\text{out}} = S^{m-1}$. We also denote by R_i the set of players that move during phase i. We also denote the upper boundary of block B_i by W_i and by W_{m+1} the lower boundary of block B_m, i.e., $W_i = U_{\max}\Delta^{1-i}$ for $i = 1, 2, ..., m+1$. So, the players of block B_i are those with maximum utility $U_j \in (W_{i+1}, W_i]$.

Lemma 2. *For every phase $i \geq 1$, it holds that*

1. $\sum_{j \in R_i} U_j \leq 10pknW_{i+1}/\varepsilon$
2. $\Phi_{R_i}(S^i) - \Phi_{R_i}(S^{i-1}) \leq 3p^2nW_{i+1}/\varepsilon$.

Proof. First observe that players not in R_i have the same set of strategies in states S^{i-1} and S^i. Furthermore, the total weight of clauses depending on variables that are controlled by players from $R_i \cap B_{i+1}$ is at most nW_{i+1}. Hence, by the definition of the subgame potential, we have that the potential of the state $(S^{i-1}_{-R_i \cap B_i}, S^i_{R_i \cap B_i})$ in which the players in $R_i \cap B_i$ play their strategies in state S^i and the remaining players play their strategies in S^{i-1} satisfies

$$\Phi_{R_i \cap B_i}(S^{i-1}_{-R_i \cap B_i}, S^i_{R_i \cap B_i}) \geq \Phi_{R_i}(S^i) - nW_{i+1}. \tag{1}$$

We will use inequality (1) in the proof of the next lemma that provides a bound on the potential $\Phi_{R_i}(S^{i-1})$ as well as later in the current proof.

Lemma 3. $\Phi_{R_i}(S^{i-1}) \leq 3pnW_{i+1}/\varepsilon$.

Proof. We assume on the contrary that $\Phi_{R_i}(S^{i-1}) > 3pnW_{i+1}/\varepsilon$ and we are going to conclude that the potential of the state $(S^{i-1}_{-R_i \cap B_i}, S^i_{R_i \cap B_i})$ satisfies $\Phi_{R_i \cap B_i}(S^{i-1}_{-R_i \cap B_i}, S^i_{R_i \cap B_i}) > \theta(q) \cdot \Phi_{R_i \cap B_i}(S^{i-1})$. By Theorem 1, this would contradict the fact that S^{i-1} is the output of phase $i-1$, i.e., a q-approximate equilibrium of the subgame among the players in $R_i \cap B_i$, since there is another q–approximate equilibrium (the one that can be reached from $(S^{i-1}_{-R_i \cap B_i}, S^i_{R_i \cap B_i})$ with q-moves by the players in $R_i \cap B_i$) with a potential that is higher than $\theta(q)$ times the potential at state S^{i-1}.

We denote by $\ell(j)$ the utility of player $j \in R_i \cap B_i$ right after she makes her last move in phase i. Then we have

$$\Phi_{R_i}(S^i) - \Phi_{R_i}(S^{i-1}) \geq (1 - 1/p) \cdot \sum_{j \in R_i \cap B_i} \ell(j). \tag{2}$$

Indeed, the last move of a player $j \in R_i \cap B_i$ increases her utility by a factor of at least p and the difference $\Phi_{R_i}(S^i) - \Phi_{R_i}(S^{i-1})$ equals to the total increase in the utility of the deviating players within the phase.

Furthermore, we claim that

$$\sum_{j \in R_i \cap B_i} \ell(j) + nW_{i+1} \geq \Phi_{R_i}(S^i). \tag{3}$$

To see why (3) is true, observe that the right-hand side is the sum of the weights of the clauses in $SAT_{R_i}(S^i)$. The term nW_{i+1} is an upper bound on the total weight of the clauses in $SAT_{R_i \cap B_{i+1}}(S^i)$. The weight of each of the remaining ones (i.e., the clauses in $SAT_{R_i}(S^i) \setminus SAT_{R_i \cap B_{i+1}}(S^i)$) is accounted for at least once in the sum $\sum_{j \in R_i \cap B_i} \ell(j)$, as part of the utility of some player from $R_i \cap B_i$ after her last move.

By (2) and (3) (i.e., by multiplying (2) by p and (3) by $p-1$ and summing them), we obtain that

$$\Phi_{R_i}(S^i) \geq p \cdot \Phi_{R_i}(S^{i-1}) - (p-1)nW_{i+1}. \tag{4}$$

Hence, using (1), (4), the definition of p, and the second inequality of Lemma 1, we obtain

$$\begin{aligned} \Phi_{R_i \cap B_i}(S^{i-1}_{-R_i \cap B_i}, S^i_{R_i \cap B_i}) &\geq \Phi_{R_i}(S^i) - nW_{i+1} \geq p \cdot \Phi_{R_i}(S^{i-1}) - pnW_{i+1} \\ &> (p - \varepsilon/3) \cdot \Phi_{R_i}(S^{i-1}) \geq \theta(q) \cdot \Phi_{R_i \cap B_i}(S^{i-1}). \end{aligned}$$

We have obtained the desired contradiction. □

Using the observation that no player in $R_i \cap B_i$ has a q-move to make at the end of phase $i-1$ (i.e., at state S^{i-1}) as well as the first inequality of Lemma 1, we obtain that

$$\begin{aligned} \sum_{j \in R_i \cap B_i} U_j &\leq \sum_{j \in R_i \cap B_i} \left(u_j(S^{i-1}) + u_j(S^{i-1}_{-j}, s'_j) \right) \leq \sum_{j \in R_i \cap B_i} (1+q) u_j(S^{i-1}) \\ &\leq (1+q)k \cdot \Phi_{R_i \cap B_i}(S^{i-1}) \leq 9pknW_{i+1}/\varepsilon. \end{aligned}$$

The proof of the first inequality in the statement of the lemma follows by observing that the total utility of the players in $R_i \cap B_{i+1}$ is at most nW_{i+1}.

In order to prove the second inequality we use inequality (1), the q-stretch bound for the subgame among the players in $R_i \cap B_i$, the fact that $\theta(q) \leq p$, the second inequality of Lemma 1, and the bound on $\Phi_{R_i}(S^{i-1})$ from Lemma 3.

$$\begin{aligned} \Phi_{R_i}(S^i) - \Phi_{R_i}(S^{i-1}) &\leq \Phi_{R_i \cap B_i}(S^{i-1}_{-R_i \cap B_i}, S^i_{R_i \cap B_i}) - \Phi_{R_i}(S^{i-1}) + nW_{i+1} \\ &\leq \theta(q) \cdot \Phi_{R_i \cap B_i}(S^{i-1}) - \Phi_{R_i}(S^{i-1}) + nW_{i+1} \\ &\leq (p-1) \cdot \Phi_{R_i}(S^{i-1}) + nW_{i+1} \\ &\leq 3p^2 nW_{i+1}/\varepsilon. \end{aligned}$$

□

The first property of Lemma 2 indicates that the total weight of the moving players in phase i is significantly smaller than the upper boundary of block B_i. In Lemma 4 we combine this with the fact that the upper boundary of subsequent blocks decreases exponentially and formally prove that, after the strategy of a player is irrevocably decided, subsequent phases may have only a negligible effect on her. Recall that θ is the stretch of the class of games to which the input game belongs to and equals $k+1$ for P_k–Flip games, 3 for nae–$(2,k)$–Flip games, and 2 for nae–$(3,k)$–Flip games, and $k-1$ for Parity–k–Flip games with odd k.

Lemma 4. *The state S_{out} is a $(\theta+\varepsilon)$–approximate pure Nash equilibrium.*

Proof. By the definition of phase $m-1$, the players in blocks B_{m-1} and B_m have no p-move to make at the end of phase $m-1$. We will consider a player j belonging to block B_t whose strategy is irrevocably decided at the end of phase t with $t \leq m-2$, and will show that she has no $(p+\varepsilon/3)$-move to make at the end of phase $m-1$ (i.e., at state $S^{m-1} = S_{\text{out}}$). The lemma will then follow since $p+\varepsilon/3 = \theta(1+\frac{\varepsilon}{3k}) + 2\varepsilon/3 = \theta+\varepsilon$.

Let s_j be the strategy used by player j at the end of phase t. Using Lemma 2 and the definition of the block boundaries, we can bound the quantity $\sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r$. Thus, we get an upper bound on the total weight of clauses with players that move in phases $t+1,...,m-1$, as follows:

$$\sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r \leq \sum_{i=t+1}^{m-1} 10pnkW_{i+1}/\varepsilon \leq \frac{10pnkW_{t+1}}{\varepsilon}\sum_{i=1}^{\infty}\Delta^{-i}$$
$$= \frac{10pnkW_{t+1}}{\varepsilon(\Delta-1)} \leq \frac{W_{t+1}\varepsilon}{10p^2}. \tag{5}$$

The last inequality follows by the definition of Δ and the fact that $\Delta-1 \geq \Delta/2$.

Now observe that since player j has no p-move at the end of phase t (i.e., at state S^t), it holds that $u_j(S^t) \geq u_j(S^t_{-j}, s'_j)/p$ and $W_{t+1} \leq u_j(S^t)+u_j(S^t_{-j}, s'_j) \leq (1+p)u_j(S^t)$, i.e., $u_j(S^t) \geq \frac{W_{t+1}}{1+p}$. Furthermore, during phases $t+1,...,m-1$, the total change in the utility of player j or in the utility player j would have by deviating is at most $\sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r$. Using these observations and inequality (5), we have

$$\begin{aligned}
u_j(S^{m-1}) &\geq u_j(S^t) - \sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r \\
&\geq \frac{p}{p+\varepsilon/3}u_j(S^t) + \frac{\varepsilon/3}{p+\varepsilon/3}\frac{W_{t+1}}{1+p} - \sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r \\
&\geq \frac{1}{p+\varepsilon/3}u_j(S^t_{-j}, s'_j) + \frac{W_{t+1}\varepsilon}{5p(p+\varepsilon/3)} - \sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r \\
&\geq \frac{1}{p+\varepsilon/3}u_j(S^{m-1}_{-j}, s'_j) + \frac{W_{t+1}\varepsilon}{5p(p+\varepsilon/3)} - \left(1+\frac{1}{p+\varepsilon/3}\right)\sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r \\
&\geq \frac{1}{p+\varepsilon/3}u_j(S^{m-1}_{-j}, s'_j) + \frac{W_{t+1}\varepsilon}{5p(p+\varepsilon/3)} - \frac{2p}{p+\varepsilon/3}\sum_{i=t+1}^{m-1}\sum_{r\in R_i} U_r \\
&\geq \frac{1}{p+\varepsilon/3}u_j(S^{m-1}_{-j}, s'_j),
\end{aligned}$$

as desired. In the third and fifth inequalities we have used the inequalities $3(1+p) \leq 5p$ and $p+1+\varepsilon/3 \leq 2p$ which follow since $p \geq 2$ and $\varepsilon \in (0,1]$. This completes the proof of the lemma. □

To conclude the proof of Theorem 2, it remains to bound the running time of the algorithm; the proof of the next statement is omitted due to lack of space.

Lemma 5. *On input of P_k–FLIP (in particular, NAE–$(\bar{k}, k)$–FLIP) game, the algorithm identifies a sequence of at most $\mathcal{O}(n^3k^7/\varepsilon^4)$ (in particular, $\mathcal{O}(n^3k^2/\varepsilon^4)$) moves.*

6 Open Problems

A challenging open problem is to improve the approximation guarantee of our algorithm. Our analysis indicates that a state with lower stretch at the beginning of each phase would allow us to use an even smaller value for parameter p and, subsequently, to obtain a better approximation guarantee. One idea that comes immediately to mind is to replace the q-moves of the players of block B_{i+1} within phase i with the execution of an algorithm that computes states with approximately–optimal potential. For example, a random assignment to players of B_{i+1} would yield a 2-approximation to the potential of the subgame among them. Furthermore, for more structured P_k–FLIP games such as cut games, one might think to use the famous algorithm of [9] that is based on semi-definite programming. Unfortunately, we do not see how to include these ideas into our algorithm at this point. The main difficulty is that the low-stretch property should hold for the subgame among the players that will move during the next phase which we do not know in advance. An algorithm that approximates the potential of all subgames simultaneously would be ideal here but, besides the local search approach implied by the q-moves, neither the random assignment nor the SDP-based algorithms satisfy this property.

Even if we could bypass these obstacle, our technique has limitations since computing states with low-stretch in P_k–FLIP games includes famous hard-to-approximate problems (e.g., see [10]). So, in order to compute almost exact equilibria, we need new techniques. Of course, we have no idea whether this is at all possible. To put the question differently, is there some inapproximability threshold for approximate equilibria? We remark that such negative statements are not known in the literature: the only known negative results are either specific to exact equilibria (such as the PLS-hardness results of [8,19]) or rule out any reasonable approximation guarantee in games with very general structure (e.g., in [20]). We believe that such questions that are related to the computational complexity of approximate pure Nash equilibria deserve further attention.

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Paths to Stable Allocations

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Abstract. The stable allocation problem is one of the broadest extensions of the well-known stable marriage problem. In an allocation problem, edges of a bipartite graph have capacities and vertices have quotas to fill. Here we investigate the case of uncoordinated processes in stable allocation instances. In this setting, a feasible allocation is given and the aim is to reach a stable allocation by raising the value of the allocation along blocking edges and reducing it on worse edges if needed. Do such myopic changes lead to a stable solution?

In our present work, we analyze both better and best response dynamics from an algorithmic point of view. With the help of two deterministic algorithms we show that random procedures reach a stable solution with probability one for all rational input data in both cases. Surprisingly, while there is a polynomial path to stability when better response strategies are played (even for irrational input data), the more intuitive best response steps may require exponential time. We also study the special case of correlated markets. There, random best response strategies lead to a stable allocation in expected polynomial time.

Keywords: Stable matching, stable allocation, paths to stability, best response strategy, better response strategy, correlated market.

1 Introduction

Matching markets without prices model various real-life problems such as, e. g., employee placement, task scheduling or kidney donor matching. Research on those markets focuses on maximizing social welfare instead of profit. Stability is probably the most widely used optimality criterion in that case.

Finding equilibria in markets that lack a central authority of control is another widely studied, challenging task. Besides modeling uncoordinated markets, like third-generation (3G) wireless data networks [11], selfish and uncontrolled agents can also represent modifications in coordinated markets, e. g., the arrival of a new participant or slightly changed preferences [4]. In our present work, those two topics are combined: we study uncoordinated capacitated matching markets.

1.1 Stability in Matching Markets

The theory of stable matchings has been investigated for decades. Gale and Shapley [10] introduced the notion of stability on their well-known *stable marriage problem*. An instance of this problem consists of a bipartite graph where color

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classes symbolize men and women, respectively. Each participant has a preference list of their acquaintances of the opposite gender. A set of marriages (a matching) is *stable*, if no pair blocks it. A *blocking pair* is an unmarried pair so that the man is single or he prefers the woman to his current wife and vice versa, the woman is single or she prefers the man to her current husband. The Gale-Shapley algorithm was the first proof for the existence of stable matchings.

A natural extension of matching problems arises when capacities are introduced. The stable allocation problem is defined in a bipartite graph with edge capacities and quotas on vertices. The exact problem formulation is provided in Section 2, and a detailed example can be found in the full version of the paper [6].

1.2 Better and Best Response Steps in Uncoordinated Markets

Central planning is needed in order to produce a stable solution with the Gale-Shapley algorithm. In many real-life situations, however, such a coordination is not available. Agents play their selfish strategy, trying to reach the best possible solution. A *path to stability* is a series of myopic operations. The intuitive picture of a myopic operation is the following. If a man and a woman block a marriage scheme, then they both agree to form a couple together, even if they divorce their current partners to that end. This step may induce new blocking pairs. Such changes are made until a stable matching is reached. Note that stability is naturally a desirable property of uncoordinated markets. A stable matching seems to be the best reachable solution for all participants, because they cannot find any partnership that could improve their own position.

The study of uncoordinated matching processes has a long history. In the case of one-to-one matchings, two different concepts have been studied: better and best response dynamics. One of the color classes is chosen to be the *active* side. These vertices submit proposals to the *passive* vertices. According to *best response dynamics*, the best blocking edge of an active vertex is chosen to perform myopic changes along. In *better response dynamics*, any blocking edge can play this role.

The core questions regarding uncoordinated processes rise naturally. Can a series of myopic changes result in returning back to the same unstable matching? If yes, is there a way to reach a stable solution? How do random procedures behave? The first question about uncoordinated two-sided matching markets was brought up by Knuth [14] in 1976. He also gives an example of a matching problem where better response dynamics cycle. More than a decade later, Roth and Vande Vate [16] came up with the next result on the topic. They show that random better response dynamics converge to a stable matching with probability one. Analogous results for best response dynamics were published in 2011 by Ackermann et al. [2]. They also show an instance in which best response dynamics cycle (see the full version of this paper [6]), give a deterministic algorithm for reaching a stable solution in polynomial time and prove that the convergence time is exponential in both random cases.

Besides these works on the classical stable marriage problem, there is a number of papers investigating variants of it from the paths-to-stability point of view. For the stable roommates problem, the non-bipartite version of the stable marriage problem, it is known that there is a series of myopic operations that leads to a stable solution, if one exists [8]. A path to stability also exists in the bipartite matching case with payments where flexible salaries and productivity are taken into account [5]. In the hospitals/residents assignment problem, when couples are present, the existence of such a path is only guaranteed if the preferences are weakly responsive [13]. Weak responsiveness ensures consistence between the preferences of each partner and the couple's preference list on pairs of hospitals. In many-to-many markets, supposing substitutable preferences on one side and responsive preferences on the other side, a path to stability can be found [15]. Both substitutable and responsive preferences are defined in instances where preferences are given on sets of vertices. Although many variants of the stable marriage problem have been studied, no paper discusses the case of allocations (instead of matchings or b-matchings), where edges are capacitated, thus, they can be partially in stable solutions. Our present work makes an attempt to fill this gap in the literature.

Structure of the paper In the next section, the essential theoretical basis is provided: stable allocations, and better and best response modifications on such instances are defined. In Section 3, a special case of allocation instances are investigated. We show that although random best response processes generally run in exponential time, in the case of correlated markets, polynomial convergence is expected. Better and best response dynamics in the general case on rational input are extensively studied in Section 4. We describe two deterministic algorithms that generalize the result of Ackermann et al. on one-to-one matching markets to stable allocation instances and also show algorithmic differences between the two strategies. In the case of random procedures, convergence is shown for both strategies. Section 5 focuses on running time efficiency. There, a better response algorithm is presented that terminates with a stable solution in $O(|V|^2|E|)$ time, even for irrational input data. A counterexample proves that such an acceleration for the best response dynamics cannot be reached.

Table 1. Our results for rational input

	shortest path to stability	random path to stability
best response dynamics	exponential length	converges with probability 1
better response dynamics	polynomial length	converges with probability 1

Applied to a matching instance, our best-response algorithm performs the same steps as the two-phase best response algorithm of Ackermann et al. Our better-response variant can also be interpreted as an extended version of the above mentioned method. The only difference is that while our first phase is better response, theirs is best response. However, this seems to be a minor difference, as their proof is also valid for a better response first phase, and our proof still holds if only best blocking edges are chosen. Moreover, stable allocations might be the most complex model in which this approach brings results. The most intuitive extension of Ackermann's algorithm for stable flows [9] does not even result in feasible myopic changes.

On the other hand, our accelerated better-response algorithm generalizes another known method. Applied directly to the instance with the empty allocation, the accelerated Phase II performs augmentations like the augmenting path algorithms of Baïou and Balinski, and of Dean and Munshi. Since our algorithm is an accelerated version of our first algorithm, our concept offers a bridge between two known methods for solving two completely different problems, providing a solution to both of them.

2 Preliminaries

2.1 Stable Allocations

The marriage problem has been extended in several directions. A great deal of research effort has been spent on *many-to-one* and *many-to-many matchings*, sometimes also referred to as b-matchings. Their extension is called the *stable allocation problem*, also known as the ordinal transportation problem, since it is a direct analog of the classical cost-based transportation problem. In this problem, the vertices of a bipartite graph $G = (V, E)$ have *quotas* $q : V \to \mathbb{R}_{\geq 0}$, while edges have *capacities* $c : E \to \mathbb{R}_{\geq 0}$. Both functions are *real-valued*, unlike the respective functions in many-to-many instances, where capacities are unit, while quotas are integer-valued. Therefore, allocations can model more complex problems, for example where goods can be divided unequally between agents. In order to avoid confusion caused by terms associated with the marriage model, we call the vertices of the first color class *jobs* and the remaining vertices *machines.* For each machine, its quota is the maximal time spent working. A job's quota is the total time that machines must spend on the job in order to complete it. In addition, machines have a limit on the time spent on a specific job; this is modeled by edge capacities. A feasible allocation is a set of contracts where no machine is overwhelmed and no job is worked on after it has been completed.

Definition 1 (allocation). *Function $x : E \to \mathbb{R}_{\geq 0}$ is called an* allocation *if both of the following hold for every edge $e \in E$ and every vertex $v \in V$ of G:*

1. $x(e) \leq c(e)$;
2. $x(v) := \sum_{e \in \delta(v)} x(e) \leq q(v)$, *where $\delta(v)$ is the set of edges incident to v.*

To define stability we need *preference lists* as well. All vertices rank their incident edges strictly. Vertex v prefers uv to wv, if uv has a lower rank on v's preference list than wv: $\text{rank}_v(uv) < \text{rank}_v(wv)$. In this case we say that *uv dominates wv* at v. A stable allocation instance consists of four elements: $\mathcal{I} = (G, q, c, O)$, where O is the set of all preference lists.

Definition 2 (blocking edge, stable allocation). *An allocation x is* blocked *by an edge jm if all of the following properties hold:*

1. $x(jm) < c(jm)$*;*
2. $x(j) < q(j)$ *or* j *prefers* jm *to its worst edge with positive value in* x*;*
3. $x(m) < q(m)$ *or* m *prefers* jm *to its worst edge with positive value in* x*.*

A feasible allocation is stable *if no edge blocks it.*

In other words, edge jm is blocking if it is unsaturated and neither end vertices of jm could fill up its quota with at least as good edges as jm. If an unsaturated edge fulfills the second criterion, then we say that it *dominates* x at j. Similarly, if the third criterion is fulfilled, then we talk about an edge dominating x at m.

Baïou and Balinski [3] prove that stable allocations always exist. They also give two algorithms for finding them, an extended version of the Gale-Shapley algorithm and an inductive algorithm. The worst case running time of the first algorithm is exponential, but the latter one runs in strongly polynomial time. Dean and Munshi [7] speed up the polynomial algorithm using sophisticated data structures: their version runs in $O(|E| \log |V|)$ time for any real-valued instance.

2.2 Better and Best Response Steps for Allocations

First, we provide some basic definitions and notations we will use throughout the entire paper. A feasible, but possibly unstable allocation x is given at the beginning, the instance can be written as $\mathcal{I}= (G, q, c, O, x)$. Increasing x along a blocking edge and possibly decreasing it along worse edges is a better response step: through this operation, both end vertices of the blocking edge come better off. The definition of better and best response strategies is not as straightforward as it is in the matching instance with unit quotas and capacities. Here, the possible outcomes of a player are ordered lexicographically.

Although lexicographical order seems to be a natural choice, it is somewhat against the convention when discussing stable allocations. In most cases, when comparing the position of an agent in two stable allocations, the so called *min-min criterion* is used [3]. According to this rule, the agent prefers the allocation in which its worst positive edge is ranked higher. In order to make use of such an ordering relation, each vertex has to have the same allocation value in all stable solutions. Therefore here, when studying and comparing arbitrary feasible allocations, this concept proves to be counter-intuitive.

In our instance $\mathcal{I}$, jobs form the active side J, machines M are passive players. For sake of simplicity we denote the residual capacity $c(jm) - x(jm)$ of edge jm by $\bar{x}(jm)$ and similarly, the residual quota $q(v) - x(v)$ of vertex v by $\bar{x}(v)$.

An active player j having some blocking edges is chosen to perform a *best response step* on the current allocation x. Amongst j's blocking edges, let jm be the one ranked highest on j's preference list. The aim of player j is to reach its best possible lexicographical position via increasing $x(jm)$. To this end, j is ready to allocate all its remaining quota $\bar{x}(j)$ to jm, moreover, it reassigns allocation from all edges worse than jm to jm. Thus, j aims to increase $x(jm)$ by $\bar{x}(j) + x(\text{edges dominated by } jm \text{ at } j)$. To preserve feasibility, $x(jm)$ is not increased by more than $\bar{x}(jm)$. The passive player m agrees to increase $x(jm)$

as long as it does not lose allocation on better edges. This constraint gives the third upper bound, $\bar{x}(m) + x(\text{edges dominated by } jm \text{ at } m)$. To summarize this, in a best response step $x(jm)$ is increased by the following amount.

$$A := \min\{\bar{x}(j) + x(\text{edges dominated by } jm \text{ at } j), \bar{x}(jm), \\ \bar{x}(m) + x(\text{edges dominated by } jm \text{ at } m)\}$$

Once this A and the new $x(jm)$ is determined, j and m fill their remaining quota, then refuse allocation on their worst allocated edges, until x becomes feasible.

Better response steps are much less complicated to describe. The chosen active vertex j increases the allocation on an arbitrary blocking edge jm. Both j and m are allowed to refuse allocation on worse edges than jm. This rule guarantees that j's lexicographical situation develops and that the change is myopic for both vertices. By definition, best response steps are always better response steps at the same time. The execution of a single better response step consists of modifications on at most $|\delta(j)| + |\delta(m)| - 1 \leq |V| - 1$ edges.

3 Correlated Markets

Before tackling the general paths to stability problem, we first restrict ourselves to instances with special preference profiles. In this section, we study the case of stable allocations on an uncoordinated market with correlated preferences. Later we will prove that the convergence time of random best and better response strategies is exponential on general instances. By contrast, here we show that on correlated markets, random best response strategies terminate in expected polynomial time, even in the presence of irrational data. At the end of this section we also elaborate on the behavior of better response dynamics.

Definition 3 (correlated market). *An allocation instance is* correlated*, if there is a function $f: E \to \mathbb{N}$ such that* $\text{rank}_v(uv) < \text{rank}_v(wv)$ *if* $f(uv) < f(wv)$ *for every $u, v, w \in V$ and no two edges have the same f value.*

Correlated markets are also called *instances with globally ranked pairs* or *acyclic markets*. The latter property means that there is no cycle of edges such that every edge is preferred to the previous one by their common vertex. Abraham et al. [1] show that acyclic markets are correlated and vice versa. Ackermann et al. [2] were the first to prove that random better and best response dynamics reach a stable matching on correlated markets in expected polynomial time. Using a similar argumentation, we extend their result to allocation instances. The detailed proof can be found in the full version of the paper [6].

Theorem 1. *On correlated allocation instances with real-valued input data, random best response dynamics reach a stable solution in expected time $O(|V|^2|E|)$.*

In order to establish a similar result for better response dynamics in real-valued instances, an exact interpretation of random events would be needed.

In the matching case, best and better response dynamics differ exclusively in the rank of the chosen blocking edge: when playing best response strategy, the best blocking edge is chosen by an active vertex j. In contrast to this, here, better response steps differ also in the amount of modification and in the edges chosen to refuse allocation along. The first factor indicates a continuous sample space.

If we assume that any better response step results in reassigning the highest possible allocation value to an arbitrary blocking edge, an analogous proof can be derived. Then, termination needs $O(|V|^3|E|)$ steps in expectation.

4 Best and Better Responses with Rational Data

In this section, the case of allocations on an uncoordinated market *with rational data* is studied. As already mentioned, better and best response dynamics can cycle in such instances. We describe two deterministic methods, a better-response and a best-response algorithm that yield stable allocations in finite time. The main idea of our algorithms is to distinguish between blocking edges based on the type of blocking at the job: dominance or free quota.

A blocking edge can be of two types. Recall point 2 of Definition 2: if jm blocks x, then $x(j) < q(j)$ or j prefers jm to its worst edge with positive value in x. We talk about *blocking of type I* in the latter case, if jm blocks x because j prefers jm to its worst edge having positive value in x. *Blocking of type II* means that j has no allocated edge worse than jm, but j has not filled up its quota yet, $x(j) < q(j)$. Note that the reason of the blocking property at m is not involved when defining the two groups.

4.1 Better Response Dynamics

First, we provide a deterministic algorithm that constructs a finite path to stability from any feasible allocation. In the first phase of our algorithm, only blocking edges of type I are chosen to perform myopic changes along. The active vertices (jobs) choose one of their blocking edges of type I, not necessarily the best one. In all cases, withdrawal is executed along worst allocated edges. The amount of allocation set to the better edge is determined in such a way that at least one edge or a vertex becomes saturated or empty. Active vertices replace their worst edges with better ones, even if they had free quota. When no blocking edge of type I remains, the second phase starts. The allocation value is increased on blocking edges of type II such that they cease to be blocking. The runtime of our algorithm is exponential. Later, in Section 5 we will also show that this algorithm can be accelerated such that a stable solution is reached in polynomial time. The detailed proof of correctness, a pseudocode and execution on a sample instance are provided in the full version of the paper [6].

Theorem 2. *For every allocation instance with rational data and a given feasible allocation x, there is a finite sequence of better responses that leads to a stable allocation.*

The main idea of the proof is the following. We need to keep track of the change in total allocation value and in the lexicographical position of the active vertices simultaneously. In one step of the first phase along edge jm, either both j and m refuse edges, thus, the allocation value $|x| = \sum_{j \in J} x(j)$ decreases, or only j does so, keeping $|x|$ and improving its situation lexicographically. Since both procedures are monotone and the second one does not impair the first one, the first phase terminates. Termination for the second phase is implied by the fact that passive vertices improve their lexicographical situation in each step.

This algorithm also proves an important result regarding rational random better response processes. If the input is rational (there is a smallest positive number that can be represented as a linear combination of all data), it is clearly worthwhile to restrict the set of feasible better response modifications to the ones that reassign a multiple of this unit. For this reason, the set of reachable allocations is finite and they can be seen as states of a discrete time Markov chain. Our algorithm proves that from any state there is a finite path to an absorbing state with positive probability.

Theorem 3. *In the rational case, random better response strategies terminate with a stable allocation with probability one.*

Polynomial time convergence cannot be shown, since better response strategies need exponential time to converge even in matching instances [2].

4.2 Best Response Dynamics

In this subsection, we derive analogous results for best response modifications to the ones established for better response strategies. The main difference from the algorithmic point of view is that instances can be found in which no series of best response strategies terminate with a stable solution in polynomial time. A small example resembles the instance given by Baïou and Balinski [3] to prove that the Gale-Shapley algorithm requires exponential time to terminate in stable allocation instances. Let G be a complete bipartite graph on four vertices, with quota $q(j_1) = N + 1, q(j_2) = q(m_1) = q(m_2) = N$ and initial allocation $x(j_1 m_1) = x(j_2 m_2) = N$ for an arbitrary large number N. If the preference profile is chosen to be cyclic, such that $\text{rank}_{j_1}(m_1) = \text{rank}_{j_2}(m_2) = \text{rank}_{m_1}(j_2) = \text{rank}_{m_2}(j_1) = 1$, the unique series of best-response steps consists of $2N$ operations. A path of exponential length to stability can still be found.

Theorem 4. *For every allocation instance with rational data and a given feasible allocation x, there is a finite sequence of best responses that leads to a stable allocation.*

A similar two-phase algorithm is constructed to find a path to stability. Details can be found in the full version of the paper [6]. The same arguments using finite Markov chains imply the result on random procedures.

Theorem 5. *In the rational case, random best response strategies terminate with a stable allocation with probability one.*

5 Irrational Data - A Strongly Polynomial Algorithm

In our previous section, we relied several times on the fact that in each step, x is changed with values greater than a specific positive lower bound. When irrational data are present, e. g., q, c or x are real-valued functions, this cannot be guaranteed. Hence, our arguments for termination are not any more valid. Moreover, both of our algorithms require exponentially many steps to terminate. In this section, we describe a fast version of our two-phase better response algorithm that terminates in polynomial time with a stable allocation also for irrational input data. In the full version of the paper [6], we give a detailed proof of correctness for the first phase and show a construction with which all Phase II steps can be interpreted as Phase I operations on a slightly modified instance.

5.1 Accelerated First Phase

The algorithm and the proof of its correctness can be outlined the following way. A helper graph is built in order to keep track of edges that may gain or lose some allocation. A potential function is also defined, it stores information about the structure of the helper graph and the degree of instability of the current allocation. In the helper graph we are looking for walks to augment along. The amount of allocation we augment with is specified in such a way that the potential function decreases and the helper graph changes. When using walks instead of proposal-refusal triplets, more than one myopic operation can be executed at a time. Moreover, we also keep track of consequences of locally myopic improvements. For example, we spare running time by avoiding reducing allocation on edges that later become blocking anyway.

First, we elaborate on the structure of the helper graph, define alternating walks and specify the amount augmentation. Whereas the method is described in details here, the proof of correctness, the pseudocode and a sample execution can be found in the full version of the paper [6].

Helper Graph. Recall that our real-valued input $\mathcal{I}$ consists of a stable allocation instance (G, q, c, O) and a feasible allocation x. First, we define a helper graph $H(x)$ on the same vertices as G. This graph is dependent on the current allocation x and will be changed whenever we modify x. The edge set of $H(x)$ is partitioned into three disjoint subsets. The first subset P is the set of Phase I blocking edges. Each job j that has at least one edge with positive x value, also has a worst allocated edge, $r(j)$. These are the edges jobs tend to reduce x along when a myopic change is made. These *refusal pointers* form R, the second subset of $E(H(x))$. We also keep track of edges that are currently not of blocking type I, but later on they may enter set P. This last subset P' consists of edges that may become blocking of type I after some myopic changes. An edge $jm \notin P$ has to fulfill three criteria in order to belong to P': 1) $c(jm) > x(jm)$; 2) m has at least one refusal edge; 3) j prefers jm to its worst allocated edge $r(j)$. Such an edge immediately becomes blocking if m loses allocation along one of its refusal edges. Edges in P' are called *possibly blocking edges*, the set $P \cup P'$ forms the set of *proposal edges*. Note that a job j may have several edges in P and P', but at most one in R. Moreover, if j has a proposal edge in $H(x)$, it also has an edge in R. Regarding the machines, if m has a P'-edge, it also has an R-edge. The following lemma provides an additional structural property of $H(x)$.

Lemma 1. *If $jm \in P$ and $j'm \in P'$, then* $\text{rank}_m(jm) < \text{rank}_m(j'm)$. *That is, machines prefer their blocking edges to their possibly blocking edges.*

Alternating Walks. Our algorithm performs augmentations along alternating walks, so that the allocation value of the refusal edges decreases, while the value of proposal edges increases. This is done in such a way that R, P or P' (and thus, $H(x)$) changes. The main idea behind these operations is the same we used in our previous proof: reassigning allocation to blocking edges from worse edges, such that the procedure is monotone. The difference between the two methods is that while our first algorithm tackles a single blocking edge in each step, here we deal with a set of blocking edges (forming the alternating walk) at once.

When choosing the alternating proposal-refusal walk W to augment along, the following rules have to be observed:

1. The first edge jm_1 is a P-edge.
2. P and P'-edges are added to W together with the refusal edge they are incident with on the active side.
3. Machines choose their best P or P'-edge.
4. W ends at m if 1) m has no proposal edge or 2) its best proposal edge goes to a vertex already visited by W.

As long as there is a blocking edge of type I, the first edge jm_1 of such a walk can always be found. Lemma 1 guarantees that point 3 is not harmed by this jm_1. After taking $r(j)$, all that remains is to continue on best proposal edges of machines and refusal edges of jobs they end at. Since G is a finite set, either of the cases listed in point 4 will appear. According to these rules, proposal-refusal edge pairs are added to the current path until 1) there is no pair to add or 2) the path reaches a vertex already visited. In the first case, W is a path. In the latter case, W is a union of a path and a cycle, connecting at exactly one vertex. This vertex is the last vertex listed on W, where our method halts, observing point 4. W can be, of course, a single path or a single cycle as well.

Before elaborating on the amount of augmentation, we emphasize that W is a *subset* of the set of edges whose x value changes during an augmentation step. The goal is to reassign allocation from refusal edges to blocking edges, until a stable solution is derived. Naturally, on an alternating walk, refusal edges lose the same amount of allocation proposal edges gain. But, except if augmentations are performed along a single cycle, there is a single machine m_1 that gains allocation in total. In order to preserve feasibility, this machine might have to refuse allocation on edges not belonging to W. The exact amount of these refusals is discussed later, together with the amount of augmentation along W. Since no other vertex gains allocation in an augmentation step, feasibility cannot be harmed elsewhere. Thus, these are the only edges not on W that are modified.

By contrast, if the augmentation is performed along a single cycle C, refusals only happen on $r(j) \in W \cap R$ edges. Even if the machine m_1 that started C has a full quota, it does not need to refuse any allocation, since $x(m_1)$ remains unchanged during the augmentation. Note that executing several local myopic steps greedily, like in our first algorithm, would lead to a different output. Then, m_1 would refuse edges, not knowing that it loses allocation later. As a result of that, m_1 would go under its quota, and would possibly create new blocking edges. Both strategies are better response, the difference is that our second algorithm keeps track of changes made as a consequence of a myopic operation.

Amount of Augmentation. Once W is fixed, the amount of allocation A has to be determined to augment with. It must be chosen so that 1) a feasible allocation is derived and 2) at least one refusal edge becomes empty or at least one proposal edge leaves $P \cup P'$. These points guarantee that $H(x)$ changes. To fulfill these two requirements, the minimum of the following terms is determined.

1. Allocation value on refusal edges along W: $x(r(j))$, where $r(j) \in W \cap R$.
2. Residual capacity on proposal edges along W: $\bar{x}(p), \bar{x}(p')$, where $p, p' \in W \cap (P \cup P')$.
3. If W is not a single cycle, m_1 may refuse sufficient amount of allocation such that jm_1 does not become saturated, but it stops dominating x at m_1. In this case, the residual quota of m_1 must be filled up and, in addition, the sum of allocation value on edges worse than jm_1 must be refused. With this, jm_1 becomes the worst allocated edge of a full machine. Until reaching this point, jm_1 may gain $\bar{x}(m_1) + x(\text{edges dominated by } jm_1 \text{ at } m_1)$ amount of allocation in total.

To summarize this, we augment with $A := \min\{x(r(j)), \bar{x}(p), \bar{x}(p') | r(j) \in W \cap R, p, p' \in W \cap (P \cup P')\}$ if W is a cycle, because then the last case with the starting vertex m_1 may not occur. Otherwise, the amount of augmentation is $A := \min\{x(r(j)), \bar{x}(p), \bar{x}(p'), \bar{x}(m_1) + x(\text{edges dominated by } jm_1 \text{ at } m_1) | r(j) \in W \cap R, p, p' \in W \cap (P \cup P')\}$.

The second phase of our method can be interpreted as the execution of the first phase on a modified instance. The modification needed consist of introducing a dummy job and swapping the roles of the active and passive color classes.

In total, the algorithm performs $O(|V||E|)$ rounds, each of them needs $O(|V|)$ time to be computed. Thus, it runs in $O(|V|^2|E|)$ time. For a detailed proof of correctness and runtime computation, see the full paper [6].

Theorem 6. *For every real-valued allocation instance and given feasible allocation, there is a sequence of better responses leading to a stable allocation in* $O(|V|^2|E|)$ *time.*

Our method resembles the well-known notion of *rotations* [12]. They can be used when deriving a stable solution from another, by finding an alternating cycle of matching and non-matching edges and augmenting along them. In our algorithm, when we are searching for augmenting cycles or walks, we use an approach similar to rotations: jobs candidate their edges better than their worst positive edge, while machines choose the best out of them. However, two differences can be spotted right away. While rotations are always assigned to a stable solution different from the job-optimal, our method works on unstable input. Moreover, besides cycles we also augment along paths and walks.

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Self-stabilizing Uncoupled Dynamics*

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Abstract. Dynamics in a distributed system are self-stabilizing if they are guaranteed to reach a stable state regardless of how the system is initialized. Game dynamics are uncoupled if each player's behavior is independent of the other players' preferences. Recognizing an equilibrium in this setting is a distributed computational task. Self-stabilizing uncoupled dynamics, then, have both resilience to arbitrary initial states and distribution of knowledge. We study these dynamics by analyzing their behavior in a bounded-recall synchronous environment. We determine, for every "size" of game, the minimum number of periods of play that stochastic (randomized) players must recall in order for uncoupled dynamics to be self-stabilizing. We also do this for the special case when the game is guaranteed to have unique best replies. For deterministic players, we demonstrate two self-stabilizing uncoupled protocols. One applies to all games and uses three steps of recall. The other uses two steps of recall and applies to games where each player has at least four available actions. For uncoupled deterministic players, we prove that a single step of recall is insufficient to achieve self-stabilization, regardless of the number of available actions.

1 Introduction

Self-stabilization is a failure-resilience property that is central to distributed computing theory and is the subject of extensive research (see, e.g., [3] for a survey). It is characterized by the ability of a distributed system to reach a stable state from every initial state. Dynamic interaction between strategic agents is a central research topic in game theory (see, e.g., [4,11]). One area of interest is *uncoupled dynamics*, in which each player's strategy is independent of the other players' payoffs [9]. Here, we bring together these two research areas and study of *self-stabilizing uncoupled dynamics* within the broader research agenda of *distributed computing with adaptive heuristics* [10]. The same questions we answer here can be asked for a broad variety of dynamics and notions of convergence and equilibria. These directions, as well as a conjecture, are discussed in Section 5.

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We focus our investigation on a bounded-recall, synchronous setting. We consider self-stabilization in a multi-agent distributed system in which, at each timestep, the agents act as strategic players in a game, simultaneously selecting *actions* from their respective finite action sets to form an *action profile*. The space of action profiles is relevant throughout this work, and we refer to its size as the *size* of the game. We study the effects of *bounded recall*, in which the state of this system at any time consists of the r most recent action profiles, for some finite r. The stable states in r-recall systems necessarily have the same action profile in r consecutive time steps. In our context, we want stable states that are robust to players acting selfishly—i.e., those where the repeated action profile is an equilibrium of the stage game. In this paper, we consider pure Nash equilibria (PNE). Thus, in our setting, dynamics self-stabilize for a given game if, from every starting state, players are guaranteed to converge to a PNE. For games without PNE, dynamics cannot self-stabilize in this sense. Throughout this paper, we say that particular dynamics *succeed* on a class of games if they self-stabilize for games in that class whenever a PNE exists.

Traditional study of convergence to equilibria in game dynamics makes various assumptions about the "reasonableness" of players' behavior, restricting them to always play the game in ways that are somehow consistent with their self-interest given their current knowledge. In contrast to these behavioral restrictions on the players, uncoupledness is an informational restriction, in that the players have no knowledge of each other's payoffs. In this situation, no individual player can recognize a PNE, so finding an equilibrium is a truly distributed task.

If uncoupledness is the only restriction on the dynamics, then the players can find a PNE through a straightforward exhaustive search. However, this changes when players' abilities to remember past actions is restricted. In a continuous-time setting, Hart and Mas-Colell [7] showed that deterministic uncoupled dynamics fail to reach a stable state for some games that have PNE if the dynamics must be *historyless*, i.e., if the state space of the system is identical to the action profile space of the game. This suggests the central question that we address:

> *On a given class of games, how much recall do uncoupled players need in order to self-stabilize whenever a PNE exists? That is, when are there successful k-recall dynamics?*

This question was answered in part by Hart and Mas-Colell [8], who showed that in a discrete-time setting, even when players are allowed randomness, no historyless uncoupled dynamics succeed on all two-player games where each player has three actions. Moreover, they showed that even for *generic* games (where at every action profile each player has a unique best response), no historyless uncoupled dynamics succeed on games with three three-action players. They also gave positive results, proving that there are historyless uncoupled dynamics that succeed on all two-player generic games, and that if the players have 2-recall (i.e., they are allowed to see the two most recent action profiles), then over every action profile space there are stochastic uncoupled dynamics that succeed on all games.

Our results. We show in Section 3 that there exist historyless uncoupled dynamics that succeed on all two-player games with a two-action player and on all three-player generic games with a two-action player (Theorems 4 and 9). In both cases, we prove that these results are tight, in that they do not hold for any larger size of game (Theorems 5 and 11). Combined with the results of Hart and Mas-Colell [8], this provides a complete characterization of the exact minimum recall needed, for any action profile space, for uncoupled dynamics to succeed on all games over that space and on generic games over that space. In Section 4, turning to deterministic dynamics, we demonstrate 3-recall deterministic uncoupled dynamics that succeed on all games (Theorem 14) and 2-recall deterministic uncoupled dynamics that succeed on all games in which every player has at least four actions (Theorem 15). We also prove for every action profile space that no historyless deterministic uncoupled dynamics succeed on all games over that space (Theorem 16). Some proofs are omitted from the proceedings version of this paper. A longer version with all proofs included can be found at `http://arxiv.org/abs/1403.5791`.

Related work. There are rich connections between distributed computing and game theory, some of which are surveyed by Halpern [5]. Jaggard, Schapira, and Wright [10] investigated convergence to pure Nash equilibria by game dynamics in asynchronous distributed systems. Most closely related to our specific setting, Hart and Mas-Colell introduced the concept of uncoupled game dynamics [7]. In addition to the results mentioned above, they also addressed convergence to mixed Nash equilibria by bounded-recall uncoupled dynamics [8]. Babichenko investigated the situation when the uncoupled players are finite-state automata, as well as *completely uncoupled dynamics*, in which each player can see only the history of its own actions and payoffs [1,2]. Young [13] and Pradelski and Young [12] gave completely uncoupled dynamics that achieve an equilibrium in a high proportion of steps but do not necessarily converge. Hart and Mansour [6] analyzed the time to convergence for uncoupled dynamics.

2 Preliminaries

We begin with definitions of the concepts used in the paper.

Games. Let $n \in \mathbb{N}$ and $(k_1, ..., k_n) \in \mathbb{N}^n$, with $n \geq 2$ and each $k_i \geq 2$. A *game* of *size* $(k_1, ..., k_n)$ is a pair (A, U), where $A = A_1 \times ... \times A_n$ such that each $|A_i| = k_i$, and $U = (u_1, ..., u_n)$ is an n-tuple of functions $u_i : A \to \mathbb{R}$. A_i and u_i are the *action set* and *utility function* of *player* i. $\Delta(A_i)$, the probability simplex over A_i, is player i's set of *mixed actions*. When n is small, we may describe a game (A, U) as a k_1-*by*-...-*by*-k_n game. Elements of A are the *(action) profiles* of the game, and A is called the *(action) profile space.* $\mathcal{U}(A)$ is the the class of all U such that each u_i takes A_i as input, so $A \times \mathcal{U}(A)$ is the class of all games with profile space A. When A is clear from context, we often identify the game with the utility function vector U.

Let $U \in \mathcal{U}(A)$. For $i \in \{1, ..., n\}$ and $a = (a_1, ..., a_n) \in A$, we say that player i is *U-best-replying* at a if $u_i(a) \geq u_i((a_1, ..., a_i', ..., a_n))$ for every $a_i' \in A_i$. We define the set of *U-best-replies* for player i at a,

$$BR_i^U(a) = \{a_i' \in A_i : i \text{ is } U\text{-best-replying at } (a_1, ..., a_i', ..., a_n)\}.$$

We omit U from this notation when the game being played is clear from context. A profile $p \in A$ is a *pure Nash equilibrium*, abbreviated *PNE*, for U if every player $i \in \{1, ..., n\}$ is best-replying at p. An action $a_i \in A_i$ is *weakly dominant* for player i if $a_i \in BR_i(x)$ for every $x \in A$; it is *strictly dominant* for player i if $BR_i(x) = \{a_i\}$ for every $x \in A$.

A game $(A, U) \in A \times \mathcal{U}(A)$ is *generic* if every player's best-replies are unique, i.e., if for every $a \in A$ and $i \in \{1, ..., n\}$, $|BR_i^U(a)|=1$. For generic games (A, U) we may abuse notation slightly by using $BR_i^U(a)$ to refer to this set's unique element. $A \times \mathcal{G}(A)$ is the class of all generic games on A.

Dynamics. We now consider the repeated play of a game. Let the profile at timestep $t \in \mathbb{Z}$ be $a^{(t)} = \left(a_1^{(t)}, ..., a_n^{(t)}\right) \in A$. The *stage game* $(A, U) \in A \times \mathcal{U}(A)$ is then played: each player i simultaneously selects a new action $a_i^{(t+1)}$ by applying an *r-recall stationary strategy* $f_i^U : A^r \to \Delta(A_i)$, where $r \in \mathbb{N}$ and A^r is the Cartesian product of A with itself r times. A *deterministic* r-recall stationary strategy mapping ranges over A_i instead of $\Delta(A_i)$. The strategy f_i^U, which is *stationary* in the sense that it does not depend on t, will take as input $(a^{(t-r+1)}, ..., a^{(t)})$, the r most recent profiles. We call this r-tuple the *state* at time t. The terms 1-*recall* and *historyless* are interchangeable. A *strategy vector* is an n-tuple $f^U = (f_1^U, ..., f_n^U)$, where each f_i^U is a strategy for player i. $\mathcal{F}(A)$ will denote the set of all strategy vectors for A.

A *strategy mapping* for A is a mapping $f : \mathcal{U}(A) \to \mathcal{F}(A)$ that assigns to each U a strategy vector f^U. A strategy mapping f is *uncoupled* if the strategy it assigns each player depends only on that player's utility function and not, e.g., on the other players' payoffs. That is, there are mappings $f_1, ..., f_n$ where each f_i maps utility functions on A to strategies for A, such that $f_i(u_i) \equiv f_i^U$ for $i = 1, ..., n$. If f_i^U is stationary, deterministic, or r-recall for $i = 1, ..., n$, then f^U is also. If every f^U has any of those properties, then f does also.

Now let $x = \left(x^{(1)}, ..., x^{(r)}\right) \in A^r$, and let f^U be an r-recall strategy vector. For $T \geq r$, a *partial f^U-run* for T steps starting from x is a tuple of profiles $\left(a^{(1)}, ..., a^{(T+r)}\right) \in A^{T+r}$ such that $x = (a^{(1)}, ..., a^{(r)})$ and for every $r < t \leq T+r$,

$$\Pr\left(f^U\left(a^{(t-r)}, ..., a^{(t-1)}\right) = a^{(t)}\right) > 0.$$

An *f^U-run* is an infinite sequence of profiles $a^{(1)}, a^{(2)}, ...$ such that every finite prefix is a partial f^U-run. We say that $y \in A^r$ is *f^U-reachable* from $x \in A^r$ if there exist a $T \in \mathbb{N}$ and a partial f^U-run $\left(a^{(1)}, ..., a^{(T+r)}\right)$ such that $x = (a^{(1)}, ..., a^{(r)})$ and $y = \left(a^{(T)}, ..., a^{(T+r)}\right)$. The state x is an *f^U-absorbing state* if for every f^U-run $a^{(1)}, a^{(2)}, ...$ beginning from x, $\left(a^{(t+1)}, ..., a^{(t+r)}\right) = x$ for every $t \in \mathbb{N}$. Notice that any f^U-absorbing state $x = \left(a^{(1)}, ..., a^{(r)}\right)$ must have $a^{(1)} = ... = a^{(r)}$. We omit the strategy vector from this notation when it is clear from context. The *game dynamics* of f consist of all pairs (U, R) such that R is an f^U-run.

Convergence. A sequence of profiles $a^{(1)}, a^{(2)}, ...$ *converges* to a profile a if there some $T \in \mathbb{N}$ such that $a^{(t)} = a$ for every $t \geq T$. If from every $x \in A^r$, some f^U-absorbing PNE is f^U-reachable, then f *self-stabilizes* on game (A, U). We say that f *succeeds* on a game U if f self-stabilizes on (A, U) or if (A, U) has no PNE. Let $\mathcal{C}(A)$ be a class of games on A. If f succeeds on every game $(A, U) \in A \times \mathcal{C}(A)$, then f *succeeds* on $\mathcal{C}(A)$.

Let $A = A_1 \times ... \times A_n$ and $B = B_1 \times ... \times B_n$ be profile spaces of the same size, in the sense that there is some permutation π on $\{1, ..., n\}$ such that $(|A_1|, ..., |A_n|) = (|B_{\pi(1)}|, ..., |B_{\pi(n)}|)$. Then we write $A \simeq B$. If f succeeds on $\mathcal{C}(A)$, then there is a strategy mapping derived from f that succeeds on $\mathcal{C}(B)$, simply by rearranging the players and bijectively mapping actions in each A_i to actions in $B_{\pi(i)}$. This new strategy mapping retains any properties of f that are of interest here (uncoupledness, r-recall, stationarity, and determinism). For this reason we define

$$\mathcal{C}(|A_1|, ..., |A_n|) = \bigcup_{B \simeq A} \mathcal{C}(B),$$

and we say that f succeeds on $\mathcal{C}(|A_1|, ..., |A_n|)$ if f succeeds on $\mathcal{C}(B)$ for some $B \simeq A$. For example, "f succeeds on $\mathcal{G}(2, 3)$" means "f self-stabilizes on every generic 2-by-3 game with a PNE (up to renaming of actions)."

3 Stochastic Uncoupled Dynamics

In this section, we determine, for every profile space A, the minimum $r \in \mathbb{N}$ such that an uncoupled r-recall stationary strategy mapping exists that succeeds on all games $(A, U) \in A \times \mathcal{U}(A)$ or all generic games $(A, U) \in A \times \mathcal{G}(A)$. Hart and Mas-Colell [8] proved that 2-recall is sufficient to succeed on all games, 1-recall is sufficient to succeed on generic two-player games, and that 1-recall is not sufficient to succeed on all games, or even all generic games. We state these results in the present setting.

Theorem 1 (Hart and Mas-Colell [8]). *For any profile space A, there exists an uncoupled 2-recall stationary strategy mapping that succeeds on all games (A, U).*

Theorem 2 (Hart and Mas-Colell [8]). *There is no uncoupled historyless stationary strategy mapping that succeeds on all 3-by-3 games, or on all 3-by-3-by-3 generic games.*

Theorem 3 (Hart and Mas-Colell [8]). *For any two-player profile space A, there is an uncoupled historyless stationary strategy mapping that succeeds on all generic games (A, U).*

We now describe the strategy mapping given in the proof of Theorem 3. Notice that for a historyless stationary strategy mapping, the state space is exactly the profile space, so the terms *state* and *profile* are interchangeable in this context.

Definition For any n-player profile space A, the *canonical* historyless uncoupled stationary strategy mapping for A is $h : \mathcal{U}(A) \to \mathcal{F}(A)$, defined as follows. Let $U = (u_1, ..., u_n) \in \mathcal{U}(A)$. Then $h(U) = (h_1^U, ..., h_n^U)$, where for $i \in \{1, ..., n\}$, $h_i^U : A \to A_i$ is given by

$$\Pr\left(h^U(a_i) = a_i \,|\, a_i \in BR_i(a)\right) = 1$$
$$\Pr\left(h^U(a_i) = b_i \,|\, a_i \notin BR_i(a)\right) = 1/k_i,$$

for all $a_i, b_i \in A_i$. That is, if player i is already best replying, then it will continue to play the same action. Otherwise, i will play an action chosen uniformly at random from its action set.

In their proof of Theorem 2, Hart and Mas-Colell make the following observation.

Observation 1 (Hart and Mas-Colell [8]). *Suppose f is an uncoupled historyless stationary strategy mapping for profile space A and f succeeds on all generic games (A, U). Then two conditions hold for every game (A, U) and $a = (a_1, ..., a_n) \in A$. First, if player i is best-replying at a, then $Pr(f_i^U(a) = a_i) = 1$. Second, if player i is not best replying at a, then $\Pr(f_i^U(a) = a_i') > 0$ for some $a_i' \in A_i \setminus \{a_i\}$.*

Informally, no player can move when it is best-replying, and each player must move w.p.p. whenever it is not best-replying. The first condition guarantees that every PNE is an absorbing state; the second guarantees that no non-PNE is an absorbing state. Implicit in the same proof is the fact that h is at least as "powerful" as any other historyless uncoupled strategy mapping.

Observation 2 (Hart and Mas-Colell [8]). *If any historyless uncoupled stationary strategy mapping succeeds on $\mathcal{U}(A)$ or on $\mathcal{G}(A)$, then h succeeds on that class.*

3.1 Stochastic Dynamics for $\mathcal{U}(A)$

We now describe the profile spaces in which there are uncoupled historyless strategy mappings that succeed on every game, or equivalently (by Observation 2), the A for which h succeeds on $\mathcal{U}(A)$. A proof that h succeeds on 2-by-k games proceeds by simple case checking.

Theorem 4. *For every two-player profile space A in which one player has only two actions, h succeeds on all games (A, U).*

It turns out that 2-by-k profile spaces are the only ones where h succeeds on all games.

Theorem 5. *Let A be a profile space. Unless A has only two players and one of those players has only two actions, no historyless uncoupled stationary strategy mapping succeeds on all games (A, U).*

We give three lemmas that will be used in the proof of Theorem 5. Informally, Lemma 6 says that additional actions do not make a profile space any "easier" in this context; the players will need at least as much recall to succeed on all games in the larger space. The proof relies on a type of reduction in which the players take advantage of a strategy mapping for a larger game by "pretending" to play the larger game. Whenever player i plays k_i, all players guess randomly whether i would have played k_i or $k_i + 1$ in the larger game.

Lemma 6. *Let $n \geq 2$, $k_1, ..., k_n \geq 2$, and $i \in \{1, ..., n\}$. If h succeeds on $\mathcal{U}(k_1, ..., k_i + 1, ..., k_n)$, then h succeeds on $\mathcal{U}(k_1, ..., k_i, ..., k_n)$.*

Lemma 7 tells us that the same is true of adding players to the game. Its proof also uses a simple reduction. The players utilize the strategy mapping for the $(n+1)$-player game by behaving as if there is an additional player who never wishes to move. This preserves genericity, so the lemma also applies to the class of generic games.

Lemma 7. *Let $n \geq 2$ and $k_1, ..., k_n, k_{n+1} \geq 2$. If h succeeds on $\mathcal{U}(k_1, ..., k_n, k_{n+1})$, then h succeeds on $\mathcal{U}(k_1, ..., k_i, ..., k_n)$. The same is true if we replace $\mathcal{U}$ with $\mathcal{G}$.*

Finally, Lemma 8 says that h does not succeed on all 2-by-2-by-2 games. An example is given in its proof of a game with a PNE where h fails to converge.

Lemma 8. *No historyless uncoupled stationary strategy mapping succeeds on $\mathcal{U}(2, 2, 2)$.*

Proof of Theorem 5. Let $A = A_1 \times ... \times A_n$. By Observation 2, it suffices to show that h does not succeed on $\mathcal{U}(|A_1|, ..., |A_n|)$. Assume that h does succeed on $\mathcal{U}(|A_1|, ..., |A_n|)$. If $n = 2$, $|A_1|, |A_2| > 2$, and h succeeds on $\mathcal{U}(k_1, k_2)$, then by repeatedly applying Lemma 6, h succeeds on $\mathcal{U}(3, 3)$. This contradicts Theorem 2. Now suppose that $n \geq 3$. If h succeeds on $\mathcal{U}(|A_1|, ..., |A_n|)$, then by repeatedly applying Lemma 7, h succeeds on $\mathcal{U}(|A_1|, |A_2|, |A_3|)$. So by repeatedly applying Lemma 6, h succeeds on $\mathcal{U}(2, 2, 2)$. This contradicts Lemma 8.

3.2 Stochastic Dynamics for $\mathcal{G}(A)$

We now turn to generic games and to describing the class of profile spaces A for which there exist historyless uncoupled strategy mappings that succeed on $\mathcal{G}(A)$. Theorem 3 tells us that h succeeds on two-player generic games. In fact, h also succeeds on three-player generic games where one player has only two options.

Theorem 9. *Let A be a three-player profile space such that one player has only two actions. Then h succeeds on all generic games (A, U).*

The proof of this theorem relies partially on an analogy between a k-by-ℓ-by-2 generic game and a $k\ell$-by-2 game that might not be generic. This requires the following technical lemma showing that under h, two players in a generic game sometimes behave similarly to a single player.

Lemma 10. *Let $k, l \in \mathbb{N}$, and let $U \in \mathcal{G}(k, \ell)$ be a game in which neither player has a strictly dominant action. For every $a, b \in A$ such that a is not a PNE for U, b is h^U-reachable from a.*

Proof of Theorem 9. Let $A = \{1, ..., k\} \times \{1, ..., \ell\} \times \{0, 1\}$ for some $\ell, k \in \mathbb{N}$. Let $U \in \mathcal{G}(A)$ and $a = (a_1, a_2, a_3) \in A$. All PNE are absorbing states under h, so it will suffice to show there is some PNE that is h^U-reachable from a.

Let $A' = \{1, ..., k\} \times \{1, ..., \ell\}$, and consider the games $U^0 = (u_1^0, u_2^0)$ and $U^1 = (u_1^1, u_2^1) \in \mathcal{G}(A')$ defined by

$$u_i^0(x_1, x_2) = u_i(x_1, x_2, 0)$$
$$u_i^1(x_1, x_2) = u_i(x_1, x_2, 1)$$

for every $x_1 \in \{1, ..., k\}$, $x_2 \in \{1, ..., \ell\}$, and $i \in \{0, 1\}$. In this proof we will repeatedly use the fact that over any finite number of steps, w.p.p. player 3 doesn't move, so if $(y_1, y_2) \in A'$ is h^{U^0}-reachable from $(x_1, x_2) \in A'$, then $(y_1, y_2, 0) \in A$ is h^U-reachable from $(x_1, x_2, 0) \in A$, and similarly for h^{U^1}.

Claim. If either player has a strictly dominant action in U^0 or U^1, then some PNE is h^U-reachable from a.

Thus we may assume that neither player has a strictly dominant action in U^0 or in U^1. Consider a two-player game $\widehat{U} = (\widehat{u}_1, \widehat{u}_2)$ on $\widehat{A} = (\{1, ..., k\} \times \{1, ..., \ell\}) \times \{0, 1\}$ given by

$$\widehat{u}_1(x) = \begin{cases} 1 & \text{if } (x_1, x_2) \text{ is a PNE for } U^{x_3} \\ 0 & \text{otherwise} \end{cases}$$
$$\widehat{u}_2(x) = u_3((x_1, x_2, x_3)),$$

for every $x = ((x_1, x_2), x_3) \in \widehat{A}$. Note that unlike U, this game is not necessarily generic. By Theorem 4, some PNE $\widehat{p} = ((p_1, p_2), p_3)$ for $\widehat{U}$ is $h^{\widehat{U}}$-reachable from $\widehat{a} = ((a_1, a_2), a_3)$.

Now let $\widehat{x} = ((x_1, x_2), x_3)$ and $\widehat{y} = ((y_1, y_2), y_3) \in \widehat{A}$ such that w.p.p. $\widehat{y} = h^{\widehat{U}}(\widehat{x})$. If $x_3 \neq y_3$, then $x_3 \notin BR_2^{\widehat{U}}(\widehat{x})$, so $x_3 \neq BR_3^U(x)$. Thus w.p.p. $h^U(x) = (x_1, x_2, y_3)$. Since $BR_3^U(x) \neq x_3 \neq y_3$ and $|A_3| = 2$, we must have $BR_3^U(x) = y_3$, so if (x_1, x_2) is a PNE for U^{y_3}, then (x_1, x_2, y_3) is a PNE for U. Otherwise, by Lemma 10 (y_1, y_2) is $h^{U^{x_3}}$-reachable from (x_1, x_2), so $y = (y_1, y_2, y_3)$ is h^U-reachable from (x_1, x_2, y_3) and therefore from x.

Applying this to the each step on the path by which $\widehat{p}$ is $h^{\widehat{U}}$-reachable from $\widehat{a}$, we see that either $p = (p_1, p_2, p_3)$ (which is a PNE for U) is h^U-reachable from a, or some other PNE for U is encountered in this process and thus h^U-reachable from a.

In fact, two-player and 2-by-k-by-ℓ are the only sizes of generic games on which h always succeeds.

Theorem 11. *Let A be a profile space. If A has more than three players, or if every player has more than two actions, then no historyless uncoupled stationary strategy mapping succeeds on all generic games (A, U).*

Before proving this theorem, we present two lemmas. Lemma 12 says that h does not succeed on all 2-by-2-by-k-by-ℓ generic games. It is proved by giving an example of such a game.

Lemma 12. *For every $k, \ell \geq 2$, h does not succeed on $\mathcal{G}(2, 2, k, \ell)$.*

Lemma 13 says that h doesn't succeed on all three-player generic games in which all players have at least three actions. This is demonstrated by simple modifications of the 3-by-3-by-3 game used by Hart and Mas-Colell in their proof of Theorem 2.

Lemma 13. *For every $k_1, k_2, k_3 \geq 3$, h does not succeed on $\mathcal{G}(k_1, k_2, k_3)$*

Proof of Theorem 11. By Observation 2, if suffices to show that h does not succeed on $\mathcal{G}(|A_1|, ..., |A_n|)$. Assume for contradiction that h does succeed on $\mathcal{G}(|A_1|, ..., |A_n|)$. If $n = 3$ and h succeeds on $\mathcal{G}(|A_1|, |A_2|, |A_3|)$, then by Lemma 13 we cannot have $|A_1|, |A_2|, |A_3| > 2$. If $n = 4$ and h succeeds on $\mathcal{G}(|A_1|, ..., |A_4|)$, then by Lemma 12 there are distinct $i, j, k \in \{1, 2, 3, 4\}$ such that $|A_i|, |A_j|, |A_k| > 2$. But by Lemma 7, h succeeds on $\mathcal{G}(|A_i|, |A_j|, |A_k|)$, contradicting lemma 13. If $n > 4$ and h succeeds on $\mathcal{G}(|A_1|, ..., |A_n|)$, then by repeatedly applying Lemma 12, h succeeds on $\mathcal{G}(|A_1|, ..., |A_4|)$, which we have already shown to be impossible.

4 Deterministic Uncoupled Dynamics

Both h and the strategy mapping used by Hart and Mas-Colell [8] to prove Theorem 1 are variations on random search. For deterministic dynamics, an exhaustive search requires more structure, and the challenge for deterministic players in short-recall uncoupled dynamics is in keeping track of their progress in the search.

We show that there are successful 3-recall deterministic dynamics by using repeated profiles to coordinate.

Theorem 14. *For every profile space A, there exists a deterministic uncoupled 3-recall stationary strategy mapping that succeeds on all games (A, U).*

Proof. Let $n \geq 2$, $k_1, ..., k_n \geq 2$, and $A = \{1, ..., k_1\} \times ... \times \{1, ..., k_n\}$. It suffices to show that such a strategy mapping exists for $\mathcal{U}(A)$. Let $\sigma : A \to A$ be a cyclic permutation on the profiles. We write $\sigma_i(a)$ for the action of player i in

$\sigma(a)$. Let $f : \mathcal{U}(A) \to \mathcal{F}(A)$ be the strategy mapping such that, for every game $U \in \mathcal{U}(A)$, player $i \in \{1, ..., n\}$, and state $x = (a, b, c) \in A^3$,

$$f_i^U(x) = \begin{cases} c_i & \text{if } b = c \text{ and } c_i \in BR_i(c) \\ \min BR_i(c) & \text{if } b = c \text{ and } c_i \notin BR_i(c) \\ \sigma_i(a) & \text{if } a = b \neq c \\ c_i & \text{otherwise.} \end{cases}$$

Informally, the players use repetition to keep track of which profile is the current "PNE candidate" in each step. If a profile has just been repeated, then it is the current candidate, and each player plays a best reply to it, with a preference against moving. If the players look back and see that some profile a was repeated in the past but then followed by a different profile, they infer that a was rejected as a candidate and move on by playing a's successor, $\sigma(a)$. Otherwise the players repeat the most recent profile, establishing it as the new candidate. We call these three types of states *query*, *move-on*, and *repeat* states, respectively. Here "query" refers to asking each player for one of its best replies to b.

Let $U \in \mathcal{U}(A)$ be a game with at least one PNE. We wish to show that f^U guarantees convergence to a PNE. Let $x = (a, b, c) \in A^3$, and let y be the next state $(b, c, f^U(x))$. If x is a repeat state, then $y = (b, c, c)$, which is a query state. If x is a move-on state, then $b \neq c$, and $y = (b, c, \sigma(a))$. If $c = \sigma(a)$, then this is a query state; otherwise, it's a repeat state, which will be followed by the query state $(c, \sigma(a), \sigma(a))$. Thus every non-query state will be followed within two steps by a query state.

Now let $x = (a, b, b) \in A^3$ be a query state, and let y and z be the next two states. If b is a PNE, then $y = (b, b, b)$, which is an absorbing state. Otherwise, $y = (b, b, c)$ for some $c \neq b$, so y is a move-on state, which will be followed by a query state $(b, \sigma(b), \sigma(b))$ or $(c, \sigma(b), \sigma(b))$ within two steps. Let p be a PNE for U. Since σ is cyclic, $p = \sigma^r(b)$ for some $r \in \mathbb{N}$. So (p, p, p) is reachable from x unless $\sigma^s(b)$ is a PNE for some $s < r$. It follows that f^U guarantees convergence to a PNE, so f succeeds on $\mathcal{U}(A)$. □

Recall that Lemma 6 says that in the stochastic setting, adding actions to a profile space A does not make success on $\mathcal{U}(A)$ any easier. In light of that result, it is perhaps surprising that we can improve on the above bound when every player has sufficiently many actions.

Theorem 15. *If A is a profile space in which every player has at least four actions, then there exists a 2-recall deterministic uncoupled stationary strategy mapping that succeeds on all games (A, U).*

Proof. Let $n \geq 2$, $k_1, ..., k_n \geq 4$, and $A = \{1, ..., k_1\} \times ... \times \{1, ..., k_n\}$. It suffices to show that such a strategy mapping exists for $\mathcal{U}(A)$.

Define a permutation $\sigma : A \to A$ such that for every $a \in A$, $\sigma(a)$ is a's lexicographic successor. Formally, $\sigma(a) = (\sigma_1(a), ..., \sigma_n(a))$ where for $i = 1, ..., n-1$,

$$\sigma_i(a) = \begin{cases} a_i + 1 \bmod k_i & \text{if } a_j = k_j \text{ for every } j \in \{i+1, ..., n\} \\ a_i & \text{otherwise,} \end{cases}$$

and $\sigma_n(a) = a_n + 1 \bmod k_n$. Observe then that σ is cyclic, and for each player i and $a \in A$, we have

$$\sigma_i(a) - a_i \bmod k_i \in \{0, 1\}.$$

We now describe a strategy mapping $f : \mathcal{U}(A) \to \mathcal{F}(A)$. To each $U \in \mathcal{U}$, f assigns the strategy vector f^U defined as follows. At state $x = (a, b) \in A^2$, f^U differentiates between three types of states, each named according to the event it prompts:

– *move-on*: If $a \neq b$ and $a_j - b_j \bmod k_j \in \{0, 1\}$ for every $j \in \{1, ..., n\}$, then the players "move on" from a, in the sense that each player i plays $\sigma_i(a)$, giving $f^U(x) = \sigma(a)$.
– *query*: If $b_j - a_j \bmod k_j \in \{0, 1, 2\}$, then we "query" each player's utility function to check whether it is U-best-replying at b. Each player i answers by playing b_i if it is best-replying and $b_i - 1 \bmod k_i$ if it is not. So at query states,

$$f_i^U(x) = \begin{cases} b_i & \text{if } b_i \in BR_i(b) \\ b_i - 1 \bmod k_i & \text{otherwise,} \end{cases}$$

for $i = 1, ..., n$.
– *repeat*: Otherwise, each player i "repeats" by playing b_i, giving $f^U(x) = b$.

Notice that because $k_1, ..., k_n \geq 4$, it is never the case that both $a_j - b_j \bmod k_j \in \{0, 1\}$ and $b_j - a_j \bmod k_j \in \{0, 1, 2\}$. Thus the conditions for the *move-on* and *query* types are mutually exclusive, and the three state types are all disjoint.

The state following $x = (a, b)$ is $y = (b, f^U(x))$. If x is a move-on state, then $y = (b, \sigma(a))$. Since for every player i, $a_i - b_i \bmod k_i \in \{0, 1\}$ and $\sigma(a)_i - a_i \bmod k_i \in \{0, 1\}$, we have $\sigma_i(a) - b_i \bmod k_i \in \{0, 1, 2\}$, so y is a query state. If x is instead a query state, then $b_i - f_i^U(x) \bmod k_i \in \{0, 1\}$ for every player i, so y is a move-on state unless $b = f^U(x)$, in which case $y = (b, b)$ is a query state. But if $b = f^U(x)$ and x was a query state, then $b_i \in BR_i(b)$ for every player i, i.e., b is a PNE. Finally, if x is a repeat state, then $y = (b, b)$ is a query state.

Thus move-on states and repeat states are always followed by query states, and ask-all states are never followed by repeat states. We conclude that with the possible exception of the initial state, every state will be a move-on or query state, and no two consecutive states will be move-on states. In particular, some query state is reachable from every initial state.

For any query state $x = (a, b)$, x will be followed by (b, b) if and only if b is a PNE, and (b, b) is an absorbing state for every PNE b. If b is not a PNE, then x will be followed will be a move-on state (b, c), for some $c \in A$. This will be followed by the *query* state $(c, \sigma(b))$. Continuing inductively, since σ is cyclic, unless the players converge to a PNE, they will examine every profile $v \in A$ with a query state of the form (u, v). Thus for every game U with at least one PNE, f^U guarantees convergence to a PNE, i.e., f succeeds on $\mathcal{U}(A)$. □

While there are deterministic uncoupled 2-recall dynamics that succeed on at least some classes that require 2-recall in the stochastic setting, historyless dynamics of this type fail on $\mathcal{U}(A)$ for every profile space A.

Theorem 16. *For every profile space A, no deterministic uncoupled historyless stationary strategy mapping succeeds on all games (A, U).*

5 Future Directions

It remains open to determine tight bounds on the minimum recall of successful deterministic uncoupled dynamics for every profile space, analogous to those given in Section 3 for stochastic dynamics. In particular, we make the following conjecture.

Conjecture 1. *There exists a profile space A such that no deterministic uncoupled 2-recall strategy mapping succeeds on all games (A,U).*

The same questions answered in this work may naturally be asked for other important classes of games (e.g., symmetric games) and other equilibrium concepts, especially mixed Nash equilibrium. More generally, the resources (e.g., recall, memory) required by uncoupled self-stabilizing dynamics in asynchronous environments should be investigated.

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Profit Sharing with Thresholds and Non-monotone Player Utilities

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Abstract. We study profit sharing games in which players select projects to participate in and share the reward resulting from that project equally. Unlike most existing work, in which it is assumed that the player utility is monotone in the number of participants working on their project, we consider non-monotone player utilities. Such utilities could result, for example, from "threshold" or "phase transition" effects, when the total benefit from a project improves slowly until the number of participants reaches some critical mass, then improves rapidly, and then slows again due to diminishing returns.

Non-monotone player utilities result in a lot of instability: strong Nash equilibrium may no longer exist, and the quality of Nash equilibria may be far away from the centralized optimum. We show, however, that by adding additional requirements such as players needing permission to leave a project from the players currently on this project, or instead players needing permission to a join a project from players on that project, we ensure that strong Nash equilibrium always exists. Moreover, just the addition of permission to leave already guarantees the existence of strong Nash equilibrium within a factor of 2 of the social optimum. In this paper, we provide results on the existence and quality of several different coalitional solution concepts, focusing especially on permission to leave and join projects, and show that such requirements result in the existence of good stable solutions even for the case when player utilities are non-monotone.

1 Introduction

Resource selection games, in which players choose which project, market, or group to participate in, and then receive utility based on the number of people who choose the same strategy as them, have been heavily studied in algorithmic game theory (see for example [2,3,11,15,17]). Closely related to such games are coalition formation games, in which players choose which coalition to participate in, and player utility depends on the members of their coalition (such utilities are often called *hedonic* [4–7,9,12,14]). In most such games, player utility is assumed to either only *increase* with the number of players that choose the same group or project, or only *decrease* with the number of players that choose the same group or project (for example in [15], where more people choosing the same project means more competition).

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In many important situations, however, player utility may not be a monotone function in the number of players who choose the same strategy. For example, consider the scenario where players choose which project to work on (or form teams in order to submit funding proposals). It is mostly true that more participants will improve the overall outcome of the project; in most existing work the overall success of the project is assumed to either be convex or concave non-decreasing as a function of the number of participants. However, many projects exhibit "threshold" or "phase-transition" behavior: until there is a critical mass of participants there will be very little progress, and after that critical mass, each additional participant only makes a marginal amount of difference to the project's success. In such a scenario, and assuming that the benefit (e.g., credit) from a project's success is divided equally among the participants, the utility of a player as a function of the number of project participants may increase until this threshold is reached, and then begin to decrease. In this paper, we study such resource selection games, but for player utility functions which do not have to be monotone, and thus are much more general.

Our Model. More concretely, consider the following simple profit sharing game in which multiple projects are available to the players. There are n identical players and m projects; the strategy set of each player is $\{\emptyset, 1, 2, \ldots, m\}$. Each project k has a payoff function $p_k : \mathbb{N}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is monotone nondecreasing in the number of players working on the project: this function represents the total benefit or success of this project. Each player selects a project and shares the reward equally with everyone else working on that project. That is, the utility of a player i that selects strategy $k \neq \emptyset$ is $u_i(\mathbf{s}) = \frac{p_k(x_k(\mathbf{s}))}{x_k(\mathbf{s})}$, where $x_k(\mathbf{s})$ is the number of players selecting k in solution $\mathbf{s}$. This payoff scheme is very robust and can model many different situations. For example, if the project payoff function is convex, then the individual reward function will also be increasing. On the other extreme, if the project payoff function is constant, then the individual reward function is strictly decreasing. Finally, if the project payoff function is a "threshold" function, then the individual reward function will alternate between being increasing and decreasing. More generally, the individual payoff function may not even be single-peaked: it can increase, decrease, and then increase again.

While games of this type with either convex or concave project payoff functions p_k have been shown to have very nice properties, arbitrary non-decreasing payoff functions cause a lot of instability. First, observe that Nash equilibrium is not a good solution concept for threshold payoff functions (and tends to perform very poorly compared to the social optimum), since players typically cannot satisfy thresholds with unilateral deviations. Just as teams form to tackle difficult projects, we need to consider group deviations. Unfortunately, unlike for the case when p_k are constant or convex, strong Nash equilibrium does not exist when p_k are threshold functions. To illustrate this, consider the following simple example of $n \geq 3$ players and two projects.

Example 1. Project A is a small project that any number of people can work on and share the credit, while project B is a large project that will only succeed with a large team. Formally, project A has total value of 2 no matter how many players choose to work on it, while project B requires n players (any fewer players working on project B would get nothing), and has a total payoff of n if there are n players working on it. In other words, p_A is a constant function, while p_B is a threshold function with the threshold being n. If everyone chooses to work on project B, then the utility of each player is 1, and so they would each rather unilaterally switch to project A. It is not difficult to verify that the only Nash Equilibrium solution is for all players to pick project A, and thus is very far from optimum. Perhaps even worse, strong Nash equilibrium does not exist: if everyone chooses project A, then all the players could together deviate to project B, and obtain 1 utility instead of $\frac{2}{n}$.

Contractual Stability. Given the motivation of players forming teams to work on projects, or forming teams to submit grant proposals, the above bad example does not seem satisfactory to the authors. While it is true that strong Nash equilibrium does not exist above, this occurs because of a strange player interaction. When all players choose project A, they together decide to switch to project B; everyone is better off. After this occurs, a single player says: "Aha! Now that you have left project A unoccupied, I will leave project B," a side-effect of which is that all the other players receive nothing, since every player is an integral part of the team. It is unusual that team members drop out once a grant proposal has been funded: this is because there is often a contractual obligation for them to perform the work, and because by dropping out they will incur the bad will and shame from the other members of the team. If players are not allowed to leave a project without the permission of the other project members, then the above example has a wonderful stable solution where everyone chooses project B. On the other hand, someone is usually not allowed to join a grant proposal or a team project without permission of its members. If this were true in the above example, then once again a coalitionally stable solution exists. This line of thought suggests the study of different coalitional stability concepts in which players must obtain permission to leave or join projects.

In this paper, we provide an analysis of this generalized profit sharing game. In particular, we provide results on the existence and quality of many different coalitional solution concepts, focusing especially on permission to leave and join projects. These ideas are captured precisely by solution concepts from *hedonic games* literature [4–7, 9, 12, 14]; to our knowledge we are the first to consider them in the context of non-cooperative games.

1.1 Our Contributions

For the Profit Sharing model defined above, we introduce stability concepts which include permission to join and permission to leave. For permission to leave, players cannot leave a project with other players remaining on it, unless the utility of those players does not decrease as a result of their leaving. Permission to leave can be thought of as the enforcement of a contract: when a player decides to work on a project with a group, she is essentially entering into a contract with them to complete the project, and she will only be let out of this contract if it benefits the rest of the participants. Using the terminology from hedonic games, we call strong Nash stability with permission to leave *strong contractual Nash stability (SCNS)*. Unlike strong Nash equilibrium, we show that *SCNS* always exists. Furthermore, we show there always exists a *SCNS* solution which is within a factor of 2 of the social optimum, thus implying that the price of stability with respect to *SCNS* is at most 2. We show that this bound is asymptotically tight. Our results indicate that adding contractual obligations to not leave projects before they are completed not only results in the creation of coalitionally stable solutions, but also in the creation of high welfare stable solutions.

For permission to join, players cannot join a project with other players already on it unless the utility of those players does not decrease as a result of their joining. Permission to join is a natural idea, since a person cannot typically work on a team with other people unless they allow her to join their group. Again borrowing the terminology from coalition formation, we call strong Nash stability along with permission to join *strong individual stability (SIS)*. We show that *SIS* solutions always exist as well. However, the quality of *SIS* can be very bad; the price of stability with respect to *SIS* can be n, the number of players. This is because a player can join a high value project by itself, and since others cannot join to lower the player's utility from that project, it has no incentive to ever switch. The other players might be left on projects with high thresholds that cannot be satisfied without everyone's cooperation, which means they are receiving very little utility potentially. Essentially, permission to join does not encourage cooperation as well as permission to leave does.

Finally, we consider what happens when both permission to leave and permission to join are in effect: we call the corresponding solution concept *strong contractual individual stability (SCIS)*. Not surprisingly, this stability notion is very strong: we show that the centrally optimal solution is always *SCIS*. However, since every *SIS* state is a *SCIS* state, the price of anarchy with respect to *SCIS* can be as high as n.

Our results show that unlike in cases where p_k is convex or concave, in profit sharing where the project benefit can contain threshold effects (or in general when player utilities are not monotone), it is crucial to have some coordinating mechanism like permission to leave a project or permission to join a project. Without such a mechanism coalitionally stable solutions are not guaranteed to exist, and Nash equilibrium can be very inefficient. Once permission to leave is added through contractual obligations, however, this results in the creation of high-quality solutions which are resilient even to deviations by coalitions.

1.2 Related Work

Our model is a generalization of the well-studied market sharing game [3,11,17]. In the market sharing game, each market has an associated value that is shared equally among the players that have selected that particular market. In other words, it is a special case of our model in which the payoff functions are constants. The threshold model can be thought of as the market sharing game in which the value of the market is not awarded to the players until a sufficient number of players have selected the market. The market sharing game is a simple example of a monotone valid-utility game [17, 19], but our model is not: the utility of a player can be less than their marginal contribution to the total welfare, and the total player utility is not submodular.

The stability concepts used to analyze our model are borrowed primarily from hedonic games literature; consult [4–7, 9, 12, 14] and their references. Hedonic games are coalition formation games in which how much a player values a group depends solely on who is in her group and is independent of how the remaining players are partitioned into groups. Our model has this property as well, since a player's payoff depends only on how many people are on her chosen project. Hedonic games differ from the non-cooperative game we study primarily in that they are cooperative games in which players form groups rather than select projects: the utility of a player depends only on the members of the group, and not on which project they select together. In fact, in many hedonic game formulations, the players have ordinal preferences over groups rather than utilities. Hedonic games literature consists primarily of results characterizing the existence of stability concepts such as the core, (strong) Nash stability, (strong) individual stability, (strong) contractual Nash stability, etc. We use these concepts to model permission to join and to leave projects in our model. However, the *quality* of stable solutions for hedonic games has received little attention; in contrast we consider the price of stability and price of anarchy [18] with respect to these stability concepts.

Numerous other games are motivated by project/group selection, and, in some cases, apply concepts from hedonic games literature to non-cooperative models. Kutten et al. [16] define several group formation games where there is only a single project with a payoff function that is essentially our threshold payoff function. Players form groups (that require permission to join) that compete for this project, but only one group receives a payoff. Chalkiadakis et al. [8] define a cooperative game that is similar to our threshold model, but it has additional constraints on the project payoff functions, and there is an infinite number of each type of project. Augustine et al. [2] define non-cooperative project selection games based on monotone convex cooperative games and provide quality of equilibria and convergence results for them. Kleinberg and Oren [15] define a project selection game with decreasing player payoff functions, unlike ours which may not be monotone, and their results focus on redesigning project payoff functions to ensure that the optimal solution is stable. Finally, Feldman et al. [10] define a class of non-cooperative games called hedonic clustering games, and provide quality of equilibrium results for them. Their model has little in common with ours, but they borrow heavily from hedonic games literature and apply concepts like price of anarchy, as we do.

2 Definitions

Model. As mentioned in the Introduction, the Profit Sharing game we consider consists of n identical players, and m projects, with each player strategy being to choose one of these projects. We let $[n] = \{1, 2, \ldots, n\}$ and $[m] = \{1, 2, \ldots, m\}$. Each project has a non-decreasing payoff function p_k. We define $x_k(\mathbf{s})$ to be the number of players who choose project k in state s; then the utility of a player i with $s_i = k$ is $u_i(\mathbf{s}) = \frac{p_k(x_k(\mathbf{s}))}{x_k(\mathbf{s})}$. We allow players to opt out of playing by selecting the null strategy $\emptyset$ where for all players i, $u_i(\emptyset, s_{-i}) = 0$.

While our results will generally hold for arbitrary non-decreasing functions p_k, we will sometimes refer to the important special case of threshold payoff functions. We define a threshold payoff function for a project k to be

$$p_k(x) = \begin{cases} 0, & \text{if } x < t_k \\ c_k, & \text{otherwise} \end{cases}$$

where $t_k \in \mathbb{Z}_{>0}$ is the threshold of project k and $c_k \in \mathbb{R}_{\geq 0}$ is the value of the project k once the threshold has been met, which will be split equally among all players on project k.

Stability Concepts. We now introduce several stability concepts. Many of these concepts are adapted from hedonic games literature [4–7,9,12,14] to fit into the framework of non-cooperative game theory. These concepts take standard non-cooperative game theory equilibrium concepts such as *Nash equilibrium* and *strong Nash equilibrium* and add permission to join and/or permission to leave. "Individually stable" refers to permission to join, and "contractually stable" refers to permission to leave a project.

A state is a *Nash equilibrium* or *Nash stable (NS)* if no player can change her strategy to improve her utility. That is, if for every player i and every strategy $s'_i \in [m]$, we have $u_i(\mathbf{s}) \geq u_i(s'_i, s_{-i})$.

A state is *individually stable (IS)* if for every player i and every strategy $s'_i \in [m]$, either $u_i(\mathbf{s}) \geq u_i(s'_i, s_{-i})$ or there exists player j with $s_j = s'_i$ such that $u_j(\mathbf{s}) > u_j(s'_i, s_{-i})$.

A state is *contractually Nash stable (CNS)* if for every player i and every strategy $s'_i \in [m]$, either $u_i(\mathbf{s}) \geq u_i(s'_i, s_{-i})$ or there exists player j with $s_j = s_i$ such that $u_j(\mathbf{s}) > u_j(s'_i, s_{-i})$.

A state is *contractually individual stable (CIS)* if for every player i and every strategy $s'_i \in [m]$, either $u_i(\mathbf{s}) \geq u_i(s'_i, s_{-i})$ or there exists player j with $s_j = s'_i$ such that $u_j(\mathbf{s}) > u_j(s'_i, s_{-i})$ or there exists player j with $s_j = s_i$ such that $u_j(\mathbf{s}) > u_j(s'_i, s_{-i})$.

A state is a *strong Nash equilibrium* or *strong Nash stable (SNS)* if for every non-empty subset of players $C \subseteq [n]$ and for every $s'_C \in [m]^C$, there exists a player $i \in C$ such that $u_i(\mathbf{s}) \geq u_i(s'_C, s_{-C})$.

A state is *strong individually stable (SIS)* if for every non-empty subset of players $C \subseteq [n]$ and for every $s'_C \in [m]^C$, there exists a player $i \in C$ such that $u_i(\mathbf{s}) \geq u_i(s'_C, s_{-C})$ or there exists player $j \notin C$ with $s_j = s'_i$ such that $u_j(\mathbf{s}) > u_j(s'_C, s_{-C})$.

A state is *strong contractually Nash stable (SCNS)* if for every non-empty subset of players $C \subseteq [n]$ and for every $s'_C \in [m]^C$, there exists a player $i \in C$

such that $u_i(\mathbf{s}) \geq u_i(s'_C, s_{-C})$ or there exists $j \notin C$ with $s_j = s_i$ such that $u_j(\mathbf{s}) > u_j(s'_C, s_{-C})$.

A state is *strong contractually individually stable (SCIS)* if for every non-empty subset of players $C \subseteq [n]$ and for every $s'_C \in [m]^C$, there exists a player $i \in C$ such that $u_i(\mathbf{s}) \geq u_i(s'_C, s_{-C})$ or there exists player $j \notin C$ with $s_j = s'_i$ such that $u_j(\mathbf{s}) > u_j(s'_C, s_{-C})$ or there exists $j \notin C$ with $s_j = s_i$ such that $u_j(\mathbf{s}) > u_j(s'_C, s_{-C})$.

Price of Anarchy and Price of Stability. The global objective function we consider (i.e., the social welfare) is simply $u(\mathbf{s}) = \sum_i u_i(\mathbf{s})$. Let $\mathcal{N}(\mathcal{G}, SC)$ denote the set of states that are stable with respect to the stability concept SC. The *price of anarchy with respect to SC* of $\mathcal{G}$ is $\frac{u(\mathbf{s}^*)}{\min_{\mathbf{s} \in \mathcal{N}(\mathcal{G},SC)} u(\mathbf{s})}$. The *price of stability with respect to SC* of $\mathcal{G}$ is $\frac{u(\mathbf{s}^*)}{\max_{\mathbf{s} \in \mathcal{N}(\mathcal{G},SC)} u(\mathbf{s})}$

3 Properties of Profit Sharing with Thresholds

First, we note that Profit Sharing model is an exact potential game [18], with the standard potential function $\Phi(\mathbf{s}) = \sum_{k \in [m]} \sum_{l=1}^{x_k(\mathbf{s})} \frac{p_k(l)}{l}$, which implies that pure Nash equilibrium always exists. As we saw in Example 1, there is no such guarantee for strong Nash equilibrium.

Claim 2. A Nash equilibrium always exists. A strong Nash equilibrium need not exist.

As mentioned in Example 1 in the Introduction, the quality of Nash equilibrium can be very bad even for threshold payoff functions. Although strong Nash equilibria are not guaranteed to exist (and in fact don't exist for even very simple cases), when they do, they are efficient. The proofs of Theorems 3 and 4 and all other omitted proofs can be found in the full version of this paper [1].

Theorem 3. The price of stability with respect to *NS* is at most n, and the price of anarchy with respect to *NS* is unbounded.

Theorem 4. The price of stability and price of anarchy with respect to *SNS* is at most 2, and this bound is tight.

3.1 Existence and Quality of Contractually Stable Solutions

We prove the main results of this paper in this section. Since strong Nash stable states do not necessarily exist, we add permission to leave to the solution concept to give us *SCNS*. This puts a restriction on the types of deviations that coalitions are allowed to make. Namely, players cannot abandon their old projects if it means the players remaining on those projects will be harmed.

We will use a lexicographic improvement argument [13] to show that a *SCNS* state always exists. In this paper, we will need two different notions of lexicographic ordering, as defined below.

Lexicographic Orderings. Define $v(\mathbf{s})$ to be the vector of player payoffs, i.e., $v(\mathbf{s}) = (u_1(\mathbf{s}), u_2(\mathbf{s}), \dots, u_n(\mathbf{s}))$. We let $\hat{v}(\mathbf{s})$ denote the sorted vector in which the entries of $v(\mathbf{s})$ are sorted in non-decreasing order, i.e., $\hat{v}_1(\mathbf{s}) \leq \hat{v}_2(\mathbf{s}) \leq \cdots \leq \hat{v}_n(\mathbf{s})$. Then we say that $\mathbf{s}$ is *min-lex* strictly larger than $\mathbf{s}'$ if there exists an index k such that for all $i < k$, $\hat{v}_i(\mathbf{s}) = \hat{v}_i(\mathbf{s}')$ and $\hat{v}_k(\mathbf{s}) > \hat{v}_k(\mathbf{s}')$. We say that a solution $\mathbf{s}$ is the *min-lex maximizer* if there does not exist a solution $\mathbf{s}'$ such that $\mathbf{s}'$ is min-lex strictly larger than $\mathbf{s}$. Similarly, we say that $\mathbf{s}$ is *max-lex* larger than $\mathbf{s}'$ if there exists an index k such that for all $i > k$, $\hat{v}_i(\mathbf{s}) = \hat{v}_i(\mathbf{s}')$ and $\hat{v}_k(\mathbf{s}) > \hat{v}_k(\mathbf{s}')$.

Intuitively, min-lex order ranks solutions by how well the players with the least utility are doing, while max-lex ranks solutions by how well the players with the most utility are doing. The min-lex maximizer maximizes the minimum utility received by any player, while the max-lex maximizer maximizes the the maximum utility received by any player.

We will see that when permission to leave a project must be granted by its participants, any improving deviation results in a min-lex improvement. That is, the minimum utility never becomes worse. Then any state that is a min-lex local maximum is *SCNS*. Thus, our model will always converge to a *SCNS* under best response dynamics.

Theorem 5. A *SCNS* state always exists.

Proof. We claim that the min-lex maximizer $\mathbf{s}$ is a *SCNS* state. For a coalition C, let $X(C)$ be the set of players j such that $s_j = s_i$ for some $i \in C$. In other words, $X(C)$ is the set of players who share projects with at least one member of C.

Suppose there exists a coalition C with an improving deviation from state $\mathbf{s}$ to state $\mathbf{s}'$, such that for every player $j \notin C$ who is on the same project as any player $i \in C$ in state s, it must be that $u_j(\mathbf{s}') \geq u_j(\mathbf{s})$. In other words, players who are not part of the coalition will allow the players in C to leave their projects, because they do not suffer due to this coalitional deviation. This would occur, for example, if the project payoff function is a threshold function, and after some players of C leave, the threshold is still satisfied. Thus, for all players $i \in C \cup X(C)$, it must be that $u_i(\mathbf{s}') \geq u_i(\mathbf{s})$.

The utility of some players not in C must have decreased in $\mathbf{s}'$ since $\mathbf{s}$ is the min-lex maximizer. Thus, there must be some project k such that $p_k(x_k(\mathbf{s}))/x_k(\mathbf{s}) > p_k(x_k(\mathbf{s}'))/x_k(\mathbf{s}')$.

Because we have permission to leave, we know for every project that had players deviate from it, the utility of the players that remained on those projects did not decrease. Then only projects that gained players from $\mathbf{s}$ to $\mathbf{s}'$ can have players who lost utility. Since every player in the deviating coalition increased her utility, it must have been the players who were already part of the project in $\mathbf{s}$ whose utility decreased (that is, there must exist at least one project that gained players and was non-empty in $\mathbf{s}$). However, for every player i whose utility decreased, there exists a player $j \in C$ who joined her project such that $u_i(\mathbf{s}') = u_j(\mathbf{s}') > u_j(\mathbf{s})$. Thus, all of the players whose utility decreased from $\mathbf{s}$ to $\mathbf{s}'$ on this project still have $> u_j(\mathbf{s})$ utility, which means that $\mathbf{s}'$ is min-lex larger than $\mathbf{s}$, a contradiction. □

We will now show that there are always good *SCNS* solutions, by demonstrating that the min-lex maximizer is always within a factor of 2 of the optimal solution. Notice that since *SNS* solutions usually do not exist, this result does not follow from Theorem 4.

Theorem 6. The price of stability with respect to *SCNS* is at most 2, and this bound is tight.

Proof. To prove the upper bound, we show that the social welfare of the min-lex maximizer is at least $\frac{1}{2}$ that of the optimal solution. Let $\mathbf{s}$ denote the min-lex maximizer. Let $\mathbf{s}^*$ denote the optimal solution. We claim that $u(\mathbf{s}) \geq \frac{1}{2}u(\mathbf{s}^*)$. Suppose, by way of contradiction, that $u(\mathbf{s}) < \frac{1}{2}u(\mathbf{s}^*)$.

Let $l = \min_{i\in[n]} u_i(\mathbf{s})$. Let A denote the set of players in $\mathbf{s}^*$ that receive $\leq l$ utility. Let B denote the set of players in $\mathbf{s}^*$ that receive $> 2l$ utility. Since $\mathbf{s}$ is the min-lex maximizer, we know that A is nonempty.

We claim that B is nonempty. We observe that

$$ln \leq \sum_{i\in[n]} u_i(\mathbf{s}) = u(\mathbf{s}) < \frac{1}{2}u(\mathbf{s}^*),$$

which implies that the average utility of a player in $\mathbf{s}^*$ is strictly larger than $2l$.

We will now reassign players in A to create a new solution such that they receive $> l$ utility using the following algorithm.

We start with $\mathbf{s}^*$. Let M' denote the set of projects that the players in B are assigned to. For each player $i \in A$, we move it to any project in M' where the utility of that player will be $> l$. That is, if a project $k \in M'$ currently has x_k players assigned to it, then we do not assign players to it if $\frac{p_k(x_k+1)}{x_k+1} < l$. We repeat this process until every player in A is assigned to a project in M' or until we cannot assign a player in A to a project in M' such that she receives $> l$ utility. We call the resulting solution $\mathbf{s}'$.

We claim that our algorithm always assigns every player in A to a project in M' such that her utility is $> l$. Suppose not. We begin by observing that

$$\begin{aligned} 2ln < u(\mathbf{s}^*) &= \sum_{i\in A} u_i(\mathbf{s}^*) + \sum_{i\in B} u_i(\mathbf{s}^*) + \sum_{i\notin A\cup B} u_i(\mathbf{s}^*) \\ &\leq |A|l + \sum_{i\in B} u_i(\mathbf{s}^*) + 2\left(n - |A| - |B|\right) l \\ \implies \sum_{i\in B} u_i(\mathbf{s}^*) &= \sum_{k\in M'} p_k\left(x_k(\mathbf{s}^*)\right) > |A|l + 2|B|l. \end{aligned}$$

Since there are players in A that could not be assigned to projects in M', it must be the case that for all $k \in M'$, $\frac{p_k(x_k(\mathbf{s}')+1)}{x_k(\mathbf{s})'+1} < l$, because otherwise we would have assigned that player to that project. Furthermore, since not every player in A was reassigned, $\sum_{k \in M'} x_k(\mathbf{s}') < |A| + |B|$. Combining these facts together allows us to derive that

$$\begin{aligned} |A| + 2|B| &> \sum_{k \in M'} x_k(\mathbf{s}') + |M'| = \sum_{k \in M'} (x_k(\mathbf{s}') + 1) \\ &> \sum_{k \in M'} \frac{p_k(x_k(\mathbf{s}') + 1)}{l} \geq \sum_{k \in M'} \frac{p_k(x_k(\mathbf{s}^*))}{l} \\ &= \frac{1}{l} \sum_{i \in B} u_i(\mathbf{s}^*) > |A| + 2|B| \end{aligned}$$

which is a contradiction. Then our algorithm terminated with every player in A reassigned to a project in M'. Then every player in $\mathbf{s}'$ has strictly larger than l utility, which contradicts that $\mathbf{s}$ is the min-lex maximizer. We conclude that $\mathbf{s}$ is within a factor of 2 of the optimal solution.

The corresponding lower bound comes from the following example: suppose there are n projects with $p_A(x) = n + \epsilon$ for some $\epsilon > 0$ and $p_k(x) = 1$, for all other projects k. The optimal solution $\mathbf{s}^*$ is assigning one player to each project, and $u(\mathbf{s}^*) = 2n - 1 + \epsilon$. The sole *SCNS* state is every player working on project A, because they are always guaranteed to receive > 1 utility by working on A. Thus, as n goes the infinity and ϵ goes to 0, the price of stability approaches 2. □

3.2 Existence and Quality of Individually Stable Solutions

We now examine what happens when we add permission to join to strong Nash stability. The resulting concept is called *SIS*. We are able to show the our game has a similar lexicographic improvement property with respect to *SIS*, which implies *SIS* states always exist. However, unlike *SCNS* which uses min-lex, *SIS* uses max-lex. In other words, the utility of the players with the most utility always increases with improving deviations.

Theorem 7. A *SIS* state always exists.

We will now see that the quality of *SIS* states is not as good as the quality of *SCNS* states. This is because dynamics that would lead to players deviating to good, high threshold projects together cannot occur. For example, if a player is alone on a project with low utility which gives slightly more than what they would receive on the high value, high threshold project, then they have no incentive to deviate. If there are no other projects to join, the remaining players have no options: they cannot meet the threshold of the high value project, nor can they join the low value project to drive the value down and force the lone player there to deviate to the high value project. Thus, many players are stranded without any projects to join.

Theorem 8. The price of stability and price of anarchy with respect to *SIS* is at most n, and this bound is tight.

3.3 Permission to Join and Leave Results

We finally analyze what happens when we must have both permission to join and permission to leave a project. We observe that the optimal solution becomes stable, but there still exist stable, low quality states since *SCIS* is more general than *SIS*.

Theorem 9. The optimal solution $\mathbf{s}^*$ is *SCIS*.

Corollary 10. The price of stability with respect to *SCIS* is 1, while the price of anarchy with respect to *SCIS* is n.

4 Conclusion

Our results show that in profit sharing games when player utilities are not monotone, it is crucial to have some coordinating mechanism like permission to leave a project or permission to join a project. Without such a mechanism no coalitionally stable solutions exist, and even Nash equilibrium can be very bad. Once permission to leave is added through contractual obligations, however, this results in the creation of high-quality solutions which are resilient even to deviations by coalitions. Making sure that people honor their commitments seems to have more effect than excluding people from joining your project, as the former will always have good stable solutions, while the latter may still have high price of stability. Finally, requiring permission to both leave and join a project is perhaps too constraining: a very large number of solutions become stable due to the paucity of allowed deviations, leading to price of stability of 1 but to high price of anarchy.

A natural extension of our model is to allow even more general project payoff functions. In particular, what if we allow the project payoff function to be non-monotone as well? For example, at a certain point, having too many people working on the same project can make communication and organization more difficult, and there may not be enough tasks for everyone, so adding more people may actually *hurt* the project. This captures the idea of the project having a capacity, for example. Due to the extremely powerful stability concept, the centrally optimal solution is still *SCIS* even for this more general case. Unfortunately, even for simple non-monotone project payoff functions, the price of stability with respect to *SCNS* can be horrible. Suppose there is a single project with the payoff function $p_A(x) = 1$ if $x < n$ and $p_A(n) = \epsilon$. Then we need one player to remain unassigned to the project, but that player will always prefer to receive some utility rather than none, so the only *SCNS* solution is having all players work on the project. One extension that seems to avoid such issues is project payoff functions with hard capacities: that is, non-decreasing functions which, after a certain point, suddenly become 0. We conjecture that the price of stability with respect to *SCNS* for this case is still small, and consider this a good future direction.

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Shared Resource Management via Reward Schemes

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Abstract. We study scenarios in which consumers have several alternatives for using a shared resource. We investigate whether rewards can be used to motivate effective usage of the resource. Our goal is to design reward schemes that, in equilibrium, minimize the cost to society and the total sum of rewards. We introduce a generic scheme *which does not use any knowledge about the valuations of the consumers*, yet its cost in equilibrium is always close to the cost of the optimal scheme that has *complete* knowledge of the consumers' valuations. We show that our scheme is essentially optimal in some settings while in others no good schemes exist.

1 Introduction

In this paper we study the management of resources that are overly demanded. Traditionally, taxes are used to control consumption, see, e.g., London's or Paris' congestion charges [3,17]. In contrast, recent experiments in traffic congestion reduction [12,11,16] take a different approach. Instead of charging users for driving during peak hours, drivers who travel during off-peak hours receive monetary compensation. While in a purely quasi-linear world taxes and rewards are equivalent, in practice they may differ greatly, as established by a large body of research in, e.g., psychology [14] and prospect theory [10]. We refer the interested reader to [2] and references within for a thorough discussion on fees vs. rewards in congestion control[1].

We study a basic problem of managing the consumption of a single public resource. We assume that there are several alternative consumption methods, such as time slots, locations, etc. The social cost is determined not only by the total consumption but also by the way that this consumption is distributed across alternatives (e.g., it is preferred that factories will consume electricity during off peak hours). Our goal is to develop mechanisms that minimize the sum of the social cost of the consumption and the total payment of the mechanism.

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[1] We emphasize that our claim is not that rewards are better than prices; just that they may serve as a powerful tool in some settings.

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Our interest is not limited only to congestion control; Similar settings include basic infrastructures such as electricity that can be consumed in peak or non-peak hours, fresh air and water, forest preservation, and more. All of which are settings in which a designer – such as a transportation authority – may work with a *small* number n of big consumers (polluting factories, large truck fleets, major electricity consumers, etc.) on a voluntary basis to improve the efficiency of using the resource[2]. Our main contributions are as follows:

1. We present a stylized model of resource management via reward schemes.
2. We introduce a novel solution concept for analyzing the performance of reward schemes: in many settings the designer either does not know the valuations of the consumers or is legally not allowed to use them for designing reward schemes. In these cases, the designer may use only his own costs. The competitive ratio of a scheme $\mathcal{R}$ is obtained by considering all possible instances $\mathcal{I}$ (i.e., all possible realizations of the valuations of the players) and taking the maximum ratio between the worst equilibrium induced by $\mathcal{R}$ and the best equilibrium induced by any other scheme $\mathcal{R}_{\mathcal{I}}$. Notice that $\mathcal{R}_{\mathcal{I}}$ may be designed using knowledge on the valuations of the consumers.
3. We present a generic reward scheme, the *square root scheme*, and show that it is competitive in many settings. In other settings, we prove that no scheme is competitive. We focus on two basic consumption models: atomic (e.g., truck drives to downtown) and non atomic (e.g., electricity consumption).

We now provide a more formal description of our model, the solution concept, and our results.

The Model. In our formal setup we have n agents (consumers), each is interested in consuming one unit of the public good (a unit could be, for example, a single drive to the city center or 1000KW of electricity). There are m different alternative ways to consume the resource, such as time slots or locations, so the strategy of each agent i is a vector $f^i = (f^i_1, \ldots, f^i_m)$ such that for each j, $f^i_j \geq 0$ and $\Sigma_j f^i_j = 1$. In the *atomic model* we always assume that $f^i_j \in \{0, 1\}$. In the *non atomic* model we do not make this assumption.

For each consumer i and alternative j, there is a function u^i_j that depends only on the use of the alternative $f_j = \Sigma_i f^i_j$. This function quantifies the gains of consumer i from consuming via alternative j. We assume that $u^i_j(f_j) = v^i_j \cdot L_j(f_j)$, where v^i_j is privately known to consumer i and the attractiveness functions $L_j(\cdot)$ are public information. The attractiveness functions are decreasing monotoni-

[2] E.g., a municipality attempting to incentivize an international company to locate its polluting factory further away from the city.

cally. The utility of consumer i is defined to be $\Sigma_j f_j^i \cdot u_j^i(f_j) + p^i_{f^1,\ldots,f^n}$, where $p^i_{f^1,\ldots,f^n} \geq 0$ is the payment that the consumer receives.

We assume that the cost to society for every unit of good consumed via alternative j is $w_j > 0$. The direct loss to society is therefore $\Sigma_j w_j \cdot f_j$. Without loss of generality we assume that $w_1 \geq \ldots \geq w_m$. The ratio $\alpha = \frac{w_1}{w_m}$ will be essential in our analysis. We assume without loss of generality that $w_m = 1$.

In this paper we seek for $p^i_{f^1,\ldots,f^n}$'s so that in equilibrium the total cost to the society $\Sigma_j w_j \cdot f_j + \Sigma_i p^i_{f^1,\ldots,f^n}$ is minimized (notice that this cost includes payments). To illustrate this point, observe that the VCG mechanism can be used to minimize the direct loss of the society ($\Sigma_j w_j \cdot f_j$). However, it is well known that in the VCG mechanism the amount that the society has pay might be huge, whereas in many settings big subsidies are unacceptable either politically or economically. A set of payment functions $p^i_{f^1,\ldots,f^n}$'s as above is called a *reward scheme* or *payment scheme*. Of particular interest are simple reward schemes where payment to each consumer i depends only on its own consumption f^i. In this paper we seek for such payment schemes that are competitive with respect to arbitrarily complicated schemes. Given a reward scheme $\mathcal{R}$ and a consumption profile f, we let $C_{\mathcal{R}}(f) = \Sigma_j w_j \cdot f_j + \Sigma_i p^i_{f^1,\ldots,f^n}$ denote the overall cost to society, i.e. the total loss plus the total payment.

The Solution Concept: Robustness in an Incomplete Information Setting

Assuming full information of the consumers' utilities is often unrealistic. For instance, it equips the designer with the unlikely power to accurately estimate the profit of a specific freight forwarder for sending a specific truck at peak hours. In some other cases, the designer might be legally prohibited from using knowledge of the consumers' valuations in a reward scheme. On the other extreme, the popular approach in the Algorithmic Mechanism Design community is to design truthful mechanisms. However, since our objective function involves payments, more often than not this leads to impossibility results. See, e.g., the various impossibility results on the design of frugal mechanisms (e.g., [7,1]). Externalities among the consumers also make the existence of satisfactory truthful mechanisms even less likely[3].

This paper takes a middle way that allows us to design payment schemes with good guarantees for games with incomplete information. The private information of the consumer i is $v^i = v^i_1, \ldots, v^i_m$. Suppose that all the private information $v = v^1, \ldots, v^n$ is known. Let $\mathcal{R}$ be a reward scheme. Together, we have a complete information game. Let f be an equilibrium in this game. The total cost of f in $\mathcal{R}$ equals $\Sigma_j w_j \cdot f_j + \Sigma_i p^i_{f^1,\ldots,f^n}$. An *optimal solution* is a

[3] For VCG, while the welfare is maximized, it is easy to see that the payment can be huge.

reward scheme $\mathcal{O}$ and an equilibrium f' in $\mathcal{O}$, such that the total cost of f' is minimal among all reward schemes and equilibria. In other words, $C_{\mathcal{O}}(f')$ denotes the total cost to society of the best equilibrium of the best reward scheme given that the values of the consumers are v.

Definition 1. (competitive scheme) *A reward scheme $\mathcal{R}$ is c-*competitive *if for every vector of values of the consumers v, equilibrium f in the game defined by v and $\mathcal{R}$, an optimal scheme $\mathcal{O}$ for v, and every equilibrium f' in the game defined by $\mathcal{O}$ and v it holds that $\frac{C_{\mathcal{R}}(f)}{C_{\mathcal{O}}(f')} \leq c$.*

Notice our guarantee: if the "market" converges to an equilibrium, then the total cost to society of the solution is not too far away from the total cost of the optimal solution – even had we known the consumers' valuations in advance. Thus, to some extent we seek for Nash implementation of reward schemes. However, while most work on Nash implementation usually considers full information games or takes a Bayesian approach, we assume private information and our guarantees are worst-case ones. In this sense, our work lies on the border of price of anarchy analysis and algorithmic mechanism design.

Related to our work is a series of works on setting tolls in congestion games to improve the efficiency of equilibrium (see, e.g., [6,8,9]). One main difference between this line of work and ours is that the aforementioned papers design tolls for a specific complete-information game. In our approach consumers have private information, yet equilibrium is *always* guaranteed to be approximately-efficient. Another key difference is that our objective function takes into account only the cost to the society and not the cost to the consumers. Finally, there are also technical differnces in the approximation notion.

Our approach is closely related to coordination mechanisms [4], and in particular to coordination mechanisms for congestion games [5]. A *coordination mechanism* is allowed to change the latencies of some edges in order to improve the price of anarchy in the network. One main difference between our work and the work on coordination mechanisms (besides considering completely different models) is that our benchmark takes into account the effort needed to change the network (in the sense of payments), while in coordination mechanisms changes in the network are "free" as long as the quality of the solution improves.

Some Comments on Our Modeling Choices. A few words are in place about our modeling choices. First, perhaps our most controversial design choice is the separation of the welfare of the consumers from the public welfare. This seems justified to us as in the applications we envision the designer works with a few big consumers on a voluntary basis (note that the consumers can always choose an option that will not give them any rewards) in order to improve the welfare of a large public. For example, consider a big freight company that has to send trucks to the center of the city. Suppose that there are only two alternatives: use peak hours (and negatively affect the commute time of many drivers) or off peak hours. We think about the w_j's (the cost of alternative j to the society) as the *marginal* loss that the rest of the drivers incur from the freight company that uses alternative j. Notice that the freight company can always decline changing and get no rewards. It is thus reasonable for the society to neglect the welfare of such a consumer and focus on the rest of the public as long as voluntary participation is kept. Nevertheless, relaxing this assumption is an interesting future direction.

We also assume that society's loss from a specific alternative is linear in the congestion. This assumption may be considered as more technical than conceptual and we leave relaxing this assumption to future work.

A Simple Example. In order to make our model and notation more concrete consider the following toy example. We will have only a single consumer ($n = 1$) and two alternatives with $w_1 = 9$ and $w_2 = 1$. Thus, $\alpha = 9$. Suppose that the consumer must choose either alternative 1 or alternative 2 but cannot split its consumption (e.g., cannot use half of alternative 1 and half of alternative 2). Notice that the attractiveness functions $L_j(\cdot)$ have no real meaning here since there is only a single consumer. Thus we assume that $L_j(.)$ is always 1.

Consider the case were no reward is given. Suppose that consumer that slightly prefers alternative 1 over alternative 2, e.g. $v_1^1 = 1 + \epsilon$ and $v_2^1 = 1$. Thus, he will choose alternative 1, i.e. $f_1^1 = 1$, and the overall cost will be 9. Paying ϵ for using alternative 2 will yield a total cost of only $1 + \epsilon$.

Consider the scheme that gives the consumer a reward of 2 for using alternative 2, and no reward for using alternative 1. We claim that the social cost of the solution in this scheme is always at most a factor of 3 away from the cost of the optimal solution, regardless of the valuation of the consumer. Since this scheme is actually a special case of the square root scheme, to be introduced shortly, we will not formally prove this but instead analyze some interesting instances.

Suppose that the customer's preferences are as before, alternative 2 will be chosen and the overall cost would be $1 + 2 = 3$ (1 for using alternative 1 and a reward of 2). However, when the value of the consumer for using alternative 1 is 1 and $1 + \epsilon$ for using alternative 2, the optimal solution is to pay nothing to the consumer, since he prefers the less costly alternative anyway. Similarly to before, the total cost is $1 + 2 = 3$ while the optimal cost is 1.

The last instance we consider is when the value of the consumer for using alternative 1 is 100 and his value for using alternative 2 is 1. The optimal solution is to pay nothing to the consumer and let him use alternative 1, since we have to pay the consumer at least 99 to prefer alternative 2. In this case the consumer will use alternative 1 and the total cost is 9. In our scheme the total cost is also 9, since the offered reward is too small for the consumer to prefer alternative 2.

Results. Our main tool is a new payment scheme – the square root scheme. Divide the alternatives to two: an alternative j is *cheap* if $w_j \leq \sqrt{\alpha}$ (recall that $w_1 = \alpha$ and $w_m = 1$), and is *expensive* otherwise. The reward for choosing a cheap alternative j is $p_j = \sqrt{\alpha} - w_j$ and 0 for expensive alternatives. Note that the more attractive an alternative is to society, the higher its payment. We pay consumer i that plays a strategy $f^i = (f^i_1, \ldots, f^i_m)$ a payment of $\Sigma_j \; f^i_j \cdot p_j$. Notice that the scheme is simple to implement: the designer announces only which alternatives are cheap, and the reward for using every cheap alternative.

We show that the square root payment scheme produces good results in some settings – in particular it is applicable to the two alternative case (e.g., peak and off-peak hours), both in the atomic and the non-atomic settings[4], and in the case of a single consumer. The later can also be applied to cases where there is no friction between consumers so they can be considered separately. In the non-atomic model, however, we prove these results only with respect to *linear* schemes – where the payment is proportional to the usage of the alternative.

Theorem: The square root payment scheme is $O(\sqrt{\alpha})$ competitive in the following settings, both in the atomic model and with respect to linear schemes in the non-atomic model: (1) a single consumer and multiple alternatives, and (2) multiple consumers and two alternatives.

The proofs are based on charging arguments comparing the contribution of each consumer to the total cost in the optimal solution and in the equilibrium produced by the scheme. We state this charging argument as one key lemma, and apply it in different settings. Proving the result in the non-atomic model is more involved and requires structural understanding of the equilibrium. For some settings we can show that the competitive ratio cannot be substantially improved:

Theorem: Every payment scheme is $\Omega(\sqrt{\alpha})$-competitive for a single consumer and many alternatives both in the atomic and the non-atomic settings. Furthermore, no payment scheme is $\frac{\sqrt{\alpha}}{n}$-competitive in the atomic model with $n \geq 2$ consumers.

When there are at least three alternatives and at least two consumers, we show that there are essentially no good schemes, at least in the atomic case and when the number of consumers is small:

Theorem: No non-increasing payment scheme is $\frac{n-1+\alpha}{n}$-competitive in the atomic model when there are at least three alternatives and $n \geq 2$ consumers.

[4] The two models may look superficially similar, but in fact give rise to different behavior.

Observe that the multi-consumer bounds hold only in the atomic model and are tight only for a small number of consumers. Extending these bounds to non-atomic settings, as well as obtaining tight bound for a large number of consumers, is an open question. On the positive side, for the non-atomic model, we show that when the consumers preferences are identical (but unknown to – or unused by – the designer) an optimal competitive ratio can be achieved:

Theorem: In the non-atomic model, when all the consumers have the same preferences, the square root scheme is $\sqrt{\alpha}$-competitive with respect to any linear anonymous reward scheme.

2 The Square Root Reward Scheme

A main tool in our constructions is the following *square root scheme*. Before introducing it, we require some notation. We say that an alternative j is *cheap* if $w_j \leq \sqrt{\alpha}$. Otherwise, the alternative is called *expensive*.

Definition 2. (square root reward scheme). *The* square root reward scheme *is defined as follows. The reward function to each consumer i is identical for all consumers, and depends only on the actions of consumer i.*

$$t_j = \begin{cases} \sqrt{\alpha} - w_j, & j \text{ is cheap;} \\ 0, & j \text{ is expensive.} \end{cases}$$

The payment is then defined by $p^i_f = \Sigma_j \, f^i_j \cdot t_j$.

Notice that in the atomic case a consumer is rewarded by $\sqrt{\alpha} - w_j$ when choosing a cheap alternative j and zero otherwise. A concrete example of the square root scheme was given in the introduction.

2.1 A Key Lemma

To facilitate the analysis of our scheme in various settings, we now provide a lemma that will be form the basis of our analysis in each proof. Roughly specking, the lemma relies on two facts. The first is that the overpayment of the square root scheme on cheap alternatives is not so big. On the other hand, if the usage of cheap alternatives is greater in the square root scheme, then the optimal scheme pays at least as the square root scheme. Let $\mathcal{C}$ denote the set of cheap alternatives. Recall that t_j denotes the threshold of the square root scheme.

Definition 3. *Let M be the square root scheme and denote by $p(\cdot)$ its payment function. Let M^* be any scheme, and $p^*(\cdot)$ its payment function. We say that M is* payment bounded *with respect to M^* if the following holds for every equilibrium f, f^* of M, M^* respectively:*

- *If there exists an expensive alternative* e *such that* $f_e > f^*_e$*, then it holds that for every consumer* i *and cheap alternative* j *such that* $f^*_j > f_j$ *we have that* M^* *pays consumer* i *at least as* M *for using alternative* j*:* $p^*_i(f^*) \geq \sum_{j \in \mathcal{C}, f^*_j \geq f_j} (f^{*i}_j - f^i_j) \cdot t_j$.

In other words, for each cheap alternative j, either the usage of j in f is greater than the usage of j in f^*, or M^*'s rewards are at least as M's for it. We now use this property to analyze the competitive ratio of the square root scheme.

Lemma 1. (key lemma) *Let* M *be the square root scheme and denote by* $p(\cdot)$ *its payment function. Let* M^* *be any scheme, and* $p^*(\cdot)$ *its payment function. If* M *is payment bounded with respect to* M^* *than for every equilibrium* f, f^* *in* M, M^* *respectively,* $C_M(f) \leq \sqrt{\alpha} \cdot C^*_M(f^*)$*.*

Proof. Let $\mathcal{C}$ and $\mathcal{E}$ denote the sets of cheap and expensive alternatives, respectively. Let $f_{\mathcal{C}} = \sum_{j \in \mathcal{C}} f_j, f_{\mathcal{E}} = \sum_{j \in \mathcal{E}} f_j$ denote the total usage of cheap and expensive alternatives respectively. We divide the proof into two different cases. In the first one, the usage of cheap alternatives in f is bigger than the the usage of cheap alternatives in f^*. We will then consider the complement case.

- **Case 1**: $f_{\mathcal{C}} \geq f^*_{\mathcal{C}}$. The square root scheme pays nothing on the expensive alternatives. On each cheap alternative j we the cost is $[(\sqrt{\alpha} - w_j) + w_j] \cdot f_j = \sqrt{\alpha} \cdot f_j$. Thus,

$$C_M(f) \leq \alpha \cdot f_{\mathcal{E}} + \sqrt{\alpha} \cdot f_{\mathcal{C}} \leq \sqrt{\alpha} \cdot (\sqrt{\alpha} \cdot f_{\mathcal{E}} + f_{\mathcal{C}}) \leq \sqrt{\alpha} \cdot (\sqrt{\alpha} \cdot f^*_{\mathcal{E}} + f^*_{\mathcal{C}}) \leq \sqrt{\alpha} \cdot opt$$

  The third inequality holds since $f_{\mathcal{C}} - f^*_{\mathcal{C}} = f^*_{\mathcal{E}} - f_{\mathcal{E}}$ so we only "shifted mass" from the cheap to the expensive alternatives (i.e., from weight $\sqrt{\alpha}$ to weight α).
- **Case 2**: $f_{\mathcal{C}} < f^*_{\mathcal{C}}$. In this case $f_{\mathcal{E}} > f^*_{\mathcal{E}}$ so there exists at least one expensive alternative e such that $f_e > f^*_e$. Thus, by the payment bound condition, for every cheap alternative j, if $f^*_j \geq f_j$ then M^* pays at least as M for using alternative j. Given a consumer i, we can thus decompose i's payment to $\sum_j t'^i_j \cdot (f^{*i}_j - f^i_j)$ such that $t'^i_j \geq t_j$ on every cheap alternative j where $f^*_j \geq f_j$. Let t'_j denote the weighted average reward (over consumers) $\sum \frac{f^{i*}_j - f^i_j}{f^*_j - f^i_j} \cdot t'^i_j$. In particular, $t'_j \geq t_j$ for all the cheap alternatives for which $f^*_j \geq f_j$. We now divide the total usage of both schemes into three groups and show the competitive ratio in each group separately.
  1. S_1 is composed of usage of expensive alternatives of size $f^*_{\mathcal{E}}$ from each of the equilibria. On f^* this comprises from all usage of expensive alternatives. On f this covers all except $f_{\mathcal{E}} - f^*_{\mathcal{E}}$ of it.

2. S_2 will be a consumption of size $f_{\mathcal{E}} - f^*_{\mathcal{E}} = f^*_{\mathcal{C}} - f_{\mathcal{C}}$. From the square root scheme M we will take this mass from the expensive consumption. From M^* we will take this mass[5] *o*nly from alternatives $j \in \mathcal{C}$ such that $f^*_j \geq f_j$ and from consumers i such that $f^{*i}_j \geq f^i_j$. From each such consumer we will take up to $f^{*i}_j - f^i_j$. This is feasible as, $f^*_{\mathcal{C}} - f_{\mathcal{C}} = \sum_{j \in \mathcal{C}, f^*_j \geq f_j} f^*_j - f_j + \sum_{j \in \mathcal{C}, f^*_j < f_j} f^*_j - f_j \leq \sum_{j \in \mathcal{C}, f^*_j \geq f_j} f^*_j - f_j \leq \sum_{i,j \ s.t. j \in \mathcal{C}, f^*_j \geq f_j, f^{*i}_j \geq f^i_j} f^{*i}_j - f^i_j$.
3. S_3 is a mass of size $f_{\mathcal{C}}$ from the cheap alternatives in both equilibria. In f this is all usage of cheap alternatives. In f^* this is the remaining usage of cheap alternatives after we took $f^*_{\mathcal{C}} - f_{\mathcal{C}}$ to S_2.

The reader may verify that these groups cover exactly the total consumption, both cheap and expensive, in both equilibria. Let $C_M(S_i), C_{M^*}(S_i)$ denote the costs of each set. Let $|S_i|$ denote the total mass of S_i. We now show that the $\sqrt{\alpha}$ ratio is preserved for each of the sets. Since the total cost of each equilibrium is the sum of its cost on the sets the lemma will follow.

For S_1, all the mass is taken from expensive alternatives. The square root scheme pays nothing for using these alternatives. Thus, $C_M(S_1) \leq \alpha \cdot |S_1|$ and $C_{M^*}(S_1) \geq \sqrt{\alpha} \cdot |S_1|$ yielding $\frac{C_M(S_1)}{C_{M^*}(S_1)} \geq \sqrt{\alpha}$.

For S_2, recall that M is payment bounded w.r.t. M^*. Since all the mass for S_2 of f^* is taken from cheap alternatives j with $f^*_j \geq f_j$, the payment decomposition defined above implies that the payment per unit t'_j for each such alternative j is at least $\sqrt{\alpha} - w_j$. (Note that the only place where we charge payment to M^* is S_2 so we can use this decomposition.) Thus, $C_{M^*}(S_2) \geq \sqrt{\alpha} \cdot |S_2|$. Since $C_M(S_2)$ is at most $\alpha \cdot |S_2|$ the ratio follows.

Observe that S_3 is fully composed of mass from cheap alternatives. The scheme pays $\sqrt{\alpha} - w_j$ on each cheap alternative j and thus its average cost per unit of usage of a cheap alternative is $\sqrt{\alpha}$. Thus $C_M(S_3) \leq \sqrt{\alpha} \cdot |S_3|$. The cost of f^* is at least $|S_3|$, and the ratio follows. This finishes the proof.

3 Atomic Consumption

This section assumes that each consumer chooses a single alternative and cannot divide his usage between several alternatives. Examples include a company that can build a new factory either close to the city or in a more remote site, a single drive that can to be made either in peak hours or in off-peak hours, etc. From a technical perspective, in this case each f^i_j is in $\{0, 1\}$.

[5] By taking mass of size m we mean that we are given a set S of pairs i, j where i is a consumer and j alternative s.t. $\sum_{(i,j) \in S} f^i_j \geq m$ and we take up to m of it, for example by letting c s.t. $c \cdot \sum_{(i,j) \in S} f^i_j = m$ and taking $c \cdot f^i_j$ from every such pair.

We first prove the existence of Nash equilibrium under some mild conditions[6]. A scheme is *non-increasing* if the reward for using an alternative j does not increase with f_j. A scheme is *alternative-based* if the reward for using alternative j depends only on f_j. The square root scheme maintains both properties.

Proposition 1. *If the reward scheme is alternative-based and non-increasing then a pure Nash equilibrium exists in the induced complete information game.*

The proof is in the full version. In the sequel we analyze the power and limitations of reward schemes. We start with analyzing the two-alternative case.

3.1 Two Alternatives

Theorem 1. *If $m = 2$ the square root payment scheme is $\sqrt{\alpha}$-competitive.*

Proof. Recall that by our convention alternative 1 is the expensive alternative and alternative 2 is the cheap one. Let M^* be some scheme. According to Lemma 3 it suffices to show that the square root scheme M is payment bounded w.r.t M^*. Let f, f^* denote equilibria in M and in M^*, respectively. If $f_1 \leq f{*}_1$ then Lemma 3 trivially holds for f and f^*, therefore assume that $f_1 > f_1^*$. Thus, $f_2 < f_2^*$. Since the attractiveness functions are non-increasing $L_1(f_1^*) \geq L_1(f_1), L_2(f_2^*) \leq L_2(f_2)$, our goal is to show that if consumer i uses alternative 2 in f^* and alternative 1 in f, then M^* pays to consumer i at least as the possible payment of M to i, which is $\sqrt{\alpha} - 1$.

Let p_j^i, p_j^{*i} denote the payment to i when choosing j in each scheme respectively (fixing the actions of the other consumers). Since M pays nothing for using alternative 1 but the consumer still uses it, $p_2^i + v_2^i \cdot L_2(f_2) \leq v_1^i \cdot L_1(f_1)$. In M^* the consumer prefers alternative 2 and therefore:

$$p_2^{*i} \geq p_1^{*i} + v_1^i \cdot L_1(f_1^*) - v_2^i \cdot L_2(f_2^*) \geq p_1^{*i} + v_1^i \cdot L_1(f_1) - v_2^i \cdot L_2(f_2) \geq p_2^i = \sqrt{\alpha} - 1.$$

Thus the conditions of Lemma 3 are met and we have that $C_f(M) \leq \sqrt{\alpha} \cdot C_f(M^*)$, and the competitive ratio follows. □

We now show that under mild conditions, this bound is essentially tight. An attractiveness function L_j is *bounded away from* 0 if for every if for any number of consumers n' and any value $t \geq 0$ there exists some v such that $v \cdot L_j(n') = t$ (i.e., $L_j(n') > 0$). The following theorem (proof in the full version) provides bounds which are tight up to a constant for the case of $m = 2$ and for the single consumer case. We will later provide stronger bounds when both the number of alternatives and the number of consumers is big.

[6] In general the existence of an equilibrium is not guaranteed: consider a scheme for two alternatives and two players that pays a huge amount for one consumer to use any alternative alone, and a huge payment for the second consumer to use any alternative with the second consumer.

Theorem 2. *Consider the case where $m \geq 2$. For all attractiveness functions that are bounded away from 0, and for all $\epsilon > 0$, no deterministic payment scheme can provide a competitive ratio better than $\frac{(n-1)+\alpha}{(n-1)+\sqrt{\alpha}}$. Furthermore, no randomized scheme can provide a competitive ratio better than $\frac{(n-1)+\alpha}{2(n-1)+\sqrt{\alpha}}$.*

3.2 Multiple Alternatives

We begin with a lower bound for multiple alternatives and consumers. Our bound is meaningful only for small number of players, and we do not know how to extend it further. After proving the lower bound we will show that if there is a single consumer the square root scheme does provide a good competitive ratio when there are multiple alternatives. Proofs appear in the full version.

Theorem 3. *When $m \geq 3$, $n \geq 2$ and the attractiveness functions that are bounded away from 0, no non-increasing reward scheme provides a competitive ratio better than $\frac{n-1+\alpha}{n}$.*

Theorem 4. *When $n = 1$, the square root payment scheme is $\sqrt{\alpha}$-competitive.*

4 Non Atomic Consumption

In this section we focus on consumption that can be split between different alternatives (e.g. a total of 1000KW has to be consumed, but some of it can be consumed in off-peak hours). From a technical point of view, the difficulty of analysis in this model stems from the fact that the attractiveness functions are functions of the overall load on the alternatives (including the consumer's own usage), so the marginal utility of a consumer from using an alternative depends not just on the total load, but also on his own usage of that alternative. Due to lack of space we postpone this section to the full version.

5 Discussion and Open Questions

We studied a basic problem of managing a single shared resource via positive incentives. We focused on the case where a central designer is interested in motivating a few big consumers to change their demand while minimizing the cost. Various concrete open questions are mentioned in the body of this paper. Perhaps the most important one is understanding the setting where there are multiple alternatives and many consumers – our lower bounds hold only when the number of consumers is small and good schemes may exist.

Many extensions of our model almost suggest themselves. For example, what happens if the consumers have combinatorial preferences over each set of alternatives (two drives in peak hours worth more than a drive in peak hours and a drive in off-peak hours). Also, broadening the set of attractiveness functions and the set of society's cost functions is a natural extension. One can also take a graph-theoretic view of our model by considering two nodes (source and target) connected by m parallel edges, each corresponding to another alternative. Analyzing different graph topologies seems like a worthy research avenue.

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Budget-Restricted Utility Games with Ordered Strategic Decisions *

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Abstract. We introduce the concept of *budget games*. Players choose a set of tasks and each task has a certain demand on every resource in the game. Each resource has a budget. If the budget is not enough to satisfy the sum of all demands, it has to be shared between the tasks. We study strategic budget games, where the budget is shared proportionally. We also consider a variant in which the order of the strategic decisions influences the distribution of the budgets. The complexity of the optimal solution as well as existence, complexity and quality of equilibria are analyzed. Finally, we show that the time an ordered budget game needs to convergence towards an equilibrium may be exponential.

1 Introduction

Recent advancements of network technology enabled and simplified outsourcing of processing and storing information to remote facilities. The offering of such services in a competitive environment has become known as cloud computing. The competitive aspect is twofold. On the one hand, customers compete over the allocation of various types of services and resources like bandwidth, memory space, computing power etc. These resources are usually limited in capacity and as soon as the demand exceeds that capacity customers' demand can only be satisfied partially. On the other hand, the service providers face strategic decisions in the markets which have to take into account the budget of their clients. As long as a client can afford all the desired products, his budget has no consequence. But once their total costs exceed his budget, he has to split it between them. When deciding to offer a product, a provider therefore has to consider the remaining budgets of the interested clients.

We study this in a game theoretic setting called *budget games* in which several tasks (or products) have a certain demand for resources (or money) and the resources (or clients) have budgets. As long as the sum of the demands does not exceed the budget, the demands can be completely satisfied, otherwise only partially. For example in scheduling, where tasks are allocated to one or more servers. Each server disposes of a limited amount of computational capacity,

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space or bandwidth and when it runs too many tasks, this capacity has to be split between them. Naturally, every job will aim to obtain as much capacity as it needs, which may vary between the different servers. Also, not every server combination may be possible for every task.

We study budget games as strategic games as well as in a variant that takes into account temporal aspect. Strategic games are often analyzed as one-shot games which do not capture situations like a new provider entering a market having a disadvantage against those already established. The clients prioritize the products they already know and spend only what may be left of their budget on what the new provider offers. As a result, he cannot gain more than what is left of a clients budget.

In the strategic game the utility of a resource is shared proportional among all tasks. In the second approach, called *ordered* budget games, we also take into account the order in which the tasks arrived. Each resource has an ordering of the tasks and its utility is allocated to the tasks in that order. If a player decides to deviate to another strategy, the tasks that are allocated to different resources are moved to the last position in the ordering of those resources.

Related Work. There are several models which share similarities to budget games. Li et al. [5] developed cost-sharing mechanisms for set cover games. Every element e_i has a coverage requirement r_i, every set S_j has a cost c_j and the multiplicity of e_i in S_j is $k_{j,i}$. The multiplicity states how many times e_i is covered by S_j. The sets are chosen on the condition that e_i has to be covered at least r_i many times. The total costs are distributed between the elements such that the result is $\frac{1}{\ln(d_{\max})}$-budget-balanced and fair under core. In [6], Li et al. analyze set cover games in which the elements are the agents and declare bids for the sets. They give mechanisms which decide which elements will be covered, which sets are used and how much each element is charged.

Other games have been defined on the facility location problem [4]. In [1], Ahn et al. studied the Voronoi game in which two players alternately choose their facilities and the space they control is determined by the nearest-neighbor rule. They give a winning strategy for player 2, although player 1 can ensure that the advantage is only arbitrarily small.

Also related to our model are congestion games. Rosenthal [10] showed that they always have a pure Nash equilibrium. Milchtaich [8] extended this result to weighted congestion games with player-specific payoff functions, where the utility of player i playing strategy j is a monotonically nonincreasing function S_{ij} of the total weight of all players with the same strategy. Mavronicolas et al. [7] considered the special case of latency functions $f_{ie} = g_e \odot c_{ie}$, where g_e is the latency function of resource e, $c_{ie} > 0$ and $\odot$ is the operation of an abelian group. A characterization of the class of congestion games with pure Nash equilibria was recently given by Gairing and Klimm [3]. They showed that the player-specific cost functions of the weighted players have to be affine transformations of each other as well as be affine or exponential. These games emphasize that the impact of the same strategic choice may vary between the players.

Finally, the strategic version of our game is a basic utility game. One property of basic utility games is that the social welfare function is submodular and non-decreasing, which is used in Section 3 to approximate the optimal solution for any of our games. Vetta [11] showed that any basic utility game has a Price of Anarchy at most 2. We prove the same for the non-strategic ordered budget games.

Our Contribution. We show that computing an optimal allocation for both variants of budget games is NP-hard in general but can be approximated within a factor of $1 - 1/e$ if the strategies of the players have a matroid structure.

In standard budget games a stable solution, i. e., a pure Nash equilibrium, might not exist and deciding if one exists is NP-hard. For ordered budget games the situation is more positive. Nash and even strong equilibria exist and can be computed in polynomial time. We show that this complexity result cannot be extended to super strong equilibria as these are NP-hard to compute. Moreover, we compare the performance of equilibria to optimal solutions and show that the price of (strong) stability is 1 and the price of (strong) anarchy is 2. Concerning the convergence of repeated improvement steps we show that the dynamic that emerges if players repeatedly make improving moves converges towards a Nash equilibrium and this is even true for simultaneous moves of several players if ties are broken in a certain way. However, there are games and initial strategy profiles in which the convergence process may take exponentially long.

2 Model

A *budget game* $\mathcal{B}$ is a tuple $(\mathcal{N}, \mathcal{R}, (b_r)_{r\in\mathcal{R}}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$, where the set of players is denoted by $\mathcal{N} = \{1, \ldots, n\}$, the set of resources by $\mathcal{R} = \{1, \ldots, m\}$, and the budget of resource r by b_r. Each player i has a set of tasks $\mathcal{T}_i = \{t_1^i, \ldots, t_{q_i}^i\}$ with $t_k^i \in \mathbb{R}^m_{\geq 0}$. For a task $t \in \mathcal{T}_i$, we use $t(r)$ to denote the demand for resource r. We say a task t is *connected* to a resource r if $t(r) > 0$. If the task demands the full resource, i. e. $t(r) = b_r$, we say that t is *fully connected* to r. Now, let $\mathcal{T} = \cup_{i\in\mathcal{N}}\mathcal{T}_i$ denote the set of all tasks. A strategy of a player is a set of tasks and $\mathcal{S}_i \subseteq 2^{\mathcal{T}_i}$ denotes the set of strategies available to player i. $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ is the set of strategy profiles and $u_i : \mathcal{S} \to \mathbb{R}_{\geq 0}$ denotes the private utility function player i strives to maximize. For a strategy profile $s = (s_1, \ldots, s_n)$, let $u_{t,r}(s) : \mathcal{S} \to \mathbb{R}_{\geq 0}$ denote the utility of t from r and $u_i(s) := \sum_{t\in s_i}\sum_{r\in\mathcal{R}} u_{t,r}(s)$. We demand that the utilities are always valid, i. e. $\sum_{i\in\mathcal{N}}\sum_{t\in s_i} u_{t,r} \leq b_r$ for every $r \in \mathcal{R}$.

We consider two different utility distribution rules and call the games *standard budget games* (or simply budget games) and *ordered budget games*. In a standard budget game, the utility of task $t \in s_i$ from resource r is defined as $u_{t,r}(s) := \min\left(t(r), {}^{b_r \cdot t(r)}/\left(\sum_{j\in\mathcal{N}}\sum_{t'\in s_j} t'(r)\right)\right)$.

In an ordered budget game, the utilities do not only depend on the current strategy profile, but also on the course of the game up to this point. To that end a strategy profile is augmented by an ordering of the tasks for each resource.

Let $\prec = (\prec_e)_{r\in\mathcal{R}}$ be a vector of total orders on the set $\mathcal{T}$. The utility of a task $t \in s_i$ in $(s, \prec)$ is $u_{t,r}(s, \prec) := t(r)$ if $\sum_{j\in\mathcal{N}} \sum_{t'\in s_j \text{ with } t'\prec_r t} t'(r) \le b_r$ and $u_{t,r}(s, \prec) := \max\left(0, b_r - \sum_{j\in\mathcal{N}} \sum_{t'\in s_j \text{ with } t'\prec_r t} t'(r)\right)$ otherwise.

When player i changes its strategy from s_i to s'_i all new tasks are moved to the end of $\prec_r$ for all resources. Let $\tau = s'_i \setminus s_i$ then the new state is $((s'_i, s_{-i}), \prec')$ with $x \prec'_r y$ if and only if $x \prec_r y$ and $x \prec'_r t$ for all $x, y \in \mathcal{T} \setminus \tau$ and $t \in \tau$. Here, the order for given tasks of the same player is arbitrary, as it does not change the utility function of the specific player. For strategy changes of a coalition $C \subseteq \mathcal{N}$ of players the definition is analogous and we set $\tau = \cup_{i\in C}(s'_i \setminus s_i)$. For the ordering between tasks in τ, we show two tie-breaking rules in Section 5.

A pure Nash equilibrium (NE) is a strategy profile s in which no player has an incentive to deviate, i. e., there is no $s'_i \in \mathcal{S}_i$ such that $u_i(s'_i, s_{-i}) > u_i(s)$ for all $i \in \mathcal{N}_i$. A strong equilibrium is a profile s in which there is no coalition $C \subseteq \mathcal{N}$ which can improve, i. e., there is no $s'_C \in \times_{i\in C}\mathcal{S}_i$ such that $u_i(s'_C, u_{-C}) > u_i(s_C, s_{-C})$ for all $i \in \mathcal{N}$. For super strong equilibrium we only demand that this inequality is strict for at least one player.

For a strategy profile s, $u(s) := \sum_{i\in\mathcal{N}} u_i(s)$ is the social welfare of s. The optimal solution of $\mathcal{B}$ is the strategy profile *opt* with $u(opt) \ge u(s)$ for every $s \in \mathcal{S}$. The price of anarchy (PoA) is defined as $\max \frac{u(opt)}{u(s)}$, the price of stability (PoS) as $\min \frac{u(opt)}{u(s)}$, where s is a NE. Analogously the price of (super) strong anarchy and stability is defined with s being a (super) strong equilibrium.

3 Complexity of the Optimal Solution

For any form of budget game, the social welfare is independent of the order of the tasks. The following results hold for both standard and ordered budget games.

Theorem 1. *Computing the optimal solution for a budget game with respect to social welfare is* NP*-hard, even if the tasks and strategy sets of all players are equal and the strategies are restricted to singletons.*

Proof. We give a reduction from the maximum set coverage problem. An instance $\mathcal{I} = (\mathcal{U}, \mathcal{W}, w)$ of this problem is given by a set $\mathcal{U}$, a collection of subsets $\mathcal{W} = \{\mathcal{W}_1, \ldots, \mathcal{W}_q\}$ with $\mathcal{W}_i \subseteq \mathcal{U}$ and an integer $w \in \mathbb{N}$. The task is to cover as many elements from $\mathcal{U}$ as possible by choosing at most w sets from $\mathcal{W}$.

From $\mathcal{I}$, we create a budget game $\mathcal{B} = (\mathcal{N}, \mathcal{R}, (b_r)_{r\in\mathcal{R}}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$. We create a number of w players, that is $\mathcal{N} = \{1, \ldots, w\}$. Now, let the set of resources correspond to the set $\mathcal{U}$, i. e. $\mathcal{R} = \mathcal{U}$, and set the budget of each resource $j \in \mathcal{R}$ to $b_j = 1$. For each player i, we define the set of tasks as $\mathcal{T}_i = \{t_{\mathcal{W}_1}, \ldots, t_{\mathcal{W}_q}\}$, where the demands of a task are set to $t_{\mathcal{W}_k}(r) = 1$ for $r \in \mathcal{W}_k$ and $t_{\mathcal{W}_k}(r) = 0$ otherwise. Note that the set of tasks is equal for all players. Finally, we set the strategy space to be $\mathcal{S}_i = \{\{t_{\mathcal{W}_k}\} \mid 1 \le k \le q\}$ for all $i \in \mathcal{N}$.

Given a strategy profile s for $\mathcal{B}$, the social welfare increases by 1 for every resource r that is used by some task. This applies if and only if there is a set $\mathcal{W}_k$

with $r \in \mathcal{W}_k$ so that the chosen strategy of some player i is $s_i = \{t_{\mathcal{W}_k}\}$. Choosing strategies for all players corresponds to choosing w sets from $\{\mathcal{W}_1, \dots, \mathcal{W}_q\}$ and thus a strategy profile for $\mathcal{B}$ also describes a solution for $\mathcal{I}$ where the number of covered elements equals the social welfare of s. In addition, every solution for $\mathcal{I}$ can be transformed into a strategy profile for $\mathcal{B}$ by assigning each chosen set $\mathcal{W}_k$ to one player i by setting $s_i = \{t_{\mathcal{W}_k}\}$. Again, the social welfare and the number of covered elements are equal. Therefore, the problems of finding an optimal solution for $\mathcal{B}$ and finding an optimal solution for $\mathcal{I}$ is equivalent. □

If the sets of strategies $\mathcal{S}_i$ correspond to bases of some matroid (with the tasks as elements), the optimal solution for a budget game can be approximated up to a constant factor, since computing an optimal solution corresponds to maximization of a submodular monotone function. A function $g : 2^{\mathcal{U}} \to \mathbb{R}$ over a set $\mathcal{U}$ is submodular if $g(X \cup \{u\}) - g(X) \geq g(Y \cup \{u\}) - g(Y)$ for $X \subseteq Y, u \notin Y$ and monotone if $g(A) \leq g(B)$ for all $A \subseteq B$. For budget games, the function mapping the set of tasks chosen by the players to the social welfare has these properties. Nemhauser et al. [9] proved that greedy maximization yields an approximation factor of $1 - \frac{1}{e}$. In our case, this means always picking the task (out of all) with the highest utility next. The resulting strategies are then valid, provided the number of tasks in each is not too large. Feige [2] showed that there is no better approximation algorithm for the maximum set coverage problem unless $\mathsf{P} = \mathsf{NP}$. Therefore, we conclude the following result.

Corollary 1. *In a matroid budget game, greedy maximization of the social welfare creates a strategy profile s with $\frac{u(opt)}{u(s)} \leq 1 - \frac{1}{e}$. This bound is tight if* $\mathsf{P} \neq \mathsf{NP}$.

4 Standard Budget Games

A (standard) budget game does not always possess a NE. In addition, the question whether a given game instance has at least one NE is NP-hard.

Theorem 2. *To decide for a budget game $\mathcal{B}$ whether it has a NE is* NP*-complete.*

Proof. We only give a short sketch of the proof. See the full version of the paper for the complete proof. The problem is obviously in NP. To show that it is also NP-hard, we reduce from the exact cover by 3-sets problem. Given an instance $\mathcal{I} = (\mathcal{U}, \mathcal{W})$ consisting of a set $\mathcal{U}$ with $|\mathcal{U}| = 3m$ and a collection of subsets $\mathcal{W} = \mathcal{W}_1, \dots, \mathcal{W}_q \subseteq \mathcal{U}$ with $|\mathcal{W}_k| = 3$ for every k, the question whether $\mathcal{W}$ contains an exact cover for $\mathcal{U}$ in which every element is covered by exactly one subset is NP-hard. We create a budget game $\mathcal{B}$ with the players $\mathcal{N} = \{1, \dots, q, A, B, C, D\}$, each having two strategies $\mathcal{S}_i = \{\{t_0^i\}, \{t_1^i\}\}$. The players $1, \dots, q$ correspond to the sets in $\mathcal{W}$. Basically, our game consists of two smaller ones. The first involves the players $1, \dots, q$ and is based on $\mathcal{I}$, the other revolves around A, B and C and is mostly constant. Player D forms a connection between the two games. The fact whether $\mathcal{I}$ has a solution determines how the

NE in the first game looks like. If $\mathcal{I}$ can be solved, there is a NE which causes D to participate in the first game. This in turn is necessary for the existence of any NE in the second game and therefore for the existence in $\mathcal{B}$ as a whole. □

As a finite strategic game, every budget game has a mixed Nash equilibrium. It is also a basic utility game and from [11] we know that the price of anarchy is at most 2 for this class of games. If a budget game has a NE, this upper bound applies as well. We can get arbitrarily close to it as shown in the following example. Let $\mathcal{B}$ be a budget game with $\mathcal{N} = \{1, \ldots, n+1\}$, $\mathcal{T}_i = \{t_0^i, t_1^i\}$ for $i = 1, \ldots, n$ and $\mathcal{T}_{n+1} = \{t^{n+1}\}$. Each player may only choose a single task, i. e. $\mathcal{S}_i = \{\{t_0^i\}, \{t_1^i\}\}$ and $\mathcal{S}_{n+1} = \{\{t^{n+1}\}\}$. There are two resources $\mathcal{R} = \{r_1, r_2\}$ with $b_1 = b_2 = 1$. The demands are $t_0^i(r_1) = \frac{1}{n+1} - \varepsilon$, $t_1^i(r_2) = b$, $t^n(r_2) = b$ and 0 else. The optimal solution is the strategy profile $opt = (t_0^1, \ldots, t_0^n, t^{n+1})$ with a social welfare of $n \cdot (\frac{1}{n+1} - \varepsilon) + 1$. The only NE is $s = (t_1^1, \ldots, t_1^n, t^{n+1})$ with a social welfare of 1.

5 Ordered Budget Games

We now turn to an extension of budget games namely ordered budget games that take into account chronological aspects. Note that ordered budget games are not strategic games as the utility of a player does not only depend on the strategy profile but also on the *order* in which they made their choices. For ordered budget games, the social welfare function is a potential function. Since every strategy change by a player (or a coalition of players) does not decrease the utility of the remaining players, it is easy to observe that every improvement step by a player (or a coalition of players where every player improves his utility) increases social welfare.

Corollary 2. *The social welfare function is a potential function as every improvement step of a player increases social welfare.*

Using this insight we derive a simple method to compute a strong equilibrium.

Theorem 3. *A strong equilibrium can be computed in time $\mathcal{O}(n)$.*

Proof. A (strong) equilibrium can be computed in time $\mathcal{O}(n)$ by inserting players one after the other. In the resulting state, no player has an incentive to deviate from its strategy as long as the players which have been inserted before him play the strategy they chose when they were inserted. □

Thus, computing both Nash and strong equilibria can be done in polynomial time. However, if we consider super strong equilibria, the situation is different. We show that finding such a state is NP-hard.

Theorem 4. *Computing a super-strong equilibrium for an ordered budget game with n players is* NP*-hard, even if the number of strategies per player is constant.*

Proof. We prove the theorem via a reduction from the monotone One-In-Three 3SAT problem. Given is a set $U = \{x_1, \ldots, x_n\}$ of variables and a collection C of clauses over U with $|c| = 3$ for each $c \in C$. In this context, monotone implies that no c contains a negated literal. We therefore call the literals just variables.

We construct an ordered budget game $\mathcal{B} = (\mathcal{N}, \mathcal{R}, (b_r)_{r\in\mathcal{R}}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ from the sets U and C. Every variable $x_i \in U$ defines a player $i \in \mathcal{N}$ with $\mathcal{T}_i = \{0_i, 1_i\}$. Every clause $c_j \in C$ defines two resources $r_{j,0}, r_{j,1} \in \mathcal{R}$ with $b_{j,0} = 2$ and $b_{j,1} = 1$. $\mathcal{S}_i = \mathcal{T}_i$ for every player i. The demands are defined as

$$0_i(r_{j,0}) = \begin{cases} 1, & \text{if } x_i \in c_j \\ 0, & \text{else} \end{cases} \qquad 1_i(r_{j,1}) = \begin{cases} 1, & \text{if } x_i \in c_j \\ 0, & \text{else} \end{cases}$$

Set the remaining demands $0_i(r_{j,1})$ and $1_i(r_{j,0})$ to 0. Let k_i be the number of clauses the variable x_i occurs in. Then each task of i has a demand of 1 on k_i many resources and a demand of 0 on all others. The highest utility the player i can obtain is also k_i. If there is a satisfying truth assignment ϕ for C, then each player can obtain this individual maximum. If $\phi(x_i) = 0$, let player i choose strategy 0_i, otherwise 1_i. ϕ has to one-in-three property, which means that in each clause, only one variable is set to 1. Thus, every resource $r_{j,1}$ is covered by exactly one task 1_i and every resource $r_{j,0}$ by exactly two tasks 0_{i_1} and 0_{i_2}. No resource experiences a demand higher than its budget, therefore the order of the tasks is not important here. In this case, the social welfare achieves a value of $\sum_{i\in\mathcal{N}} k_i$. If there exists a strategy profile with this social welfare in $\mathcal{B}$, then it induces in turn a satisfying truth assignment ϕ for C. Note that if such a strategy profile exists, it is also the only super-strong equilibrium of the game. In each other state, all players can form a coalition to collectively assume this strategy profile without reducing their utility. Therefore, computing a super-strong equilibrium for $\mathcal{B}$ determines whether C can be satisfied or not. □

Since the optimal solution of an ordered budget game is a NE and even a super-strong equilibrium, we obtain the following bound on the price of (super strong) stability.

Corollary 3. *The price of (super strong) stability of ordered budget games is* 1.

For the price of anarchy we obtain the following, nearly tight bound.

Theorem 5. *For every ordered budget game, the price of anarchy is at most* 2. *For every* $\varepsilon > 0$*, there exists an ordered budget game with* $PoA = 2 - \varepsilon$.

Proof. We begin by upper bounding the price of anarchy of an ordered budget game $\mathcal{B}$. Let $(s, \prec)$ be a NE of $\mathcal{B}$ and s^* be the strategy profile with the maximal social welfare. Note that the ordering of the players is irrelevant for the social welfare. To simplify notation we will use s and (s_{-i}, s_i^*) as a shorthand for $(s, \prec)$ and $((s_{-i}, s_i^*), \prec')$ with $\prec'$ the new ordering as defined in Section 2. We can lower bound the social welfare of a NE s as follows.

$$\sum_{i\in\mathcal{N}} u_i(s) = \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\sum_{t\in s_i} u_{t,r}(s)$$

$$\geq \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\sum_{t\in s_i^*} u_{t,r}(s_{-i}, s_i^*) \tag{1}$$

$$\geq \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\sum_{t\in s_i^*} \min\left(t(r), b_r - \sum_{i'\neq i}\sum_{t'\in s_{i'}} u_{t',r}(s)\right) \tag{2}$$

$$\geq \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\sum_{t\in s_i^*} \min\left(u_{t,r}(s^*), b_r - \sum_{i'\neq i}\sum_{t'\in s_{i'}} u_{t',r}(s)\right)$$

$$\geq \sum_{r\in\mathcal{R}_1}\sum_{i\in\mathcal{N}}\sum_{t\in s_i^*} \min\left(u_{t,r}(s^*), b_r - \sum_{i'\neq i}\sum_{t'\in s_{i'}} u_{t',r}(s)\right) + \sum_{r\in\mathcal{R}_2}\sum_{i\in\mathcal{N}}\sum_{t\in s_i^*} u_{t,r}(s^*) \tag{3}$$

$$\geq \sum_{r\in\mathcal{R}_1}\left(b_r - \sum_{i'\in\mathcal{N}}\sum_{t'\in s_{i'}} u_{t',r}(s)\right) + \sum_{r\in\mathcal{R}_2}\sum_{i\in\mathcal{N}}\sum_{t\in s_i^*} u_{t,r}(s^*)$$

$$\geq \sum_{r\in\mathcal{R}_1}\left(\sum_{i\in N}\sum_{t\in s_i^*} u_{t,r}(s^*) - \sum_{i\in\mathcal{N}}\sum_{t'\in s_i} u_{t',r}(s)\right) + \sum_{r\in\mathcal{R}_2}\sum_{i\in\mathcal{N}}\sum_{t\in s_i^*} u_{t,r}(s^*)$$

$$\geq \sum_{r\in\mathcal{R}}\sum_{i\in N}\sum_{t\in s_i^*} u_{t,r}(s^*) - \sum_{r\in\mathcal{R}_1}\sum_{i\in\mathcal{N}}\sum_{t'\in s_i} u_{t',r}(s)$$

$$\geq \sum_{i\in\mathcal{N}} u_i(s^*) - \sum_{r\in\mathcal{R}}\sum_{i\in\mathcal{N}}\sum_{t'\in s_i} u_{t',r}(s)$$

$$\geq \sum_{i\in\mathcal{N}} u_i(s^*) - \sum_{i\in\mathcal{N}} u_i(s) \tag{4}$$

Observe that (1) follows from the Nash inequality and (2) from the definition of the utility functions. In (3) we partition $\mathcal{R}$ into $\mathcal{R}_1$ and $\mathcal{R}_2$ where $\mathcal{R}_1$ contains all resources with at least one task that evaluates the min statement to the second expression. That is there is a $i \in \mathcal{N}$ and a $t \in s_i^*$ with $u_{t,r}(s^*) > b_r - \sum_{i'\neq i}\sum_{t'\in s_{i'}} u_{t',r}(s)$. Adding $\sum_{i\in\mathcal{N}} u_i(s)$ to both sides at (4) shows that the price of anarchy is bounded by 2.

For a lower bound, consider the game $\mathcal{B} = (\mathcal{N}, \mathcal{R}, (b_r)_{r\in\mathcal{R}}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$ with $\mathcal{N} = \{1, 2\}$ with $\mathcal{T}_1 = \{t_1^1, t_2^1\}$ and $\mathcal{T}_2 = \{t^2\}$, $\mathcal{R} = \{r_1, r_2\}$ with $b_1 = b$ and $b_2 = b(1-\varepsilon)$. Set the demands to $t_1^1(r_1) = b$, $t_2^1(r_2) = b(1-\varepsilon)$, $t^2(r_2) = b$ and all others to 0. Set $\mathcal{S}_1 = \{\{t_1^1,\}, \{t_2^1\}\}$ and $\mathcal{S}_1 = \{\{t^2\}\}$. In the optimal solution, the social welfare is $u_1(opt) + u_2(opt) = b - b\cdot\varepsilon + b = 2b - b\cdot\varepsilon$. If player 1 is inserted first, his best response is to open task t_1^1. This leads to a NE in which the utility of player 2 is 0 and the price of anarchy $2-\varepsilon$. $\mathcal{B}$ can be extended to n players by using $\frac{n}{2}$ instances of the two-player version. □

In contrast to the fact that one can easily construct an equilibrium in n steps by inserting players one after the other, the situation is different when starting in an arbitrary situation. We now study the dynamic that emerges if players repeatedly perform strategy changes that improve their utilities. This may also lead to situations in which a resource is simultaneously newly allocated by two tasks of different players which necessitates the existence of a tie-breaking rule. We introduce two tie-breaking rules which guarantee that the game still converges towards an equilibrium. For an ordered budget game $\mathcal{B}$, let $p : \mathcal{N} \to \mathbb{N}$ be an injective function that assigns a unique priority to every player i. Whenever simultaneous strategy changes occur, they are executed sequentially, in decreasing order of the priorities of the players involved. This corresponds to setting $t_1 \prec_r t_2$ for all resources r and all pairs of tasks where the priority of the player with t_1 was higher than the priority of the player with t_2. For p_{fix}, the priorities are fixed. For $p_{\max}$, they change over time, with $p_{\max}(i_0) > p_{\max}(i_1)$ if $u_{i_0}(s) > u_{i_1}(s)$ for the current strategy profile s. Any ties may be broken arbitrarily.

Theorem 6. *Let $\mathcal{B}$ be an ordered budget game which allows multiple simultaneous strategy changes. If $\mathcal{B}$ uses either p_{fix} or p_{max} to set the priorities of the players, then it reaches a NE after finitely many improvement steps.*

Proof. Let s be the current strategy profile $\mathcal{B}$ and $\overrightarrow{u}(s) \in \mathbb{R}^n_{\geq 0}$ the vector containing the current utilities of all players. We call $\overrightarrow{u}(s)$ the utility vector of $\mathcal{B}$ under s. We always sort $\overrightarrow{u}(s)$ in decreasing order of the player priorities, i. e. the player at position i has a higher priority than the player at position $i+1$. For $p_{\max}$, this order may change over time. Let $N \subseteq \mathcal{N}$ be the set of players who are simultaneously performing a strategy change. Each player would improve his utility if he were the only player in N. Let s' be the resulting strategy profile. Note that $\overrightarrow{u}(s) <_{\text{lex}} \overrightarrow{u}(s')$ for both priority functions, where $<_{\text{lex}}$ is the lexicographical order. Let $i \in N$ be the player with the highest priority among those in N. For p_{fix}, i receives exactly the utility increase he expected from the strategy change. From all the players in N, he is also the one with the smallest index in both $\overrightarrow{u}(s)$ and $\overrightarrow{u}(s')$. This alone warrants that $\overrightarrow{u}(s) <_{\text{lex}} \overrightarrow{u}(s')$. For $p_{\max}$, the same argumentation holds if the position of i in the utility vectors does not change. Otherwise, his index in $\overrightarrow{u}(s)$ is now occupied by a player i' with $u_i(s) < u_i(s') < u_{i'}(s')$. Again, we have $\overrightarrow{u}(s) <_{\text{lex}} \overrightarrow{u}(s')$. Since the utility vectors are strongly monotonely increasing, but bounded by the vectors containing the maximal utility of each player, a NE is reached after finitely many steps. □

For the following, we assume that p_{fix} is used as tie-breaking rule and that the priority of a player corresponds to her index. We show that the number of improvement steps towards an equilibrium may be exponential in the number of players, even if the number of strategies per player is constant.

Theorem 7. *For any n, there is an ordered budget game $\mathcal{B}_n$ with polynomial description length in n and a strategy profile s_0 so that the number of best-response improvement steps from s_0 to any NE s of $\mathcal{B}_n$ is exponential in n.*

Proof. We give a recursive construction of the game $\mathcal{B}_n$. $\mathcal{B}_n$ contains the game $\mathcal{B}_{n-1}$, for which there is exactly one path of best-response improvement steps of length $\mathcal{O}(2^{n-1})$. $\mathcal{B}_{n-1}$ is executed once. Then it is reset to its original state and executed once more along the same path. In the end, $\mathcal{B}_n$ has reached a NE after $\mathcal{O}(2^n)$ steps. Each player has only two tasks and as a strategy, she can choose one of them, i. e. $\mathcal{S}_i = \{\{t_1^i\}, \{t_2^i\}\}$. Labeling the strategies of player i with 0_i and 1_i, each strategy profile can be written as a binary number. The initial strategy profile s_0 can be regarded as 0 and the first execution of $\mathcal{B}_{n-1}$ counts up to $2^{n-1}-1$. The reset of $\mathcal{B}_{n-1}$ corresponds to increasing that value by 1 to 2^{n-1} and the second iteration of $\mathcal{B}_{n-1}$ continues counting up to $2^n - 1$. In the final state of $\mathcal{B}_n$, every player i plays strategy 1_i. Since the strategies contain only single tasks, the ordering of these tasks on the resources is also an ordering of the players and we can abuse notation and say $i_1 \prec i_2$ for players i_1 and i_2 if $t_1 \prec_r t_2$ holds for any pair of tasks $t_1 \in \mathcal{T}_{i_1}$ and $t_2 \in \mathcal{T}_{i_2}$ and any resource r.

In the following construction, for any pair of task t and resource r, t is either fully connected to r or not connected to r at all. Thus, $t(r)$ is either b_r or 0. In the following, *connecting* a task t to a resource r means setting $t(r) := b_r$.

We need a few new notations for our proof. The only NE that is reached in our construction is the state where every player i plays strategy 1_i and in which the players reach their final state in descending order, i. e. $i_1 \prec i_2$ for $i_1 > i_2$. Let p_n^i be the utility of player i in that NE for $\mathcal{B}_n$. For the ordered budget game $\mathcal{B}_n$, let s_0^n be the initial strategy profile in which it is started. Let $i_1 \prec i_2$ for $i_1 > i_2$ in s_0^n. Intuitively, this means that players with a higher index get prioritized.

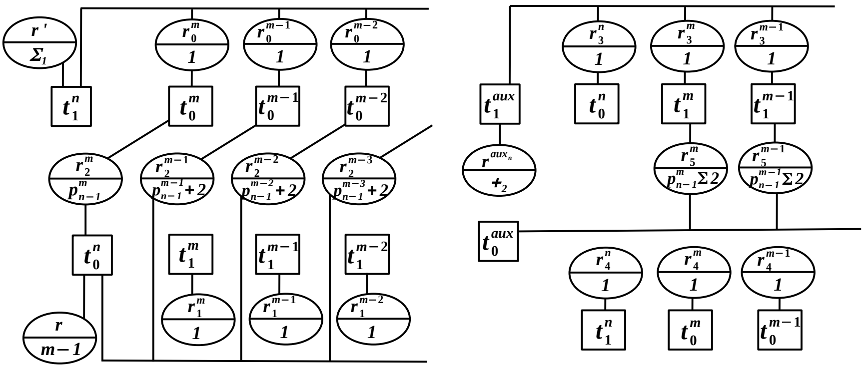
For $n = 1$, we build an instance $\mathcal{B}_1$ with a single player: $\mathcal{N} = \{1\}$ with tasks $\mathcal{T}_1 = \{t_1^1, t_2^1\}$, strategy space $\mathcal{S}_1 = \{\{t_0^1\}, \{t_1^1\}\}$ and two resources $\mathcal{R} = \{r_1, r_2\}$ with $b_1 = 1$, $b_2 = 2$. We connect t_1^1 to r_1 and t_2^1 to r_2. The initial strategy profile is $s_0^1 = (\{t_1^1\})$ and after one improvement step, $\mathcal{B}_1$ is in an equilibrium.

For $n > 1$, we extend the game $\mathcal{B}_{n-1}$. Let m denote the number of players in $\mathcal{B}_{n-1}$. We split the rest of the proof in two parts. First, we explore how to reset $\mathcal{B}_{n-1}$ to s_0^{n-1}. The structure is sketched in Figure 1(a). We introduce a new player n with $\mathcal{T}_n = \{t_0^n, t_1^n\}$ and $\mathcal{S}_n = \{\{t_0^n\}, \{t_1^n\}\}$. The initial strategy of each i is $\{t_0^i\}$. We now add several new resources.

Resource	$r_0^1, \ldots, r_0^m$	$r_1^1, \ldots, r_1^m$	$r_2^i, i = 1, \ldots, m$	r	r'
Budget	1	1	$p_{n-1}^i + 2$	$m - 1$	$\sum_{i=1}^m p_{n-1}^i + 2m$

For $i = 1, \ldots, m$, we connect r_0^i to task t_0^i and r_1^i to task t_1^i. Since all budgets are 1, this does not influence the game $\mathcal{B}_{n-1}$. We connect all r_0^i to t_1^n and r to t_0^n. Now, the initial utility of player n is $u_n(s_0^n) = m - 1$ and when all other players i play strategy $\{t_1^i\}$, player n can improve her utility by 1 by switching to $\{t_1^n\}$.

It remains to extend the current game such that once player n has switched to t_1^n, the remaining players $m, m-1, \ldots, 1$ also switch their strategy to recreate s_0^{n-1}. For $i = 1, \ldots, m$, connect each resource r_2^i to t_0^n and t_0^i. This increases the utility of t_0^n by $\sum_{i=1}^m p_{n-1}^i + 2m$. As a compensation, we connect the resource r' to t_1^n. When n switches to t_1^n, all the budgets of the resources r_2^i become available again. This will cause the players $1, \ldots, m$ to change their strategies to t_0^i. Before switching, the utility of player i is $p_{n-1}^i + 1$ due to the connection between r_1^i



(a) extension of $\mathcal{B}_{n-1}$ that is necessary to reset $\mathcal{B}_{n-1}$

(b) extension of $\mathcal{B}_{n-1}$ that is necessary to restart $\mathcal{B}_{n-1}$

Fig. 1. Construction of the ordered budget game $\mathcal{B}_n$. Figure 1(a) shows the extension of $\mathcal{B}_{n-1}$ necessary to reset $\mathcal{B}_{n-1}$. Once all players i are playing $\{t_1^i\}$, the player n changes its strategy, creating for all others the incentive to go back to $\{t_0^i\}$. Here, we set $\Sigma_1 = \sum_{i=1}^m p_{n-1}^i + 2m$. Figure 1(b) shows the extensions of $\mathcal{B}_{n-1}$ which restart $\mathcal{B}_{n-1}$. The principle is the same, this time the player *aux* is used to *create* the new budget available. We set $\Sigma_2 = \sum_{i=1}^m (p_{n-1}^i) + m$.

and t_1^i. Switching the strategy improves this value by at least 1. By definition of strategy changes of a coalition with p_{fix}, we have $i_1 \prec i_2$ for all $i_1 > i_2$ and thus the resulting strategy profile is identical to the initial one for players $0, \dots, m$.

To restart the game $\mathcal{B}_{n-1}$, we apply a similar trick as before. The construction is sketched in Figure 1(b). We introduce an auxiliary player aux_n with $\mathcal{T}_{aux_n} = \{t_0^{aux_n}, t_1^{aux_n}\}$, $\mathcal{S}_{aux_n} = \{\{t_0^{aux_n}\}, \{t_1^{aux_n}\}\}$ and the following resources.

Resource	$r_3^1, \dots, r_3^m, r_3^n$	$r_4^1, \dots, r_4^m, r_4^n$	$r_5^i, i = 1, \dots, m$	r^{aux_n}
Budget	1	1	$p_{n-1}^i + 2$	$\sum_{i=1}^m (p_{n-1}^i) + m$

In s_0^n, we set $aux_n \prec n$ and initially, her strategy is $\{t_0^{aux_n}\}$. We connect t_0^{aux} to r_5^i for all $i \in \{1, \dots, m\}$. Now, this auxiliary player starts the game with a utility of $\sum_{i=1}^m p_{n-1}^i + 2m$. We also connect t_1^{aux} to r^{aux_n} and to all resources r_3^i for $i \in \{1, \dots, m, n\}$. Finally, for every player $i = 1, \dots, m$, we connect r_4^i to t_0^i, r_4^i to t_1^i and r_5^i to t_1^i. For player n, we establish these connections the other way around, such that r_3^n is connected to t_0^n and r_4^n to t_1^n.

Again, the effects of r_3^i and r_4^i regarding $\mathcal{B}_{n-1}$ cancel out. Only when every player i in $\mathcal{B}_{n-1}$ plays strategy $\{t_0^i\}$ and player n plays strategy $\{t_1^n\}$, the auxiliary player will change to t_1^{aux} and obtain a utility of $\sum_{i=1}^m (p_{n-1}^i) + 2m + 1$. This frees the budget of all resources r_5^i and the utility of every task t_1^i in $\mathcal{B}_{n-1}$ is increased by the same amount we increased the utility of t_0^i in the first part of the construction. The game $\mathcal{B}_{n-1}$ is executed once more, only the player n remains idle. When all players i are playing strategy $\{t_1^i\}$, $\mathcal{B}_n$ has reached a NE.

Thus, together with the auxiliary players we get a total of $2 \cdot n - 1$ players in $\mathcal{B}_n$. and the number of steps to reach the NE is at least $2^n - 1$. At the same time, the number of tasks and resources is polynomial in n. □

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Influence Maximization in Switching-Selection Threshold Models

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Abstract. We study influence maximization problems over social networks, in the presence of competition. Our focus is on diffusion processes within the family of threshold models. Motivated by the general lack of positive results establishing monotonicity and submodularity of the influence function for threshold models, we introduce a general class of switching-selection threshold models where the switching and selection functions may also depend on the node activation history. This extension allows us to establish monotonicity and submodularity when (i) the switching function is linear and depends on the influence by all active neighbors, and (ii) the selection function is linear and depends on the influence by the nodes activated only in the last step. This implies a $(1 - 1/e - \varepsilon)$-approximation for the influence maximization problem in the competitive setting. On the negative side, we present a collection of counterexamples establishing that the restrictions above are essentially necessary. Moreover, we show that switching-selection threshold games with properties (i) and (ii) are valid utility games, and thus their Price of Anarchy is at most 2.

1 Introduction

A large part of recent research on social networks concerns the design of marketing strategies for advertising new products over a network. The focus of these efforts is on exploiting viral effects for the spread of new ideas and technologies among networks of friends, colleagues, relatives or other circles. The algorithmic question that naturally arises under such diffusion processes is then the following: find a subset of "most influential" nodes to target (i.e., advertise the new product to or even give it for free), so as to maximize the expected number of product adoptions, subject to a budget constraint.

This problem was initially formalized and studied by Domingos and Richardson [3] and by Kempe et al. [9], who focused on two of the most popular families of stochastic diffusion processes, namely the so-called *threshold* models [6,12] and *cascade* models [4]. Finding the optimal set of influential nodes under this framework is an NP-hard problem, and the work of [9] proposed an approximation algorithm, achieving a guarantee of $1 - 1/e$. The algorithm is based on the observation that the function quantifying the total influence of a set of early adopters is a monotone and submodular function, and thus, the classical greedy approach for maximizing such set functions applies [11].

The models above however, do not take into account the presence of multiple competing products in a market. In real networks, customers (i.e., nodes) end up choosing

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a product among various alternatives. To take the simplest possible scenario, suppose there are two firms, R and B (standing for the red and blue product respectively), trying to promote their product over a social network. A convenient way to model the process now is by viewing this setting as a 2-player game, with the strategy space being the subsets of nodes that can be targeted subject to each firm's budget constraint.

Within this game-theoretic framework, interesting research questions arise. First, one can have a natural extension of the problem studied in [9] for a single product, as follows: Given a strategy of firm B, find the best subset of nodes for firm R, so as to maximize the expected number of product adoptions in her favor. In other words, find an algorithm to compute the best response of a player to a strategy of her competitor. At first sight, it may appear that the problem under competition may not differ significantly from that without competition. For certain cascade models, this is indeed the case, see e.g. [1]. Interestingly enough however, this does not hold for threshold models. In [2], several extensions of the threshold model were presented where the best response function is nonmonotone and/or nonsubmodular and the techniques used in [9] cannot be employed to obtain a good approximation. It is still a major open problem in the area to understand for which diffusion models, one can compute (near) optimal strategies efficiently. Moreover, apart from best responses, another direction is to study further the properties of Nash equilibria of the game and quantify their performance, as was done recently in [7,5]. For example, one can study the Price of Anarchy of such games, or other criteria, such as the Budget Multiplier, introduced in [5].

Our Contribution: Motivated by the lack of positive results establishing monotonicity and submodularity of the influence function in competitive threshold models, we embark on a more systematic study of this question. On the conceptual side, we introduce a fairly general class of threshold models that belong to the family of *switching-selection* models. Under these models, a node first makes a decision on whether to adopt some product (i.e., whether to switch to being activated) and then makes a separate decision on which product to adopt (selection process). These two steps are determined by a *switching* and a *selection* function. Our class is essentially a threshold version of the models studied recently in [5] and [7], generalizing at the same time some of their aspects. In particular, we do not restrict the switching and selection functions to depend only on the set of currently active neighbors. Instead, we let them depend on the whole activation history, i.e., on the sets of active nodes at every time step. This extension allows for a careful investigation of properties that lead to a monotone and submodular influence function, and we obtain both positive and negative results under this class.

On the positive side, our main technical contribution is a set of conditions on the switching and selection functions that lead to monotonicity and submodularity and thus, enable us to obtain an $(1-1/e-\varepsilon)$-approximation for the influence maximization problem in the competitive setting, for any $\epsilon > 0$. Specifically, our main result (Section 3) is that the best response of a switching-selection threshold model is monotone and submodular if (i) the switching function is linear and depends on the weight of all active neighbors of a node, and (ii) the selection function is linear and depends on the weight of the nodes activated in the last step (i.e., the most recent buyers are the ones to influence the actual product selection). For the proof, we first establish the equivalence of such models with a generalization of the "live edges" approach [9], applicable to this particular setting, and then we develop quite delicate coupling arguments for establishing monotonicity and submodularity. Moreover, we conjecture that our positive results extend to the case where the switching function is any nondecreasing concave function of the weight of all active neighbors (see the discussion at the end of Section 3).

On the negative side, we present (Section 4) a comprehensive collection of counterexamples establishing that the restrictions above are essentially necessary. Regarding the switching process, we present examples showing that the influence function may not be monotone and submodular if the switching function is either not monotone or not concave, or it allows for the influence to decrease over time. For the selection process, we have analogous counterexamples when the selection function depends on the weight of neighbors activated in steps before the last one, or when it deviates from linearity.

Finally, we also study the performance of Nash equilibria of the underlying game, motivated by the properties established for the models in [5,7]. We show (Section 5) that switching-selection threshold games with the properties identified above are valid utility games, and thus their Price of Anarchy is at most 2.

2 The Model

In this section, we define the class of *Switching-Selection Threshold Models*. This is essentially a "threshold" version of the *Switching-Selection Model* introduced in [5], generalizing at the same time some of its aspects, as we clarify later within this section.

Social Networks. We model a social network by a directed graph $G(V, E)$, $|V| = n$. Each edge (u, v) has a weight $w_{uv} \in [0, 1]$, specifying the degree of influence of node u towards node v. For any node v, we denote by $N(v)$ the set of in-neighbors of v, and we require that the sum of the weights of the edges towards v is no more than 1: $\sum_{u \in N(v)} w_{uv} \leq 1$.

We consider a 2-player game between two competing firms that try to promote their product over the network (in fact our results generalize to games with more players, as we state later on, but for simplicity the presentation in Section 3 is for 2 players). We denote the two players by R and B standing for the red and blue product respectively. Each player $p \in \{R, B\}$ has a budget $K_p \in \mathbb{N}_+$, which they will use to target selected nodes in the network. The decision that the firms need to make is to choose how to disperse their budget to the n nodes, hence the strategy space for each firm p consists of all vectors (i.e., multisets) in the form $a_p = (a_{1p}, a_{2p}, \ldots, a_{np})$, where $a_{jp} \in \mathbb{N}$ and $\sum_{j=1}^{n} a_{jp} \leq K_p$.

Once the firms make a choice, the spread of the two products is modeled by a stochastic diffusion process that takes as input the strategies of the 2 firms, a_R, a_B. We describe next a family of such processes that we are interested in.

Switching-Selection Diffusion Processes. The process that determines the eventual adoptions, takes place in discrete steps. The state $s_{ut} \in \{\mathit{R(ed)}, \mathit{B(lue)}, \mathit{U(ncolored)}\}$, of node u, denotes whether node u has adopted a product at step t and, if yes, which product it adopted. As with the majority of the literature, we assume that the process is progressive, i.e., once a node is colored, it never changes its state afterwards. The process evolves as follows:

- At time step $t = 0$, the initialization takes place. For every node u:
  - A threshold θ_u is selected uniformly at random in $[0, 1]$.
  - Given the strategies a_R, a_B of the 2 firms, if $(a_{uR} = 0 \wedge a_{uB} = 0)$ then $s_{u0} = U$.
  - Otherwise $s_{u0} = R$ with probability $\frac{a_{uR}}{a_{uR}+a_{uB}}$ and $s_{u0} = B$ w.p. $\frac{a_{uB}}{a_{uR}+a_{uB}}$
- At any time step $t > 0$, each uncolored node u decides:
  1. whether to adopt some product based on the decisions of its neighbors up until step $t-1$, on its threshold, θ_u, and on a *switching function* f, described below.
  2. which product to adopt, in case that it decided to adopt some product. The choice of product is determined by a *selection function* g, also described below.

  Clearly, the process can last for at most $n - 1$ steps. We allow for fairly general functions f and g. In particular, let R_t (respectively B_t) denote the set of red (blue) nodes at step t and let $A_t = R_t \cup B_t$. Similarly, let $W_{ut}(R)$ (respectively $W_{ut}(B)$) denote the total weight of the edges (v, u) such that $v \in R_t$ (resp. B_t). Let also $W_{ut} = W_{ut}(R) + W_{ut}(B)$.
  1. The switching function applied at step t to node u, takes as argument the vector $\boldsymbol{C}_{ut} = (W_{u0},\dots, W_{u,t-1})$, i.e., the whole history of how the cumulative weight of active neighbors has evolved in the previous steps. Node u switches from uncolored to colored at time t if
  $$f(\boldsymbol{C}_{ut}) \geq \theta_u$$
  2. The selection function takes as arguments the vectors $\boldsymbol{C}_{ut}(R) = (W_{u0}(R),\dots, W_{u(t-1)}(R))$, and $\boldsymbol{C}_{ut}(B) = (W_{u0}(B), \dots, W_{u,t-1}(B))$, i.e., the histories for the total weight of red and blue neighbors in the previous steps. Then with probability
  $$g(\boldsymbol{C}_{ut}(R), \boldsymbol{C}_{ut}(B)),$$
  node u selects the red product and $s_{ut} = R$. Else $s_{ut} = B$.

Note that the model can be easily extended to the case of $k > 2$ players.

Comparisons with related models: The model encompasses some families that have already been described before. For example, for linear f and g, and with $f(\boldsymbol{C}_{ut}) := f(W_{u,t-1})$, and $g(\boldsymbol{C}_{ut}(R), \boldsymbol{C}_{ut}(B)) := g(\frac{W_{u,t-1}(R)}{W_{u,t-1}})$, we have the *Weight-Proportional Competitive Linear Threshold Model* studied in [2].

Our model can be viewed as a threshold version of the models studied in [7,5]. We allow more general switching and selection functions, in the sense that these functions can depend on how the total weight evolves over time. In [7,5], these functions depend only on the active nodes at step $t-1$, when applied for step t. Finally, another technical difference is that we do not have any update schedule determining the order of updates. Instead, we consider that at each step any node that can switch to a colored state will do so by taking into account what has happened up until time $t - 1$.

Best Response Computation. As with other competitive diffusion models, such as [1,2], our primary focus is on the problem of computing the best strategy for a firm, given its opponent's strategy. Suppose we take the viewpoint of the Red firm. Given strategies a_R, a_B, we let $\sigma(a_R, a_B)$ denote the expected number of red nodes at the end of the diffusion process. The expectation here is over both the tie-breaking rule in the initialization phase and over the probabilistic choice of thresholds. We take this as the utility function of the red firm under this game.The problem we are interested then is:

The Influence Maximization Problem: Given a diffusion process, specifying the functions f and g, and given the strategy of the blue firm, a_B, find a strategy a_R for the red firm so as to maximize $\sigma(a_R, a_B)$.

3 Dependence of Selection Function only on New Influencers

In this section, we will focus on the case where

– The switching function f depends only on the aggregate weight of all the colored neighbors, up until the previous step. Hence, to check if a node u becomes colored at step t, we check if $f(W_{u,t-1}) \geq \theta_u$.
– The selection function depends on the set of nodes that became active exactly at the previous step of the process. In particular, at step t, the function g depends on the aggregate weights of colored nodes at the previous 2 steps, in the form $g := g(\frac{W_{u,t-1}(R)-W_{u,t-2}(R)}{W_{u,t-1}-W_{u,t-2}})$, and we also require that g is a linear function.

To see the motivation behind these types of switching and selection functions, one can think of the competition between two smartphones. The choice of the switching function is quite natural, and follows the recent works in the literature. E.g., the decision on whether to buy a smartphone or not, is affected by the set of all neighbors who have already bought one, regardless of which of the two products they have chosen. As for the selection function, the rationale is that a node may be more heavily influenced by the most recent buyers, i.e., the nodes that became active at the previous step in our model. If in the recent past more people made a choice towards one of the two products, then the node will have a higher probability to select the same product as well.

As we will see in Section 4, significant deviations from these assumptions make the algorithmic considerations that we are interested in more challenging.

Linear Switching Functions. Our positive results concern the case where f is a linear function. In fact, we can assume WLOG that f and g are the identity function. We will refer to this as the LSMSTM model (*Linear Switching-Marginal Selection Threshold Model*). We conclude this section with a discussion regarding non-linear switching functions.

From now on, fix a strategy of the blue firm, say $a_B = (a_{1B}, a_{2B}, \ldots, a_{nB})$. We want to find a strategy $a_R = (a_{1R}, a_{2R}, \ldots, a_{nR})$ so as to maximize $\sigma(a_R, a_B)$. We will provide an approximation algorithm to this problem by using the standard tools of optimizing monotone and submodular functions.

Definition 1. *Consider a function $h : \mathbb{Z}^n \to \mathbb{R}$. Let $x, y \in \mathbb{Z}^n$ be two vectors with $x_j \leq y_j$ for every $j = 1, ..., n$. Let also $e_j \in \mathbb{R}^n$ be the unit vector with $e_j(j) = 1$ and $e_j(k) = 0$, for $k \neq j$. We will say that h is*

– *monotone, if $h(x) \leq h(y)$,*
– *submodular, if $h(x + e_j) - h(x) \geq h(y + e_j) - h(y)$ for $j = 1, ..., n$.*

Note that this is a generalization of the standard definition of submodularity, to the case of functions defined over multisets rather than sets, as defined also in [8].

We are interested in the expected number of red nodes at the end of the diffusion process as a function of the red firm's strategy, i.e., $\sigma(a_R, a_B)$ viewed as a function of a_R only. Our main result is the following:

Theorem 1. *Under the LSMSTM model, and for any given strategy a_B of the blue firm, the function $\sigma(a_R, a_B)$ is monotone and submodular.*

In order to use the machinery of [11] or [8] (for multisets), we also need to be able to compute the expectation $\sigma(a_R, a_B)$, for any strategies a_R, a_B. We can use sampling methods to approximate this value within any accuracy and as explained in [9], this suffices for the greedy algorithm of [11]. This implies the following corollary:

Corollary 1. *Under the LSMSTM model, and for any $\epsilon > 0$, there is a $(1 - 1/e - \epsilon)$-approximation algorithm for computing the best response of any player against her competitor.*

To prove Theorem 1, we will begin by showing that LSMSTM is equivalent to another model, which we will refer to as *Single Incoming Edge Analog* (SIEA). This is in a similar spirit as the approach via "live edges" in [9].

Definition 2. *(SIEA) Under this stochastic process, given a_R, a_B, the initialization phase is exactly the same as in LSMSTM. Then, for each node u we preserve at most one incoming edge. Node u selects the edge $e = (v, u)$ with probability $w_{v,u}$ and no edge w.p. $1 - \sum_{v \in N(u)} w_{v,u}$. We refer to the selected edges as live edges. Afterwards the contagion process works deterministically. At step $t = 1$, any node that has an incoming live edge from a colored neighbor, obtains the color of its neighbor. Continuing in this manner, at step t, any node that has an incoming edge from a colored node, becomes colored with the color of that node.*

A crucial observation is the following:

Lemma 1. *Given a pair of strategies a_R, a_B, the distributions over red-colored sets and blue-colored sets derived from running LSMSTM are the same as the distributions produced by SIEA.*

The proof of Lemma 1 is based on similar techniques as the proof of Claim 2.6 in [9]. From now on and till the end of the proof of Theorem 1, we will work only with the SIEA model. We first prove monotonicity[1].

Lemma 2. *Let $a_R, a'_R \in \mathbb{Z}^n$ such that $a_R \leq a'_R$. Under SIEA, and for any a_B, $\sigma(a_R, a_B) \leq \sigma(a'_R, a_B)$.*

Proof. Consider 2 SIEA processes, π_1 and π_2 with $a_R^{\pi_1} = a_R$, $a_R^{\pi_2} = a'_R$, and $a_B^{\pi_1} = a_B^{\pi_2} = a_B$. We will prove that the expected number of red nodes at π_2 is at least as high as that in π_1.

[1] Note that in the case of a single product, monotonicity is trivial. This is not always the case in threshold models with at least two competing products. See e.g. [2] for some examples.

We define a coupling between π_1 and π_2, and prove the lemma using induction on the number of steps. We consider the following coupled processes, which by slight abuse of notation, we will keep denoting by π_1 and π_2: We first pick randomly the set of live edges, as described in the SIEA model, which we take to be the same for both processes. At step $t = 0$, for every node u, where $a_{uR}^{\pi_1} + a_{uB} > 0$, we pick a number uniformly at random in $[0, 1]$ and we decide on the color of u at each process, based on the following 3 intervals of $[0, 1]$.

- with probability $\frac{a_{uB}}{a_{uR}^{\pi_2}+a_{uB}}$, we color u blue in both processes.
- with probability $\frac{a_{uB}}{a_{uR}^{\pi_1}+a_{uB}} - \frac{a_{uB}}{a_{uR}^{\pi_2}+a_{uB}}$, we color u blue in π_1 and red in π_2.
- with probability $1 - \frac{a_{uB}}{a_{uR}^{\pi_1}+a_{uB}}$, we color u red in both processes.

Any other node can be colored with no ambiguity in π_1 and π_2 or remain uncolored in one or both of the processes (e.g., if $a_{uR}^{\pi_2} = a_{uB} = 0$). The next steps in both processes continue as in the original SIEA processes (but note that both processes will use the same set of live edges).

It is quite straightforward to see that this is a valid coupling, since it produces the same distribution of blue and red nodes at each step t, as if we run the original processes. Indeed, at step $t = 0$, the probability that in π_1 a node u is colored blue is the probability that the result of the coin flip falls in one of the first two cases and hence equal to:

$$\frac{a_{uB}}{a_{uR}^{\pi_2} + a_{uB}} + \left(\frac{a_{uB}}{a_{uR}^{\pi_1} + a_{uB}} - \frac{a_{uB}}{a_{uR}^{\pi_2} + a_{uB}}\right) = \frac{a_{uB}}{a_{uR}^{\pi_1} + a_{uB}}$$

This is precisely the same for the original π_1 process without coupling. The same is true for the process π_2 and by induction we can then prove that the distributions of red and blue nodes is the same as in the uncoupled processes.

Coupling helps us in establishing the following claim, which trivially then implies monotonicity:

Claim. For the coupled processes π_1 and π_2, for every step t and for every node u, it holds that if $s_{ut}^{\pi_1} = R$, then $s_{ut}^{\pi_2} = R$.

Proof. We proceed by induction on the number of steps.
Induction basis: This is trivial by the construction of the coupling.
Induction step: Suppose that the claim holds until step $t - 1$. We will show that it holds for step t. For an arbitrary node u, suppose $s_{u,t}^{\pi_1} = R$. If it is the case that the node was colored in previous steps, then we would also have $s_{u,t-1}^{\pi_1} = R$. But by the induction hypothesis, then $s_{u,t-1}^{\pi_2} = R$, and hence, $s_{u,t}^{\pi_2} = R$. Now consider the case where node u becomes red in π_1 exactly at step t. This means that there is a live edge from a node v, and also $s_{v,t-1}^{\pi_1} = R$. But then by the induction hypothesis, $s_{v,t-1}^{\pi_2} = R$. Recall now that the coupled processes use the same set of live edges, and also that there can be at most one incoming live edge to a node u. Hence, node u cannot have possibly been colored in π_2 by some other live edge before step t. Thus u is uncolored in π_2 at step $t - 1$, and it will become red in π_2 as well, at step t. □

We established that for any random selection of live edges, the number of red nodes at the end of π_2 is at least as high as those in π_1. Hence the expected number of red nodes will also have the same property, i.e., the SIEA model satisfies monotonicity. □

We now proceed to prove submodularity for our model.

Lemma 3. *Let $a_R, a'_R \in \mathbb{Z}^n$ such that $a_R \leq a'_R$. Under SIEA, and for any a_B, and any $j \in \{1, ..., n\}$, $\sigma(a_R + e_j, a_B) - \sigma(a_R, a_B) \geq \sigma(a'_R + e_j, a_B) - \sigma(a'_R, a_B)$.*

Proof. The proof is based on more involved coupling arguments than the case of monotonicity.

Consider 4 processes, π_1,π_2, π_3 and π_4 with the following features:

- $a_B^{\pi_1} = a_B^{\pi_2} = a_B^{\pi_3} = a_B^{\pi_4} = a_B$,
- $a_R^{\pi_1} = a_R$, and $a_R^{\pi_2} = a'_R$,
- $a_R^{\pi_3} = a_R^{\pi_1} + e_j$, and $a_R^{\pi_4} = a_R^{\pi_2} + e_j$.

Let $p_i = \frac{a_{uR}^{\pi_i}}{a_{uR}^{\pi_i} + a_{uB}}$ be the probability that node i is colored red at the initialization phase of process π_i, $i \in \{1, 2, 3, 4\}$. We consider now the following coupling between these processes: We pick at random a set of live edges as described under the SIEA model, which will be the same for all the processes. Then at step $t = 0$, for a node u with $a_{uR}^{\pi_1} + a_{uB} > 0$, we pick uniformly at random a number in [0, 1] and we decide on the color of u at each of the coupled processes, based on whether the number falls in one of 5 subintervals of $[0, 1]$, with lengths as defined below. In particular,

- With probability p_1, we paint node u red in all processes.
- With probability $(p_2 + p_3) - (p_1 + p_4) : s_{u0}^{\pi_2} = s_{u0}^{\pi_3} = s_{u0}^{\pi_4} = R \wedge s_{u0}^{\pi_1} = B$.
- With probability $p_4 - p_3 : s_{u0}^{\pi_2} = s_{u0}^{\pi_4} = R \wedge s_{u0}^{\pi_1} = s_{u0}^{\pi_2} = B$.
- With probability $p_4 - p_2 : s_{u0}^{\pi_3} = s_{u0}^{\pi_4} = R \wedge s_{u0}^{\pi_1} = s_{u0}^{\pi_3} = B$.
- With probability $1 - p_4$: we color u blue in all processes.

We can easily see that the probabilities above sum up to 1. It is also easy to check that this is indeed a valid coupling that produces the same distribution of blue and red nodes at each step t as if we run the original processes. For example, at step $t = 0$, the probability that in π_4 a node u is colored red is the probability that the result of the coin flip falls in one of the first four cases above and hence equal to:

$$p_1 + (p_2 + p_3) - (p_1 + p_4) + (p_4 - p_3) + (p_4 - p_2)$$

The above is equal to p_4, as desired. The same holds for the other processes as well. For nodes where, $a_{uR}^{\pi_1} + a_{uB} = 0$, we need to have an analogous (but simpler) construction, and the same holds for the case where $a_{uR}^{\pi_2} + a_{uB} = 0$. We omit the details for handling these simpler cases from this version.

The claim that we need in order to conclude our proof is the following:

Claim. For the coupled processes, for every step t and for every node u, it holds:

- $(s_{ut}^{\pi_4} = R) \Rightarrow (s_{ut}^{\pi_2} = R) \vee (s_{ut}^{\pi_3} = R)$.
- $(s_{ut}^{\pi_1} = R) \Rightarrow (s_{ut}^{\pi_2} = R) \wedge (s_{ut}^{\pi_3} = R)$.

Proof. Induction basis: The properties hold by the construction of the coupling.
Inductive step: Suppose the claim holds for step $t - 1$. To see the first part of the claim, consider a node u with $s_{u,t}^{\pi_4} = R$. If it is the case that the node was colored in previous

steps, then we would also have $s^{\pi_4}_{u,t-1} = R$. But by the induction hypothesis, then either $s^{\pi_2}_{u,t-1} = R$, and hence, $s^{\pi_2}_{u,t} = R$ or $s^{\pi_3}_{u,t-1} = R$, and hence, $s^{\pi_3}_{u,t} = R$. Now consider the case where node u becomes red in π_4 exactly at step t. This means that there is a live edge from a node v, and also $s^{\pi_4}_{v,t-1} = R$. But then by the induction hypothesis, $s^{\pi_2}_{v,t-1} = R$, or $s^{\pi_3}_{v,t-1} = R$. Recall now that the coupled processes use the same set of live edges, and also that there can be at most one incoming live edge to a node u. This means that node u cannot have possibly been colored in both π_2 and π_3 by some other live edge up until step $t-1$. Hence u will become red in π_2 or π_3 at t. This establishes the first part of the claim. The second part is established in a very similar way. □

It is easy to see that the claim implies submodularity of $\sigma(a_R, a_B)$. Hence this completes the proof. □

Remark 1. We can generalize the above results (the equivalence to SIEA as well as monotonicity and submodularity) for the case of $k > 2$ players. The selection function would still retain the same form, taking into account in the denominator the weight of all neighbors that were colored in the last step. To prove the same results say for player 1, we only need to consider that there is one Blue opponent with budget for node u equal to the sum of all other players' budgets for $i = 2, \ldots, k$. The intuition behind this is that for each player, the identity of her opponents does not make a difference. Hence, it is as if playing versus one Blue player that is the union of all other players.

Discussion about Non-linear Switching Functions. In the absence of competition, when the switching function is concave (and there is no selection function), monotonicity and submodularity hold [10]. This gives some indication that with such a switching function and with a linear selection function that is implemented just on the new adopters, the same properties may also hold. However, in the competitive setting, concave switching functions make the problem more challenging.

Firstly, the *live-edge* technique cannot be used in the case of a concave switching function. The reason for this is that the model ceases to be equivalent to *SIEA*. The activation time is more crucial now, and the unconditional probability of a node influencing a neighbor, depends on the order with which it will become active. The later it becomes active, the smaller the influence it will exert.

Secondly, the technique used by Mossel and Roch in [10] for the single product case cannot apply here. Their proof relies on the so called *antisense coupling* technique. A crucial point for the technique to apply is that the ordering with which the neighbors will get colored does not affect the outcome. This is not the case in the competitive setting as the nodes might get painted with different colors and the ordering affects the probability of a node getting colored with a particular color.

Despite the technical difficulty of dealing with this case, we conjecture that monotonicity and submodularity hold in the case of concave switching functions along with a linear selection function depending solely on new adopters. This would provide an interesting generalization of [10] in the concave setting with competition.

4 Necessity of Assumptions

Next, we justify the assumptions behind LSMSTM, by demonstrating that they are essentially necessary for the monotonicity and the submodularity of the influence

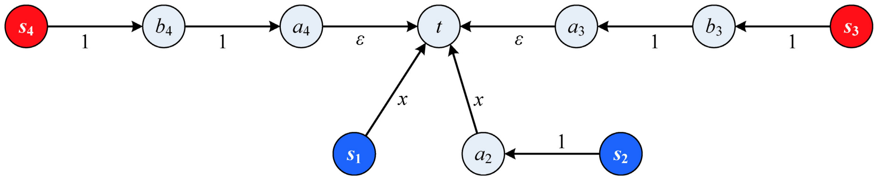

Fig. 1. Using this social network, we show that the utility function may not be submodular if the edge weights decrease in the diffusion process

functions. Specifically, we present examples showing that any significant deviation from LSMSTM yields a utility function that is nonmonotone or nonsubmodular (or both).

Monotonicity and Concavity of the Switching Function. Clearly, if the switching function is nonmonotone, the utility function need not be monotone. We also show here that the submodularity of the utility function requires that the switching function should be concave. For simplicity, we focus on the monopoly case with one product. Let the social network consist of 3 nodes s_1, s_2 and t and of two directed edges (s_1, t) and (s_2, t) with weights w_1 and w_2. Then, if the switching function f is strictly convex at some point, i.e., if there are w_1 and w_2 such that $f(w_1 + w_2) < f(w_1) + f(w_2)$, then the utility of the firm is not subadditive, and thus not submodular in such an instance.

Influence from the Neighbors in the Switching Function. Next, we show that if the edge weights decrease by an additive term of ε in the k-th step after their infection, the utility function is nonsubmodular. Thus, we demonstrate that submodularity requires that the edge weights, as taken into account by the switching function, should not decrease over time. Since we focus on models that do not depend on the node identities, we assume that this decrease takes place in any edge in the k-th step after its infection.

Let us consider the network in Fig. 1 where the blue firm selects nodes s_1 and s_2 and the red firm selects nodes s_3 and s_4. We assume that $k = 2$, i.e., the weight of each edge decreases by an additive term of ε in the second step after the edge's infection, that f satisfies $f(2x) < f(2x + \varepsilon)$, and that the selection function g is linear. Then, if t has not become blue by step 2 of the process, its threshold is larger than $f(2x)$. Then, in the third step, the weight of (s_1, t) decreases by ε and the total switching influence on t is $2x + \varepsilon$, if both s_3 and s_4 are selected by the red firm from the beginning, and at most $2x$, otherwise. Therefore, the probability that t becomes red is positive iff the red firm selects both s_3 and s_4 from the beginning. Thus, the utility function of the red firm is nonsubmodular in this case. Connecting s_2 to a_2 by a $(k - 1)$-chain of unit weight edges and connecting s_3 to a_3 and s_4 to a_4 by a k-chain of unit weight edges, we can generalize this example to the case where the edge weights decrease in the k-th step after their infection, for any $k \geq 2$. In fact, using similar in spirit (but more complicated) constructions, we can generalize this example to the case where the weight of each edge can decrease by a time dependent quantity in each step after the edge's infection.

Dependence of Selection Function on Previously Colored Nodes. Since we do not differentiate the nodes based on their identities, we can only differentiate them based on activation time. If the selection function considers not only the nodes colored in the last step, but also the nodes colored in previous steps, we can adjust the example in [2, Section 2] and show that the utility function may be nonmonotone and nonsubmodular.

(Almost) Linearity of the Selection Function. Finally, we observe that if the selection function g is highly convex at some point, i.e., if there exist some x_1, x_2, x_3 such that

$$g\left(\frac{x_1}{x_1+x_3}\right) f(x_1+x_3)+g\left(\frac{x_2}{x_2+x_3}\right) f(x_2+x_3) < g\left(\frac{x_1+x_2}{x_1+x_2+x_3}\right) f(x_1+x_2+x_3)\,, \quad (1)$$

then the utility function may not be submodular. This follows directly from (1) applied to a simple network with 4 nodes t, s_1, s_2, and s_3, and 3 directed edges (s_1, t), (s_2, t), and (s_3, t), with weights x_1, x_2, and x_3, respectively, where the blue firm selects s_3. The same argument shows that the selection function (of the red firm) $g(x)$ should not be highly concave, since otherwise, the selection function $1 - g(x)$ of the blue firm would be highly convex. Therefore, the selection function should be almost linear.

5 Performance of Equilibria

We conclude our work with a different and orthogonal question, namely studying the performance of Nash equilibria of the underlying game. We will present the analysis directly for an arbitrary number of competing firms, say k of them. For ease of presentation, we consider the case where the players choose a set rather than a multiset as their strategy to seed nodes.

Viewing the process as a game, we take as the utility of player i the expected number of nodes adopting product i at the end of the process. For a strategy profile $\boldsymbol{S} = (S_1, ..., S_n)$, we denote the payoff of i by $\sigma_i(\boldsymbol{S})$. Note that the nature of our switching function is such that the number of colored nodes at the end (independently of what color they chose), when starting from a strategy profile $\boldsymbol{S} = (S_1, ..., S_n)$ only depends on the set $S = \cup S_i$. Hence our social utility function can be defined simply over subsets of seeded nodes $S \subseteq V$, i.e., as $\gamma(S) = \gamma(\boldsymbol{S}) = \sum_i \sigma_i(\boldsymbol{S})$ where $\boldsymbol{S}$ can be any strategy profile that results in seeding S at step $t = 0$.

To quantify the Price of Anarchy of this game, we need to compare the values of $\gamma(\cdot)$ at the optimal seeding set against that at an equilibrium. For this we will use the approach of Vetta regarding utility games [13], also used by [1,7]. We start with the definition of a utility game.

Definition 3. *Consider a game with k players, and a ground set V, so that the strategy space of each player are the subsets of V. Let $\gamma(S)$ be a social welfare function. A game is defined to be a utility game if it satisfies the following three properties:*

1. *The social utility function $\gamma(\cdot)$ is submodular.*
2. *Given a profile $\boldsymbol{S}$ resulting in a seeding set S, the total value for all the players is less than or equal to the total social value: $\sum \sigma_i(\boldsymbol{S}) \leq \gamma(S)$.*
3. *The value for a player i is at least her added value for the society: $\sigma_i(\boldsymbol{S}) \geq \gamma(\boldsymbol{S}) - \gamma(\boldsymbol{S}_{-i})$*

Theorem 2. *The LSMSTM model induces a utility game.*

The proof of this theorem is by establishing the three properties listed above. Note that Property 2 in Definition 3 is trivial. In fact, in our case it holds with equality. Hence, the main part of the proof is to ensure the first and the third property as well. For this we use the equivalence with the SIEA model, which facilitates the analysis (note that according to Remark 1, this equivalence holds for an arbitrary number of players). We omit further details from this version.

From the above theorem, using [13], we have:

Corollary 2. *The Price of Anarchy even for coarse correlated equilibria is at most* 2.

A modification of the tight example in [7] shows that our upper bound is tight as well.

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Network Cost-Sharing without Anonymity

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Abstract. We consider network cost-sharing games with non-anonymous cost functions, where the cost of each edge is a submodular function of its users, and this cost is shared using the Shapley value. The goal of this paper is to identify well-motivated equilibrium refinements that admit good worst-case approximation bounds. Our primary results are tight bounds on the cost of strong Nash equilibria and potential function minimizers in network cost-sharing games with non-anonymous cost functions, parameterized by the set $\mathcal{C}$ of allowable submodular cost functions. These two worst-case bounds coincide for every set $\mathcal{C}$, and equal the *summability* parameter introduced in [31] to characterize efficiency loss in a family of cost-sharing mechanisms. Thus, a single parameter simultaneously governs the worst-case inefficiency of network cost-sharing games (in two incomparable senses) and cost-sharing mechanisms. This parameter is always at most the kth Harmonic number $\mathcal{H}_k \approx \ln k$, where k is the number of players, and is constant for many function classes of interest.

1 Introduction

We consider network cost-sharing games with non-anonymous cost functions. Such a game takes place in a directed graph $G = (V, E)$ and has k players. Player i has a source $s_i \in V$ and a sink vertex $t_i \in V$, and its strategy set is the s_i-t_i paths of the graph.[1] Outcomes of the game correspond to path vectors $(P_1, \ldots, P_k)$, with the semantics that the subnetwork $(V, \cup_{i=1}^k P_i)$ gets formed.

Each edge e has a cost function C_e, specifying the total cost incurred on edge e as a function of its users — the players S_e that pick a path that includes e. The function $C_e(S_e)$ models the infrastructure or service cost of supporting the users S_e between e's endpoints. We always assume that $C_e(\emptyset) = 0$ and that C_e is monotone, meaning $S_e \subseteq T_e$ implies $C_e(S_e) \leq C_e(T_e)$. For most of the paper, we assume that C_e is submodular, meaning it exhibits diminishing costs in the following sense:

$$C_e(T_e \cup \{i\}) - C_e(T_e) \leq C_e(S_e \cup \{i\}) - C_e(S_e)$$

for all i and $S_e \subseteq T_e$. Almost all previous work on network cost-sharing games, beginning with [1], considers only *anonymous* cost functions, where the cost of an edge depends solely on the number of users. For anonymous cost functions,

[1] The main results of this paper continue to hold, with the same proofs, when the strategy set of a player i is an arbitrary subset of 2^E.

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submodularity is equivalent to non-increasing marginal costs. Non-anonymous cost functions model asymmetries between players, which can arise from different bandwidth requirements, durations of use, services needed, and so on.

Example 1 (Weighted Players). For a simple example of a non-anonymous cost function, suppose each player i has a positive weight w_i. The joint cost function $C_e(S_e)$ depends on the set S_e of users only through the sum of their weights $\sum_{i \in S_e} w_i$. If $C_e(S_e) = f(\sum_{i \in S_e} w_i)$ for a nondecreasing concave function f, then C_e is monotone and submodular.

Example 2 (Coverage Functions). For a more general class of non-anonymous cost functions, consider a ground set X of services, where supporting a service $j \in X$ imposes a weight of w_j on the service provider. Each player i requires a set $A_i \subseteq X$ of services. The cost C_e of supporting all of the services required by a set S_e of users is $C_e(S_e) = f(\sum_{j \in \cup_{i \in S_e} A_i} w_j)$. Provided f is a monotone concave function, C_e is a monotone submodular function. Example 1 corresponds to the special case in which the players require disjoint sets of services. If all of the A_i's coincide, we recover the special case of (anonymous) constant cost functions.

To complete the description of the game, we need to define players' costs. We assign a *cost share* $\chi_e(i, S_e)$ to each user $i \in S_e$ of each edge e. The cost $c_i(S)$ of a player i in a strategy profile $S = (P_1, \ldots, P_k)$ is then

$$c_i(S) = \sum_{e \in P_i} \chi_e(i, S_e),$$

where $S_e = \{j : e \in P_j\}$ is the set of users of e.

With anonymous cost functions, the natural cost shares proposed in [1] are the equal cost shares: $\chi_e(i, S_e) = C_e(S_e)/|S_e|$. With non-anonymous cost functions, however, such cost shares are not as well motivated. We extend the idea of equal cost-sharing to non-anonymous cost functions by taking $\chi_e(i, S_e)$ to be i's Shapley value in the cooperative game induced by C_e and S_e. In more detail, for a permutation σ of the players of S_e, let $\Delta_\sigma(i)$ denote the increase $C(S_{\sigma(1..i-1)} \cup \{i\}) - C(S_{\sigma(1..i-1)})$ in cost due to i's arrival, where $S_{\sigma(1..i-1)}$ is the set of players that precede i in σ. Then, $\chi_e(i, S_e)$ is defined as the expected value of $\Delta_\sigma(i)$, where the expectation is over the (uniform at random) choice of σ. It is easy to verify that: (i) these cost shares coincide with equal cost-sharing when C_e is anonymous; (ii) the joint cost is shared fully across the players, with $\sum_{i \in S_e} \chi_e(i, S_e) = C_e(S_e)$; and (iii) a submodular cost function C_e leads to positive externalities, in the sense that $\chi_e(i, S_e) \leq \chi_e(i, T_e)$ whenever $i \in T_e \subseteq S_e$. Properties (ii) and (iii) are called *budget-balance* and *cross-monotonicity.* These cost shares also ensure that every network cost-sharing game has a potential function and therefore admits at least one pure Nash equilibrium (see Section 2).

Example 3 (Weighted Players Revisited). Consider two players with weights 1 and 3 and the cost function $C_e(S_e) = (\sum_{i \in S_e} w_i)^{1/2}$. The joint cost of the two players is 2. The players' cost shares are their Shapley values, namely $\frac{1}{2}(3-\sqrt{3}) \approx .635$ and $\frac{1}{2}(1+\sqrt{3}) \approx 1.365$, respectively.

1.1 Measures of Equilibrium Inefficiency

The primary goal of this paper is to characterize the inefficiency of equilibria in network cost-sharing games with non-anonymous cost functions, in as many senses as possible. We define the cost $C(S)$ of a strategy profile S of a network cost-sharing game as the sum of players' costs:

$$C(S) = \sum_{e \in E} C_e(S_e) = \sum_{i=1}^{k} c_i(S), \tag{1}$$

and take (1) as our objective function. Recall that a pure Nash equilibrium (PNE) is a strategy profile from which no player can decrease its cost via a unilateral deviation. To what extent do PNE minimize the cost (1)?

A Non-starter: The Price of Anarchy. It is well known that network cost-sharing games can have multiple PNE of wildly varying quality. The canonical example in the basic model with constant cost functions [1,2] posits k players, each choosing between an edge e_1 with fixed cost $1+\epsilon$ and an edge e_2 with fixed cost k. Since all players using the second edge is a PNE, the ratio between the worst PNE and an optimal solution (i.e., the "price of anarchy") can be as large as k.

Equilibrium Refinements. The bad equilibrium identified above does not imply that network cost-sharing games are uninteresting — just that, to reason meaningfully about the quality of their equilibria, a more fine-grained approach is required. Recall that an *equilibrium refinement* defines a subset of equilibria. We are interested in equilibrium refinements with the following two properties:

1. All equilibria in the refined set have cost close to optimal.
2. There is a plausible explanation why equilibria in the refinement are more "important" or "likely" than those outside the set.

Previous work on network-cost sharing games with anonymous submodular cost functions can be interpreted as proposing two refinements with these two properties: potential function minimizers and strong Nash equilibria.

Potential Function Minimizers and the Price of Stability. Anshelevich et al. [1] proposed circumventing bad PNE by studying the price of stability, defined as the ratio between the minimum-cost PNE and that of an optimal outcome. They prove that the worst-case price of stability in network cost-sharing games with anonymous submodular cost functions is exactly the kth Harmonic number $\mathcal{H}_k = \sum_{i=1}^{k} \frac{1}{i} = \ln k + \Theta(1)$. We can interpret the upper bound in [1] as a worst-case bound for an equilibrium refinement by examining its proof. The first step in [1] constructs a potential function [28] for every network cost-sharing game with anonymous cost functions — a function Φ such that the change in Φ under a unilateral deviation by player i equals the change in i's cost. The PNE correspond to the local minimizers (under unilateral deviations) of Φ. The second step of the proof shows that every global minimizer of Φ has cost at most $\mathcal{H}_k$ times that of an optimal outcome. The global minimizers of Φ therefore form an equilibrium refinement of the PNE that satisfies the first property in Section 1.1. As for the second property, global potential function optimizers have been previously proposed as a plausible equilibrium refinement in the game theory and economics literature, together with supporting theoretical [3,8,34] and experimental evidence [13].

Strong Nash Equilibria. Epstein et al. [17] studied an incomparable equilibrium refinement in network cost-sharing games with anonymous concave cost functions. Recall that a strong Nash equilibrium (SNE) [4] is a strategy profile such that no coalition of players can deviate in a coordinated way to strictly decrease all of their costs. Robustness to coalitional deviations provides good motivation for favoring strong Nash equilibria over non-strong PNE.

Global potential function minimizers need not be SNE — indeed, SNE are not guaranteed to exist in general [17] — and SNE need not minimize the potential function, except in very simple networks [21]. Nevertheless, [17] proved that the worst-case ratio between an SNE and an optimal outcome is also precisely $\mathcal{H}_k$.

1.2 Contributions and Paper Organization

The goal of this paper is to identify well-motivated equilibrium refinements that admit good worst-case approximation bounds. Our primary positive results are characterizations of the worst-case inefficiency of both potential function minimizers and strong Nash equilibria in network cost-sharing games, as a function of the class $\mathcal{C}$ of allowable submodular cost functions. Despite their instance-by-instance incomparability, we prove in Section 3 that the worst-case approximation ratios of these two equilibrium refinements are identical for every set $\mathcal{C}$ of allowable cost functions, and equal the worst-case *summability* of a function in $\mathcal{C}$.[2] This bound is always at most $\mathcal{H}_k$, (Proposition 1), and is constant for many cost function classes of interest. For example, consider weighted players (Example 1) and a polynomial cost function $C_e(S_e) = (\sum_{i \in S_e} w_i)^d$. For $d \in (0, 1]$, this function is at most $\frac{1}{d}$-summable, independent of the number of players and their weights (Example 5). This yields a constant-factor approximation guarantee for potential function minimizers and strong Nash equilibria in games with such cost functions.

Additionally, in Section 4 we extend the equivalence of summability and worst-case approximation bounds for strong Nash equilibria to network cost-sharing games with non-submodular cost functions and arbitrary cross-monotonic cost-sharing methods. At this level of generality the games typically have no potential function, so the first refinement is not well defined.

Our results in Sections 3 and 4 effectively isolate the key features of the standard network cost-sharing model that drive inefficiency bounds. While there

[2] Our upper bound for strong Nash equilibria meets the "coalitional smoothness" criterion of [5], and therefore extends to additional solution concepts.

exist properties that require the full symmetry of anonymous cost functions (we consider these in the full version of this paper), tight worst-case bounds for potential function minimizers only rely on cross-monotonicity of the underlying cost-sharing method and (obviously) the existence of a potential function, and tight bounds for strong Nash equilibria do not even require a potential function.

1.3 Further Related Work

This paper contributes to the literature on network cost-sharing games that was initiated in [1,2]. Subsequent works on these and related models include [18,27,7,15,6,25,10,9,35,11,12,26]. All of these papers study only anonymous cost functions. There are two previous papers that treat network cost-sharing games with non-anonymous cost functions. Gopalakrishnan et al. [19] characterize cost-sharing rules that guarantee the existence of PNE and do not consider equilibrium inefficiency. Von Falkenhausen and Harks [37] consider machine scheduling games and design cost sharing methods that yield small PoA bounds.

Many other models of network formation have been studied; see [36] for a survey. For a comprehensive treatment of game-theoretic models of network formation, see [22].

The idea of using the Shapley value to define cost-sharing methods with good properties is not new; see [30] and the references therein. In network cost-sharing games, the weighted Shapley value [33,24] was first introduced in [12] for games with constant cost functions to characterize cost-sharing rules that guarantee the existence of pure Nash equilibria. This characterization was recently generalized by [19] to all cost functions. Shapley value-based cost shares have also been used in congestion games, which can be thought of as the "negative externality version" of network cost-sharing games [26].

2 Preliminaries

Summability. We recall the summability parameter introduced in [31] to characterize efficiency loss in cost-sharing mechanisms [30]. Let $C(\cdot)$ denote a cost function defined on a ground set U and χ a cost-sharing method – a function from player sets to cost shares. We call χ *α-summable for C* if for every $S \subseteq U$ and every ordering σ of the players of S:

$$\sum_{\ell=1}^{|S|} \chi(i_{\sigma(\ell)}, S_{\sigma(1..\ell)}) \leq \alpha \cdot C(S), \tag{2}$$

where $i_{\sigma(\ell)}$ denotes the ℓth player, and $S_{\sigma(1..\ell)}$ denotes the set of the first ℓ players in the ordering σ. In words, we begin with the empty set and add players of S one-by-one according to σ. Letting X_ℓ denote the cost share of the ℓth player (according to χ) when the player is first added, the cost-sharing method χ is α-summable for C if the sum $\sum_\ell X_\ell$ only overestimates the cost of $C(S)$ by an α factor (for a worst-case choice of the subset S and the ordering of the players). Let $\alpha(C)$ denote the smallest α such that C is α-summable and $\alpha(\mathcal{C}) = \sup_{C\in\mathcal{C}} \alpha(C)$. The Shapley value is never worse than $\mathcal{H}_k$-summable for a monotone cost function C.

Proposition 1 (Roughgarden & Sundararajan [31]). *For the Shapley cost-sharing method and a set $\mathcal{C}$ of monotone cost functions, $\alpha(\mathcal{C}) \leq \mathcal{H}_k$.*

Moreover, for many cost functions, the summability is constant in k.

Example 4 (Polynomial Cost Functions). Consider anonymous polynomial cost functions, of the form $C(S) = |S|^d$ with $d \in (0, 1]$. Since the cost function is anonymous, the players have equal cost shares: $|S|^d/|S| = |S|^{d-1}$. Since the summands are decreasing, we can upper bound the sum that ranges from 1 to $|S|$ by an integral that ranges from 0 to $|S|$:

$$\sum_{i=1}^{|S|} i^{d-1} \;\leq\; \int_0^{|S|} t^{d-1}dt \;=\; d^{-1}t^d\Big|_0^{|S|} \;=\; d^{-1}|S|^d,$$

which shows that $\alpha(C) \leq 1/d$, independent of k.

More interesting is the following computation for the non-anonymous case. The summability remains $\frac{1}{d}$ with weighted players and a polynomial cost function.

Example 5 (Weighted Players). Suppose every player i has a weight w_i and $C(S) = \left(\sum_{i\in S} w_i\right)^d$ with $d \in (0, 1]$. Shapley cost sharing yields different costs for players with different weights. However, the summability is highest with equal-weight players (see the full paper for a proof). Therefore, the summability with weighted players is bounded by our previous example, with $\alpha(C) \leq 1/d$.

The Shapley Value Yields Potential Games. Every network cost-sharing game with Shapley cost shares is a potential game in the sense of [28]. First we define the *ordered potential* $\Phi_\sigma(S)$ with respect to an ordering σ of the players:

$$\Phi_\sigma(S) = \sum_{e\in E}\sum_{\ell=1}^{|S_e|} \chi_e(i_{\sigma(\ell)}, S_{e,\sigma(1..\ell)}) \tag{3}$$

where $S_{e,\sigma(1..\ell)}$ is S_e restricted to the first ℓ in players σ. Remarkably, when the Shapley value is used as the cost-sharing method, the ordered potential (3) is the same for every ordering σ, even though individual summands generally differ [20,26]. We can therefore define $\Phi(S)$ as the value in (3) for an arbitrary choice of the ordering σ. The next proposition notes that the "order-independence" property of Φ implies that it is a potential function.

Proposition 2 (e.g. [26]). *For every network cost-sharing game with Shapley cost sharing, and for every pair S and $S' = (S'_i, S_{-i})$ of strategies that differ only in their ith component,*

$$\Phi(S') - \Phi(S) = c_i(S') - c_i(S).$$

The order-independence property also implies that when χ is the Shapley value, the left-hand side of the summability equation (2) is independent of the ordering. The more general cross-monotonic cost-sharing methods we consider in Section 4 do not have this order-independence property, but summability (2) will continue to characterize the inefficiency of strong Nash equilibria.

3 Summability Characterizes Worst-Case Inefficiency

In this section we give tight bounds for the strong price of anarchy and potential function minimizers for network cost-sharing games with submodular cost functions. The proofs appear in the full version of this paper.

3.1 Strong Price of Anarchy Upper Bound

The strong price of anarchy is the worst-case ratio between the cost at a strong Nash equilibrium and the optimal cost. For network cost-sharing games, this is bounded by the summability.

Theorem 1. *The strong price of anarchy in a network cost-sharing game $\mathcal{I}_{\mathcal{C}}$ with submodular cost functions and Shapley cost-sharing is at most the summability $\alpha(\mathcal{C})$.*

Smoothness frameworks extend price of anarchy bounds automatically to more general solution concepts. The coalitional smoothness framework of [5] applies to equilibria that are resistant to deviations of a group of players, and the proof of Theorem 1 can be recast as a coalitional smoothness argument.

Proposition 3. *Network cost-sharing games with monotone submodular cost functions and Shapley cost-sharing are $(\alpha(\mathcal{C}), 0)$-coalitionally smooth.*

Since network cost-sharing games are coalitionally smooth, we inherit the extensions described in [5] to strong correlated equilibria [29], strong coarse correlated equilibria [32], and coalitional sink equilibria [5].

Corollary 1. *For network cost-sharing games, the expected cost at*

- *any strong correlated equilibrium is at most $\alpha(\mathcal{C})$ times the optimal.*
- *any strong coarse correlated equilibrium is at most $\alpha(\mathcal{C})$ times the optimal.*
- *a coalitional sink equilibrium is at most $\mathcal{H}_k \cdot \alpha(\mathcal{C})$ times the optimal.*

In Section 4, we consider cost-sharing methods other than the Shapley value, and prove that the strong price of anarchy remains bounded by the summability α even in games without potential functions.

3.2 Potential Function Minimizer Inefficiency Upper Bound

We now consider the second equilibrium refinement, potential function minimizers. Given the existence of a potential function (Proposition 2), it is straightforward to apply the potential function method (see [36]) to bound the cost of its minimizers in terms of the summability. When we consider cross-monotonic cost-sharing methods other than the Shapley value in the next section, this method will no longer apply.

Theorem 2. *The inefficiency of the potential function minimizer in a network cost-sharing game $\mathcal{I}_\mathcal{C}$ with submodular cost functions is at most the summability $\alpha(\mathcal{C})$.*

3.3 Lower Bounds

We now show how to adapt a well-known instance from [1] to show simultaneously that our two upper bounds are tight for every set $\mathcal{C}$ of monotone submodular cost functions. The key reason that the lower bound instance from [1] remains relevant in the present more general context is the cross-monotonicity of the Shapley value for a submodular cost function.

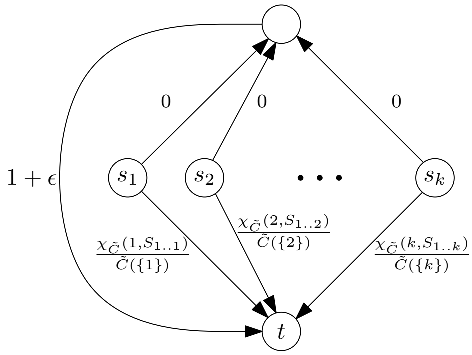

Fig. 1. A network cost-sharing game that shows matching lower bounds

In Figure 1 we have a network with k players that start at s_i and have a common sink t. The cost functions on all edges are scalar multiples of $\tilde{C}(S_e)$, where $\tilde{C} \in \mathcal{C}$ is a cost function with $\alpha(\tilde{C}) = \alpha(\mathcal{C})$ (or arbitrarily close to $\alpha(\mathcal{C})$); the numbers on the edges denote the multiples.

As ϵ goes to 0, the total cost is minimized when all players share the cost of the top route. This is not a Nash equilibrium: player k pays slightly less by taking its own personal shortcut to t. This is true no matter what the other players do, by cross-monotonicity. Given that player k takes the shortcut, player $k-1$ has a (conditional) dominant strategy to take its shortcut, and so on until player 1 does the same. None of the players have an incentive to deviate from this strategy, so this is the unique Nash equilibrium in this network. The equilibrium is strong as it consists of (conditional) dominant strategies, and it is the potential function minimizer since it is the only equilibrium. The cost that each player pays is $\chi_{\tilde{C}}(i, S_{1..i})$, so by the definition of summability, the total cost of the equilibrium is $\alpha(\mathcal{C})$ worse than the optimal cost $\tilde{C}(S)$.

Proposition 4. *For every set of submodular cost functions $\mathcal{C}$, there exists a network cost-sharing game $\mathcal{I}_{\mathcal{C}}$ in which the strong Price of Anarchy is $\alpha(\mathcal{C})$.*

Proposition 5. *For every set of submodular cost functions $\mathcal{C}$, there exists a network cost-sharing game $\mathcal{I}_{\mathcal{C}}$ in which the potential function minimizer yields an equilibrium with cost $\alpha(\mathcal{C})$ times that of an optimal solution.*

Recall that the summability parameter that we use was originally proposed in the context of cost-sharing mechanisms. The connection is stronger: it is possible to give a one-to-one mapping from Moulin (cost-sharing) mechanisms to network cost-sharing games with the structure given in Figure 1. Therefore, an alternative way of giving these lower bounds, is using the known lower bound instances from the Moulin mechanism literature [31].

4 Beyond Submodular Cost Functions

The previous section considered submodular cost functions and cost sharing according to the Shapley value. This section considers two natural directions for further generalization. Section 4.1 retains Shapley cost shares (and hence a potential function) but relaxes submodularity, with the consequence that Shapley cost shares are not generally cross-monotonic. Section 4.2 considers cross-monotonic (and hence non-Shapley) cost-sharing methods for non-submodular cost functions.

4.1 Non-submodular Costs with Shapley Cost Sharing

If we use Shapley cost-sharing with non-submodular (monotone) cost functions, the corresponding network cost-sharing games continue to have a potential function (Proposition 2), the summability remains bounded by $\mathcal{H}_k$ (Proposition 1), and the summability continues to upper bound the worst-case approximation ratio of potential function minimizers (Theorem 2). However, without cross-monotonicity the lower bound in Proposition 5 no longer holds. The reason is that we use cross-monotonicity to argue conditional dominant strategies. The last player has a dominant strategy to take their lower path, because she knows that her most advantageous situation is when all players share the cost of the top path, and in that case she pays less by taking the bottom path. However, if her cost could decrease by another player choosing to deviate to their lower path, this is no longer a dominant strategy for her, and the equilibrium will depend on the cost function. Indeed, there are examples of monotone non-submodular cost functions such that the approximation ratio of potential function minimizes is strictly better than the summability (see [16] for one).

Also, without cross-monotonicity, the upper bound (Theorem 1) for the strong price of anarchy fails to hold. The proof of Theorem 1 uses cross-monotonicity to upper bound the cost for a player in an entangled strategy profile. Indeed, Theorem 1 is probably false for non-cross-monotonic cost-sharing methods (see [14, Theorem 5.1] for an example). The lower bound argument (Proposition 4) for the price of anarchy of strong Nash equilibria breaks down for the same reasons as for potential function minimizers.

In summary, relaxing cross-monotonicity constraint has significant consequences: none of the bounds for strong Nash equilibria carry over, and only an upper bound that we know is loose carries over for potential function minimizers.

4.2 Non-shapley Cross-Monotonic Cost Sharing

This section explores non-Shapley-value cost-sharing methods that are cross-monotonic; the design of such methods has been explored extensively in the context of cost-sharing mechanisms (see [23] for a survey). Network cost-sharing games with arbitrary cross-monotonic cost-sharing methods no longer admit potential functions in general, and even the best PNE can be arbitrarily bad (see [11] for an example), so we focus on strong Nash equilibria. Despite the non-existence of a potential function, we can still prove that whenever a strong Nash equilibrium exists, it is approximately optimal.

We also allow non-budget-balanced rules; this relaxation permits cross-monotonic cost-sharing rules for many non-submodular cost functions (see [23]). We call a cost-sharing method *β-budget balanced* if it recoups at least a β fraction of the cost from its players.

Let $\mathcal{F}$ denote the class of all admissible cost functions (including non-submodular functions), and let $\mathcal{I}_{\mathcal{F}}$ denote an instance of a network cost-sharing game with cost functions from $\mathcal{F}$. Similarly, $\alpha(\mathcal{F})$ denotes the summability of $\mathcal{F}$. Since we no longer use the Shapley value for cost sharing, the cost-sharing method χ is no longer order-independent, and summability is defined to hold for all orders σ as in (2). Finally, we also revert to the *ordered potential Φ_σ* in the original definition in (3).

Theorem 3. *For arbitrary monotone cost functions $\mathcal{F}$, and a cost-sharing method χ that is cross-monotonic and β-budget balanced, the strong Price of Anarchy in a network cost-sharing game $\mathcal{I}_{\mathcal{F}}$ is at most $\beta \cdot \alpha(\mathcal{F})$.*

The smoothness proof (Proposition 3) and therefore the extension to additional solution concepts (Corollary 1) also carry over to the present more general setting, with a loss of β in the approximation factors.

While the summability of a cost-sharing method other than the Shapley value can be larger than $\mathcal{H}_k$, for many non-submodular cost functions, there exist $O(1)$-budget-balanced cross-monotonic cost-sharing methods with summability $O(\log k)$ or $O(\log^2 k)$; see [31] for a survey.

5 Conclusion

This paper studied systematically network cost-sharing games with non-anonymous cost functions; the large literature on network cost-sharing games has confined its attention almost entirely to the anonymous case. Non-anonymous cost functions arise naturally when, for example, different players have different sizes or service requirements. Two well-studied equilibrium refinements, strong Nash equilibria and potential function minimizers, are near-optimal with non-anonymous cost functions as long as the functions are submodular and costs are shared using the Shapley value. Strong Nash equilibria remain near-optimal, when they exist, provided costs are shared by a cost-monotonic cost-sharing method. All of these worst-case approximation guarantees equal the summability of the cost-sharing method used for the set of allowable cost functions.

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Recognizing 1-Euclidean Preferences: An Alternative Approach

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Abstract. We consider the problem of detecting whether a given election is 1-*Euclidean*, i.e., whether voters and candidates can be mapped to points on the real line so that voters' preferences over the candidates are determined by the Euclidean distance. A recent paper by Knoblauch [14] shows that this problem admits a polynomial-time algorithm. Knoblauch's approach relies on the fact that a 1-Euclidean election is necessarily single-peaked, and makes use of the properties of the respective candidate order to find a mapping of voters and candidates to the real line. We propose an alternative polynomial-time algorithm for this problem, which is based on the observation that a 1-Euclidean election is necessarily singe-crossing, and we use the properties of the respective voter order to find the desired mapping.

1 Introduction

There are many settings where agents express their preferences over a finite set of candidates by submitting full rankings of the candidates. Often, the set of candidates has a special structure, which influences the agents' preferences. For instance, it may be the case that voters and/or candidates can be mapped to points on the real line so that the agents' preferences are consistent with this mapping. Different instantiations of this idea give rise to such well-known preference domains as *single-peaked* preferences [4], *single-crossing* preferences [15], and 1-*dimensional Euclidean*, or 1-*Euclidean*, preferences [13].

In this paper, we study 1-Euclidean elections (though we will also discuss single-peaked and single-crossing elections, as all three domains are closely interrelated). These are elections that can be succinctly described by embedding both voters and candidates in the real line so that each voter prefers an alternative that is closer to her to the one that is further away. A typical situation that results in 1-Euclidean preferences is facility location on a line: there is a single facility (such as a bus stop, a playground, or a library) to be constructed in one of several possible locations along a street, and the voters (who are the residents of that street) want the facility to be located as close to them as possible. 1-Euclidean preferences can also arise in settings where the structure of the alternative space is not immediately obvious: for instance, political elections may turn out to be 1-Euclidean when the voters rank the candidates according to some combination of factors that happens to map onto the real line. It is then natural to ask whether, given an election, we can uncover its hidden metric structure, i.e., decide whether it is 1-Euclidean. This is the question that is the main focus of this paper.

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Before we discuss 1-Euclidean preferences in further detail, let us review the relationship between, on the one hand, 1-Euclidean preferences and, on the other hand, single-peaked and single-crossing preferences, as the nature of this relationship will play an important role for our algorithmic results. Recall that the agents' preferences are said to be *single-peaked* when the candidates can be ordered on the line so that each voter, when comparing two candidates located on the same side of her favorite point, prefers one that is closer to her top choice to the one that is further away from it; in contrast with 1-Euclidean domain, voter's preferences concerning two candidates located on different sides of her favorite point are unconstrained. On the other hand, a preference profile is said to be *single-crossing* if the voters can be ordered so that for each pair of candidates a, b, the "trajectories" of a and b in the voters' preferences cross at most once, i.e., if the first voter prefers a to b, then all voters who prefer a to b precede all voters who prefer b to a. Both of these domains have received a considerable amount of attention in social choice literature, as they have a number of desirable properties: for instance, both single-peaked and single-crossing elections are guaranteed to have a Condorcet winner, and admit a non-trivial strategyproof social choice rule [16,17,1]. Recently, it has also been shown that some of the algorithmic problems related to elections (such as winner determination, manipulation, bribery, and control) become easier if the voters' preferences belong to one of these domains [11,5,3,18]. Further, there are polynomial-time algorithms for checking whether an election is single-peaked [2,10] or single-crossing [6,9], and, if this is the case, finding an ordering of candidates or voters witnessing this. It is not hard to see that 1-Euclidean elections are both single-peaked and single-crossing [12]; however, the converse is not true, i.e., there are single-peaked single-crossing elections that are not 1-Euclidean [8,7].

It is fairly easy to see that, given an ordering of voters and candidates in an election E, we can efficiently check whether there is a mapping that places the voters and the candidates on the real line in a way that is consistent with this ordering and witnesses that E is 1-Euclidean; indeed, this question can be captured by a simple linear feasibility program (a variant of this observation is due to Knoblauch [14]; see also Proposition 3). Thus, checking whether an election is 1-Euclidean can be reduced to finding an appropriate ordering of voters and candidates. Since 1-Euclidean elections are single-peaked and single-crossing, given an input election E with a candidate set C and a voter set V, it is natural to first check whether E is single-peaked and single-crossing, and, if so, use the respective orderings of C and V to construct the required ordering of $C \cup V$. Indeed, a variant of this approach has been recently pursued by Knoblauch [14], who used an ordering of candidates witnessing that E is single-peaked as a starting point for her algorithm for checking whether E is 1-Euclidean. We discuss Knoblauch's algorithm in more detail in Section 5; at this point, we would like to mention that a single-peaked ordering of candidates is not unique, which causes considerable complications.

In contrast, in this paper we start with an ordering of voters witnessing that a given election $E = (C, V)$ is single-crossing, and show how to extend it to an ordering of $C \cup V$ witnessing that E is 1-Euclidean. The advantage of this approach is that, if E is single-crossing, there is effectively a unique ordering of voters that certifies this (this is shown in Proposition 1). As a result, we construct an algorithm for recognizing 1-Euclidean preferences that is arguably simpler than that of Knoblauch.

The rest of the paper is organized as follows. After introducing the necessary notation and formally defining single-peaked, single-crossing and 1-Euclidean preferences (Section 2), we state a few basic observations about the 1-Euclidean domain (Section 3), followed by a presentation of our algorithm (Section 4). We then provide an overview of Knoblauch's algorithm (Section 5). We conclude the paper by discussing topics for future research (Section 6).

2 Preliminaries

For every positive integer s, we let $[s]$ denote the set $\{1, \dots, s\}$. An *election* is a pair $E = (C, V)$ where $C = \{c_1, \dots, c_m\}$ is a set of candidates and $V = (v_1, \dots, v_n)$ is an ordered list of voters. Each voter $v_i \in V$ has a *preference order*, or *vote*, $\succ_i$, i.e., a linear order over C that ranks all the candidates from the most desirable one to the least desirable one. We refer to the list V as the *preference profile*. In what follows, we use the terms "election","preferences" and "preference profile" interchangeably.

We denote the most preferred candidate in a vote v_i by $\mathrm{top}(v_i)$. Given an election $E = (C, V)$ and a subset of candidates $D \subset C$, we denote by $V|_D$ the restriction of the preferences of the voters in V to D. Given two sets $A, B \subset C$, we write $\dots \succ A \succ B \succ \dots$ to denote a vote where all candidates in A appear above all candidates in B.

Euclidean, Single-Crossing and Single-Peaked Profiles. We will now define three important preference domains that will be considered in this paper.

Perhaps the most intuitive of the three is the domain of Euclidean preferences: both voters and candidates are identified with points on the real line (or, more generally, in $\mathbb{R}^d$), and the voters' preferences are determined by the Euclidean distance to the candidates.

Definition 1. *An election $E = (C, V)$ is said to be d-*Euclidean *if there is a mapping $x : C \cup V \to \mathbb{R}^d$ such that for every voter $v \in V$ and every pair of candidates $a, b \in C$ it holds that $a \succ_v b$ if and only if $\|x(v) - x(a)\|_d < \|x(v) - x(b)\|_d$, where $\| \cdot \|_d$ is the Euclidean norm on $\mathbb{R}^d$, i.e., for every vector $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d$ we have $\|\mathbf{u}\|_d = \left(u_1^2 + \dots + u_d^2\right)^{1/2}$.*

Note that in Definition 1 a voter cannot be equidistant from two distinct candidates a and b. One might argue that such a situation should be allowed, and the voter can then be indifferent between a and b, or break the tie arbitrarily. However, the former interpretation would not fit our model of preference orders being strict, and the latter would render the notion of d-Euclidean elections useless: Every election would be d-Euclidean (for each $d \in \mathbb{N}$), since we could map all voters and all candidates to a single point. One can deal with this objection by requiring that the positions of all candidates are distinct; the resulting model (and the associated algorithmic problem), while non-standard, deserves future study (see Section 6).

The notion of single-crossing preferences (sometimes also called *intermediate* preferences) dates back to the work of Mirrlees [15].

Definition 2. *An election $E = (C, V)$, where C is a set of candidates and $V = (v_1, \dots, v_n)$ is an ordered list of voters, is* single-crossing with respect to V *if for every pair of candidates $a, b \in C$ such that $a \succ_1 b$, there exists a $t \in [n]$ such that $\{i \in [n] \mid a \succ_i b\} = [t]$.*

Definition 2 refers to the ordering of the voters given by V. Alternatively, we could say that an election is single-crossing if the voters can be reordered so that the condition of Definition 2 is satisfied. However, from the algorithmic perspective, this distinction is not essential: one can compute an order of the voters that makes an election single-crossing or decide that such an order does not exist, in polynomial time [9,6].

Another relevant concept is that of single-peaked preferences [4].

Definition 3. *Let $\succ$ be a preference order over a candidate set C and let $\lhd$ be an order over C. We say that $\succ$ is* single-peaked with respect to $\lhd$ *if for every triple of candidates $a, b, c \in C$ such that $a \lhd b \lhd c$ or $c \lhd b \lhd a$ it holds that $a \succ b$ implies $b \succ c$. An election $E = (C, V)$ is* single-peaked with respect to an order $\lhd$ over C *if the preference order of every voter $v \in V$ is single-peaked with respect to $\lhd$. An election $E = (C, V)$ is* single-peaked *if there exists an order $\lhd$ over C with respect to which it is single-peaked.*

If E is single-peaked with respect to some order $\lhd$ then we call $\lhd$ a *societal axis* for E. There are polynomial-time algorithms that, given an election E, decide if it is single-peaked and if so, compute a societal axis for it [2,10].

It is not hard to show that 1-Euclidean elections are both single-peaked and single-crossing, see, e.g., [12]. However, there exist elections that are both single-peaked and single-crossing, but not 1-Euclidean [7,8].

3 Basic Observations

We will now present some simple observations that will be used in the analysis of our algorithm.

Our first observation is that for single-crossing elections there is effectively a unique ordering of voters that witnesses that it is single-crossing, up to flipping the entire ordering and reordering identical voters.

Proposition 1. *Consider an election $E = (C, V)$ that is single-crossing with respect to $V = (v_1, \dots, v_n)$. If the preferences of the voters in V are pairwise distinct, then the only other order of the voters witnessing that E is single-crossing is $(v_n, \dots, v_1)$.*

Proof. The proof is by induction on n. For $n = 1$ and $n = 2$, our claim is trivially true. Now suppose that it is true for $n - 1$, where $n \geq 3$; we will show that it is true for n.

Since $v_1 \neq v_2$, there is a pair of candidates a, b such that v_1 prefers a to b, but v_2 prefers b to a. Since E is single-crossing, all voters other than v_1 prefer b to a. Therefore, if E is single-crossing with respect to some order $\widehat{V}$, then voter v_1 has to be first or last in $\widehat{V}$.

Now, consider the election (C, V'), where $V' = (v_2, \dots, v_n)$. It is single-crossing and has $n - 1$ voters, so by the induction hypothesis the voters in V' have to be ordered as $(v_2, \dots, v_n)$ or $(v_n, \dots, v_2)$.

It remains to argue that E is not single-crossing with respect to $(v_2, \dots, v_n, v_1)$ and $(v_1, v_n, \dots, v_2)$. It suffices to consider the first of these two orderings. Since $n \geq 3$ and we assume that all preferences are pairwise distinct, there is a pair of candidates c, d such that v_2 prefers c to d, but v_n prefers d to c. Since the original election is single-crossing, it has to be the case that v_1 prefers c to d. But this means that E is not single-crossing with respect to $(v_2, \dots, v_n, v_1)$. $\square$

In contrast, there can be exponentially many axes witnessing that a given election is single-peaked [10]. For instance, a unanimous election where all voters order the candidates as $c_1 \succ \cdots \succ c_m$ is single-peaked with respect to 2^{m-1} axes (any subset of $\{c_2, \ldots, c_m\}$ can appear to the left of c_1 on the axis).

We will now list some useful properties of 1-Euclidean elections.

Proposition 2. *Let $E = (C, V)$ be a 1-Euclidean election with $V = (v_1, \ldots, v_n)$ such that all votes are pairwise distinct, and let $x : C \cup V \to \mathbb{R}$ be some mapping witnessing that E is 1-Euclidean. Assume without loss of generality that $x(v_1) < x(v_n)$. Suppose that $\mathrm{top}(v_1) = a$, $\mathrm{top}(v_n) = b$ (and hence $x(a) < x(b)$). Let*

$$C_L = \{c \in C \mid x(c) < x(a)\},$$
$$C_M = \{c \in C \mid x(a) \leq x(c) \leq x(b)\},$$
$$C_R = \{c \in C \mid x(c) > x(b)\}.$$

Then

(1) $C_M = \{c \in C \mid c \succ_1 b$ and $c \succ_n a\} \cup \{a, b\}$.
(2) For every pair of candidates $c, d \in C$, if $c \succ_1 d$, but $d \succ_n c$, then $c \in C_L \cup C_M$ and $d \in C_M \cup C_R$.

Proof. Obviously, $a, b \in C_M$. If $c \in C_M$ and $c \neq a, b$, then v_1 ranks c above b and v_n ranks c above a. On the other hand, if $c \in C_R$ then v_1 ranks c below b and if $c \in C_L$, then v_n ranks c below a. This proves our first claim.

To prove the second claim, consider a pair of candidates $c, d \in C$ such that v_1 prefers c to d, but v_n prefers d to c. If $c \in C_R$, we have $x(v_1) < x(v_n) \leq x(c)$. Now, if v_1 prefers c to d, we have $x(c) < x(d)$, which implies that v_n, too, prefers c to d, a contradiction. If $d \in C_L$, we obtain a contradiction as well by a similar argument. □

Finally, once we have an ordering of candidates, finding a mapping x that witnesses that an election is 1-Euclidean and is consistent with this ordering reduces to solving a system of linear inequalities. A variant of this observation is due to Knoblauch [14]; however, Knoblauch's reduction produces a system of *strict* inequalities, and standard tools of linear programming are not directly applicable to such systems. The following proposition shows that for a fixed ordering of candidates our problem can be reduced to a system of *non-strict* inequalities.

Proposition 3. *There exists a polynomial-time algorithm that, given an election $E = (C, V)$ and an ordering of candidates $\lhd$, decides whether there exists a mapping $x : C \cup V \to \mathbb{R}$ that witnesses that E is 1-Euclidean and respects $\lhd$, i.e., such that for every pair of candidates $a, b \in C$ it holds that $x(a) < x(b)$ if and only if $a \lhd b$.*

Proof. We introduce a real variable x_v for each $v \in V$ and a real variable x_c for each $c \in C$; these variables encode the positions of voters and candidates on the real line. For every pair of candidates $a, b \in C$ such that $a \lhd b$ we introduce the inequality $x_a + 1 \leq x_b$. Also, for every voter $v \in V$, if v prefers a to b, we introduce the inequality $x_v + 1 \leq (x_a + x_b)/2$, and if v prefers b to a, we introduce the inequality $x_v \geq (x_a + x_b)/2 + 1$. Thus, altogether we introduce $(n + 1)m(m - 1)/2$ inequalities. It is easy to see that every feasible solution to this linear program (LP) describes a mapping x that respects $\lhd$ and witnesses that E is 1-Euclidean.

Conversely, if E is 1-Euclidean and this is witnessed by a mapping x that respects $\lhd$, this LP has a feasible solution. Indeed, set $\delta_1 = \min_{a,b \in C} |x(a) - x(b)|$, $\delta_2 = \frac{1}{2} \min_{a,b \in C, v \in V} |2x(v) - x(a) - x(b)|$. Note that $\delta_1 > 0$, since otherwise there would be a pair of candidates a, b with $x(a) = x(b)$ and then the voters would be indifferent between a and b. Similarly, $\delta_2 > 0$ since otherwise there would be a pair of candidates a, b and a voter v such that $|x(v) - x(a)| = |x(v) - x(b)|$, i.e., voter v would be indifferent between a and b. We can now set $\delta = \min\{\delta_1, \delta_2\}$ and $x_z = x(z)/\delta$ for all $z \in C \cup V$; it is easy to see that this provides a feasible solution to our LP. □

In what follows, we refer to the algorithm that takes a candidate set C, a voter list V and an ordering $\lhd$ of C as its input, and returns a mapping $x : C \cup V \to \mathbb{R}$ that corresponds to a feasible solution to our LP (or $\perp$ if this LP admits no feasible solution) as LP$(C, V, \lhd)$.

4 Algorithm

We are now ready to present our algorithm.

Theorem 1. *Given an election $E = (C, V)$ with $C = \{c_1, \ldots, c_m\}$, $V = (v_1, \ldots, v_n)$, we can decide in time polynomial in n and m whether E is 1-Euclidean, and, if so, construct a mapping x that witnesses this.*

Proof. We can assume without loss of generality that the voters' preferences are pairwise distinct. Indeed, if this is not the case, we can simply remove the "duplicate" voters: the resulting election is 1-Euclidean if and only if the original one is. Also, we can assume that $n > 1$, since otherwise the election is clearly 1-Euclidean.

We first verify that E is single-crossing and output $\perp$ (indicating that E is not 1-Euclidean) if this is not the case. From now on, we assume that E is single-crossing with respect to the voter order $(v_1, \ldots, v_n)$; note that by Proposition 1 this order is unique up to a reversal. We then execute Algorithm 1. This algorithm consists of three main stages. First, it colors the candidates red, green, blue, or grey based on the preferences of voter 1 and voter n (lines 2–14). We will argue that this is done in such a way that the set of red candidates is exactly C_M, blue candidates are contained in C_R, and green candidates are contained in C_L (see Proposition 2 for definitions of these sets). Then our algorithm defines a complete order $\lhd$ on the set C^+ that consists of all non-grey candidates (lines 16–24). This order is passed to the algorithm described in Proposition 3, which places the voters and the candidates in C^+ on the real line (lines 25–28). The result of this step is a mapping $x : V \cup C^+ \to \mathbb{R}$. Finally, the algorithm inserts the grey candidates. To this end, it partitions the grey and non-grey candidates into groups according to the order of their appearance in the preferences of the first voter (line 29). After placing the voters and the first group of each type (lines 32–33), it "stretches" x to ensure that different non-grey groups are well-separated (lines 34–39), and inserts the grey candidates into the appropriate spaces (lines 40–41).

Algorithm 1: 1-Euclidean

Input: a single-crossing election $E = (C, V)$.
Output: a mapping $y : C \cup V \to \mathbb{R}$ witnessing that E is 1-Euclidean, or $\bot$ if E is not 1-Euclidean.

$c^- \leftarrow \mathrm{top}(v_1)$, $c^+ \leftarrow \mathrm{top}(v_n)$;
foreach $c \in C$ **do**
  if $(c \succ_1 c^+$ **and** $c \succ_n c^-)$ **or** $c \in \{c^-, c^+\}$ **then**
    $\gamma(c) \leftarrow$ red
  else
    $\gamma(c) \leftarrow$ grey
foreach $a, b \in C$ **do**
  if $(a \succ_1 b$ **and** $b \succ_n a)$ **then**
    if $\gamma(a) =$ blue **or** $\gamma(b) =$ green **then**
      return $\bot$
    if $\gamma(a) =$ grey **then**
      $\gamma(a) \leftarrow$ green
    if $\gamma(b) =$ grey **then**
      $\gamma(b) \leftarrow$ blue
$C^+ \leftarrow \{c \mid \gamma(c) \neq \text{grey}\}$, $C^- \leftarrow \{c \mid \gamma(c) = \text{grey}\}$;
foreach $a, b \in C^+$ **do**
  if $(\gamma(a) =$ green **and** $\gamma(b) =$ red$)$ **or**
    $(\gamma(a) =$ red **and** $\gamma(b) =$ blue$)$ **or**
    $(\gamma(a) =$ green **and** $\gamma(b) =$ blue$)$ **then**
    set $a \lhd b$
  if $(\gamma(a) = \gamma(b) =$ red **and** $a \succ_1 b)$ **or**
    $(\gamma(a) = \gamma(b) =$ blue **and** $a \succ_1 b)$ **or**
    $(\gamma(a) = \gamma(b) =$ green **and** $b \succ_n a)$ **then**
    set $a \lhd b$
if $\mathrm{LP}(C^+, V|_{C^+}, \lhd) = \bot$ **then**
  return $\bot$
else
  $x \leftarrow \mathrm{LP}(C^+, V|_{C^+}, \lhd)$
Represent $\succ_1$ as $F_1 \succ_1 G_1 \succ_1 \ldots \succ_1 F_k \succ_1 G_k$, where $F_i \subseteq C^+$, $G_i \subseteq C^-$, $F_i \neq \emptyset$ for all $i \in [k]$, $G_i \neq \emptyset$ for all $i \in [k-1]$, $G_i = \{g^i_1, \ldots, g^i_{s_i}\}$, where $g^i_1 \succ_1 \ldots \succ_1 g^i_{s_i}$ for all $i \in [k]$;
$x^L \leftarrow \min_{t \in V \cup F_1} x(t)$, $x^R \leftarrow \max_{t \in V \cup F_1} x(t)$;
$\Delta \leftarrow \max_{t, t' \in V \cup C^+} |x(t) - x(t')|$;
foreach $t \in V \cup F_1$ **do** $y(t) \leftarrow x(t)$
foreach $g^1_\ell \in G_1$ **do** $y(g^1_\ell) \leftarrow x^R + 6\Delta + \frac{\ell}{m}\Delta$
foreach $i = 2, \ldots, k$ **do**
  foreach $c \in F_i$ **do**
    if $x(c) < x^L$ **then**
      $y(c) \leftarrow x(c) - (i+1)^2\Delta$
    if $x(c) > x^R$ **then**
      $y(c) \leftarrow x(c) + (i+1)^2\Delta$
  foreach $\ell = 1, \ldots, s_i$ **do**
    $y(g^i_\ell) \leftarrow x^R + (i+1)^2\Delta + 2\Delta + \frac{\ell}{m}\Delta$
return y;

The algorithm then returns the resulting mapping y. Note that the algorithm returns $\perp$ if the input election is not single-crossing, or the coloring stage cannot be completed (line 10), or the linear program does not have a feasible solution (line 26).

We will now argue that this algorithm is correct. Suppose that E is 1-Euclidean, and consider an arbitrary mapping z with $z(v_1) < z(v_n)$ that witnesses this. Let $c^- = \text{top}(v_1)$, $c^+ = \text{top}(v_n)$, and observe that $z(c^-) < z(c^+)$. Let $C_L = \{c \mid z(c) < z(c^-)\}$, $C_M = \{c \mid z(c^-) \leq z(c) \leq z(c^+)\}$, $C_R = \{c \mid z(c) > z(c^+)\}$. The first claim of Proposition 2 implies that the set of all red candidates is exactly C_M. Further, the second claim of Proposition 2 implies that if the conditions in line 8 are satisfied then $a \in C_L \cup C_M$ and $b \in C_M \cup C_R$. This explains why our algorithm outputs $\perp$ in line 10: if we have already decided that a is blue (and hence $a \in C_R$) or that b is green (and hence $b \in C_L$), we obtain a contradiction. Also, it follows that every green candidate belongs to C_L and every blue candidate belongs to C_R. Indeed, when we color a green in line 12, we know from line 8 that $a \in C_L \cup C_M$, and if a were in C_M, it would have been colored red already; the argument for blue candidates is similar.

Now, suppose that the algorithm has not output $\perp$ in line 10, i.e., we have consistently colored some set of candidates C^+ red, green, and blue. Clearly, we have $z(a) < z(b) < z(c)$ for any $a \in C_L$, $b \in C_M$, $c \in C_R$. Moreover, if $a, b \in C_M \cup C_R$ then $z(a) < z(b)$ if and only if $a \succ_1 b$ and if $a, b \in C_L$ then $z(a) < z(b)$ if and only if $b \succ_1 a$. Thus, the ordering $\lhd$ constructed in lines 16–24 is consistent with z. Note that z is an arbitrary mapping witnessing that E is 1-Euclidean such that $z(v_1) < z(v_n)$; our argument shows that every such mapping orders C^+ in the same way.

We now invoke Proposition 3. If E is 1-Euclidean, then so is $(C^+, V|_{C^+})$, and our mapping z witnesses this fact (clearly, z is consistent with $\lhd$). Thus, LP$(C^+, V|_{C^+}, \lhd)$ outputs some such mapping $x : V \cup C^+ \to \mathbb{R}$. On the other hand, if LP$(C^+, V|_{C^+}, \lhd) = \perp$, then there is no mapping witnessing that E is 1-Euclidean that is consistent with $\lhd$, and hence, as argued above, E is not 1-Euclidean.

Now, suppose that LP$(C^+, V|_{C^+}, \lhd)$ returned a mapping $x : V \cup C^+ \to \mathbb{R}$ witnessing that $(C^+, V|_{C^+})$ is 1-Euclidean. In line 29 we represent the preference ordering of the first voter as an alternating sequence of non-grey and grey blocks; the first block is non-grey since $c^- = \text{top}(v_1)$ is red. Since all blocks, except possibly the last grey block, are required to be non-empty, this representation is unique. If $G_1 = \emptyset$, we have $C = C^+$, so we are done. Thus, assume that $G_1 \neq \emptyset$. The following lemmas present some useful observations about the sets F_i and G_i for $i \in [k]$.

Lemma 1. *We have* $F_1 \succ_i G_1 \succ_i \ldots \succ_i F_k \succ_i G_k$ *for all* $i \in [n]$. *Moreover, if* $a, b \in C^-$ *then* $a \succ_i b$ *if and only if* $a \succ_1 b$.

Proof. Consider a candidate $a \in C^-$. Since a remained grey by the end of the coloring stage, v_1 and v_n agree on all comparisons involving a. Since E is single-crossing with respect to $(v_1, \ldots, v_n)$, this means that for every candidate $b \neq a$ either all voters prefer a to b or all voters prefer b to a. This immediately implies our second claim. For the first claim, consider a voter v_i and a pair of candidates $a \in F_j$, $b \in G_j$ for some $j \geq 1$. We have $a \succ_1 b$, so, by the argument above, $a \succ_i b$. Similarly, if $a \in G_j$, $b \in F_{j+1}$ for some j, $1 \leq j < k$, then $a \succ_1 b$, so, by the argument above, $a \succ_i b$. Now our claim follows by induction on j. □

Lemma 2. *For all $j = 2, \ldots, k$ and all $c \in F_j$ we have $x(c) \notin [x^L, x^R]$, where x^L, x^R are defined in line 30 of our algorithm.*

Proof. Fix a j with $2 \leq j \leq k$ and a candidate $c \in F_j$. Assume for the sake of contradiction that $x(c) \in [x^L, x^R]$. We consider the following four cases.

– $x^L = x(a)$, $x^R = x(b)$ for some $a, b \in F_1$. Then either $x(v_1) \leq x(c)$, in which case v_1 prefers c to b, or $x(v_1) > x(c)$, in which case v_1 prefers c to a.
– $x^L = x(a)$, $x^R = x(v_i)$ for some $a \in F_1$, $v_i \in V$. Then $c \succ_i a$.
– $x^L = x(v_i)$, $x^R = x(b)$ for some $b \in F_1$, $v_i \in V$. Then $c \succ_i b$.
– $x^L = x(v_i)$, $x^R = x(v_\ell)$ for some $v_i, v_\ell \in V$. Pick some $a \in F_1$ (note that $F_1 \neq \emptyset$). If $x(a) < x(c)$, then v_ℓ prefers c to a, and if $x(a) > x(c)$, then v_i prefers c to a.

In all cases, we obtain a contradiction, since by Lemma 1 we have $a \succ_i c$ for all $a \in F_1$ and all $i \in [n]$. □

Now, consider the mapping y returned in line 42. Fix an arbitrary voter v_i and a pair of candidates $a, b \in C$. To complete the proof of correctness, we will show that $|y(v_i) - y(a)| < |y(v_i) - y(b)|$ if and only if $a \succ_i b$. Note that the quantity Δ defined in line 31 satisfies $\Delta > 0$, since otherwise it would be the case that $x(v_1) = x(v_n)$, in which case all voters have the same preference order, and we assumed that this is not the case.

It suffices to consider the following six cases; the remaining cases follow by the transitivity of $\succ_i$.

– $a, b \in F_1$. Then we have $y(a) = x(a)$, $y(b) = x(b)$, $y(v_i) = x(v_i)$. Since $a, b \in C^+$ and x is a witness that $(C^+, V|_{C^+})$ is 1-Euclidean, our claim follows.
– $a, b \in F_j$, $j \geq 2$. We have $y(v_i) \in [x^L, x^R]$, and $y(a), y(b) \notin [x^L, x^R]$ by Lemma 2. Thus,

$$|y(v_i) - y(a)| = |x(v_i) - x(a)| + (j+1)^2\Delta, \ |y(v_i) - y(b)| = |x(v_i) - x(b)| + (j+1)^2\Delta$$

(lines 35–39). Again, our claim follows, since $a, b \in C^+$ and x is a witness that $(C^+, V|_{C^+})$ is 1-Euclidean.
– $a, b \in G_j$, $j \geq 1$. Assume without loss of generality that $a \succ_i b$. Then by Lemma 1 we have $a \succ_1 b$ and hence $x^R < y(a) < y(b)$ (lines 33 and 41). Since $x(v_i) \leq x^R$, our claim follows.
– $a \in F_1$, $b \in G_1$. By Lemma 1 we have $a \succ_i b$. On the other hand, $y(a) \in [x^L, x^R]$, $y(v_i) \in [x^L, x^R]$, whereas $y(b) > x^R + 6\Delta$. Since $|x^R - x^L| \leq \Delta$, the claim follows.
– $a \in F_j$, $b \in G_j$, $j > 1$. By Lemma 1 we have $a \succ_i b$. Since $j > 1$, we have $x(a) \in [x^L - \Delta, x^R + \Delta] \setminus [x^L, x^R]$, so we have

$$\begin{aligned} y(a) &\in [x^R + (j+1)^2\Delta, x^R + (j+1)^2\Delta + \Delta] \\ &\quad \cup [x^L - (j+1)^2\Delta - \Delta, x^L - (j+1)^2\Delta], \\ y(b) &\in [x^R + (j+1)^2\Delta + 2\Delta + \frac{\Delta}{m}, x^R + (j+1)^2\Delta + 3\Delta]. \end{aligned}$$

Since $y(v_i) \in [x^L, x^R]$, we have

$$|y(a) - y(v_i)| \le (j+1)^2\Delta + 2\Delta, \quad |y(b) - y(v_i)| > (j+1)^2\Delta + 2\Delta,$$

and our claim follows.

– $a \in G_j$, $b \in F_{j+1}$, $1 \le j < k$. Again, by Lemma 1 we have $a \succ_i b$, and $x(b) \in [x^L - \Delta, x^R + \Delta] \setminus [x^L, x^R]$, so

$$\begin{aligned} y(a) &\in [x^R + (j+1)^2\Delta + 2\Delta + \frac{\Delta}{m}, x^R + (j+1)^2\Delta + 3\Delta], \\ y(b) &\in [x^R + (j+2)^2\Delta, x^R + (j+2)^2\Delta + \Delta] \\ &\quad \cup [x^L - (j+2)^2\Delta - \Delta, x^L - (j+2)^2\Delta]. \end{aligned}$$

Hence,

$$|y(a) - y(v_i)| \le (j+1)^2\Delta + 4\Delta, \quad |y(b) - y(v_i)| \ge (j+2)^2\Delta;$$

since $(j+1)^2 + 4 < (j+2)^2$ for all $j \ge 1$, our claim follows.

It remains to analyze the running time of our algorithm. One can check whether an election is single-crossing and, if so, determine the voter order that witnesses this, in time $O(nm^2)$ [6]. Further, our procedures for coloring the candidate set and constructing the order $\lhd$ run in time $O(m^2)$. The algorithm described in the proof of Proposition 3 is based on solving a linear program with coefficients in $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$, $O(n+m)$ variables, and $O(nm^2)$ constraints. Finally, the mapping y is computed by performing a constant number of arithmetic operations for each voter or candidate, and these operations involve numbers that form a feasible solution to our linear program. Thus, the overall running time of our algorithm is polynomial in n and m, and is dominated by the time needed to solve the linear program. □

Remark 1. Note that, while we chose to place the candidates in G_i to the right of the candidates in F_i, we could have also placed them to the left of the candidates in F_i. Further, instead of dealing with an entire grey block in a single step, we could have processed the grey candidates one by one. This shows that, after the end of the coloring stage, we can arbitrarily color all grey candidates green or blue, use this coloring to construct an order $\lhd$ on C, and apply Proposition 3 to E and $\lhd$. While the resulting algorithm is simpler, it may require solving a much larger linear program.

Remark 2. The reader may wonder if stretching x (lines 34–39) is necessary to place the candidates in C^-: perhaps we can find suitable positions for them without modifying the positions of the candidates in C^+? The following example shows that this is not always the case. Consider an election $E = (C, V)$ with $C = \{a, b, c, d, e\}$, $V = (u, v)$, where u ranks the candidates as $a \succ b \succ c \succ d \succ e$ and v ranks the candidates as $b \succ a \succ c \succ e \succ d$. For this election we have $F_1 = \{a, b\}$, $G_1 = \{c\}$, $F_2 = \{d, e\}$ and the ordering $\lhd$ over C^+ is given by $d \lhd a \lhd b \lhd e$. A feasible solution to the corresponding linear program is $x_u = -2$, $x_v = 2$, $x_a = -6$, $x_b = 6$, $x_d = -12$, $x_e = 12$. Now, suppose that we want to place c on the real line without changing the positions of other points. Since our construction is symmetric, we can assume without loss of generality that c should be placed to the right of 0. Since v prefers a to c, it has to be the case that $x_c > 10$. However, this means that u prefers d to c, a contradiction.

However, one can eliminate the stretching steps by adding constraints saying that different non-grey blocks are well-separated to the linear program itself. Then each grey block can be simply placed in the middle of the respective interval.

5 An Overview of Knoblauch's Algorithm

The main difference between our algorithm and that of Knoblauch is that the latter uses a single-peaked ordering of the candidates as its starting point. It then partitions the candidates into groups so that, for each group, the ordering of the candidates in this group is the same (up to reversal) for all societal axes witnessing that the election is single-peaked. This partition is fairly straightforward to derive from the votes, and can be shown to be a refinement of the partition $\{F_1, G_1, \ldots, F_k, G_k\}$ implicitly constructed by our algorithm. The groups that correspond to subsets of C^+ are then "glued together", i.e., the algorithm defines an ordering on C^+; this procedure is the heart of the algorithm, and is quite complicated. From this point on, Knoblauch's algorithm proceeds in the same manner as our algorithm: it uses a linear program to embed C^+ and V into the real line, and then places the candidates from C^-. However, both of these steps are implemented somewhat differently. In more detail, Knoblauch's linear program only has variables for elements of C^+, and the number of inequalities in it is bounded by $O(nm^4)$; effectively, it is obtained from our linear program by variable elimination. While it uses strict inequalities, it is not hard to modify it so that only non-strict inequalities are used (see Proposition 3). Finally, Knoblauch's algorithm places the candidates in C^- one by one rather than in blocks; whenever a candidate in C^- is placed, some of the candidates in C^+ are shifted by an unspecified "large enough" amount. As a consequence, candidates in C^+ may be shifted multiple times.

In terms of performance, neither algorithm has a clear advantage over the other: the running time of both algorithms is dominated by solving a linear program, and the two linear programs are closely related. Thus, our main contribution is conceptual: we provide a quick and simple method for obtaining an ordering of the non-grey candidates that is based on the single-crossing property of 1-Euclidean elections. We find it remarkable that our algorithm does not use the fact that a 1-Euclidean election is single-peaked, whereas Knoblauch's paper does not mention single-crossing elections at all; thus, the two approaches provide very different perspectives on the problem at hand.

6 Future Work

We have presented an alternative algorithm for recognizing whether an election is 1-Euclidean. Both our algorithm and that of Knoblauch rely on solving a linear program. A natural question is whether this step can be eliminated, i.e., whether our problem admits a purely combinatorial algorithm.

Further, it would be interesting to see if our algorithm (or that of Knoblauch) can be extended to higher dimensions, i.e., the problem of recognizing whether an election is d-Euclidean for $d > 1$. We remark that, while d-Euclidean elections with $d > 1$ are not particularly attractive from a purely social choice-theoretic perspective (e.g., such elections are not guaranteed to have a Condorcet winner), it is plausible that they may admit efficient algorithms for problems in computational social choice that are known to be hard on the general domain.

Another promising direction is to explore whether our ideas can be used to identify elections that are close to being 1-Euclidean, for an appropriate notion of distance. A challenge that one would need to cope with in this context is that an "almost Euclidean" election need not be single-peaked or single-crossing. One can also consider a variant of the 1-Euclidean model where a voter can be equidistant from two different candidates, in which case she may prefer either of these candidates, but the positions of all candidates are required to be pairwise distinct, and ask whether preference profiles that are 1-Euclidean in this sense can be recognized in polynomial time.

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Clearing Markets via Bundles

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Abstract. We study algorithms for combinatorial market design problems, where a set of heterogeneous and indivisible objects are priced and sold to potential buyers subject to equilibrium constraints. Extending the CWE notion introduced by Feldman et al. [STOC 2013], we introduce the concept of a *Market-Clearing Combinatorial Walrasian Equilibium* (MC-CWE) as a natural relaxation of the classical Walrasian equilibrium (WE) solution concept. The only difference between a MC-CWE and a WE is the ability for the seller to bundle the items prior to sale. This innocuous and natural bundling operation imposes a plethora of algorithmic and economic challenges and opportunities. Unlike WE, which is guaranteed to exist only for (gross) substitutes valuations, a MC-CWE always exists. The main algorithmic challenge, therefore, is to design computationally efficient mechanisms that generate MC-CWE outcomes that approximately maximize social welfare. For a variety of valuation classes encompassing substitutes and complements (including super-additive, single-minded and budget-additive valuations), we design polynomial-time MC-CWE mechanisms that provide tight welfare approximation results.

1 Introduction

The *resource allocation* problem lies at the heart of theoretical economics: how should scarce resources be allocated among individual agents with competing interests? Since the emergence of the Internet, which enables complex resource allocation on a grand scale, this has naturally become a central problem in computer science as well. Economists generally approach this problem by adopting the notion of *market equilibrium*. Broadly speaking, a market equilibrium is a set of resource prices that are stable in the sense that all agents are maximally happy with their allocations and no resources are left unallocated. A long line of work has been dedicated to addressing the existence of equilibrium prices, and it has been shown (see, e.g., [2]) that market equilibria exist very generally, as long as the market is *convex*.

While this result sounds appealing in its generality, the convexity assumption usually requires that resources be infinitely divisible. In many applications of interest, especially those with a computational aspect, resources are indivisible; in these cases the convexity assumption is inapplicable. Do the results from the convex environments carry over to non-convex environments? In general the answer is no: the existence of equilibrium prices is not guaranteed. As a result, the study of markets for indivisible goods tends to focus on specific cases for which such prices exists, such as when buyer values satisfy the *gross substitutes* condition.

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To be more precise, consider the following model. There are m indivisible, heterogeneous items to be allocated among n agents. Each agent $i \in [n]$ has preferences over bundles of items in the form of a valuation function v_i that maps every subset S of items into the value $v_i(S)$ that agent i derives from the bundle S (in monetary terms). Given a price vector $\mathbf{p} = (p_1, \ldots, p_m)$, a bundle S is said to be in agent i's *demand set* if S maximizes i's *utility* given $\mathbf{p}$, defined as the difference between $v_i(S)$ and $\sum_{j \in S} p_j$.

A *Walrasian equilibrium* (WE) is an assignment of *item prices* to the m items, and an assignment of the objects to the agents, such that: (1) every agent is allocated a bundle in his demand set, and (2) the market *clears*; i.e., all items are allocated. Such a solution is appealing; every agent is maximally happy despite competing preferences, the market clears, and the pricing structure is natural, simple, and transparent. Unfortunately, WE do not exist in general. A WE is guaranteed to exist only for the rather narrow class of gross substitutes (GS) valuations (a strict subset of submodular functions) [11]. This eliminates any hope for the applicability of WE to environments with valuations that exhibit complementarities, and many forms of substitutes as well.

Recently, [8] proposed a relaxation of WE, termed a *combinatorial Walrasian equilibrium* (CWE). In a CWE, the seller can choose to *bundle* objects prior to assigning prices. This is a natural power to afford the seller, since as the owner of the resources he has some inherent ability to define what is meant by an "item." The generated bundles induce a reduced market — a market in which the items for sale are the bundles generated by the seller. In addition to the bundling operation, the CWE further relaxes the WE notion in that it allows for items to remain unallocated (even when they are priced above zero). A CWE exists for any valuation profile, since the seller could bundle all objects into a single item. The important issue, then, is whether there exists a CWE that is (approximately) efficient with respect to social welfare. Indeed, [8] show there always exists a CWE with at least half of the optimal (unconstrained) welfare.

The CWE notion relaxes the WE notion in two ways: (i) it allows bundling, (ii) it does not require market clearance. While the bundling relaxation is central to the notion of CWE, the second relaxation warrants some discussion. The relaxation of market clearance is somewhat at odds with the notion of a two-sided market equilibrium: prices might not be stable from the seller's perspective. After all, if an object (i.e. bundle) does not sell, the seller may be tempted to decrease its price in order to to sell it and increase revenue. The concept of CWE therefore implicitly requires that the seller pre-commit to (sub-optimal) prices, in addition to committing to a bundling of the items. With this in mind, we consider whether the relaxation of market clearance is truly necessary. It is easy to see that the bundling relaxation alone is enough to guarantee existence of an equilibrium, so the question becomes one of welfare. Can we hope to achieve the welfare bound of [8] without relaxing market clearance?

To answer this question we define *Market-clearing CWE* (MC-CWE), which allows the bundling operation, but requires market clearance. A MC-CWE is precisely a WE over the reduced market; it differs only in the ability of the seller to pre-bundle the items, and in particular it is a stronger (more restrictive) concept than CWE.

For a number of valuation classes, encompassing both substitutes and complements, we provide two types of results. The first finds the fraction of the optimal social welfare that can be obtained in a MC-CWE outcome. The second addresses the same problem but under the additional requirement of operating in polynomial time. Note that the approximation result established in [8] is only semi-computational — given the optimal allocation, it finds in polynomial time a CWE outcome that gives at least a half of the optimal welfare. Here, we devise polynomial approximation algorithms that do not need any initial allocation. Moreover, all of our approximation results match the computational lower bounds for their corresponding valuation classes.

We note that while the focus of our paper is welfare maximization, our analysis and results have some immediate implications on revenue maximization. In particular, all of our approximation results carry over to revenue approximation, since for each of our mechanisms the bound we obtain on the social welfare is precisely the revenue extracted at equilibrium. Further discussion of MC-CWE revenue appears in the full version of the paper.

1.1 Our Results and Techniques

	Uniform BA identical budgets	Uniform BA	Super additive	Single minded
MC-CWE gap	1	$\geq$8/7 $\leq$2	1	1
Poly-time MC-CWE approx.	$\leq 4/3$	$\leq 8/3$	$O(m/\sqrt{\log m})$ [value] $\theta(\sqrt{m})$ [demand]	$\theta(\sqrt{m})$

Fig. 1. Summary of our approximation results. The columns correspond to valuation classes. The first column corresponds to uniform budget additive valuations with identical budgets, and the second column corresponds to uniform budget additive valuations with arbitrary budgets. The first row corresponds to the gap in social welfare due to MC-CWE, disregarding computational considerations. The second row corresponds to the approximation that can be achieved with a MC-CWE poly-time mechanism. All approximation results assume the value-query model, unless otherwise stated. Note that m is the number of items for sale.

Super-additive valuations. In the case where agent valuations are super-additive, we show that there always exists a MC-CWE that maximizes social welfare. Note that it is not always possible to maximize social welfare without bundling, even if the market clearance requirement is relaxed: there exist input instances in which all bidders are single-minded, but every outcome with item pricing obtains only an $O(\sqrt{m})$ fraction of the optimal social welfare [8]. The use of bundling is therefore crucial in generating a socially efficient equilibrium outcome.

We next turn to computational algorithms. We show how to construct a MC-CWE that obtains an $O(\sqrt{m})$ approximation to the optimal social welfare in a polynomial number of *demand queries*[1], matching known lower bounds [16]. Our mechanism proceeds by first crafting an $O(\sqrt{m})$-approximate allocation and prices, then applying local search to repeatedly satisfy agent demands (bundling objects and/or raising prices in the process) until every agent obtains a demanded set at the given prices. Our construction makes use of demand queries in a way similar to that of [8]: rather than querying demand sets over the original space of objects, we query demand over bundles of objects

[1] A demand query returns the utility-maximizing set given a vector of item prices.

(under linear prices). With *value queries*, we show that the $O(m/\sqrt{\log m})$-approximate mechanism due to [13] satisfies the MC-CWE property. We also show that in the case of *single-minded valuations*, our demand-query mechanism can be modified to achieve an $O(\sqrt{m})$ approximation in a polynomial number of value queries.

Sub-additive valuations: uniform budget additive. Since efficient WE exist for the class of GS valuations, efficient MC-CWE exist for this class as well. We therefore consider a class of non-GS valuations: those that are *uniform budget-additive*. In this class, each item a has a value v_a, and each agent values the item at either 0 or v_a. Furthermore, each bidder has a budget that limits his value for any set of items. For this class, we demonstrate that WE do not necessarily exist. Moreover, we provide an instance in which no MC-CWE can achieve more than a $7/8$ fraction of the optimal welfare. On the other hand, we show that any allocation can be converted (in polynomial time) into a MC-CWE outcome that achieves at least half of the original social welfare. Thus, at least half of the optimal welfare can always be achieved in a MC-CWE outcome.

Turning to computational considerations, the welfare-maximization problem for this valuation class is known to be APX-hard. The best-known algorithm achieves an approximation of $4/3$ (see [1,3,18,6]). Combined with the aforementioned algorithm, which converts every outcome to a MC-CWE outcome with at least half of the welfare, this implies a MC-CWE mechanism that achieves a $8/3$ approximation. Our analysis is based on the observation that an outcome can be implemented at MC-CWE if and only if that outcome is an optimal solution (among all fractional solutions) to a certain linear program: the *configuration LP* for the assignment problem restricted to the bundles in the outcome allocation. Our construction is based upon local search, but of a different nature than our super-additive mechanism. Rather than attempting to improve social welfare, we repeatedly bundle objects to *reduce* the optimal *fractional* welfare, shrinking the gap between fractional and integral solutions to the configuration LP.

If agents have identical budgets, the factor-2 loss disappears: any allocation can be made MC-CWE without loss in social welfare. Yet, even within this restricted class, a Walrasian equilibrium may not exist. These results are driven by connections between MC-CWE and the configuration LP for the combinatorial assignment problem.

1.2 Relation to Prior Work

There is a long line of work studying pricing equilibria in theoretical economics. Walrasian equilibria (i.e., market-clearing prices) in the market assignment problem were studied by Shapley and Shubik [17]. Further characterizations of existence of Walrasian equilibria were studied in, for example, [14,4,11]. Our work is motivated by the non-existence of Walrasian equilibrium in general combinatorial markets.

The problem of *algorithmic pricing* [12] is to find revenue-optimal item prices for unit-demand buyers. In contrast, we are primarily interested in maximizing social welfare for more general valuations, and we permit bundling of items before setting prices.

An alternative to Walrasian equilibrium is to allow a seller to set (non-linear) prices on arbitrary bundles. Such package auctions were formalized by Bikhchandani and Ostroy [5]. Our notion of MC-CWE differs from package auctions in that the seller commits to a partition of the objects, then sets linear prices over those bundles.

The concept of MC-CWE can be viewed as a strengthening of envy-freeness since, in particular, we require that no agent envies any *subset* of other agents. Fiat et al. [9] study this extension of envy-freeness directly, which they term multi-envy-freeness.

Fu, Kleinberg and Lavi [10] introduce the notion of *conditional equilibrium*, where no buyer wishes to add additional items to his allocation under given prices. Our equilibrium concept MC-CWE differs in that it is based on reducing the space of objects via bundling, rather than directly relaxing the demand-satisfaction condition of WE.

2 Preliminaries

The auction setting considered in this work consists of a set M of m indivisible objects and a set of n agents. Each agent has a valuation function $v_i(\cdot) : 2^M \to \mathbb{R}_{\geq 0}$ that indicates his value for every set of objects, is non-decreasing (i.e., $v_i(S) \leq v_i(T)$ for every $S \subseteq T \subseteq M$) and is normalized so that $v_i(\emptyset) = 0$. The profile of agent valuations is $\mathbf{v} = (v_1, \ldots, v_n)$, and an auction setting is defined by $A = (M, \mathbf{v})$.

A price vector $\mathbf{p} = (p_1, \ldots, p_m)$ consists of a price p_j for each object $j \in M$. An *allocation* is a vector of sets $\mathbf{x} = (x_0, x_1, \ldots, x_n)$, where $x_i \cap x_k = \emptyset$ for every $i \neq k$, and $\bigcup_{i=0}^n x_i = M$. In the allocation $\mathbf{x}$, for every $i \in N$, x_i is the bundle assigned to agent i, and x_0 is the set of unallocated objects; i.e., $x_0 = M \setminus \bigcup_{i=1}^n x_i$.

We assume that each agent has a quasi-linear utility function; i.e., the utility of agent i being allocated bundle x_i under prices $\mathbf{p}$ is $u_i(x_i, \mathbf{p}) = v_i(x_i) - \sum_{j \in x_i} p_j$. Given prices $\mathbf{p}$, the *demand correspondence* $D_i(\mathbf{p})$ of agent i contains all sets of objects that maximize agent i's utility: $D_i(\mathbf{p}) = \left\{S^* : S^* \in \text{argmax}_{S \subseteq M}\{u_i(S, \mathbf{p})\}\right\}$. A tuple $(\mathbf{x}, \mathbf{p})$ is said to be *buyer stable* for auction $A = (M, \mathbf{v})$ if $x_i \in D_i(\mathbf{p})$ for every $i \in N$. A tuple $(\mathbf{x}, \mathbf{p})$ is said to be *seller stable* for auction $A = (M, \mathbf{v})$ if for every $j \in x_0$, $p_j = 0$. The seller stability condition is also known as *market clearance*.

A tuple $(\mathbf{x}, \mathbf{p})$ is said to be a *Walrasian equilibrium* (WE) for auction $A = (M, \mathbf{v})$ if it is both buyer stable and seller stable. The class of *gross substitutes* (GS) valuations is a maximal class that admits Walrasian equilibria [11]. This class is a strict subset of submodular valuations. It is also known that if a WE exists, it is economically efficient (i.e., maximizes social welfare — the sum of agents' valuations).

We now define *Market-Clearing CWE* (MC-CWE). The crux of the concept is that items are pre-partitioned into indivisible bundles. The constructed bundles are treated as indivisible objects, and the MC-CWE notion reduces to WE over the bundles. Crucially, although prices are now associated with bundles (of the original market), unlike previous notions, prices are linear once bundles are fixed. A formal definition follows.

For a partition $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_k)$ of the item set M we slightly abuse notation and denote by $\mathbf{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$ the reduced set of items, where the valuation of each agent i of a subset $S \subseteq \mathbf{A}$ is $v_i(\bigcup_{j: \mathbf{A}_j \in S} \mathbf{A}_j)$. We denote by $A_{\mathbf{A}}$ an auction over the reduced set of items $\mathbf{A}$ with the induced valuation profile.

Every allocation $\mathbf{x}$ induces a partition $\mathbf{A}(\mathbf{x}) = (x_0, \ldots, x_n)$. A tuple $(\mathbf{x}, \mathbf{p})$, where $\mathbf{x} = (x_0, \ldots, x_n)$, and p_i is the price of x_i for every $x_i \neq \emptyset$, is a *Market-Clearing CWE* (MC-CWE) if $(\mathbf{x}, \mathbf{p})$ is a WE in the auction $A_{\mathbf{A}(\mathbf{x})}$. Allocation $\mathbf{x}$ is said to be MC-CWE if it admits a price vector $\mathbf{p} \in \mathbb{R}_{\geq 0}^{n+1}$ such that $(\mathbf{x}, \mathbf{p})$ is MC-CWE. A mechanism is MC-CWE if it maps every valuation profile $\mathbf{v}$ to an MC-CWE outcome $(\mathbf{x}, \mathbf{p})$.

A tuple $(\mathbf{x}, \mathbf{p})$, where $\mathbf{x} = (x_0, \ldots, x_n)$, and p_i is the price of x_i for every $x_i \neq \emptyset$, is a CWE if $(\mathbf{x}, \mathbf{p})$ is buyer stable in the auction $A_{\mathbf{A}(\mathbf{x})}$. Note that a MC-CWE is weaker than a WE, since it allows for a pre-sale partition of the goods. On the other hand, MC-CWE is stronger than a CWE, since it requires seller stability on top of buyer stability.

2.1 Characterization

The characterization of a CWE allocation is closely related to the characterization of an allocation that can be supported in a WE [4]. A similar observation was already given in [8], but we state it here for completeness, as it is used in later sections.

For a given partition $\mathbf{A}$ of the objects, the allocation of $\mathbf{A}$ to N can be specified by a set of integral variables $y_{i,S} \in \{0, 1\}$, where $y_{i,S} = 1$ if the set $S \subseteq \mathbf{A}$ is allocated to agent $i \in N$ and $y_{i,S} = 0$ otherwise. These variables should satisfy the following conditions: $\sum_S y_{i,S} \leq 1$ for every $i \in N$ (each agent is allocated to at most one bundle) and $\sum_{i,S \supseteq \mathbf{A}_j} y_{i,S} \leq 1$ for every $\mathbf{A}_j \in \mathbf{A}$ (each element of the partition is allocated to at most one agent). A *fractional allocation* of $\mathbf{A}$ is given by variables $y_{i,S} \in [0, 1]$ that satisfy the same conditions and intuitively might be viewed as an allocation of divisible items. The configuration LP for $A_{\mathbf{A}}$ is given by the following linear program, which computes the fractional allocation that maximizes social welfare.

$$\begin{aligned} \max \quad & \sum_{i,S} v_i(S) \cdot y_{i,S} \\ \text{s.t.} \quad & \sum_S y_{i,S} \leq 1 \text{ for every } i \in N \\ & \sum_{i,S \supseteq \mathbf{A}_j} y_{i,S} \leq 1 \text{ for every } \mathbf{A}_j \in \mathbf{A} \\ & y_{i,S} \in [0, 1] \text{ for every } i \in N, S \subseteq \mathbf{A} \end{aligned}$$

The characterization given in [4] states that a WE exists if and only if the optimal fractional solution to the allocation LP occurs at an integral solution. This characterization of a WE allocation can be used to derive a characterization of a MC-CWE allocation.

Recall that every allocation $\mathbf{x}$ induces a partition $\mathbf{A}(\mathbf{x}) = (x_0, \ldots, x_n)$. The WE characterization implies the following MC-CWE characterization.

Claim. An allocation $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is a MC-CWE for A iff the configuration LP for $A_{\mathbf{A}(\mathbf{x})}$ has an integral optimal solution that sets $y_{i,x_i} = 1$ for all $i \in N$.

By the last claim, finding a MC-CWE allocation is equivalent to finding bundles such that the optimal welfare generated by fractional and integral allocations *of those bundles* are identical, then returning an efficient allocation of those bundles.

Gap in social welfare due to MC-CWE: a lower bound. In [8], an example was given in which the welfare of any CWE allocation is at most $2/3$ of the welfare of an optimal (non-CWE) allocation. We note that the same example carries over to the stronger notion of MC-CWE; i.e., the welfare of an MC-CWE allocation is at most $2/3$ of the optimal welfare. The established lower bound holds for general valuations, but disappears for some families of valuation functions (as will be shown in the sequel).

3 Super-additive Valuations

A valuation v is super-additive if, for all sets of items S and T, $v(S) + v(T) \geq v(S \cup T)$. In this section we study MC-CWE outcomes when valuations are super-additive. Missing proofs from this section appear in the full version. We first show that there is no loss in efficiency due to MC-CWE: every efficient allocation is MC-CWE.

Allocation $\mathbf{x}$ is *bundle-efficient* for $\mathbf{v}$ if, for all functions $\beta : [n] \to [n]$, we have $\sum_i v_i(x_i) \geq \sum_i v_i \left(\bigcup_{j \in \beta^{-1}(i)} x_j \right)$. That is, a bundle-efficient allocation $\mathbf{x}$ maximizes social welfare among all ways to allocate the bundles $x_1, \ldots, x_n$ to the agents.

Theorem 1. *If agents are super-additive and* $\mathbf{x}$ *is a bundle-efficient allocation, then the price vector* $p_i = v_i(x_i)$ *is such that* $(\mathbf{x}, \mathbf{p})$ *is MC-CWE.*

Proof. Pick any agent i and set of agents S. If $i \in S$, then

$$v_i(x_i) - p_i = 0 \geq v_i(\cup_{j \in S} x_j) - \sum_{j \in S} v_j(x_j) = v_i(\cup_{j \in S} x_j) - \sum_{j \in S} p_j,$$

where the inequality follows from the efficiency of allocation $\mathbf{x}$. Consider next the case where $i \notin S$. It holds that

$$v_i(x_i) + \sum_{j \in S} v_j(x_j) \geq v_i(\cup_{j \in S \cup \{i\}} x_j) \geq v_i(x_i) + v_i(\cup_{j \in S} x_j),$$

where the first inequality follows from the efficiency of allocation $\mathbf{x}$, and the second inequality follows from the super additivity of v_i. It follows that $v_i(x_i) - p_i = 0 \geq v_i(\cup_{j \in S} x_j) - \sum_{j \in S} v_j(x_j) = v_i(\cup_{j \in S} x_j) - \sum_{j \in S} p_j$, as desired. ■

A consequence of Theorem 1 is that the full surplus can be extracted as revenue.

The use of bundling is necessary for the statement of Theorem 1, even if we relax the market-clearing requirement of Walrasian equilibrium and even for single-minded bidders. There are instances for which, given any set of item prices for which agent demand sets are disjoint, the social welfare of the resulting allocation is an $O(\sqrt{m})$ fraction of the optimal social welfare [8]. We describe an example in the full version.

3.1 Polynomial-Time Mechanisms

We next study the power of poly-time approximation mechanisms for maximizing social welfare in MC-CWE outcomes (compared to the optimal welfare that can be achieved by any mechanism, poly-time or not, MC-CWE or not). Of particular interest is the question whether the MC-CWE requirement entails an additional loss on top of the loss incurred due to the poly-time requirement alone. In our analysis, we distinguish between mechanisms that operate in the value-query and demand-query models, as is standard in the literature. We find that in both models, it is possible to construct a MC-CWE mechanism with an approximation factor matching that of the best-known approximation algorithms for welfare maximization. In particular, there exist poly-time

MC-CWE mechanisms that achieve an $O(\sqrt{m})$ approximation under the demand-query model, and an $O(m/\sqrt{\log m})$ approximation under the value-query model.

We first present a MC-CWE approximation mechanism for superadditive valuations using demand queries. This mechanism, which we call SuperAdditiveMC-CWE, proceeds in two phases. In the first phase, it builds a preliminary solution by repeatedly allocating the set that maximizes *value density*. That is, set S is allocated to agent i so that $v_i(S)/|S|$ is maximized, and this process is then iterated on the remaining items. A bidder can be allocated to multiple times in this phase, in which case she is allocated the union of the assigned sets. After all objects have been allocated, we check whether the welfare can be improved by allocating all objects to a single player; if so, we do so and the mechanism ends. Otherwise we proceed to phase 2, where we repeatedly apply local improvements to the allocation. Specifically, if we write $(x_1, \ldots, x_n)$ for the tentative allocation, we look for circumstances in which some player i has more value for a set of bundles from among $\{x_1, \ldots, x_n\}$ than the players to whom those bundles were previously assigned; if such a case exists, we bundle all of these items together and reallocate them to player i, then repeat the process with this updated tentative allocation. Note that this step amounts to repeatedly satisfying the demand of a player in the market with items $\{x_1, \ldots, x_n\}$ and prices $p_i = v_i(x_i)$, until no further demands are made (which must occur since these prices only increase). When this process terminates we return the resulting allocation. The following theorem establishes the $O(\sqrt{m})$ approximation and polynomial run time of the algorithm. Due to space constraints, we defer its proof, along with a pseudocode listing of SuperAdditiveMC-CWE, to the full version of the paper.

Theorem 2. *Algorithm SuperAdditiveMC-CWE returns a MC-CWE outcome that $O(\sqrt{m})$-approximates the optimal social welfare over all assignments. Furthermore, it can be implemented in a polynomial number of demand queries.*

We next move to the value-query model. We show that the $O(m/\sqrt{\log m})$ approximation mechanism due to [13] is guaranteed to generate CWE outcomes. We note that it nearly matches the lower bound of $\Omega(m/\log m)$ on the approximability of CAs with superadditive bidders (using value queries) [15].

Theorem 3. *If agents are super-additive, then there exists a mechanism that makes a polynomial number of value queries and generates a CWE outcome that achieves a $O(m/\sqrt{\log m})$ approximation to the optimal social welfare.*

Proof. The mechanism groups the objects into $\log m$ bundles, each of size $m/\log m$, arbitrarily. It then returns the bundle-efficient allocation over those bundles. This achieves a $O(m/\sqrt{\log m})$ approximation, and can be implemented in a polynomial number of value queries [13]. Since the allocation is bundle-efficient, it is CWE. ■

3.2 Single-minded Valuations and Value Queries

In the special case in which agents are single-minded, Algorithm SuperAdditiveMC-CWE can be improved to run in a polynomial number of *value* queries, obtaining an

$O(\sqrt{m})$ approximation to the optimal welfare. This mechanism is new; as far as we are aware, existing $O(\sqrt{m})$-approximation mechanisms do not satisfy MC-CWE.

Our algorithm, which we call SingleMindedMC-CWE, proceeds as follows. We split the bidders into two groups: those with "small" desired sets (of size at most $\sqrt{m}$) and those with larger desired sets. We first generate a provisional allocation that only includes the bidders with small desired sets. We construct this preliminary allocation greedily: we order players from largest value to smallest, then allocate to players in this order if their desired set is available. Any object that is left unallocated is then added to an arbitrary *non-empty* allocation. Then, in the second phase of the algorithm, we consider those bidders with large desired sets. We order these large-set bidders from highest value to smallest, and for each bidder i in this order, say with value v_i for set S_i, we ask whether v_i is greater than the sum of values of all players whose allocations intersect set S_i. If so, we take all of those intersecting allocations from their respective bidders and allocate them all to player i. After this operation has been completed for every large-set bidder (in order from highest value to smallest), we return the resulting allocation. The following theorem establishes the $O(\sqrt{m})$ approximation and polynomial run time of the algorithm. Due to space constraints, we defer its proof, along with a pseudocode listing of SingleMindedMC-CWE to the full version of the paper.

Theorem 4. *When agent valuations are single-minded, Algorithm SingleMindedMC-CWE returns a MC-CWE outcome that $O(\sqrt{m})$-approximates the optimal social welfare over all assignments. Furthermore, it can be implemented in a polynomial number of value queries.*

4 Uniform Budget-Additive Valuations

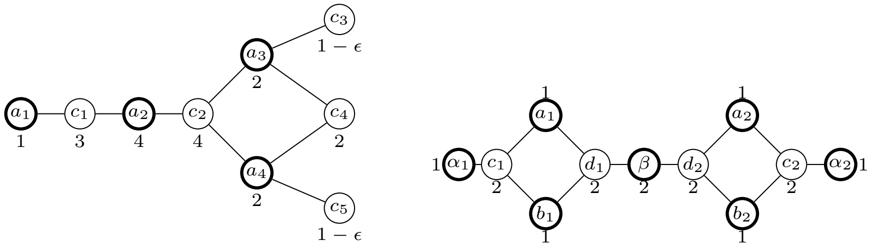
A budget-additive valuation is specified by budget B and item values v_j, $j \in M$. The value of set S is then $v(S) = \min\{B, \sum_{j \in S} v_j\}$. Note that the uniform-valuation case implies $v_i(j) \leq B_i$ for all $i \in [n], j \in M$.

The problem of maximizing social welfare with budget-additive valuations has been extensively studied in recent years from a computational perspective [6,18,7,1,3]. It is known to be APX-hard [6], and the best known approximation ratio is $4/3$ (achieved via iterative rounding or primal-dual algorithms). This factor also matches the integrality gap of the corresponding linear program. In this section we consider problem instances in which agents have *uniform* budget-additive valuations [1], in the sense that for each object j there is a value v_j such that, for all i, $v_i(j) \in \{0, v_j\}$. In other words, each object has a fixed value v_j; each player with non-zero value for j values it at v_j. Here too, the best approximation known is $4/3$.

4.1 Arbitrary Budgets

We first give an example in which no MC-CWE allocation can achieve more than an $7/8$ fraction of the optimal welfare.

Claim. There is a profile of uniform budget-additive valuations for which no MC-CWE allocation achieves more than $7/8$ of the optimal welfare.



(a) An instance of uniform budget additive bidders that admits a gap in welfare due to MC-CWE.

(b) An instance of uniform budget additive bidders with identical budgets that admits no Walrasian equilibrium.

Fig. 2. Instances of uniform budget-additive agents. Agents and items are represented by thin and thick nodes, respectively. An agent and item are connected if the agent values the item. Values written next to agents and items correspond to budgets and (uniform) values, respectively.

The full proof is deferred to the full version of the paper. An illustration of the instance that realizes this gap is given in Figure 2(a) with 5 players, $\{c_i\}_{i=1..5}$ (thin nodes), and 4 items, $\{a_i\}_{i=1..4}$ (thick nodes). In this example, the optimal fractional assignment has social welfare 8. However, the optimal integral assignment obtains value $8 - \epsilon$. Therefore, in order to get a MC-CWE outcome, one must bundle some items together. We show that, for any bundling choice, one cannot achieve a higher welfare than 7.

We next present an algorithm that converts any allocation into a MC-CWE allocation while preserving at least half of the original welfare. The algorithm, which we call UniformBudgetAdditiveMC-CWE, proceeds as follows. Given an allocation profile $\mathbf{x} = (x_1, \ldots, x_n)$, the algorithm checks whether there is an instance in which $v_j(x_i) > v_i(x_i)$. Note that this can occur only if $\sum_{k \in x_i} v_k > B_i$ and $B_j > B_i$. If there is no such instance, then the algorithm terminates and the current allocation is returned. Otherwise, if there is a pair of agents i and j with $v_j(x_i) > v_i(x_i)$, then the item of lowest value in x_i is removed from x_i and added to x_j. The algorithm then repeats. Note that the algorithm must terminate, since every iteration results in an item being shifted from one agent to another agent with strictly larger budget. Due to space considerations, a pseudocode listing of the algorithm is deferred the full version of the paper.

The intuition behind UniformBudgetAdditiveMC-CWE is that we would like to reduce the social welfare of the optimal *fractional* allocation of the bundles in $\mathbf{x}$. Indeed, if there are no instances in which $v_j(x_i) > v_i(x_i)$, then it must be that the assignment $\mathbf{x}$ is an optimal *fractional* allocation, and hence by Claim 2.1 is MC-CWE. Thus, by transforming the input allocation into an allocation that satisfies this property, we are reducing the integrality gap of our reduced market to 1. In the full version we show that this iterative procedure does not drastically reduce the welfare of the allocation.

Theorem 5. *Suppose* $\mathbf{x}$ *is an arbitrary allocation with welfare* $SW(\mathbf{x})$*. Given* $\mathbf{x}$*, Algorithm UniformBudgetAdditiveMC-CWE returns, in polynomial time, a MC-CWE outcome* $\mathbf{x}'$ *such that* $SW(\mathbf{x}') \geq \frac{1}{2} SW(\mathbf{x})$.

As a corollary, the known $4/3$ approximation algorithm for budget-additive valuations can be turned into an $8/3$ approximation MC-CWE mechanism in the case of uniform values. Furthermore, the allocation returned by Algorithm UniformBudgetAdditiveMC-CWE has the property that the full surplus can be extracted from the MC-CWE outcome as revenue. This result is established formally in the full version of the paper.

4.2 Identical Budgets

In this section we study the restriction to identical budgets. The welfare maximization problem is NP-hard even under this restriction, as it includes PARTITION as a special case[2]. Moreover, we shall show that there are input instances in this class for which Walrasian equilibria do not exist[3]. Nevertheless, we shall show that any allocation can be transformed into a MC-CWE allocation with no loss to the social welfare.

We first show that there exist input instances for which no Walrasian equilibrium exists. Consider the instance given in Figure 2(b) with 4 players and 7 items. In this example, the optimal fractional assignment has social welfare 8. For example, the fractional assignment in which c_i gets sets $\{a_i, \alpha_i\}$ and $\{b_i, \alpha_i\}$ with probability $1/2$, and d_i gets sets $\{a_i, b_i\}$ and $\{\beta\}$ with probability $1/2$, for $i \in \{1, 2\}$, achieves welfare 8. However, the optimal integral assignment obtains value at most 7. Thus the optimal fractional welfare does not occur at an integral solution, and hence a WE does not exist.

We now show that, for any allocation, there exists a MC-CWE allocation that obtains at least as much social welfare, and moreover it can be found efficiently.

Theorem 6. *Suppose* $\mathbf{x}$ *is an arbitrary allocation. There exists MC-CWE allocation* $\mathbf{x}'$ *with* $SW(\mathbf{x}') \geq SW(\mathbf{x})$*, and* $\mathbf{x}'$ *can be found in polynomial time given* $\mathbf{x}$.

Proof. Given $\mathbf{x}$, we construct $\mathbf{x}'$ by taking any object j such that $j \in x_i$ with $v_i(j) = 0$ and re-allocating it to an arbitrary agent with non-zero value for it. This can be done in polytime and can only increase the social welfare of the resulting allocation. It remains to show that this allocation is MC-CWE.

Suppose $\mathbf{x}'$ has the property that $j \in x'_i$ implies $v_i(j) > 0$. In this case, $v_i(x'_i) = \min\{B, \sum_{j \in x'_i} v_j\} = \max_\ell v_\ell(x'_i)$ for each i. The social welfare of allocation $\mathbf{x}'$ is therefore $\sum_i \max_\ell v_\ell(x'_i)$, which is an upper bound on the value of any fractional assignment of bundles $x'_1, \ldots, x'_n$. Thus the optimal fractional allocation of these bundles obtains the same welfare as $\mathbf{x}'$, and hence $\mathbf{x}'$ is MC-CWE as required. ■

As a corollary, known $4/3$-approximate algorithms for budget-additive valuations can be made MC-CWE for uniform values and identical budgets. In contrast, when budgets are identical but item values are non-uniform, there are instances where no MC-CWE allocation achieves optimal welfare. We present an example in the full version of the paper.

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[2] For a given instance of PARTITION with integers $a_1, \ldots, a_n$ such that $\sum_{j \in [n]} a_j = 2B$, construct an instance of our problem with two agents, each with budget B, and n items, with item j having value a_j.

[3] While this is well known for the case of arbitrary budget additive valuations, here we consider the further restricted class of uniform values and identical budgets.

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Walrasian Equilibrium with Few Buyers

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Abstract. We study the existence and the properties of Walrasian equilibrium (WEQ) in combinatorial auctions, under two natural classes of valuation functions. The first class is based on additive capacities or weights, and the second on the influence in a social network. While neither class holds the gross substitutes condition, we show that in both classes the existence of WEQ is guaranteed under certain restrictions, and in particular when there are only two competing buyers.

1 Introduction

In a combinatorial auction, multiple items are for sale, and the utility of a buyer may depend on the particular set of acquired items in some composite manner.

In the most general case, a buyer may assign an arbitrary value to any group of items. However, typically there is some structure to the value function that is derived from the context. A trivial example is when the value for the buyer only depends on the number of acquired items, which are all identical.

In the following scenario, buyers' valuations can still be described by a relatively simple function. Consider a game where we can attribute some fixed capacity or weight to each item. The value to each buyer is then some function of the total (additive) capacity of its acquired items.

Concrete examples for such a scenario are when the items are storage devices, discrete time intervals for advertising, routers with certain throughput and so on. The utility of each buyer is increasing in the total storage/time/throughput, regardless of how it is divided among the purchased items.

Another interesting scenario is when buyers are firms recruiting influential nodes of a social network (such as news sites or popular blogs), trying to promote a product. The value for a firm in this case is proportional to the joint influence of the recruited nodes, which depends on the network structure and dynamics.

The most fundamental question in combinatorial auctions is regarding the expected outcomes, i.e. how will items be divided among buyers, and at what price. A standard solution is to ask whether there are prices such that if each buyer independently selects her optimal bundle of items, a valid allocation of items will arise. Such prices—if exist—are known as *Walrasian equilibrium* (WEQ). It is known that the allocation under Walrasian prices maximizes the social welfare (sum of buyers' utilities). A seemingly different combinatorial setting is that of

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a *labor market*, where buyers are firms competing on hiring workers, which are strategic agents rather than passive "items" [10]. However it is known that the models are in fact equivalent, and induce the same equilibrium outcomes. For details see [8], as well as the full version of this paper [14]. The labor market interpretation is very natural in both of the scenarios we consider: the "weight" of a worker corresponds to his productivity level (e.g., the number of images that an Amazon Turk user can tag in an hour). Similarly, the workers can be particularly influential members in a social network.

It is therefore of great interest to study the conditions under which Walrasian equilibria exist. Kelso and Crawford [10] provide sufficient conditions for the existence of a WEQ—namely, that all buyers' value functions hold a technical property called *gross substitutes.*

A characterization of all games that have a WEQ (see Section 3.2) was given in a classic paper by Bikhchandani and Mamer [2], which also commented that they *"...have been less successful at identifying sufficient conditions [for existence of a WEQ] on agents' preferences"* (p. 403).

Our contribution Our primary goal is to characterize the conditions under which a WEQ exists. In particular, we are interested in extending the results of Kelso and Crawford on existence of equilibrium for buyers with value functions based on capacities or social connections, two important cases that violate the gross-substitutes condition. We start by formally defining new classes of valuation functions inspired by the examples in the introduction.

In Section 4 we study games with additive capacities (weights) and show that an equilibrium always exists with two buyers. In games with arbitrary capacities and more than two buyers, a WEQ may not exist. We show through analysis and simulations that particular heuristic prices are usually quite stable, and that a WEQ exists in almost every instance. Further, simple heuristics can be applied to find prices that are usually almost market-clearing.

In the social network model, studied in Section 5, we prove the existence of equilibrium for two buyers when the network is sufficiently sparse. We also demonstrate that our conditions for existence are minimal, in the sense that by relaxing any of them we can construct a game with no WEQ. Results are summarized in Table 1 (page 180).

All of the omitted proofs and examples, as well as the details of our simulations, are available in the full version of this paper.[1]

2 Preliminaries

We denote vectors by bold lowercase letters, e.g. $\mathbf{a} = (a_1, a_2, \ldots)$. Sets are typically denoted by capital letters, e.g. $B = \{1, 2, \ldots\}$. When $\mathbf{a}$ is a vector of indexed elements and B is a set of indices, we use the shorthand notation $a(B) = \sum_{b \in B} a_b$.

[1] A version with the proofs is available from `http://tinyurl.com/melp84c`. See [14] for an earlier version that contains further discussions and computational details.

Combinatorial auctions We consider a set K of k *items*, and a set N of n *buyers*, where $n \geq 2$. Every buyer $i \in N$ is associated with a non-decreasing value function $v_i : 2^K \rightarrow \mathbb{R}_+$, where $v_i(\emptyset) = 0$.

Given an auction $G = \langle K, N, (v_i)_{i \in N} \rangle$, a *valid outcome* is a pair $(P, \mathbf{x})$, where $P = (S_0, S_1, S_2, \ldots, S_n)$ is a partition of K among the buyers, where S_0 contains unallocated items. $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is a price vector.

$(P, \mathbf{x})$ is a *Walrasian equilibrium* (WEQ), if for every buyer, $S_i = \text{argmax}_{S \subseteq K}$ $(v(S) - x(S))$, and unsold items have price 0. That is, market is cleared and the bundle of every buyer maximizes her utility under prices $\mathbf{x}$. The profit (utility) of buyer i is denoted by $r_i(S_i) = v_i(S_i) - x(S_i)$. The *social welfare* of a partition P is the sum of buyers values, i.e. $\sum_{i \in N} v_i(S_i)$. Note that the social welfare does not depend on prices.

2.1 Value Functions

We use the notation m_i for the marginal value of an item to buyer i. For every $S \subseteq K$ and $j \notin S$, $m_i(j, S) = v_i(S \cup \{j\}) - v_i(S)$.

v_i is *submodular* (a.k.a. *concave*), if $v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + v_i(T)$ for all $S, T \subseteq K$. It is *strictly submodular* if the inequality is strict. Equivalently, v_i is submodular if the marginal contribution is nonincreasing. That is, for all $T \subseteq S$ and $j \notin S$, $m_i(j, T) \geq m_i(j, S)$. If the [strict] inequality holds only when S, T are disjoint, then we say that v_i is *[strictly] subadditive.* All value functions studied in this paper are submodular, i.e. have decreasing marginal returns. This assumption is standard in the economics literature [4,12].

Types We say that items j and j' are of the same *type* if all buyers are indifferent between them. That is, if for all $i \in N$, $S \subseteq K \setminus \{j, j'\}$, $v_i(S \cup \{j\}) = v_i(S \cup \{j'\})$. Similarly, we say that buyers i and i' have the same type, if $v_i \equiv v_{i'}$. Games where all buyers are of the same type are called *symmetric* games. In the simplest form of games, called *homogeneous games*, all items are of the same type. In such games each $v_i : [k] \rightarrow \mathbb{R}_+$ is a function of the *number* of the items used by buyer i, i.e. $v_i(S) = v_i(|S|)$.

Weighted games The primary type of value functions we consider in this work is based on capacities, or *weights*. Every item has some predefined nonnegative integer weight w_j, and the value of a set S depends only on its total weight. Thus each v_i is a function $v_i : \mathbb{N} \rightarrow \mathbb{R}_+$, where $v_i(S) = v_i\left(\sum_{j \in S} w_j\right)$.

A game where all value functions are weight-based is called a *weighted* game. Homogeneous games are a special case of weighted games, where all items have the same weight (w.l.o.g. weight 1).

A partition $P = (S_0, S_1, \ldots, S_n)$ of items in a weighted game is *balanced*, if the total weight of items that each buyer gets is the same, i.e. $w(S_i) = w(S_{i'})$ for all $i, i' \in N$. A partition is *almost balanced* if the total weight of any S_i and $S_{i'}$ (except S_0) differ by at most 1.

Influence in social networks Another value function we consider is inspired by social networks. Every social network then induces a game, where the items are some particular set of influential nodes (news sites, blogs, influential writers, etc.), and buyers are firms trying to purchase or hire nodes with *maximal influence* in the network (following [7,11]).

Synergy graphs Simple synergies between items can be represented by a weighted undirected graph, where every vertex is an item, and the value of a set is the total weight of edges linked to vertices in the set. This includes edges between vertices in the set, and edges between these vertices and external vertices.

All classes described above—except homogenous valuations—may violate the gross-substitute condition (see full version for examples).

Fair pricing We also consider other natural requirements from prices. A price vector $\mathbf{x} = (x_1, \ldots, x_k)$ is *fair* if for every pair j, j' of the same type, it holds that $x_j = x_{j'}$. In *weighted games*, a price vector $\mathbf{x}$ is *proportional*, if for all j, j' it holds that $\frac{x_j}{x_{j'}} = \frac{w_j}{w_{j'}}$. Any proportional price vector is fair.

Note that we do not externally enforce fairness nor proportionality.

3 Properties of Equilibrium Outcomes

A WEQ has many desired properties, which motivate the search for such outcomes. In addition, some of these properties will be used as tools in the next sections to prove existence and non-existence of WEQ in various games.

The first property, which follows directly from the definition of WEQ, means that a buyer does not prefer a bundle with one additional item or one item less.

Lemma 1. *Let $(P, \mathbf{x})$ be a WEQ outcome in game G. Then (1) for all $i \in N, j \in S_i$, $x_j \leq m_i(j, S_i \setminus \{j\})$; and (2) for any $i' \neq i \in N, j \in S_i$, $x_j \geq m_{i'}(j, S_{i'})$.*

3.1 Individual Rationality, Fairness and Envy Freeness

We next present three simple observations (phrased as lemmas), showing that a WEQ outcome is always individually rational, envy free, w.l.o.g. fair.

Lemma 2 (Individual rationality). *Let $(P, \mathbf{x})$ be a WEQ outcome in game G, then (1) $x_j \geq 0$ for all $j \in K$; and (2) $v(S_i) - x(S_i) \geq 0$ for all $i \in N$.*

We say that buyer i *envies* buyer t in an outcome $(P, \mathbf{x})$, if i wants to trade items and payments. That is, if $v_i(S_t) - \sum_{j \in S_t} x_j > v_i(S_i) - \sum_{j \in S_i} x_j = r_i(P, \mathbf{x})$. An outcome is *envy-free* if no buyer envies any other buyer.

Lemma 3 (Envy freeness). *Let $(P, \mathbf{x})$ be a WEQ outcome in game G. Then $(P, \mathbf{x})$ is envy-free.*

The above holds because an envious buyer i can always forgo its current items S_i and buy S_t instead.

Lemma 4 (Fairness). *If $(P, \mathbf{x})$ is a WEQ in game G, then there is a* fair *outcome $(P, \mathbf{x}^*)$ that is a WEQ in G, where the profit of each buyer remains the same.*

3.2 LP Formulation and the Welfare Theorems

Computational schemes for representing combinatorial markets and to solve them, as well as the properties of Walrasian equilibria, have been thoroughly studied (see Blumrosen and Nisan [3] for a detailed review). Briefly, there is a standard Integer Linear Program, denoted $ILP(G)$, whose solutions describe the optimal partitions in the game. The *Linear Program Relaxation* of $ILP(G)$ is denoted by $LPR(G)$.

Two fundamental properties state that every WEQ is efficient, and characterize the conditions for existence [2,3].

First Welfare Theorem (FWT). *Every Walrasian equilibrium, if exists, is optimal in terms of the social welfare.*

Second Welfare Theorem (SWT). *A Walrasian equilibrium exists if and only if the integrality gap of $ILP(G)$ is zero, i.e. if the solution quality of $ILP(G)$ and $LPR(G)$ is the same.*

Moreover, in such cases it is known that the solutions to the dual linear program of $LPR(G)$ yield the market clearing prices $\mathbf{x}$, which in the labor market interpretation represent workers' salaries under WEQ.

4 Weighted Games

The first class of value functions we study is based on capacities, or weights. Recall that a weight based value function $v : \mathbb{N} \to \mathbb{R}_+$ is a subadditive function, which maps the total weight of a set of items $w(S)$ to utility. This implies submodularity of $v(S)$ (see full version). We sometimes write v as a vector of $w(K)+1$ entries $(v(0), v(1), \ldots, v(w(K)))$, where $v(0) = 0$.

Before continuing to our existence results, we observe that without the subadditivity assumptions, weighted games may not posses a WEQ even in a most simple scenario

Homogeneous games Suppose that all items have unit weight. Kelso and Crawford [10] show that in such games the core is always non-empty. It follows that a WEQ always exists. Moreover, when buyers are symmetric, then such a WEQ has a particularly simple form.

Let $q = \lfloor k/n \rfloor$ and $\delta = v(q+1) - v(q)$. By FWT, in any WEQ every buyer has either q or $q+1$ items. Also, by Lemma 1, there is a WEQ where the price of every item is δ.

4.1 The Case of Two Buyers

Consider a symmetric weighted game with two buyers and only two items $K = \{h, l\}$, where $w_h \geq w_l$. This simple case can be solved as follows. Let $x_l = v(w_l + w_h) - v(w_h)$, and $x_h = v(w_l + w_h) - v(w_l)$, i.e. we set the payment of each item to be its own marginal contribution to the set K. Then, for a partition P where $|S_1| = |S_2| = 1$, $(P, \mathbf{x})$ is a WEQ.

However, the described prices are not necessarily proportional. A proportional outcome, which is also a WEQ, would be to pay $x_j = v(w_j)$ for each item

$j \in \{l, h\}$. If v is strictly subadditive, then there are also other proportional WEQs where the buyers keep some of the profit. We next show that we can always find a proportional WEQ for two buyers and any number of items. Note that we only require that each value function will be subadditive.

Our main result for the weighted setting is as follows.

Theorem 5. *Let $G = \langle K, N, \mathbf{w}, v_1, v_2 \rangle$ be a weighted game with two buyers, then G admits a proportional WEQ.*

The proof hinges on the idea of computing the marginal value of a *unit of weight*. However in the general case this is an evasive notion that requires a nontrivial case analysis (see full version). We bring here a simplified proof of the symmetric unbalanced case, and explain the main ideas of the general case.

Proof (sketch). Indeed, let $P = (S_1^*, S_2^*)$ be an optimal partition. For the symmetric case, denote $H = w(S_1^*)$, $L = w(S_2^*)$, and suppose that $L < H$. By optimality, the gap $H - L$ is minimal. We define $\delta = \frac{v(H)-v(L)}{H-L}$, and argue that it induces a WEQ $(P, \mathbf{x})$, where $\mathbf{x} = \delta \cdot \mathbf{w}$. Indeed, the profit of a buyer with items of total weight q is $r(q) = v(q) - \delta q$. Since $v(q)$ is concave, $r(q)$ is also concave, with maximum in $q^* \in \{\lfloor W/2 \rfloor, \lceil W/2 \rceil\}$ (since δ is between $m(\lfloor SW/2 \rfloor, 1)$ and $m(\lceil SW/2 \rceil, 1)$). Also, in P we have

$$\begin{aligned}(H - L)r_1 &= (H - L)(v(H) - H\delta) = (H - L)v(H) - H(v(H) - v(L)) \\ &= H \cdot v(L) - L \cdot v(H) = (H - L)(v(L) - L\delta) = (H - L)r_2\end{aligned}$$

This means that $r_1 = r_2$, and by minimality of the gap, buyers cannot get closer to the theoretical optimal profit $r(q^*)$: for any set of items S_i', we have $|w(S_i') - q^*| \geq |w(S_i^*) - q^*|$, and thus $r_i(q_i') \leq r(q_i)$.

For the general case, assume that both S_1^*, S_2^* are non-empty (otherwise the solution is fairly easy). Let $q_i = w(S_i^*)$, $\check{w}_i = \min_{j \in S_i^*} w_j$. Since value functions are different, the marginal contribution of each unit of weight to each buyer may also be different. We therefore replace the "slope" δ with four different quantities y_1, z_1, y_2, z_2. We denote by y_i, z_i the *normalized* marginal value of the "lightest" item below and above the threshold q_i, respectively. Formally, $y_i = \frac{1}{\check{w}_i}(v_i(q_i) - v_i(q_i - \check{w}_i))$, and $z_i = \frac{1}{\check{w}_{-i}}(v_i(q_i + \check{w}_{-i}) - v_i(q_i))$.

Next, set $z_i^* = \max_{d \geq 1} \left\{ \frac{v_i(q_i+d)-v_i(q_i)}{d} \leq y_{-i} \right\}$. Intuitively, z_i^* is the *closest slope* to y_{-i} that we can get by adding weight above the threshold q_i. By its definition, $y_i, y_{-i} \geq z_i^* \geq z_i$.

To complete the construction, we use z_i^* as a proxy for the marginal value of a unit. We define $\delta = \max\{z_1^*, z_2^*\}$, and set the (proportional) prices $x_j = \delta w_j$. The proof proceeds by showing that to improve the profit of a buyer, the total weight of acquired items changes by less than $\check{w}_i$. On the other hand, we show that if such a small change is possible, a better partition than P can be constructed, which is a contradiction to FWT. □

4.2 More than Two Buyers

A question that naturally arises is whether we can generalize Theorem 5, i.e. prove that a WEQ always exists for any number of buyers, perhaps even with the additional requirement of proportionality. A result by Gul and Staccheti [8] shows that whenever there is a buyer whose value function violates gross-substitutes, it is possible to construct an example (with additional unit-demand buyers) where an equilibrium does not exist. While we cannot apply their construction directly, it gives little hope that a WEQ exists in the general case of weighted games.

Indeed, Proposition 6 below shows that a WEQ is not guaranteed for multiple buyers. We first show nonexistence of *proportional* WEQ.

Example 1. Consider a symmetric game G^* where $w_1 = 5, w_2 = 6, w_3 = 7$, and $v(5) = 5, v(6) = 6, v(7) = 6$. Clearly in the optimal partition each buyer has a single item. Suppose there is some proportional WEQ, where $x_j = \delta \cdot w_j$ for all j. Then by Envy-freeness (Lemma 3), all three buyers make the same profit, i.e.

$$r_1 = r_2 = r_3 \Rightarrow v(5) - 5\delta \stackrel{(\#1)}{=} v(6) - 6\delta \stackrel{(\#2)}{=} v(7) - 7\delta.$$

By Eq. (#1), $\delta = v(6) - v(5) = 1$, whereas by (#2), $\delta = v(7) - v(6) = 0$. A contradiction. ♢

Proposition 6. *For any $n \geq 3$, there is a symmetric weighted game with n buyers that does not have a WEQ at all.*

An example proving the proposition appears in the full version. We hereby construct an example with $n = 4$ buyers.

Example 2. Our game G has 9 items in total, where the weights are $(2, 2, 2, 2, 2, 3, 3, 3, 5)$. We define $v(w) = \min\{w, 6\}$. We claim that the optimal partition must be either $P_1 = (\{5\}, \{3, 2, 2\}, \{3, 3\}, \{2, 2, 2\})$ or $P_2 = (\{3, 2\}, \{5, 2\}, \{3, 3\}, \{2, 2, 2\})$ (up to permutations of items of the same type). See Fig. 1.

By the shape of v, the optimal partition minimizes $|\{i \in N : w(S_i) < w'\}|$ for $w' = 1$, then for $w' = 2, 3, 4$, etc. Indeed, the total weight is 24, but it cannot be divided in a balanced way. Thus we get that every optimal partition has total weights of $(5, 7, 6, 6)$. P_1 and P_2 are the only ways to construct such a partition. Next, we want to assign payments.

By Lemma 4, we can set a uniform payment to each type of item, thus we should determine the values of x_2, x_3 and x_5. Now, by Envy-freeness (Lemma 3), $r_3 = r_4$, and thus $0 = r_3 - r_4 = (v(6) - 3x_2) - (v(6) - 2x_3) = 3x_2 - 2x_3$. Similarly, buyers 1 and 2 can trade the 5-item for the set $\{2, 3\}$. Thus $x_5 = x_2 + x_3$. We get that $\mathbf{x}$ must be a proportional payment vector, where $\mathbf{x} = \delta\mathbf{w}$ for some unit payoff δ. However, buyers 1,2 and 3 have a total weight of 5,7 and 6 respectively, exactly as in the game G^* in Example 1. As we show above, such a proportional payment vector cannot be stable.[2] ♢

[2] The integrality gap of this example is $\frac{LPR(G)}{ILP(G)} = \frac{23.5}{23} = \frac{47}{46}$.

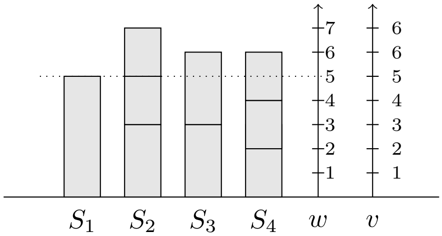


Fig. 1. The optimal partition P_1

We argue that while examples without WEQ exist even for symmetric buyers, proportional payoffs can be derived from an optimal partition using simple and heuristics. We show that the stability of these payoffs is inversely related to the gap between the total weight buyers acquire, supporting this claim with a formal argument and with simulations. First, we prove that if there exists a partition that is *almost balanced*, then together with the proportional payoffs it forms a WEQ, regardless of the value function v. Note that when weights are small integers, there is typically an almost-balanced partition.[3] We also generate weighted games from a simple distribution, showing that the incentive of buyers to deviate grows as the partition becomes less balanced (see full version).

Proposition 7. *Let $G = \langle K, N, \mathbf{w}, v\rangle$ be a weighted game with n symmetric buyers. (a) If there is an almost-balanced partition of* $\mathbf{w}$*, then G admits a proportional WEQ. (b) symmetry is a necessary condition.*

5 Games Over a Social Network

5.1 Network Model

Consider a social network $H = \langle V, E_H\rangle$ (a directed graph), and a subset of "influencers" $K \subseteq V$. Given some diffusion scheme in the network, every set $S \subseteq K$ influences some portion of the nodes V, whose size is denoted by $I_H(S)$.

Given such a social network H and a set of buyers (firms) N, we define a symmetric game where buyers bid over a set of influential nodes K (the items), trying to advertise to as many people (all nodes of H) as possible. The value function of every buyer is thus $v_i(S) = v(S) = I_H(S)$.[4]

We apply one of the most widely known diffusion schemes, called the *independent cascade model*, which has been suggested by Goldenberg et al. [7] and further promoted in [11]. We briefly describe the diffusion process.

In the Independent Cascade model, every edge in the network H has an attached *probability* $p_{u,u'}$. Once a node u is activated, it tries to activate once each neighbor u', and succeeds w.p. of $p_{u,u'}$, independently of the state of any other node. Once a node is activated, it remains active. The influence of a set S, denoted by $I_H(S)$, is the expected number of nodes that end up as active if we activate the set S.

[3] For example, if there are at least $n \cdot \max_j w_j$ items with weight 1, then an almost-balanced partition *must* exist.

[4] We can define asymmetric games by using a different network H_i for each buyer.

$$v_i(S) = v(S) = I_H(S) = \sum_{u \in V} pr(u \text{ is activated}|S \text{ is active}).$$

This is equivalent to summing the probabilities of all percolations (subgraphs of H) in which there is a directed path from some node in S to u. We should note that $I_H(S)$ is a submodular function [11]. However, $I_H(S)$ does not necessarily hold the gross-substitute condition.

While the independent cascade model seems to be more powerful than the weighted model studied in the previous section, it turns out that no model generalizes the other. Indeed, we show in the full version a weighted value function over 3 homogeneous items, that cannot be represented as the influence in any graph H. Therefore, weighted value functions and influence value functions are two different classes of submodular valuations. A natural question is whether a WEQ always exists with two buyers in the independent cascade model. Unfortunately, the answer is negative in the general case, but we can prove existence is a special case of interest.

We say that a network $H = \langle V, E_H \rangle$ is t-sparse (w.r.t the set $K \subseteq V$), if every node $u \in V$ can be reached by at most t items from K.

Intuitively, t-sparsity means that the influence cones of different items hardly intersect. A 1-sparse network means that the cones of influence are pairwise mutually exclusive and thus that the influence is completely additive (a trivial case). A 2-sparse network means that two cones may intersect, but never three or more. A k-sparse network puts us back in the general case. In order to analyze sparse networks, it will be useful to formally define synergy graphs.

5.2 Synergy Graphs and Sparse Social Networks

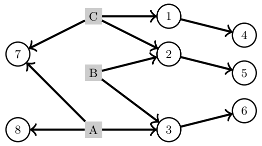
A *synergy graph* is an undirected graph $M = \langle K, E_M \rangle$ with non-negative weights, where self-edges are allowed. It can thus be represented as a symmetric matrix, which is also denoted by M. Every synergy graph M induces a value function v_M, where the value of a subset $S \subseteq K$ is the sum of weights of edges between items in the subset (including self edges), and edges going outside the subset. That is,

$$v_M(S) = \sum_{j \in S} M(j,j) + \sum_{j,j' \in S, j<j'} M(j,j') + \sum_{j \in S} \sum_{j'' \notin S} M(j,j'').$$

Lemma 8. *A value function v over a set of items K can be described by a synergy graph if and only if it can be described as the influence in a 2-sparse network. I.e. there is M s.t. $v = v_M$ iff there is H s.t. $v = I_H$.*

As an intuition, the mapping is constructed s.t. $M(j,j')$ equals the expected number of nodes that are influenced by *both* item j and item j' (i.e. the intersection of their influence cones). Fig. 2 demonstrates a network H and its corresponding synergy graph M. Our main positive result in the network model is the following.

(a) The network H



(b) The matrix M

	A	B	C
A	1	2	1
B	2	0	2
C	1	2	2
total	4	4	5

(c) The synergy graph M

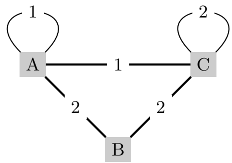


Fig. 2. An example of a sparse network / synergy graph with three items A, B and C. The maximal cut in M, which is also the optimal partition, is $P^* = (\{A, C\}, \{B\})$. Then $SW(P^*) = v(\{A, C\}) + v(\{B\}) = 8 + 4 = 12$.

Theorem 9. *Let $G = \langle K, N, v_M \rangle$ be a symmetric game with 2 buyers over a synergy graph M (or, equivalently, a 2-sparse network). Then G has a WEQ.*

The outline of the proof is as follows. The maximal cut in M is the optimal partition in G, and we set the payoff of each item j to be the average of her total influence and her exclusive influence (i.e., $x_j = \frac{v(\{j\})+M(j,j)}{2}$), and show that $\mathbf{x}$ induce a WEQ.

Unfortunately, if we relax any of the conditions in Theorem 9 (symmetry, number of buyers, or sparsity) then the existence of a WEQ is no longer guaranteed.

6 Discussion

We considered combinatorial auctions without gross substitutes, and showed that a Walrasian equilibrium (or, equivalently, a pure Nash equilibrium of the first price auction) is guaranteed to exist under certain restrictions, with a special focus on the case of few buyers.

In games based on capacities (weights) with subadditive production, we proved that a WEQ must exist if there are two buyers or if there is an almost balanced partition.

Finally, we showed that a WEQ always exists in a particular case of the network model, when the network is sparse and featuring two identical competing buyers Unfortunately, there may not be a WEQ if any of these conditions is violated. Our results are summarized in Table 1.

Related work and implications Following Kelso and Crawford's seminal paper, multiple authors studied the implications of the gross substitutes restriction. In particular, Lehmann et al. showed that only a tiny fraction of all submodular value functions are gross substitutes [12]. Further, Gul and Stacchetti proved that for *any* value function without this property (and with additional unit-demand buyers), it is possible to construct a market with no WEQ [8]. However, the construction by Gul and Stacchetti used an unbounded number of buyers/firms (one for every item/worker).

Some recent papers show that despite the Gul and Stacchetti negative result, the gross substitutes condition can be slightly relaxed. This is either by allowing

Table 1. Existence results for the cases where a WEQ is guaranteed to exist are marked with V. Cases marked with X mean that there are instances where *no WEQ* exists.

<table>
<tr><th rowspan="2"># of buyers</th><th rowspan="2">homogeneous games</th><th colspan="2">weighted games</th><th colspan="2">network games</th></tr>
<tr><th>near-balanced</th><th>any</th><th>2-sparse (syn. graphs)</th><th>any</th></tr>
<tr><td>$n = 2$, symm.</td><td rowspan="4">V (Kelso & Crawford [10])</td><td rowspan="2">V (⇐)</td><td rowspan="2">V (T. 5)</td><td>V (T. 9)</td><td>X</td></tr>
<tr><td>$n = 2$, asym.</td><td>X</td><td rowspan="3">X (⇒, ⇓)</td></tr>
<tr><td>$n \geq 3$, symm.</td><td>V (P. 7a)</td><td>X (P. 6)</td><td>X</td></tr>
<tr><td>$n \geq 3$, asym.</td><td>X (P. 7b)</td><td>X (⇓)</td><td>X (⇓)</td></tr>
</table>

very restricted complementarities [15], or by introducing a modified version of unit-demand [1]. Our results demonstrate that there are entire natural classes of value functions where items are neither substitutes not complements, yet existence of equilibrium can be guaranteed if the number of firms is low (and in particular for two firms).

Integrality gap Dobzinski and Schapira [5] study upper and lower bounds on the integrality gap of various submodular value functions. The integrality gap is an important factor in the construction of efficient approximation algorithms that find optimal allocations in combinatorial auctions. For general submodular functions, they show that the (maximal) integrality gap is between $\frac{8}{7}$ and $\frac{4}{3}$. Since our construction implements theirs, we get that the lower bound of $\frac{8}{7}$ still applies for value functions based on sparse networks. As for weighted functions, the integrality gap of Example 2 is $\frac{47}{46}$, which gives us a lower bound. An interesting challenge is to find the maximal integrality gap of instances that correspond to weighted or network games. In particular, it is an open question whether tighter bounds can be proved compared to general submodular functions.

Possible extensions While most real world networks are not 2-sparse, many of them demonstrate other forms of sparsity [6,13]. We would like to develop a heuristic solution similar to the one suggested for weighted functions, and test whether it provides us with an exact or approximate WEQ in a more general class of sparse networks.

Another direction concerns the context of the competition. In our network model the firms compete only for the services of the influencing nodes, in separation from other arenas in which they might affect one another. However, if companies are also competing for market share, then there are externalities: users in the network that are exposed to an ad of one company may become less likely to purchase the products of its competitor. Refining the model can contribute to the literature on competitive diffusion in networks (see e.g. [9]).

Other natural directions would be to study the relation between the integrality gap of various classes of valuation functions (such as those studied in [5,3]), and the number of buyers. In particular, it would be interesting to find new classes of valuations where the integrality gap with few buyers is trivial—and thus a Walrasian equilibrium exists.

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Multimarket Oligopolies with Restricted Market Access

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Abstract. We study the existence of Cournot equilibria in multimarket oligopolies under the additional restriction that every firm may sell its product only to a limited number of markets simultaneously. This situation naturally arises if market entry is subject to a valid license and each firm holds a fixed number of licenses only, or, equivalently, if the firms' short-term assets only suffice to serve up to a certain number of markets. We allow for firm-specific market reaction functions modeling heterogeneity among products. As our main result, we show the existence of a Cournot equilibrium under the following assumptions stated informally below: (i) cost functions are convex; (ii) the marginal return functions strictly decrease for strictly increased own quantities and non-decreased aggregated quantities; (iii) for every firm, the firm-specific price functions across markets are identical up to market-specific shifts. While assumptions (i) and (ii) are frequently imposed in the literature on single market oligopolies, only assumption (iii) seems limiting. We show, however, that if it is violated, there are games without a Cournot equilibrium.

1 Introduction

Cournot's work on industrial organization [3] doubtlessly represents a landmark of economic theory and is one of the earliest reference points of game theory. To date, his model of oligopolistic competition remains a corner stone of empirical and mathematical analysis in these fields. Most of the work on the existence of equilibria in Cournot oligopolies has to make strong assumptions on the topological properties of the firms' strategy sets and their utility functions. Commonly it is assumed that the strategy space of each firm corresponds to a closed interval on the real line (and, thus, forms a convex and compact subset of a one-dimensional Euclidean space) and utilities are continuous and quasi-concave. This way, classical fixed point theorems of Kakutani [7] and adapted versions (cf. Debreu [4], Glicksberg [5]) can be applied. In the past decades, the assumptions on the quasi-concavity of the utility functions have been considerably relaxed, see Vives [18] for an excellent survey. E.g., Novshek [11] only requires that the marginal revenue of each firm is decreasing in the aggregate quantities of other firms. Starting with Topkis [15] several works (cf. Amir [1],

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Kukushkin [8], Milgrom and Roberts [9], Milgrom and Shannon [10], Topkis [16], Vives [17]) discovered that Tarski's fixed-point theorem (cf. [14]) yields the existence of an equilibrium if the underlying game is supermodular, that is, the strategy space forms a lattice and the marginal utility of each firm is increasing in any other firm's output. Like this, one can obtain existence results without requiring quasi-concavity of utilities.

In *multimarket oligopolies*, firms may produce quantities for a *set* of markets; see Bulow et al. [2] and Topkis [16, §4.4.3]. In the classical model of Bulow et al., each firm has a firm-specific set of markets on which positive quantities of a homogeneous good can be offered. The utility of each firm equals the profit from selling the produced goods on the markets minus the total production cost. Similar to the single market case, under the assumption that the utilities of the firms are continuous and quasi-concave in the outputs, the existence of an equilibrium follows by standard fixed-point theorems in the spirit of Kakutani. Analogously, if the underlying game is supermodular, the application of Tarski's fixed-point theorem yields the existence of an equilibrium; see Topkis [16, §4.4.3].

In this paper, we study multimarket oligopolies in which firms may only offer positive quantities on a *limited* number of markets. Such situations arise for instance if governmental policies oblige each firm to be engaged in at most a fixed number of markets at a time, e.g., by issuing a limited number of licenses to enter a market (see, e.g., Stähler and Upmann [13]). Another typical situation in which support constraints occur is when the firms' short-term assets only suffice to serve a certain number of markets.[1] We model these situations by assuming that every firm i may only choose positive production quantities for up to $k_i \in \mathbb{N}$ many markets out of a firm-specific set of markets. Formally, the restriction of serving at most k_i markets at a time with positive quantities imposes a support restriction on the vector of production quantities of firm i.

In previous work (cf. Harks and Klimm [6]), we considered a class of games in which a strategy of a player can be represented as a tuple consisting of an action and a (one-dimensional) demand quantity. Under certain regularity assumptions on the allowable class of utility functions, the main result establishes the existence of a pure Nash equilibrium. As a special case of this result it is shown that multimarket oligopolies in which each firm procures a homogeneous product only on a *single* market at a time possess an equilibrium provided that market price functions are *equal* across markets, see [6, §4]. Regarding multimarket oligopolies, in this paper we prove a much more general result showing that there exist (pure) equilibria even for *general support constraints* and *player-specific* market reaction functions (allowing for heterogeneous products). These generalizations also require a substantially different proof technique. The main proof idea of the result in [6] crucially relies on the decoupled structure of strategies. As each firm uses only a single market at a time, there are only two local effects whenever a firm changes its strategy: only the quantities of the "new"

[1] As an illustration, think of a company running several ice cream vans that sell ice cream on local beaches. In the short term, the number of vans is fixed and their number imposes an upper bound on the number of beaches that can be visited.

and the "old" market change. In the case of multimarket oligopolies with *general* support constraints, however, whenever a firm changes the current support set (even if it adds only a single market at a time to its support set), the deviating firm will adapt the quantities on all markets contained in its support set because of the coupling in the production cost. This adaption may trigger global cascading effects on possibly all markets since those firms having support sets intersecting with that of the deviating firm will change their quantities, which, in turn, triggers the adaption of the production quantities by further firms.

Our Results. We study multimarket oligopolies with restrictions on the number of markets on which each firm can offer positive quantities. In our model, every firm has access to a firm-specific set of markets and is associated with a firm-specific non-decreasing and convex cost function. We allow for firm-specific market price functions modeling the price effect of heterogeneous products. The main result of this paper is an existence theorem for Cournot equilibria in multimarket oligopolies assuming that (i) the firm-specific price functions are non-increasing; (ii) the marginal return functions strictly decrease for strictly increased own quantities and non-decreased aggregated quantities; (iii) for every firm the firm-specific price functions per markets are identical up to market specific shifts. The proof of our existence result relies on a combination of ideas. We first show that for any strategy profile, if a firm can improve, there is always a *restricted improvement* that only adds a single new market to the support but also yields an improvement. We further introduce the notion of a *partial equilibrium*, a strategy profile that is stable against unilateral quantity deviations assuming *fixed* support sets. We show (using Kakutani's fixed point theorem) that partial equilibria always exist. Based on these two properties, we design an algorithm that computes an equilibrium. Our algorithm relies on iteratively computing a *partial equilibrium* and, whenever a firm can improve, this firm deviates to a *restricted best reply* defined as the best restricted improvement. After such a restricted best reply, the algorithm recomputes the partial equilibrium and reiterates. We prove that a firm-specific load vector of the partial equilibria lexicographically decreases in every iteration and, thus, the algorithm terminates. The key for proving this is to derive certain monotonicity properties of the computed partial equilibrium after executing a restricted best reply. It might seem surprising that there is enough structure on the thus computed partial equilibria given they are computed only implicitly using Kakutani's fixed-point theorem as a black box. We finally show that our existence result is "tight" in the sense that if the requirement of having essentially "identical markets per firm" is dropped, there is a game without an equilibrium. We conclude the paper by outlining an important generalization of our model.

2 The Model

In a multimarket oligopoly, there is a non-empty and finite set N of firms and a non-empty and finite set M of markets each endowed with a non-increasing firm-specific market reaction function $p_{i,m}, m \in M, i \in N$. A strategy of firm $i \in N$

is to choose a production quantity $x_{i,m}$ for each market m. Given a vector $\mathbf{x}_i = (x_{i,m})_{m\in M}$ of production quantities of firm i, the support of firm i is $S(\mathbf{x}_i) = \{m \in M : x_{i,m} > 0\}$. We impose two restrictions on the support of each firm i in each strategy profile. First, we assume that each firm is associated with a subset $M_i \subseteq M$ of markets that it can potentially procure, i.e., we require that $S(\mathbf{x}_i) \subseteq M_i$. Furthermore, we assume that there is an upper bound $k_i \in \mathbb{N}$ with $k_i \le |M_i|$ on the number of markets that firm i may serve in a strategy profile, i.e, we require that $|S(\mathbf{x}_i)| \le k_i$.

Formally, we derive a strategic game as follows. The set X_i of feasible strategies of firm i is defined as

$$X_i = \big\{\mathbf{x}_i = (x_{i,m})_{m\in M} \in \mathbb{R}^m_{\ge 0} : S(\mathbf{x}_i) \subseteq M_i \text{ and } |S(\mathbf{x}_i)| \le k_i\big\}.$$

The Cartesian product $X = \bigtimes_{i\in N} X_i$ of the firms' sets of feasible strategies is the joint strategy space. An element $\mathbf{x} = (\mathbf{x}_i)_{i\in N} \in X$ is called a strategy profile. With a slight abuse of notation, for a firm i and one of its strategies $\mathbf{x}_i \in X_i$, we write $x_i = \sum_{m\in M} x_{i,m}$ for the total production quantity of firm i. Analogously, for a market m, and a strategy profile $\mathbf{x} \in X$, we write $x_m = \sum_{i\in N} x_{i,m}$ for the total quantity offered on market m under strategy profile $\mathbf{x}$. The utility of firm i under strategy profile $\mathbf{x} \in X$ is then defined as $u_i(\mathbf{x}) = \sum_{m\in M} p_{i,m}(x_m)\, x_{i,m} - c_i(x_i)$. In the remainder of the paper, we will compactly represent the strategic game by the tuple

$$G = (N, (M_i)_{i\in N}, (p_{i,m})_{i\in N, m\in M_i}, (c_i)_{i\in N}, (k_i)_{i\in N}).$$

We call G a *multimarket oligopoly with support constraints.*

We use standard game theory notation. For a player $i \in N$ and a strategy profile $\mathbf{x} \in X$, we write $\mathbf{x}$ as $(\mathbf{x}_i, \mathbf{x}_{-i})$. A Cournot equilibrium is a strategy profile $\mathbf{x} \in X$ such that no firm can improve its utility by a unilateral deviation, i.e., $u_i(\mathbf{x}) \ge u_i(\mathbf{y}_i, \mathbf{x}_{-i})$ for all $i \in N$ and $\mathbf{y}_i \in X_i$.

We impose the following assumptions on the market reaction and cost functions, respectively.

Assumption 1. *For each firm $i \in N$ the cost function $c_i : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is non-decreasing, convex and differentiable.*

Assumption 2. *For all $i \in N, m \in M_i$, the market reaction function $p_{i,m} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ has the following properties:*

(a) *The market reaction function $p_{i,m}$ is non-increasing.*
(b) *The return function $x \mapsto p_{i,m}(x + x_0)\, x$ is differentiable with respect to x for all residual quantities $x_0 \in \mathbb{R}_{\ge 0}$.*
(c) *There exists $\bar{x}_{i,m} > 0$ with $p_{i,m}(\bar{x}_{i,m}) = 0$.*
(d) *For all $x, x', x_0, x_0' \in \mathbb{R}_{\ge 0}$ with $x < x'$ and $x + x_0 \le x' + x_0' \le \bar{x}_{i,m}$ we have $\frac{\partial}{\partial x}\big(p_{i,m}(x + x_0)\, x\big) > \frac{\partial}{\partial x'}\big(p_{i,m}(x' + x_0')\, x'\big)$.*

The above Assumption 2 implies that the marginal profits of every firm are decreasing in both, the own quantity and the aggregate quantities of the competitors. Bulow et al. [2] call this property *strategic substitutes*: A more aggressive play of one firm leads to a quantity reduction of the other competing firms. Assumption 2(c) implies that for every firm, there is an upper bound on the quantity that a firm will produce, hence, the space of feasible quantity vectors can be bounded. It is a simple observation that Assumption 2(d) is, e.g., satisfied if the market reaction functions are concave, decreasing and differentiable.

Finally, we require that the set of firm-specific market reaction functions consists of "compatible" functions, that is, up to market-specific shifts the firm-specific market reaction functions must be identical.

Assumption 3. *For every market $m \in M$ there is a constant $x_{0,m} \in \mathbb{R}_{\geq 0}$ such that for all $i \in N$, there is a function $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $p_{i,m}(x) = p_i(x+x_{0,m})$ for all $m \in M_i$.*

This last assumption is restrictive since it requires that different markets have identical player-specific responses for equal aggregated quantities (or identical player-specific responses up to market-specific shifts). In Section 5, we show, however, that this assumption is necessary in the sense that, if it is relaxed, there are multimarket oligopolies with support constraints not possessing a Cournot equilibrium.

In the remainder of this paper, whenever a game satisfies Assumption 3, we slightly abuse notation as we set $x_m = x_{0,m} + \sum_{i \in N} x_{i,m}$ for all $m \in M$. This allows us to write the utility of player i in strategy profile $\mathbf{x}$ as $u_i(\mathbf{x}) = \sum_{m \in M} p_i(x_m)\, x_{i,m} - c_i(x_i)$.

3 Existence of a Cournot Equilibrium

To show the existence of a Cournot equilibrium in multimarket oligopolies with support constraints we first introduce a relaxation of the Cournot equilibrium concept which we call *partial equilibrium.* Roughly speaking, a strategy profile is a partial equilibrium, if it is a Cournot equilibrium in a game in which the set of accessible markets of each firm i is restricted to a certain *active set* $Q_i \subseteq M_i$ of cardinality $|Q_i| = k_i$. We show – using standard fixed point arguments – that partial equilibria always exist and are essentially unique when fixing the underlying active set vector $(Q_i)_{i \in N}$.

For a given partial equilibrium, we further show that if a firm can improve, there is always a *restricted improvement* that exchanges only one market in the deviating firm's active set. On these two ingredients we design an algorithm that iteratively computes a *partial equilibrium* and, whenever possible let a firm deviate to a restricted improvement. The main result of this paper shows that the algorithm terminates after finitely many iterations and outputs a Cournot equilibrium.

Partial Equilibria. For a firm i, we call a set $Q_i \subseteq M_i$ of k_i markets an *active set* of firm i.

Definition 1 (Partial Equilibrium). *A strategy profile* $\mathbf{x}$ *is a* partial equilibrium*, if for each* $i \in N$ *there is an active set* Q_i *such that* $S_i(\mathbf{x}_i) \subseteq Q_i$ *and* $u_i(\mathbf{y}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$ *for all* $\mathbf{y}_i \in X_i$ *with* $S(\mathbf{y}_i) \subseteq Q_i$*.*

If $\mathbf{x}$ and $(Q_i)_{i\in N}$ satisfy the conditions of Definition 1, we active sets say that $\mathbf{x}$ is a partial equilibrium for the active set vector $(Q_i)_{i\in N}$. We proceed to prove that for each active set vector $(Q_i)_{i\in N}$ there is a partial equilibrium for $(Q_i)_{i\in N}$.

Lemma 1 (Existence of a Partial Equilibrium). *Let* G *be a multimarket oligopoly with support constraints for which Assumptions 1 and 2 hold. For each vector of active sets* $(Q_i)_{i\in N}$*, there is a partial equilibrium* $\mathbf{x}$ *for* $(Q_i)_{i\in N}$*.*

The proof follows by applying classical fixed point results for concave games with convex and compact strategy spaces [5,7].

We proceed to prove that for a fixed active set vector $(Q_i)_{i\in N}$, the partial equilibria for $(Q_i)_{i\in N}$ are essentially unique. In order to prove this result, we need the following lemma that expresses necessary optimality conditions for a partial equilibrium.

Lemma 2. *Let* G *be a multimarket oligopoly with support constraints for which Assumptions 1 and 2 hold. Let* $\mathbf{x}$ *be a partial equilibrium for* $(Q_i)_{i\in N}$*. Then, the following conditions hold for all* $i \in N$ *and all* $m \in Q_i$*:*

(a) $\frac{\partial}{\partial x_{i,m}} u_i(\mathbf{x}) \leq 0$.
(b) $\frac{\partial}{\partial x_{i,m}} u_i(\mathbf{x}) = 0$*, if* $x_{i,m} > 0$.

We are now ready to prove that a given vector $(Q_i)_{i\in N}$ of active sets, the partial equilibrium for the active sets $(Q_i)_{i\in N}$ is essentially unique in the sense that all such equilibria give rise to the same aggregated production quantities on all markets.

Lemma 3 (Uniqueness of Partial Equilibria). *Let* G *be a multimarket oligopoly with support constraints for which Assumptions 1 and 2 hold. Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two partial equilibria for the active set vector* $(Q_i)_{i\in N}$*. Then,* $x_m = y_m$ *for all* $m \in M$*.*

The proof can be found in the full version.

Restricted Improvements. For a partial equilibrium $\mathbf{x}$ for the active sets vector $(Q_i)_{i\in N}$, we introduce the notion of a *restricted improvement* from $\mathbf{x}$.

Definition 2 (Restricted Improvement, Restricted Best Reply). *Let* G *be a multimarket oligopoly with support constraints and let* $\mathbf{x}$ *be a partial equilibrium for the vector of active sets* $(Q_i)_{i\in N}$*.*

1. *A* restricted improvement *for firm* i *is a strategy* $\mathbf{z}_i \in X_i$ *with* $|S(\mathbf{z}_i)\backslash Q_i| \leq 1$ *and* $u_i(\mathbf{z}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$*.*
2. *A* restricted best reply *maximizes* $u_i(\cdot, \mathbf{x}_{-i})$ *among all restricted improvements.*

Note that a restricted best reply of player $i \in N$ to $\mathbf{x}$ need not always exist. If it exists, say $\mathbf{z}_i \in X_i$, it always satisfies $u_i(\mathbf{z}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$.

Equilibrium Existence. We are now ready to state an algorithm that actually computes an equilibrium for multimarket oligopolies with support constraints provided Assumptions 1−3 are satisfied.

The algorithm starts with arbitrary active set vector $(Q_i)_{i \in N}$ and computes a partial equilibrium $\mathbf{x}$ for $(Q_i)_{i \in N}$. Here, we assume that an oracle outputs an equilibrium (or, we apply Rosen's continuous best response dynamics, which are guaranteed to converge under rather mild conditions on utility functions [12]).

As long as there is a player i that can improve its utility by deviating from $\mathbf{x}$, the algorithm computes a restricted best reply in which only one market enters the active set of firm i. Then, a partial equilibrium is recomputed and the algorithm reiterates.

Algorithm 1.
Input: $G = (N, (M_i)_{i \in N}, (p_i)_{i \in N}, (c_i)_{i \in N}, (k_i)_{i \in N})$
Output: Cournot equilibrium $\mathbf{x}$

1. Choose an active set vector $(Q_i)_{i \in N}$ arbitrarily.
2. Compute a partial equilibrium $\mathbf{x}$ for $(Q_i)_{i \in N}$.
3. If there is a firm $i \in N$ who can improve unilaterally,
   (a) compute a restricted best reply $\mathbf{z}_i \in X_i$
   (b) choose an active set $Q_i' \supseteq S(\mathbf{z}_i)$ with $|Q_i' \setminus Q_i| = 1$ arbitrarily
   (c) $Q_i \leftarrow Q_i'$
   (d) proceed with (2).
4. Else, output $\mathbf{x}$.

Theorem 4. *Let G be a multimarket oligopoly with support constraints for which market reaction functions and cost functions satisfy Assumptions 1−3. Then, Algorithm 1 computes a Cournot equilibrium for G.*

4 Proof of the Theorem

In this section, we present a formal proof of Theorem 4. The proof consists of two steps showing that Algorithm 1 is correct and that it terminates.

For the remainder of this section we consider a multimarket oligopoly with support constraints G that satisfies Assumptions 1−3. Recall that by Assumption 3, for all $i \in N$ and $m \in M_i$ we can represent the market price function $p_{i,m}$ by a single function p_i (see the input of Algorithm 1).

Correctness of the Algorithm. For the correctness of the algorithm, we only have to show that Step (3a) is well-defined, i.e., whenever there is a unilateral improvement for some firm $i \in N$, then, there is also a restricted improvement for firm i. The proof can found in the full version.

Lemma 4 (Existence of restricted improvements). *Let* $\mathbf{x}$ *be a partial equilibrium of* G *for the active set vector* $(Q_i)_{i\in N}$*. If firm* i *can improve unilaterally, then there exists a restricted improvement* $\mathbf{z}_i \in X_i$*.*

Termination of the Algorithm. We finally have to show that Algorithm 1 terminates. We do this by proving a series of lemma showing that whenever a partial equilibrium or a restricted best response is computed, a vector-valued potential monotonically decreases.

First, we show that whenever a firm plays a restricted best reply in which one market enters its support, then the total production quantity on the market that entered the support after the best reply is strictly smaller than the production quantity on the market that left the active set. Furthermore, for all markets that are contained in the support set of the deviating firm both before and after the best reply, the quantity offered by the deviating firm may only decrease.

Lemma 5. *Let* $\mathbf{x}$ *be a partial equilibrium of* G *for the active set vector* $(Q_i)_{i\in N}$*. Let* $\mathbf{y}_i$ *be a restricted improvement of firm* i *and let* $r \in M_i$*,* $s \in Q_i$ *be such that* $S(\mathbf{y}_i) \subseteq (Q_i \setminus \{s\}) \cup \{r\}$*. Then, the following properties hold:*

(a) $x_r - x_{i,r} + y_{i,r} < x_s$,
(b) $x_m - x_{i,m} + y_{i,m} \leq x_m$ *for all* $m \in M \setminus \{r, s\}$.

Proof. We first prove (a). For a contradiction, assume $x_r - x_{i,r} + y_{i,r} \geq x_s$. We distinguish the following three cases:

First case $y_{i,r} > x_{i,s}$. As $\mathbf{x}$ is a partial equilibrium Lemma 2 implies $0 \geq \frac{\partial}{\partial x_{i,s}}\big(p_i(x_s)\,x_{i,s}\big) - c_i'(\mathbf{x}_i)$. Since the strategy $\mathbf{y}_i$ is a restricted best reply and $y_{i,r} > x_{i,s} \geq 0$, we get $\frac{\partial}{\partial y_{i,r}}\big(p_i(x_r - x_{i,r} + y_{i,r})y_{i,r}\big) = c_i'(y_i)$. We obtain

$$c_i'(x_i) \geq \frac{\partial}{\partial x_{i,s}}\big(p_i(x_s)\,x_{i,s}\big) > \frac{\partial}{\partial y_{i,r}}\big(p_i(x_r - x_{i,r} + y_{i,r})\,y_{i,r}\big) = c_i'(y_i). \tag{1}$$

Using that c_i is convex, we derive that $y_i < x_i$. If $k_i = 1$, this is a contradiction to $y_{i,r} > x_{i,s}$. If, on the other hand, $k_i > 1$ there is another market $\tilde{m} \in Q_i \setminus \{s\}$ with $y_{i,\tilde{m}} < x_{i,\tilde{m}}$. We then obtain along the same lines

$$c_i'(y_i) \geq \frac{\partial}{\partial y_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}})\,y_{i,\tilde{m}}\big) > \frac{\partial}{\partial x_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}})\,x_{i,\tilde{m}}\big) = c_i'(x_i),$$

which contradicts (1).

Second case $y_{i,r} = x_{i,s}$. We first show that firm i does not change its supplied quantity on all markets used in both strategies $\mathbf{x}_i$ and $\mathbf{y}_i$, i.e., $x_{i,m} = y_{i,m}$ for all markets $m \in Q_i \setminus \{s\}$. For the sake of a contradiction, let us assume that there is a market $\tilde{m} \in Q_i \setminus \{s\}$ with $x_{i,\tilde{m}} \neq y_{i,\tilde{m}}$. We distinguish two cases. If $x_{i,\tilde{m}} < y_{i,\tilde{m}}$, we use that $\mathbf{x}$ is a partial equilibrium and that $\mathbf{y}_i$ is a restricted improvement and obtain

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial x_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}})\,x_{i,\tilde{m}}\big) - c_i'(x_i) \\ &> \frac{\partial}{\partial y_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}})\,y_{i,\tilde{m}}\big) - c_i'(y_i) = 0, \end{aligned} \tag{2}$$

which is a contradiction! If, on the other hand, $x_{i,\tilde{m}} < y_{i,\tilde{m}}$, we obtain the same contradiction as in (2), but with all inequality signs reversed. We conclude that $x_{i,m} = y_{i,m}$ for all markets $m \in Q_i \setminus \{s\}$. This implies

$$u_i(\mathbf{y}_i, \mathbf{x}_{-i}) - u_i(\mathbf{x}) = p_i(x_r - x_{i,r} + y_{i,r})\, y_{i,r} - p_i(x_s)\, x_{i,s} \leq 0.$$

Thus, firm i does not improve, a contradiction to the fact that $\mathbf{y}_i$ is a restricted best response of firm i.

Third case $y_{i,r} < x_{i,s}$. Consider the strategy $\mathbf{w}_i = (w_{i,m})_{m\in M}$ in which firm i plays as in strategy $\mathbf{y}_i$ except that the quantity $y_{i,r}$ is put on market s instead of market r and market r is not served at all. Formally,

$$w_{i,m} = \begin{cases} y_{i,m}, & \text{if } m \in Q_i \setminus \{s\} \\ y_{i,r}, & \text{if } m = s \\ 0, & \text{otherwise.} \end{cases}$$

We observe that $x_s - x_{i,s} + w_{i,s} < x_s$ as $w_{i,s} = y_{i,r} < x_{i,s}$. We obtain

$$\begin{aligned} u_i(\mathbf{w}_i, \mathbf{x}_{-i}) &= \sum_{m\in M} p_i(x_m - x_{i,m} + w_{i,m})\, w_{i,m} - c_i(\mathbf{w}_i) \\ &= u_i(\mathbf{y}_i, \mathbf{x}_{-i}) - p_i(x_r - x_{i,r} + y_{i,r})\, y_{i,r} + p_i(x_s - x_{i,s} + w_{i,s})\, w_{i,s} \\ &> u_i(\mathbf{y}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x}), \end{aligned} \tag{3}$$

where the first inequality in (3) follows from

$$x_s - x_{i,s} + w_{i,s} < x_s \leq x_r - x_{i,r} + y_{i,r}$$

and the assumption that market reaction functions are non-increasing. As $S(\mathbf{y}_i') \subseteq S(\mathbf{x}_i)$ this is a contradiction to the fact that $\mathbf{x}$ is a partial equilibrium.

We proceed to show part (b) of the statement of the lemma. Let us assume for a contradiction that there is a market $\tilde{m} \in M \setminus \{r, s\}$ with $x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}} > x_{\tilde{m}}$ and, hence, $y_{i,\tilde{m}} > x_{i,\tilde{m}}$. It follows that

$$c_i'(x_i) \geq \frac{\partial}{\partial x_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}})\, x_{i,\tilde{m}}\big) > \frac{\partial}{\partial y_{i,\tilde{m}}}\big(p_i(x_{\tilde{m}} - x_{i,\tilde{m}} + y_{i,\tilde{m}})\, y_{i,\tilde{m}}\big) = c_i'(y_i), \tag{4}$$

which implies together with the convexity of c_i that $y_i < x_i$. This implies that at least one of the following two cases holds: (i) $y_{i,r} < x_{i,s}$; or (ii) there is a market $m \in Q_i \setminus \{s\}$ with $y_{i,m} < x_{i,m}$.

We proceed to derive contradictions for both cases. First, suppose that case (i) holds. Using $x_r - x_{i,r} + y_{i,r} < x_s$ from the first part of the statement of this lemma, we obtain

$$c_i'(y_i) \geq \frac{\partial}{\partial y_{i,r}}\big(p_i(x_r - x_{i,r} + y_{i,r})\, y_{i,r}\big) > \frac{\partial}{\partial x_{i,s}}\big(p_i(x_s)\, x_{i,s}\big) = c_i'(x_i), \tag{5}$$

a contradiction to (4).

Next, suppose that (ii) holds, i.e., there is a market $m \in Q_i \setminus \{s\}$ with $y_{i,m} < x_{i,m}$ and, thus, $x_m - x_{i,m} + y_{i,m} < x_m$. The same calculations as in (5) where we replace r and s by m give a contradiction to (4). □

Second, we show that whenever a firm plays a restricted improvement in which one market enters its active set, then after recomputing a partial equilibrium, the total quantity offered on each market may only decrease. The proof can be found in the full version.

Lemma 6. *Let* $\mathbf{x}$ *be a partial equilibrium of* G *for the active set vector* $(Q_i)_{i\in N}$*,* $\mathbf{y}_i$ *be a restricted improvement of firm* i *with* $S(\mathbf{y}_i) \subseteq Q'_i$*,* $Q'_i = (Q_i \cup \{r\}) \setminus \{s\}, s \in Q_i,\ r \in M_i \setminus Q_i$*. Let* $(\tilde{\mathbf{y}}_i, \tilde{\mathbf{x}}_{-i})$ *be a partial equilibrium for the active set vector* $(Q'_i)_{i\in N}$ *where* $Q'_j = Q_j$ *for all* $j \in N \setminus \{i\}$*. Then, the following two properties hold:*

(a) $\tilde{x}_r - \tilde{x}_{i,r} + \tilde{y}_{i,r} \leq x_r - x_{i,r} + y_{i,r}$.
(b) $\tilde{x}_m - \tilde{x}_{i,m} + \tilde{y}_{i,m} \leq x_m$ *for all* $m \in M \setminus \{r\}$.

We are now ready to prove that Algorithm 1 terminates.

Lemma 7 (Termination). *Algorithm 1 terminates.*

Proof. To prove that Algorithm 1 terminates, we consider the function

$$L : \bigtimes_{i\in N} 2^{M_i} \to \bigtimes_{i\in N} \mathbb{R}^2_{\geq 0},\ (Q_j)_{j\in N} \mapsto \big(L_i((Q_j)_{j\in N})\big)_{i\in N},\ \text{with}$$

$$L_i\big((Q_j)_{j\in N}\big) = \Big(L_i^1\big((Q_j)_{j\in N}\big), L_i^2\big((Q_j)_{j\in N}\big)\Big) = \Big(\max_{m\in Q_i} x_m,\ \big|\arg\max_{m\in Q_i} x_m\big|\Big),$$

for all $i \in N$, where $\mathbf{x}$ is an arbitrary partial equilibrium for the active set vector $(Q_i)_{i\in N}$. In words, L maps each active set vector $(Q_i)_{i\in N}$ to the vector that contains for each player (under a partial equilibrium $\mathbf{x}$ for the active set vector $(Q_i)_{i\in N}$) the tuple of the maximum aggregated supply that firm i experiences among the markets contained in Q_i and the number of markets for which this maximum is attained. Note that L is well-defined as, by Lemma 3, for a given active set vector $(Q_i)_{i\in N}$ the aggregated demands for all markets are unique for all partial equilibria for $(Q_i)_{i\in N}$.

Let us denote by $\tilde{L}\big((Q_j)_{j\in N}\big)$ the vector that contains the $|N|$ tuple of $L\big((Q_j)_{j\in N}\big)$ in non-decreasing lexicographical order, i.e.,

$$\begin{aligned} &\tilde{L}_i^1\big((Q_j)_{j\in N}\big) \geq \tilde{L}_{i+1}^1\big((Q_j)_{j\in N}\big),\\ \text{and}\quad &\tilde{L}_i^2\big((Q_j)_{j\in N}\big) \geq \tilde{L}_{i+1}^2\big((Q_j)_{j\in N}\big),\ \text{if } \tilde{L}_i^1\big((Q_j)_{j\in N}\big) = \tilde{L}_{i+1}^1\big((Q_j)_{j\in N}\big). \end{aligned}$$

We claim that $\tilde{L}$ decreases lexicographically during the execution of Algorithm 1. To see this, fix an active set vector $(Q_i)_{i\in N}$ with $Q_i \subseteq M_i$ for all $i \in N$ and an arbitrary partial equilibrium $\mathbf{x}$ for $(Q_i)_{i\in N}$. If there is no firm with a profitable unilateral deviation, then there is nothing left to show as we have reached a Cournot equilibrium. So, let us assume that there is a firm i with a strategy $\mathbf{y}_i \in X_i$ such that $u_i(\mathbf{y}_i, \mathbf{x}_{-i}) > u_i(\mathbf{x})$. Lemma 4 implies that the strategy $\mathbf{z}_i$ chosen in Line (3a) of Algorithm 1 yields also an improvement of firm i. We denote the partial equilibrium recomputed in Line (2) of Algorithm 1 by $(\tilde{\mathbf{z}}_i, \tilde{\mathbf{x}}_i)$.

For all $m \in M \setminus \{r\}$ we obtain $\tilde{x}_m - \tilde{x}_{i,m} + \tilde{z}_{i,m} \leq x_m$ using Lemma 6. Furthermore, we obtain

$$\tilde{x}_r - \tilde{x}_{i,r} + \tilde{z}_{i,r} z \leq x_r - x_{i,r} + y_{i,r} < x_s, \tag{6}$$

where the first inequality follows from Lemma 6 and the second inequality follows from Lemma 5. For firm i, the tuple $L_i\big((Q_i)_{i\in N}\big)$ does decrease because firm i leaves market s with maximal aggregated quantity and using (6) the aggregated quantity on the new market r settles strictly below the old aggregated quantity on s. For all firms $j \in N \setminus \{i\}$ with $\max_{m\in Q_j} x_m \geq x_s$, we conclude that the tuple $L_j\big((Q_i)_{i\in N}\big)$ does not increase since the maximum was not attained at s and s is the only market on which the aggregated quantity increases. Finally, for all firms $j \in N \setminus \{i\}$ with $\max_{m\in Q_j} x_m < x_s$ we observe that $\max_{m\in Q_j} x_m < x_s$ since the aggregated quantity on r does not increase beyond x_s as shown in (6). We conclude that $\tilde{L}$ decreases lexicographically.

Since there are only finitely many active set vectors $(Q_i)_{i\in N}$, Algorithm 1 terminates after a finite number of steps and outputs a Cournot equilibrium. □

5 Violation of Assumptions

In this section, we show that our assumptions on the market price functions are necessary conditions in the sense that if one of them is violated, a Cournot equilibrium may fail to exist. Since Assumption 2 frequently appears in the literature on Cournot equilibria and some kind of regularity of the market price functions is already necessary for games with a single market (cf. Novshek [11]), we here show only the necessity of the critical Assumption 3.

Proposition 1. *There is a multimarket oligopoly with support constraints for which market reaction functions and cost functions satisfy Assumptions 1–2 that does not admit a Cournot equilibrium.*

The proof can be found in the full version.

6 Conclusions

We studied multimarket oligopolies in which players face a bound on the number of markets they can be engaged in simultaneously. We assumed that the firms' cost functions are convex and the player-specific market reaction functions are concave. We proved that a Cournot equilibrium is guaranteed to exist provided that the player-specific market reaction functions on the markets are identical up to a market-specific shift in the argument. While this condition seems may seem very demanding, we further showed that if this assumption is violated, a Cournot equilibrium need not exist. We see this as a first step towards a better understanding of multimarket oligopolies with market access restrictions.

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Value of Targeting

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Abstract. We undertake a formal study of the value of targeting data to an advertiser. As expected, this value is increasing in the utility difference between realizations of the targeting data and the accuracy of the data, and depends on the distribution of competing bids. However, this value may vary non-monotonically with an advertiser's budget. Similarly, modeling the values as either private or correlated, or allowing other advertisers to also make use of the data, leads to unpredictable changes in the value of data. We address questions related to multiple data sources, show that utility of additional data may be *non*-monotonic, and provide tradeoffs between the quality and the price of data sources. In a game-theoretic setting, we show that advertisers may be worse off than if the data had not been available at all. We also ask whether a publisher can infer the value an advertiser would place on targeting data from the advertiser's bidding behavior and illustrate that this is impossible.

1 Introduction

Good targeting is paramount to successful advertising: showing the right ad to the right person is beneficial to all parties involved. On the other hand, poor targeting is wasteful: knowing about the latest car is of limited use to someone who only commutes by bike; ads for umbrellas are spam to someone who lives in the desert. In the past few years several companies have begun assisting advertisers in their targeting efforts. Firms like BlueKai and eXelate build profiles of web users and classify them into different interest categories, such as those interested in buying a new car, traveling to Barcelona, or with an obsession over the latest gadget. Data management is a multi billion dollar business, but there has been little analysis characterizing the value data to an advertiser.

An advertiser usually knows which segments of the population she wants to pitch her products to. However, these may not be perfectly aligned with the classification available from the firms above. For example, a hotel owner in Milan is not explicitly interested in Americans going to Barcelona, but does know that they may respond better than the average individual to her pitch since they have already expressed willingness to travel internationally. How much would the hotel owner be willing to pay for the "travel to Barcelona" demographic? On the other hand, suppose she instead appeals to people interested in fashion. Although not all of them have expressed interest to travel, they may be tempted by a fashion week promotion that she is running. How much should she pay for this segment? What if she buys the two segments together?

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At first glance, the answer is relatively simple: by using the additional targeting the advertiser increases the value of the "typical" user seeing her pitch, and thus the value for the targeting data is bounded above by the difference in the two values. In a similar fashion, the quality of targeting, which accounts for the alignment between the segmentation provided by the data provider and that desired by the buyer also comes into play, and higher quality (better aligned) targets should garner higher prices. However, for online advertisements, the impressions will be sold at auction, making the analysis more subtle. We show that the actions of other bidders as well as the broader competitive landscape play a big role in determining the value of the data. We outline our results below:

Model and Basics (Sections 2 and 3). We formalize the problem and present basic findings. As expected, we prove that the value for the data is higher for data segments that occur more frequently, are closely aligned with the targeting criteria, and have a larger impact on whether the advertiser wins the auction.

Budgets (Section 3.3). We show that the addition of a budget constraint changes the nature of the optimization problem, as bidders no longer bid their true value. This leads to potentially counterintuitive behavior, where an increase in a budget results in a lower valuation for the data in an optimal solution.

Private and Correlated Values (Section 3.4). The presence of correlated values also changes the underlying optimization problem. We show that the value of data crucially depends on the competition, and one may do better trying to entice others to buy the data rather than spending the money yourself.

Game Theoretic Setting (Section 4). We characterize the nature of equilibria in a game where multiple buyers simultaneously decide whether to purchase targeting data. We show that a pure equilibrium need not exist, and advertisers can be worse off, compared to a situation where no targeting data is present.

Multiple Data Sources (Section 5). We study the problem of selecting from multiple independent data sources. We show that the value of the data is non-monotone in the number of the sources, even if they are all identical, and give a prior-independent bound on the value of data as a function of its quality.

Value to Publishers (Section 6). We prove that simply observing an advertiser's bids on a large number of heterogeneous impressions does not provide enough information for a publisher to infer the advertiser's value for the data.

1.1 Related Work

A large body of existing work studies the effect of targeting in auctions, but most of this work is from the perspective of the publisher. [15] addresses the question of whether a seller should enable buyers to improve targeting in a correlated values auction by revealing information about the quality of the seller's goods. More recently, [9] and [14] consider the question of when improved targeting increases a seller's revenue in an auction where the buyers have independent private values. These papers all suggest that it is often beneficial for a seller to enable buyers to more finely target in an auction. Finally, [1] analyzes how revenue is affected by the asymmetries in information possessed by different participants and finds that such asymmetries can sometimes lead to adverse revenue effects.

A related thread of analysis considers revenue optimization strategies for an auctioneer with targeting data. [7] explores how an auctioneer selling a probabilistic good might reveal partial information to maximize its revenue. [12] considers a similar problem in a discrete setting with many goods. And [2] studies the mechanism design problem where a seller might use the asymmetry of information to maximize his revenue. All of these results describe the effects of additional data on the seller's revenue and the overall welfare. In this work we tackle the converse problem of the value of data to an individual buyer.

There have also been papers on how targeting affects market equilibria when there are multiple publishers. [10] considers the question of how cookie-matching affects the market equilibrium in a model in which advertisers can use a cookie from one publisher to better target users on other publishers. And [3] investigates the question of how enabling advertisers to target certain segments of the population would affect the market equilibrium for advertising in a model of informative advertising. This work differs from our work in that it does not consider the underlying auctions that are used to sell advertising opportunities.

Another line of work is on advertiser optimization. Here the focus is either on getting a fair or representative allocation [11], or on finding bidding strategies that work well with poor forecasts [4,6,13]. These approaches are data agnostic and do not explore advertiser actions when she can use better targeting.

Our approach is quite different from any of the above papers, as we focus on what factors affect the advertiser's utility for targeting data in an auction setting. None of the papers considered above address this question.

2 Preliminaries

We consider the setting of a single agent (advertiser) buying items (impressions) in a second price auction. The items are heterogeneous, and different items are valued differently by the buyer. Let $T = \{t_1, t_2, \ldots\}$ be the partition of items into types. For an item of type $t \in T$ (*e.g.*, impressions from Texas), we say that the buyer draws his value from a distribution with cdf G_t and corresponding pdf g_t. In the simplest case when G_t is a point distribution, we denote it by v_t. We denote by π_t the *prior* that the buyer has on the item being of type t. This is the probability that a random item is of type t; thus $\sum_t \pi_t = 1$.

In addition to knowing her prior, the buyer has access to additional data sources (signals) about the impression type. We assume that each signal is drawn from a distribution that depends only on the impression type and that conditional on the impression type, the signals are independent from the buyer's prior and from each other. The data sources may be imperfect.

For part of the manuscript we will focus on the special case in which a data source only identifies whether a user is in one of two subsets of the population, which we denote by H and L (high and low). The advertiser then has a prior π for the high type, and $1 - \pi$ for the low type, and values them at v_H and v_L respectively. In this setting we also assume that a signal has the same benefit of predicting both types. Formally, we model this by saying that each signal s_i has a *quality* $q_i \in [\frac{1}{2}, 1]$, which represents the probability of the signal being correct, $q_i = Pr[s_i = t]$; the signal is then incorrect with probability $1 - q_i$.

The items are sold in a second price auction, and we let f (F) denote the pdf (cdf) of the distribution of the highest competing bid. In the independent private value (IPV) setting, the distributions are the same for all of the different types. A natural generalization is the correlated value setting, where items of different types sell for different prices, whose densities and cumulative distribution functions we denote by f_t and F_t respectively. For the rest of the paper, we will assume the IPV setting unless explicitly specified otherwise.

We work with the standard quasilinear utility model, with buyers acting to maximize the difference between their value and the price paid. When the type is known to the buyer, she maximizes her utility by bidding her value for each impression type. If, on the other hand, she only has the prior information about the item type, she bids $\bar{v} = \sum_t \pi_t v_t$ in the IPV setting.

3 Data Basics

We begin by considering simple settings to develop some intuition about the value of targeting data. We first illustrate that the buyer's value for the data results both from buying more desired items and not overpaying for lower quality items. We extend this analysis to show that the value inherently depends on four quantities: the buyer's prior information, the quality of the signal, the difference in values for the different types, *and* the competitive landscape, expressed as the additional fraction of impressions the buyer can win with the value.

3.1 Binary User Types

To develop intuition we first consider the setting in Section 2 with two types of users, H and L. The advertiser has access to a noisy data source that will either assume the value h or ℓ. In particular, if the user is of type H (*resp.*, L), then the data source produces a signal h (*resp.*, ℓ) with probability $q > 1/2$ and ℓ (*resp.*, h) with probability $1 - q$.

In this case, if the advertiser has access to the data, she updates her prior based on the signal. The probabilities that a user is of type H upon receiving signals of h and ℓ respectively are:

$$\pi|h = \frac{\pi q}{\pi q + (1-\pi)(1-q)} \qquad \pi|\ell = \frac{\pi(1-q)}{\pi(1-q) + (1-\pi)q}$$

Thus the expected value of an advertising opportunity upon receiving a signal of h or ℓ respectively is $v|h = \pi|h \cdot v_H + (1-\pi|h)v_L$ and $v|\ell = \pi|\ell \cdot v_H + (1-\pi|\ell)v_L$. We now precisely characterize the value of the signal to the advertiser:

Lemma 1. *The advertiser's value for this noisy signal is*

$$\int_{\bar{v}}^{v|h}(\pi q(v_H-p)+(1-\pi)(1-q)(v_L-p))f(p)dp-\int_{v|\ell}^{\bar{v}}(\pi(1-q)(v_H-p)+(1-\pi)q(v_L-p))f(p)dp.$$

For complete proofs see the full version of this paper [5]. While Lemma 1 gives an exact expression for the advertiser's value for a noisy signal, it may be not be immediately transparent how the different parameters in the model affect it. Our next result gives a simpler expression which bounds this value and shows that value can be decomposed into four independent factors.

Theorem 1. *The advertiser's value for the data can be bounded from above by* $(v_H - v_L)\pi(1-\pi)(2q-1)(F(v|h) - F(v|\ell))$.

Theorem 1 provides some basic rules of thumb for valuing the data. If there is not much difference between the advertiser's values for advertising to different types of users, $(v_H - v_L)$, then the advertiser will not care much whom she advertises to, and will have little value for the data. Thus the advertiser's value for the data is increasing in this quantity. It is also intuitive that the advertiser's value for the data is increasing in the accuracy of the signal, $(2q-1)$.

Additionally, if competing buyers rarely place a bid that falls between the values the advertiser may have for the different types of users, an advertiser's ability to adjust her bid in response to the different possible realizations of the targeting data will rarely have an effect on whether she wins the advertising opportunity. Thus the value of the data is increasing in the likelihood of a competing advertiser placing a bid between the advertiser's possible values for the different types of advertising opportunities, $(F(v|h) - F(v|\ell))$.

Finally, it makes sense that the advertiser's value for the data is single-peaked in π. If π is very close 0 or 1, then the data almost always takes on the same value, and there is little gain to seeing it. By contrast, when π is closer to $1/2$, there is more uncertainty in the true type of the item, and thus more heterogeneity in the different realizations of the targeting data, so the data is more valuable.

3.2 General Distributions of Valuations

We now move to a more general model, where we consider what happens when an advertiser's best estimate for his value for advertising to a particular user may assume a large number of distinct values. We show formally that more "refined" signals on the value of the item are more valuable to the advertiser.

To model this setting, suppose that when an advertiser has access to data, she learns that the best estimate for her value for advertising to a particular user is v, which is a random draw from some distribution $G(\cdot)$ with corresponding pdf $g(\cdot)$. If an advertiser has no access to data, then the advertiser simply knows her expected value, $E[v|v \sim G] = \bar{v}$. The utility gain from the data is then $\int_0^\infty \int_{\bar{v}}^v (v-p)f(p)dp\, g(v)dv$.

Now suppose there are two different data sources that an advertiser might use, with distributions G and H. If they are unbiased, the distributions will satisfy $E_G[v] = E_H[v]$. We address the question of when one data source would be more useful to an advertiser than another:

Theorem 2. *Consider two data sources with corresponding cdfs G and H satisfying $E_G[v] = E_H[v]$. Then if $H(\cdot)$ second order stochastically dominates $G(\cdot)$, the advertiser has more value for the data source G than the data source H.*

3.3 Budgets

Throughout the analysis so far we have assumed that an advertiser does not face any budget constraints, but in some settings an advertiser only has a fixed budget for advertising. It is natural to ask how this possibility would affect an advertiser's value for targeting data. We address this possibility in this section.

We again consider the binary case, with types H and L and corresponding price distributions $F_H(\cdot)$ and $F_L(\cdot)$. Let $f(\cdot) \equiv \pi f_H(\cdot) + (1-\pi) f_L(\cdot)$ be the density of the highest competing bid unconditional on type. In addition, let B denote the maximum amount the advertiser can spend per advertising opportunity. Note that if the advertiser does not have access to the data, then in every auction she makes a bid b satisfying $\int_0^b pf(p)dp = B$.

If the advertiser has access to a perfectly informative data source, then the advertiser bids $b_L(b_H)$ when the user is type $L(H)$, where the values of b_L and b_H must satisfy $\pi \int_0^{b_H} pf_H(p)dp + (1-\pi)\int_0^{b_L} pf_L(p)dp = B$. The advertiser then chooses the values of b_L and b_H to maximize $\pi \int_0^{b_H} v_H f_H(p)dp + (1 - \pi)\int_0^{b_L} v_L f_L(p)dp - B$ subject to this budget constraint.

Now we address how small changes in the budget (which induce small changes in b, b_L, and b_H) affect the advertiser's value for the data. One might think that an advertiser's value for the data would always increase with the advertiser's budget because the advertiser would be better able to exploit the targeting information. However, this is not the case as the following theorem illustrates:

Theorem 3. *The advertiser's value for the data is not monotone in her budget.*

While it is intuitive that an advertiser's value for the data may increase with the advertiser's budget, it may be less obvious why an advertiser's value for the data might decrease with the advertiser's budget. To see why this might arise, suppose the advertiser always has a larger value for all advertising opportunities than the competing advertisers. In this case, if an advertiser has a large budget, the data hardly has any effect on the impressions that the advertiser purchases since she would purchase almost all impressions anyway. However, if the advertiser has a smaller budget, then the targeting data may have a significant effect on which advertising opportunities the advertiser wins. Thus the advertiser's value for the data may be decreasing in the size of the advertiser's budget.

3.4 Correlated Value Setting

In Sections 3.1 and 3.2 we restricted attention to scenarios in which there is no correlation between an advertiser's value for an advertising opportunity and the highest competing bid. However, in practice this assumption does not always hold because user characteristics like income make advertising opportunities more or less valuable to multiple advertisers at the same time. In this section, we address how this possibility affects an advertiser's value for targeting data.

To illustrate our results, we again consider the binary case, with users of type H and L. However, we now assume that if the user is of type $H(resp., L)$, then the highest bid placed by a competing advertiser is a random draw from the distribution $F_H(\cdot)(resp., F_L(\cdot))$ with pdf $f_H(\cdot)(resp., f_L(\cdot))$.

Before we can figure out how the advertiser would value targeting data, we must first figure out the bidding strategy that the advertiser would use when she does not have access to this targeting data. With continuous densities, in equilibrium, she will bid some amount $v^* \in [v_L, v_H]$ that satisfies[1]

$$v^* = \frac{\pi f_H(v^*)v_H + (1-\pi)f_L(v^*)v_L}{\pi f_H(v^*) + (1-\pi)f_L(v^*)}. \tag{1}$$

Thus if the advertiser does not have access to the data, her expected payoff is $u_{ND} = \pi \int_0^{v^*} (v_H - p)f_H(p)dp + (1-\pi) \int_0^{v^*} (v_L - p)f_L(p)dp$. However, if the advertiser has access to the data, then she places a bid of $v_H(v_L)$ for users of type $H(L)$. The advertiser's expected payoff is then $u_D = \pi \int_0^{v_H} (v_H - p)f_H(p)dp + (1-\pi) \int_0^{v_L} (v_L - p)f_L(p)dp$. Her value for the data is the difference between the two expressions and it depends on similar things in the correlated value setting as in the private value setting. In particular, we obtain the following result:

Theorem 4. *An advertiser's value for targeting data is increasing in her utility difference between advertising to different users ($v_H - v_L$), increasing in the likelihood of a competing advertiser placing a bid between these possible values ($f_H(p)$ for $p \in [v^*, v_H]$ and $f_L(p)$ for $p \in [v_L, v^*]$), and is single-peaked in the relative likelihoods of the different realizations of the targeting data (π).*

While the advertiser's values for the data in the correlated and private value settings depend on similar terms, the two values are incomparable: the value is not always higher in one setting than the other. In the correlated value framework, it is entirely possible for the advertiser to have zero value for the data, as the advertiser may be able to exploit the fact that the competing advertisers are perfectly segmenting the market, so that she always wins high value impressions and always loses on the low value impressions. In these circumstances, the advertiser's value for the data will be lower under the correlated value framework than in the private value framework.

At the same time, it is also possible that the advertiser could have greater value for the data in the correlated value framework than in the private value framework. If the competing advertisers are making bids that are strongly correlated with the advertiser's value for the advertising opportunity, then the advertiser may not be able to profitably bid in the auction without access to the data. In this case, the data is especially valuable for the advertiser in the correlated value framework, and the data may be more valuable under correlated values than under private values. We make this precise below.

Observation 1. *The advertiser's value for the data may be either greater or lower in the correlated value setting than in the private value setting.*

[1] We defer the proof of this claim to [5].

4 Game Theoretic Setting

So far we have focused on the value a particular advertiser would place on targeting data while ignoring the possibility that other advertisers may also use this data. We now ask how this possibility affects an individual advertiser's value for the data in a game-theoretic setting.

4.1 Value of Data

We show that when there are two bidders, each bidder prefers the competitor to buy the data *regardless* of her own actions. However, the exact value an advertiser places on the data may go up or down as a function of the other bidder's actions, and there are situations where there are no pure strategy equilibria to the game where advertisers simultaneously decide whether to purchase the data.

Consider the case with two advertisers. Each advertiser $i \in \{1, 2\}$ has a value v_i for a given advertising opportunity, where each v_i is an independent draw from the distribution $F_i(v)$ with pdf $f_i(v)$. If advertiser i has access to the data, then advertiser i knows her value for a particular advertising opportunity. If not, then this advertiser only knows the distribution from which her value is drawn. Naturally each advertiser obtains positive value from having access to the data regardless of whether the other advertiser also has access to the data. Less obviously, each advertiser also has preferences over whether the other advertiser has access to the data. In particular, we obtain the following result:

Theorem 5. *When there are two advertisers, each advertiser prefers that the other advertiser have access to the data regardless of whether she has access to the data herself.*

To understand the intuition behind this result, first note that when neither advertiser has access to the data, then some advertiser (say advertiser 2) never wins the auction. However, if advertiser 1 has access to the data, then she may sometimes bid less than advertiser 2 is bidding, which leads to advertiser 2 earning positive profit. Similarly, if advertiser 2 has access to the data, then advertiser 1 will sometimes pay less for advertising opportunities that she would have won anyway (because advertiser 2 may discover that she values these impressions for less than she originally thought), but advertiser 1 will not have to pay for impressions where advertiser 2 learns that she values these impressions for more than advertiser 1. Thus advertiser 1 is also better off when advertiser 2 has access to the data.

This result does not necessarily extend when there are more than two advertisers. Nonetheless, it does illustrate that an advertiser can have preferences over whether competing advertisers have access to targeting data, and that these preferences may be the opposite of what one might conjecture naively.

An advertiser's value for the data can also depend on whether the other advertiser has access to the data. However, there is no general relationship as to how an advertiser's value for the data depends on whether the other advertiser has access to the data, even with two bidders. This is a corollary of Theorem 4: by purchasing the data the second advertiser may affect the competing bid distribution in an arbitrary manner, thereby changing the value of the data.

Observation 2. *An advertiser's value for the data may either increase or decrease as a result of the other bidder having access to the data in both the private and correlated value framework.*

4.2 Data Buying Equilibria

We now turn to the question of whether there exists a pure strategy equilibrium to the game in which advertisers simultaneously decide whether to purchase data and then bid in the auction. In general the fact that one advertiser may prefer everyone to have the data, whereas another may prefer to be the unique holder of the data, means that pure strategy equilibria to this game need not exist.

Theorem 6. *There may be no pure strategy equilibrium to the game in which advertisers simultaneously decide whether to purchase a data source.*

While it is generally the case that there will not exist a pure strategy equilibrium to the game in which advertisers simultaneously decide whether to purchase a given source of data, there are important special cases under which such pure strategy equilibria exist. In particular, in a pure private values setting with symmetric bidders, we can prove that pure strategy equilibria exist.

Theorem 7. *Suppose each advertiser's estimate of her value is an independent draw from the distribution $H(\cdot)$ if the advertiser purchases the data and from the distribution $G(\cdot)$ otherwise. Then there is a pure strategy equilibrium to the game in which advertisers simultaneously decide whether to purchase the data.*

Finally, we present a result on the value of data when multiple symmetric bidders are all given access to the same data. In general, sources of data may refine an advertiser's estimate of her value by helping an advertiser learn that her true value for an advertising opportunity is $v + \epsilon$ for some mean-zero random variable ϵ, where v denotes the advertiser's original estimate of her value. It is interesting to ask how the value of a data source depends on whether it helps advertisers distinguish amongst high or low value advertising opportunities. We show below that the data is more valuable for picking out high valued impressions when multiple advertisers are given access to the same data.

To illustrate this, we consider a situation in which each advertiser's initial estimate of her value is an independent and identically distributed draw from the distribution $F(\cdot)$. We call a data source a (v^*, δ)-*refinement* signal, if, given an initial estimate of an advertiser's value v:

- it provides no information for $v \notin [v^* - \delta, v^* + \delta]$,
- refines the value to $v + \epsilon$ when $v \in [v^* - \delta, v^* + \delta]$ for some ϵ that is a random draw from some distribution with mean zero and narrow support.

Theorem 8. *Suppose there are at least three advertisers, and all advertisers are given access to a (v^*, δ) refinement signal for some v^* in the support of $F(\cdot)$ and an arbitrarily small value of δ. Then there is a threshold τ in the interior of the support of $F(\cdot)$ such that the value of the data source to the advertisers is positive for values of $v^* > \tau$ and negative for values of $v^* < \tau$.*

To understand this result, note that if a data source helps advertisers distinguish amongst high value advertising opportunities, then this data is likely to help an advertiser identify a high-value advertising opportunity that the advertiser might not have won in the absence of the data. Such a possibility is beneficial for the advertiser. But if a data source helps advertisers distinguish amongst low value advertising opportunities, this data is unlikely to help an advertiser much, because advertisers with low values are unlikely to win the auction. Instead all this data source is likely to do is increase the expected price paid by the winner of the auction. Thus a data source that helps advertisers distinguish amongst high-value advertising opportunities increases welfare when multiple advertisers have this data, but a data source that only helps advertisers distinguish amongst low-value advertising opportunities decreases welfare.

5 Working with Multiple Signals

In the previous sections we derived the value of a single data source to the buyer in several diverse settings. We now address questions related to multiple data sources. Will additional data sources become more or less valuable when an advertiser already has access to other data sources? How should an advertiser resolve the trade-off between the cost of a data source and its quality in deciding which of several possible data sources to purchase?

One might think intuitively that when an advertiser is buying multiple independent signals that the marginal value of additional signals would be decreasing in the number of signals that an advertiser has already purchased because each additional signal would do less to refine the advertiser's assessment of the user's type. However, this need not be the case because the advertiser's value for targeting data depends crucially on the landscape of competing bids, and as a result, a second signal may be much more likely to have an effect on whether the advertiser wants to win the auction than the first signal:

Observation 3. *In the independent private values setting with two types of users, the marginal value of an additional signal need not vary monotonically with the number of signals the advertiser already has access to.*

While the value of the data depends on what other signals the advertiser has access to, below we derive a bound on the value solely as a function of the quality of the data, independent of what other data sets are present:

Theorem 9. *In the independent private value model with two user types, H and L, an advertiser's value for a signal with quality q is bounded by $(v_H - v_L)^2 \bar{f}\left(\frac{2}{3}q^3 - \frac{1}{2}q^2 + \frac{1}{24}\right)$, where $\bar{f} \equiv \sup_{p \in [v_L, v_H]} f(p)$.*

By applying very similar logic we can illustrate bounds on the maximum additional amount that an advertiser would be willing to pay for a more accurate signal regardless of the advertiser's prior. This is done below:

Corollary 1. *Suppose an advertiser has the option of buying two different types of signals with qualities* q_1 *and* q_2 *respectively, where* $q_1 > q_2$*. Then if the cost of the first signal is more than* $\bar{f}(v_H - v_L)^2[\frac{2}{3}(q_1^3 - q_2^3) - \frac{1}{2}(q_1^2 - q_2^2)]$ *greater than the cost of the second signal, the advertiser will always prefer to purchase the second signal regardless of the advertiser's prior.*

The results in this section illustrate that there are natural bounds on the price of the data that one can use to quickly rule out whether certain sources of data are cheap enough to be worthwhile, without knowing the advertiser's prior or the entire distribution of competing bids. If the cost of the data exceeds the bounds in the previous theorems, an advertiser will never want to purchase it.

6 Reverse Engineering the Value of Data

So far, we have focused on how an advertiser values targeting data. However, it is also natural to inquire about the value of data from the publisher's perspective. If a publisher is supplying data that maps the user to a specific population segment, can he deduce how much an advertiser would value this data solely by observing the advertiser's average bidding behavior for the different realizations of the data? Unfortunately the answer to this question is no. We prove this formally below, but first provide some intuition behind the result.

Suppose the publisher supplies data that identifies whether a user lives in California, and further suppose that the advertiser is making a higher average bid on users from California. There are several possible ways for the advertiser to exhibit this aggregate bidding behavior. First, it is possible that the only factor that influences how much the advertiser values a particular advertising opportunity is whether the user lives in California. In this case, the targeting information that the publisher supplies is truly valuable to the advertiser.

Another possibility is that the advertiser only values advertising to some subset of the users in California that she can already target on, and values all other users equally. In this case, the publisher's data is worthless to the advertiser because the advertiser would make the same bids without this data, yet the average bids for users in and outside of California are the same as before. Thus it is also possible for the advertiser to have zero value for this data.

Furthermore, by taking convex combinations of the two extremes, it is possible to construct scenarios where the advertiser's value for the data can assume any value in this range. We thus obtain the following result:

Theorem 10. *It is not possible to infer an advertiser's value for targeting data if the publisher only observes the advertiser's average bids for different realizations of the targeting data as well as the distribution of the highest competing bids. Given these average bids, the advertiser's value for the targeting data may range anywhere from zero to some maximal value.*

7 Conclusion

In this paper we extensively analyzed the problem of valuing targeting data to an advertiser, and showed precisely how it depends both on the advertiser's parameters (value, budget, etc.) and the actions of other players (competing bids, participation, etc.). A natural next step is to design truthful and welfare maximizing mechanisms that can aid in a creation of an effective marketplace.

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Welfare Guarantees for Proportional Allocations*

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Abstract. According to the proportional allocation mechanism from the network optimization literature, users compete for a divisible resource – such as bandwidth – by submitting bids. The mechanism allocates to each user a fraction of the resource that is proportional to her bid and collects an amount equal to her bid as payment. Since users act as utility-maximizers, this naturally defines a proportional allocation game. Recently, Syrgkanis and Tardos (STOC 2013) quantified the inefficiency of equilibria in this game with respect to the social welfare and presented a lower bound of 26.8% on the price of anarchy over coarse-correlated and Bayes-Nash equilibria in the full and incomplete information settings, respectively. In this paper, we improve this bound to 50% over both equilibrium concepts. Our analysis is simpler and, furthermore, we argue that it cannot be improved by arguments that do not take the equilibrium structure into account. We also extend it to settings with budget constraints where we show the first constant bound (between 36% and 50%) on the price of anarchy of the corresponding game with respect to an effective welfare benchmark that takes budgets into account.

1 Introduction

The *proportional allocation mechanism*, introduced by Kelly [11], is fundamental in the network optimization literature. According to this mechanism, a divisible resource — such as bandwidth of a communication link — is allocated to users as follows. Each user submits a bid to the mechanism; this corresponds to the user's *willingness-to-pay* for sharing the resource. The mechanism allocates to each user a fraction of the resource that is equal to the ratio of her bid over the total amount of bids. It also receives a payment from each user that is equal to her bid. This naturally defines a *proportional allocation game* among the users who act as players; each player has a (typically concave, non-negative, and non-decreasing) valuation function for the resource share she receives and aims to

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maximize her utility, i.e., her value for the resource share minus her payment to the mechanism. As it is typically the case in games, the *social welfare* (i.e., the total value of the players for the resource share they receive) at *equilibria* is, in general, suboptimal.

We aim to quantify this inefficiency of equilibria by bounding the *price of anarchy* [12] of proportional allocation games. Besides the well-known work of Johari and Tsitsiklis [9] who considered pure Nash equilibria in the full information setting, there has been surprisingly little focus on price of anarchy bounds over more general equilibrium concepts. The only exception we are aware of is the recent work of Syrgkanis and Tardos [22] who studied proportional allocation as part of a broader class of mechanisms. Motivated by their work, we present new bounds on the price of anarchy of proportional allocation under general equilibrium concepts, such as *coarse-correlated* equilibria in the full information setting and *Bayes-Nash* equilibria in the incomplete information setting. In particular, we prove that the social welfare at equilibrium is at least 1/2 of the optimal social welfare. The bound holds for coarse-correlated and pure Bayes-Nash equilibria in the full information and Bayesian setting, respectively, and improves the bound of 26.8% of [22]. The proof is conceptually simple and is obtained by bounding the utility of every player at equilibrium by the utility this player would have by deviating to a particular deterministic bid.

We also consider the scenario where players have budget constraints representing their *ability-to-pay*. Here, each player has a budget and is never allowed to bid above it. We assess the quality of equilibria in this case in terms of an *effective welfare* benchmark — proposed in previous work but further refined here — that takes budgets into account. We show that the effective welfare at equilibrium is at least a constant fraction of the optimal one. To the best of our knowledge, this is the first constant price of anarchy bound (in particular, between 36% and 50%) with respect to this benchmark[1]. Again, our proofs follow by considering a single deterministic deviation for each player, defined in a slightly different way compared to the deviation we consider in our bound on the social welfare.

Related Work. The proportional allocation mechanism and its variations have received significant attention in the network optimization literature. Proportional allocation games have been considered in [7,13,14] where the existence and uniqueness conditions for pure Nash equilibria are proved. Variations of the mechanism with different definitions for the allocation rule or the payments have been considered in [15,16,17,20] (see also the discussion in [8]).

Johari and Tsitsiklis [9] were the first who assessed the quality of proportional allocations in terms of the social welfare. They focused on pure Nash equilibria and proved a lower bound of 3/4 on their price of anarchy. Their analysis is based on the important observation that a pure Nash equilibrium in a proportional allocation game is also a pure Nash equilibrium in a game where each player

[1] Previously, Syrgkanis and Tardos [22] had shown that the social welfare at equilibrium is at least $2-\sqrt{3} \approx 26.8\%$ of the optimal effective welfare. Our techniques can be used to improve this particular guarantee to 1/2.

has a *linear* valuation function with slope equal to the derivative of the original valuation function at the share value they get at equilibrium. The optimal social welfare in the new game is not smaller than the original one and this allows them to consider the significantly simpler case of linear valuations in their analysis. Then, the price of anarchy bound is obtained by solving a linear program. An alternative proof to the result of [9] without using this argument is presented in [18] (see also [8]).

Unfortunately, this transformation does not apply to more general equilibrium concepts since the resource share each player receives is, in general, a random variable. This is a rather common difficulty that manifests itself in the analysis of games, as we depart from pure Nash equilibria and full information. In particular, Bayes-Nash equilibria have such an extremely rich structure that, typically, the price of anarchy analysis assesses their quality by rather ignoring this structure. Instead, it resorts to bounding the utility of each player by appropriately selected deviations which reveal a relation between the social welfare at equilibrium and the optimal social welfare. This approach has been used in a series of papers that mostly focus on auctions (e.g., see [1,2,3,6,10,19,22]) and is actually the approach we follow in the current paper as well.

Syrgkanis and Tardos [22] present a general analysis framework for the broad class of smooth mechanisms. Among other results, they show a price of anarchy lower bound of 26.8% over coarse-correlated and mixed Bayes-Nash equilibria of proportional allocation games. In their analysis, they bound the utility of each player by the utility she would have by deviating to an appropriately defined *randomized* bid (an approach that has also been used in different contexts in [2,10,21,23]) with a probability distribution that depends only on the optimal allocation and the valuation function of the player. In contrast, the deviating bid we consider depends on the bid strategies at equilibrium (this is in the same spirit as the recent analysis of Feldman et al. [6]) and, more interestingly, it is *deterministic*. In particular, it is defined as the product of the (expected) resource share a bidder receives in the optimal allocation and the expectation of bids of the other players at equilibrium.

Budget constraints are well-motivated in auction settings. In a slightly different context than ours, the effective welfare benchmark is considered by Dobzinski and Paes Leme, who call it *liquid* welfare in [5]. In proportional allocation, Syrgkanis and Tardos [22] prove that the social welfare at equilibrium is a constant fraction of the optimal effective welfare. Note that our guarantee is considerably stronger as we compare directly the effective welfare at equilibrium with its optimal value.

Roadmap. The rest of the paper is structured as follows. We begin with preliminary definitions in Section 2. Our price of anarchy bounds in terms of the social welfare are proved in Section 3. There, we also argue that in order to improve our analysis, radically new ideas are required. The budget-constrained setting is studied in Section 4. We remark that we have not mentioned mixed Bayes-Nash equilibria in the above presentation of our results. Actually, we have observed that such equilibria coincide with pure ones even in the budget-constrained setting.

We discuss related issues as well as additional open problems in Section 5. Due to lack of space, many proofs have been omitted; they will appear in the full version of the paper.

2 Preliminaries

Each player (henceforth called *bidder*) i in a proportional allocation game has a concave[2] non-decreasing valuation function $v_i : [0,1] \rightarrow \mathbb{R}^+$. A strategy for bidder i is simply a non-negative bid. Given a bid vector $\mathbf{b} = (b_1, b_2, ..., b_n)$, with one bid per bidder, the proportional allocation mechanism allocates to each bidder a fraction of the resource that is proportional to the bid submitted by her. Denoting by d_i the resource share that is allocated to bidder i, it is $d_i = \frac{b_i}{\sum_j b_j}$. We often use the notation B_{-i} to denote the sum of bids of all bidders besides i (hence, $d_i = \frac{b_i}{b_i + B_{-i}}$). The utility of bidder i from an allocation is simply the difference of her value for the fraction of the resource she gets minus her bid, i.e., $u_i(\mathbf{b}) = v_i(d_i) - b_i$.

A bid vector $\mathbf{b}$ is a pure equilibrium if the utility of all bidders is maximized, given the bid strategies of the other bidders. So, in a pure equilibrium, no bidder has any incentive to deviate to another strategy. Denoting by $(b_i', \mathbf{b}_{-i})$ the bid vector that is obtained from $\mathbf{b}$ when bidder i unilaterally deviates to bid strategy b_i', we can express this condition as $u_i(\mathbf{b}) \geq u_i(b_i', \mathbf{b}_{-i})$.

The social welfare of an allocation d is the total value of bidders for the resource shares they receive, i.e., $SW(d) = \sum_i v_i(d_i)$. We denote by SW^* the maximum value of the social welfare over all possible allocations. The price of anarchy over pure Nash equilibria is defined as the minimum value of the social welfare among all pure Nash equilibria divided by the optimal social welfare.

The bid strategy of a bidder i can be randomized. In this case, b_i is a random variable and the bidder aims to maximize her expected utility $\mathbb{E}\left[u_i(\mathbf{b})\right]$. The bid strategies of different bidders can be independent or correlated. A vector of independent randomized bid strategies is called a mixed Nash equilibrium if it simultaneously maximizes the expected utility of each bidder, given the bid strategies of the other bidders. More generally, coarse-correlated equilibria are solution concepts that capture correlated bid strategies. A vector of (possibly correlated) bid strategies is called a coarse-correlated equilibrium if no bidder has any incentive to unilaterally deviate to any deterministic bid strategy in order to improve her expected utility (again, given the strategies of the other bidders). The notion of the price of anarchy naturally extends to these solution concepts as well. For example, the price of anarchy over correlated equilibria is defined as the minimum value of the expected social welfare among all coarse-correlated equilibria divided by the optimal social welfare.

The above setting is known as the full (or complete) information setting. We consider the incomplete information (or Bayesian) setting as well; in this case,

[2] Very recently, Correa et al. [4] studied proportional allocation games in the less standard scenario of non-concave valuation functions.

the valuation function $\mathbf{v}_i$ of each bidder i is drawn randomly (and independently from the other bidders) from a probability distribution $\mathbf{F}_i$ over concave, non-decreasing, and non-negative functions in $[0, 1]$. Again, bidder i aims to maximize her expected utility for each possible valuation function v_i drawn from $\mathbf{F}_i$. In the incomplete information setting, each bidder i bases her decision on her exact valuation v_i and on the probability distributions according to which other bidders draw their valuations (and their corresponding bid strategies); these distributions are common knowledge.

So, the bid strategy of bidder i is a (possibly random) bid function $b_i(\mathbf{v}_i)$. A vector with one such strategy per bidder (with independence between bid strategies of different bidders) is called a mixed Bayes-Nash equilibrium if no bidder has any incentive to deviate to some other bid for any valuation function drawn from $\mathbf{F}_i$. In pure Bayes-Nash equilibria, bidders use deterministic bid functions. The price of anarchy over Bayes-Nash equilibria is defined as the minimum value of the expected social welfare among all Bayes-Nash equilibria divided by the expectation of the optimal social welfare. With some abuse in notation, we also use SW^* to denote the expectation of the optimal social welfare in the Bayesian setting.

We also extend the above model by adding budget constraints to the bidders. In this setting, each bidder i has a non-negative budget c_i and she is never allowed to bid above her budget. This restriction can result to equilibria that have extremely low social welfare compared to the optimal one (whose definition does not take budgets into account). Following [22] and [5], we use the effective welfare benchmark in order to assess the quality of equilibria with budget-constrained bidders. The effective welfare of a (deterministic) allocation $d = (d_1, d_2, ..., d_n)$ is defined as $EW(d) = \sum_i \min\{v_i(d_i), c_i\}$. Note that the definition is similar to the definition of the social welfare; the important difference is that the value of each bidder is capped by her budget. We extend this definition to random allocations d as $EW(d) = \sum_i \min\{\mathbb{E}[v_i(d_i)], c_i\}$. We denote by EW^* the maximum value of the effective welfare over all allocations. The price of anarchy with respect to the effective welfare benchmark (over equilibria in a given class) is the minimum value of the effective welfare (among all allocations induced by equilibria in the class) divided by the optimal effective welfare.

In the Bayesian setting, both the budget $\mathbf{c}_i$ of bidder i and her valuation $\mathbf{v}_i$ are drawn randomly according to the probability distribution $\mathbf{F}_i$. We refine the effective welfare benchmark in this case as

$$EW(d) = \sum_i \mathbb{E}_{(\mathbf{v}_i, \mathbf{c}_i) \sim \mathbf{F}_i} \left[\min\{\mathbb{E}_{(\mathbf{v}_{-i}, \mathbf{c}_{-i}) \sim \mathbf{F}_{-i}} [\mathbf{v}_i(d_i)], \mathbf{c}_i\} \right],$$

where the inner expectation is taken over the valuation-budget value pairs of the other bidders once the pair for bidder i has been fixed (and over the corresponding bid strategies). In order to simplify notation in the proofs below, we will not explicitly use the subscripts in the expectations.

3 Bounding the Social Welfare of Equilibria

In this section, we prove the price of anarchy bounds with respect to the social welfare. We consider both coarse-correlated equilibria in the full information setting as well as pure Bayes-Nash equilibria in the Bayesian setting. Our proofs use the following lemma (its proof is omitted) which bounds the utility of a bidder at a deterministic deviation. We also use this lemma later in Section 4 where we study budget-constrained bidders.

Lemma 1. *Consider a bidder with a concave and non-decreasing valuation function $v : [0,1] \to \mathbb{R}^+$ and let Γ be the random variable denoting the sum of bids of the other bidders. Then, for every $z \in [0,1]$ and for every $\mu > 0$, the expected utility the bidder would have by deviating to the deterministic bid $\mu z \mathbb{E}[\Gamma]$ is at least $\frac{3\mu-1}{4\mu} v(z) - \mu z \mathbb{E}[\Gamma]$.*

We are ready to prove our price of anarchy bounds. We begin with the case of coarse-correlated equilibria in the full information setting which is much simpler.

Theorem 1. *The price of anarchy of proportional allocation games over coarse-correlated equilibria is at least* $1/2$.

Proof. Consider a full information proportional allocation game with n bidders in which bidder i has valuation function v_i and denote by x_i the resource fraction bidder i gets in the optimal allocation. Let $\mathbf{b}$ be a coarse-correlated equilibrium that induces a random allocation $d = (d_1, ..., d_n)$ and let $B = \sum_i b_i$ be the random variable denoting the sum of bids of all bidders, with B_{-i} being the sum of bids of all bidders besides bidder i. Since $\mathbf{b}$ is a coarse-correlated equilibrium, bidder i has no incentive to deviate to any deterministic bid (including the deviating bid $x_i \mathbb{E}[B_{-i}]$). By applying Lemma 1 for bidder i with $z = x_i$, $\mu = 1$ and $\Gamma = B_{-i}$, we obtain that

$$\mathbb{E}[u_i(\mathbf{b})] \geq \mathbb{E}[u_i(x_i \mathbb{E}[B_{-i}], \mathbf{b}_{-i})] \geq \frac{1}{2} v_i(x_i) - x_i \mathbb{E}[B_{-i}].$$

Summing over all bidders and using the fact that $B_{-i} \leq B$ for every bidder i, we have

$$\begin{aligned} \sum_i \mathbb{E}[u_i(\mathbf{b})] &\geq \frac{1}{2} \sum_i v_i(x_i) - \sum_i x_i \mathbb{E}[B_{-i}] \qquad (1) \\ &\geq \frac{1}{2} \sum_i v_i(x_i) - \sum_i x_i \mathbb{E}[B] \\ &= \frac{1}{2} SW^* - \mathbb{E}[B]. \end{aligned}$$

The theorem follows by this inequality since the social welfare equals the sum of bidders' utilities plus their bids, i.e., $\mathbb{E}[SW(d)] = \sum_i \mathbb{E}[u_i(\mathbf{b})] + \mathbb{E}[B]$. □

The last step of the proof above begins with inequality (1). Essentially, this inequality has the form

$$\sum_i \mathbb{E}\left[u_i(\mathbf{b})\right] \geq \lambda SW^* - \mu \sum_i x_i \mathbb{E}\left[B_{-i}\right].$$

The price of anarchy bound of [22] follows after first proving an inequality of this type and then concluding to a price of anarchy bound of $\frac{\lambda}{\max\{1,\mu\}}$. The smoothness arguments of [22] lead to a version of this inequality with $\lambda = 2-\sqrt{3}$ and $\mu = 1$. Here, we have been able to improve the parameters to $\lambda = 1/2$ and $\mu = 1$. The next lemma demonstrates that these parameters cannot be improved further; the proof is omitted.

Lemma 2. *For every $\epsilon > 0$, there exists a proportional allocation game such that for every λ, μ satisfying*

$$\sum_i u_i(\mathbf{b}) \geq \lambda SW^* - \mu \sum_i x_i B_{-i} \tag{2}$$

where x_i is the resource fraction of bidder i in the optimal allocation and B_{-i} is the sum of bids of all bidders besides bidder i at a (pure Nash) equilibrium, it holds that $\frac{\lambda}{\max\{1,\mu\}} \leq \frac{1}{2} + \epsilon$.

The proof for Bayes-Nash equilibria (omitted) follows the same general approach with that of Theorem 1.

Theorem 2. *The price of anarchy of proportional allocation games over pure Bayes-Nash equilibria is at least* $1/2$*.*

4 Budget-Constrained Bidders

In this section, we consider budget-constrained bidders and prove a lower bound of approximately 36% and an upper bound of 50% on the price of anarchy in terms of the effective welfare benchmark. Here, we prove Theorem 3 for Bayes-Nash equilibria only; the (simpler) proof for coarse-correlated equilibria is omitted. Our upper bound (Theorem 4) applies even to pure Nash equilibria.

Before proceeding to the presentation of our bounds for budget-constrained bidders, we remark that minor modifications of the proofs in the previous section can show that the social welfare over equilibria with budget-constrained bidders is at least 1/2 of the optimal effective welfare, improving a corresponding bound of 26.8% from [22]. The necessary modifications are as follows. First, we need to define the deviating bids in terms of the resource shares in the allocation that maximizes the effective welfare. Then, there is a subtle case where Lemma 1 cannot be used, namely when the deviating bid for a bidder exceeds her budget. Fortunately, the inequality provided by Lemma 1 follows trivially in this case (actually, we use this argument in the proof below). By repeating the analysis in the proofs of Theorems 1 and 2, we can conclude that the social welfare at equilibrium is at least 1/2 of the social welfare of the allocation that maximizes the effective welfare. The bound then follows by observing that the effective welfare of this allocation is upper-bounded by its social welfare.

Theorem 3. *The price of anarchy of proportional allocation games with budget-constrained bidders over coarse-correlated or Bayes-Nash equilibria is at least* 0.3596*.*

Proof. Let $\mu \in (1/3, 1]$ be a parameter whose exact value will be defined later. Consider an incomplete information proportional allocation game with n bidders in which the valuation function $\mathbf{v}_i$ and the budget $\mathbf{c}_i$ of bidder i are drawn from the probability distribution $\mathbf{F}_i$, independently for each bidder. Let x_i be the random variable denoting the resource fraction bidder i gets in the allocation that maximizes the effective welfare. Let $\mathbf{b}$ be a pure Bayes-Nash equilibrium that induces a random allocation $d = (d_1, ..., d_n)$ and B be the random variable denoting the sum of bids of all bidders; again, B_{-i} denotes the sum of bids of all bidders besides bidder i. We denote by A_i the set that contains all pairs of a valuation function and a corresponding budget value (v_i, c_i) that are drawn from the probability distribution $\mathbf{F}_i$ and satisfy $\mathbb{E}\left[\mathbf{v}_i(d_i)|v_i\right] \leq c_i$. Consider a bidder i with valuation-budget pair $(v_i, c_i) \notin A_i$. By the definition of A_i, we have

$$\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)|v_i\right], c_i\} \geq \min\{\mathbb{E}\left[\mathbf{v}_i(x_i)|v_i\right], c_i\}.$$

By considering all valuation-budget pairs not belonging to A_i, we obtain

$$\mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \geq \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right],$$

and summing over all bidders, we have

$$\begin{aligned} &\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \\ &\geq \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right]. \end{aligned} \tag{3}$$

Now consider a valuation-budget pair $(v_i, c_i) \in A_i$ for bidder i that is drawn from $\mathbf{F}_i$. If $\mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right] \leq c_i$, we can bound the expected utility $\mathbb{E}\left[u_i(\mathbf{b})|v_i\right]$ by considering the deviation of bidder i to bid $\mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right]$ (which is within bidder i's budget c_i). By Lemma 1, we have

$$\begin{aligned} \mathbb{E}\left[u_i(\mathbf{b})|v_i\right] &\geq \frac{3\mu - 1}{4\mu} v_i(\mathbb{E}\left[x_i|v_i\right]) - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right] \\ &\geq \frac{3\mu - 1}{4\mu}\mathbb{E}\left[v_i(x_i)|v_i\right] - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right] \\ &\geq \frac{3\mu - 1}{4\mu}\min\{\mathbb{E}\left[v_i(x_i)|v_i\right], c_i\} - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right]. \end{aligned}$$

The second inequality follows by Jensen's inequality and by the fact $\mathbb{E}\left[B_{-i}|v_i\right] = \mathbb{E}\left[B_{-i}\right]$. Otherwise, if $\mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right] > c_i$, the same inequality follows easily since

$$\begin{aligned}\mathbb{E}\left[u_i(\mathbf{b})|v_i\right] &\geq 0 \\ &> c_i - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B_{-i}|v_i\right] \\ &\geq \frac{3\mu-1}{4\mu}\min\{\mathbb{E}\left[v_i(x_i)|v_i\right], c_i\} - \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right].\end{aligned}$$

Hence, when $(v_i, c_i) \in A_i$, we have

$$\mathbb{E}\left[u_i(\mathbf{b})|v_i\right] + \mu\mathbb{E}\left[x_i|v_i\right]\mathbb{E}\left[B\right] \geq \frac{3\mu-1}{4\mu}\min\{\mathbb{E}\left[v_i(x_i)|v_i\right], c_i\}.$$

By considering all valuation-budget values belonging to A_i, we have

$$\begin{aligned}&\mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]\mathbb{E}\left[B\right] \\ \geq\ &\frac{3\mu-1}{4\mu}\mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right].\end{aligned}$$

Using the obvious fact that $\mathbb{E}\left[x_i\right] \geq \mathbb{E}\left[x_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]$ and the above inequality, we obtain that

$$\begin{aligned}&\mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\right]\mathbb{E}\left[B\right] \\ \geq\ &\mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]\mathbb{E}\left[B\right] \\ \geq\ &\frac{3\mu-1}{4\mu}\mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], c_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]. \qquad (4)\end{aligned}$$

Now, we have

$$\begin{aligned}&\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] \\ &+\mu\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \\ \geq &\sum_i \mathbb{E}\left[(u_i(\mathbf{b}) + b_i)\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\sum_i \mathbb{E}\left[b_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \\ \geq &\sum_i \mathbb{E}\left[(u_i(\mathbf{b}) + \mu b_i)\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\sum_i \mathbb{E}\left[b_i\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \\ = &\sum_i \mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[B\right] \\ = &\sum_i \left(\mathbb{E}\left[u_i(\mathbf{b})\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] + \mu\mathbb{E}\left[x_i\right]\mathbb{E}\left[B\right]\right) \\ \geq &\frac{3\mu-1}{4\mu}\sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right]. \qquad (5)\end{aligned}$$

The first inequality follows since the quantity $\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}$ equals $\mathbb{E}\left[\mathbf{v}_i(d_i)\right]$ when $(\mathbf{v}_i, \mathbf{c}_i) \in A_i$ and $\mathbf{c}_i$ otherwise; in the latter case, the budget is clearly not smaller than the bid of bidder i. The second inequality follows since $\mu \leq 1$, the two equalities are obvious, and the last inequality follows by (4). Now, using (3) and (5), we have

$$\begin{aligned}
EW(d) &= \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\right] \\
&= \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] \\
&\quad +\mu \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \\
&\quad +(1-\mu) \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(d_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \\
&\geq \frac{3\mu-1}{4\mu} \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \in A_i\right] \\
&\quad +(1-\mu) \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\mathbb{1}(\mathbf{v}_i, \mathbf{c}_i) \notin A_i\right] \\
&\geq \min\left\{\frac{3\mu-1}{4\mu}, 1-\mu\right\} \sum_i \mathbb{E}\left[\min\{\mathbb{E}\left[\mathbf{v}_i(x_i)\right], \mathbf{c}_i\}\right] \\
&= \min\left\{\frac{3\mu-1}{4\mu}, 1-\mu\right\} EW^*.
\end{aligned}$$

Hence, the price of anarchy with respect to the effective welfare benchmark is bounded by the quantity $\min\left\{\frac{3\mu-1}{4\mu}, 1-\mu\right\}$ which is maximized to $\frac{7-\sqrt{17}}{8} \approx 0.3596$ for $\mu = \frac{1+\sqrt{17}}{8}$. □

We conclude this section by presenting our upper bound on the price of anarchy; note that it holds even for pure Nash equilibria.

Theorem 4. *For every $\epsilon > 0$, there exists a proportional allocation game among budget-constrained bidders with price of anarchy at most $1/2 + \epsilon$ over pure Nash equilibria, with respect to the effective welfare benchmark.*

5 Discussion and Open Problems

Our work leaves the obvious open problem of computing the tight bound on the price of anarchy over coarse-correlated and Bayes-Nash equilibria. So far, the only upper bound that is known is the counter-example of 3/4 from [9] for pure Nash equilibria. Is 3/4 the tight bound for all equilibrium concepts considered in the current paper? Actually, we have not been able to identify any coarse-correlated equilibrium in the full information model that is non-pure. Do such equilibria really exist? Interestingly, we can show that mixed Nash equilibria coincide with pure ones (see the statement in Lemma 3). More generally, this statement applies to mixed Bayes-Nash equilibria in the budget-constrained setting. Does it extend to coarse-correlated ones? We believe that this is an interesting open problem.

Lemma 3. *The set of mixed Bayes-Nash equilibria in any proportional allocation game (possibly with budget-constrained bidders) coincides with that of pure Bayes-Nash equilibria.*

In the Bayesian setting, we have not considered more general equilibrium concepts such as coarse-correlated Bayesian equilibria. The main reason is that our analysis requires that the expectation of the sum of bids of the other bidders is the same for any possible valuation bidder i can draw from her distribution; this property is not satisfied by more general equilibrium concepts. What is the price of anarchy in this case? Interestingly, the answer cannot be 3/4 as our next counter-example indicates; the proof is omitted due to lack of space.

Lemma 4. *There exists a proportional allocation game that has price of anarchy at most* 0.7154 *over coarse-correlated Bayesian equilibria.*

Also, recall that we have assumed that bidders have independent valuations. This is a typical assumption in the Bayes-Nash price of anarchy literature [1,3,6,10,19,21,22] with [2] being the only exception we are aware of. Unfortunately, our proof of the pure Bayes-Nash price of anarchy bound does not carry over to the case of correlated valuations either (for the same reason mentioned above). Still, we have not been able to find any counter-example with non-constant price of anarchy in this setting. Again, what is the price of anarchy in this case? These questions are interesting in the budget-constrained setting as well.

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Strong Price of Anarchy, Utility Games and Coalitional Dynamics

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Abstract. We introduce a framework for studying the effect of cooperation on the quality of outcomes in utility games. Our framework is a coalitional analog of the smoothness framework of non-cooperative games. Coalitional smoothness implies bounds on the strong price of anarchy, the loss of quality of coalitionally stable outcomes. Our coalitional smoothness framework captures existing results bounding the strong price of anarchy of network design games. Moreover, we give novel strong price of anarchy results for any monotone utility-maximization game, showing that if each player's utility is at least his marginal contribution to the welfare, then the strong price of anarchy is at most 2. This captures a broad class of games, including games that have a price of anarchy as high as the number of players. Additionally, we show that in potential games the strong price of anarchy is close to the price of stability, the quality of the best Nash equilibrium.

We also initiate the study of the quality of coalitional out-of-equilibrium outcomes in games. To this end, we define a coalitional version of myopic best-response dynamics, and show that the bound on the strong price of anarchy implied by coalitional smoothness, also extends with small degradation to the average quality of outcomes of the given dynamic.

1 Introduction

We introduce a framework for studying the effect of cooperation on the quality of outcomes in games. In the past decade we have developed a good understanding of the degradation in social welfare in games due to selfish play, quantified by the price of anarchy. There are known tight bounds on the price of anarchy in a range of games from routing, to network design, to various scheduling games. Much less is understood about outcomes of games where players may cooperate.

In many settings players do cooperate, and cooperation can help improve the outcome. The worst possible Nash equilibrium is a very pessimistic prediction

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of the outcome in games that are not strictly competitive, and where cooperation may improve the utility for all participants. A key issue in understanding cooperative outcomes is the extent to which players can transfer utility among each-other. The two dominant notions of cooperative outcomes considered in the literature are the strong Nash equilibrium of Aumann [5] assuming no utility transfer between players, and the transferable utility notion of the core (see [13] for a survey). Allowing utility transfers between the players leads to an extremely demanding form of equilibrium, a solution is unstable in this sense, if there is a possible joint deviation for a group that improves the total utility of a group, even if this is not improving the utility of every single player. A stable outcome in this sense is automatically socially optimal (otherwise the grand coalition can deviate). An outcome is a strong Nash equilibrium if it is stable subject to coalitional deviations, meaning that no group of players can jointly deviate to improve the solution for every member of the coalition. Strong Nash equilibria do not imply the optimality of the outcome. We identify properties of a game such that a stable outcome cannot be too far from the social optimum. An even less restrictive notion, coalition proof equilibrium, was introduced by Moreno et al [14] requiring extra conditions from a coalitional deviation to be feasible, such as the non-existence of subsequent unilateral deviations. We focus here mainly on strong Nash equilibria and on randomized versions.

The strong price of anarchy was introduced by Andelman et al. [2] and measures the quality degradation of strong Nash equilibria in games. One of the most compelling examples is the class of cost-sharing games, where players choose costly resources and equally share the cost with other users of each resource. Anshelevich et al. [3] showed that the price of anarchy in this class of games with n players can be as bad as n, but showed a tight $H_n = O(\log n)$ bound on the price of stability, the quality loss in the best Nash equilibria compared to the socially optimal solution. While the worst Nash equilibria seems too pessimistic a prediction for the outcome, the best Nash equilibria is potentially too optimistic: while significant cooperation is needed to identify and reach this solution, the stability concept used is that of Nash equilibria, assuming that only individual players can deviate, and not groups. Epstein et al. [8] showed an H_n bound on the the strong price of anarchy, matching the price of stability bound.

For the case of worst-case Nash equilibria, Roughgarden [16] introduced the framework of *smooth games*, encompassing most price of anarchy bounds. However, such a unified framework does not exist for worst-case coalitionally stable outcomes. We propose a smoothness framework that captures efficiency in most well-established cooperative equilibrium solution concepts such as the strong Nash equilibrium and randomized versions of it. We show how our framework implies existing results on the strong price of anarchy of cost-sharing games and we generate new results on the strong price of anarchy of utility maximization games and of potential games.

The second goal of our paper is to initiate a study of outcomes of dynamic cooperative play and of efficiency of non-equilibrium solution concepts. A key point in the wide applicability of the price of anarchy analysis and of the smoothness framework of [16] is that bounds proved via smoothness automatically extend to coarse correlated equilibria, which are outcomes of no-regret learning by each player [7]. Extending the price of anarchy results to no-regret outcomes is appealing as it is a natural model of player behavior, and no-regret can be achieved via simple strategies.

Studying the efficiency of out-of-equlibrium solution concepts that capture cooperation is even more compelling in the strong price of anarchy analysis, as strong Nash equilibria, unlike for instance mixed Nash equilibria, are not guaranteed to exist, and do not exist in even small and simple cost-sharing games [8]. Hence, we need to identify properties of games that would not only imply approximate efficiency of coalitionally stable equilibrium outcomes, but whose efficiency implications would directly extend with very small degradation even to out-of-equilibrium cooperative dynamic solution concepts.

To this end, we propose a coalitional version of myopic best-response dynamics and we analyze the average welfare of such dynamics in the long run. Our dynamics can be viewed as a coalitional version of sink equilibria proposed by Goemans et al. [9]. More importantly, we show that the efficiency guarantees implied by our coalitional smoothness framework directly extend with small loss to this form of out-of-equilibrium cooperative dynamics. These out-of-equilibrium outcomes always exist and are meaningful even in games that do not admit a strong Nash equilibrium and thereby the direct extension is rather appealing as it provides an efficiency bound that is not conditional on existence.

Our Results. We propose a framework for quantifying the quality of strong Nash equilibria by introducing the notion of coalitional smoothness. We show how coalitional smoothness captures existing results on network design games, we give new results on the strong price of anarchy in utility maximization games, and show that coalitional smoothness in such games implies high social welfare at coalitional sink equilibria, which we define as the out-of-equilibrium myopic behavior as defined by a natural coalitional version of best-response dynamics.

- We define the notion of a (λ, μ)-*coalitionally smooth games* and show that the strong price of anarchy of a (λ, μ)-coalitionally smooth game is bounded by $\lambda/(1+\mu)$ in utility games and $\lambda/(1-\mu)$ in cost minimization games.
- We show that the cost-sharing games of [8] as well as network contribution games [4] studied in the literature are coalitionally smooth.
- We show that in any monotone utility-maximization game, if each player's utility is at least his marginal contribution to the welfare then the strong price of anarchy is at most 2, while the price of anarchy in this class of games can be as high as n. This result complements the results of [22,9] who studied the price of anarchy of utility-maximization games that have submodular social welfare function.
- In potential games, such as the cost-sharing game of [8], the potential minimizer is a Nash equilibrium of high quality. This equilibrium is typically used to bound the price of stability by showing that the social welfare function is similar to the potential function, namely $\lambda \cdot SW(s) \leq \Phi(s) \leq \mu \cdot SW(s)$,

implying a bound of λ/μ on the price of stability. We show that in utility games this condition also implies that the game is (λ, μ)-coalitionally smooth implying a $\frac{\lambda}{1+\mu}$ bound on the strong price of anarchy, and give conditions for a similar bound in cost-minimization games, extending the work of [8].

- Strong price of anarchy bounds via coalitional smoothness also extend to the notions of strong correlated equilibria (see e.g. Moreno et al. [14]) and strong coarse correlated equilibria of [19], which correspond to randomized outcomes where no group of players C has a joint distribution of strategies $\tilde{D}_C$ that each member of the group has regret for. Though there exist games with no strong Nash, that admit such randomized strong equilibria, unfortunately, there are no learning algorithms that guarantee this coalitional no-regret property, and in fact, these concepts may not exist in some games.
- We define a natural *coalitional best response dynamic* and the corresponding *coalitional sink equilibria*, the analog of the notion of myopic sink equilibria introduced by Goemans et al. [9] for coalitional dynamics. While myopic sink equilibria correspond to steady state behavior of the Markov chain defined by iteratively doing random unilateral best respond dynamics, coalitional sink equilibria are the steady state under our coalitional best response dynamic. We do not explicitly model how players choose to transfer utility to each other. However, our dynamic assumes that when a group cooperates, then they can also transfer utility, and hence will choose to optimize the total utility of all group members. We show that in (λ, μ)-coalitionally smooth utility games the social welfare of any coalitional sink equilibrium is at least a $\frac{1}{H_n}\frac{\lambda}{1+\mu}$ fraction of the optimal; extending our analysis of outcomes of coalitional play to games when strong Nash equilibria do not exist.

Related Work. The study of efficiency of worst-case Nash equilibria via the price of anarchy was initiated by [12], and has triggered a large body of work. Roughgarden [16] introduced a canonical way of analyzing the price of anarchy by proposing the notion of a (λ, μ)-smooth game and showing that most efficiency proofs can be cast as showing that the game is smooth. Most importantly, [16] showed that any efficiency proven via smoothness arguments directly extends to outcomes of no-regret learning behavior. Recently, similar frameworks have been proposed for games of incomplete information [17,20,21] and games with continuous strategy spaces [18]. However, these frameworks do not take into account coalitional robustness and no canonical way of showing efficiency bounds for coalitional solution concepts existed prior to our work.

The most established coalitionally robust solution concept is that of the strong Nash equilibrium introduced by Aumann [5]. The study of the efficiency of the worst strong Nash equilibrium (strong price of anarchy) was introduced in [2], and follow-up research mostly focused on specific cost minimization games such as network design games [3,1,8]. Our coalitional smoothness framework captures some of the results in this literature and gives a generic condition under which the strong price of anarchy is bounded.

For utility maximization games Vetta [22] defined the class of valid-utility games, which are utility maximization games with a monotone and submodular welfare function and where each player's utility is at least his marginal contribution to the welfare. Vetta [22] showed that every Nash equilibrium of a valid utility game achieves at least half of the optimal welfare. Later these games were analyzed from the perspective of best response dynamics by [9], who introduced the notion of a sink equilibrium (i.e. steady state distribution of the Markov chain defined by best-response dynamics) and showed that for a subclass of valid-utility games the half approximation is achieved after polynomially many rounds, while for the general class, the sink equilibria can have an efficiency that degrades linearly with the number of players. In this paper, we show that without the assumption of submodularity every monotone utility maximization game that satisfies the marginal contribution condition has good strong price of anarchy. Additionally, we define a coalitional version of sink equilibria of [9] and show that for any coalitionally smooth game the efficiency bound at these out-of-equilibrium dynamics degrades from the strong price of anarchy bound by a factor that is only logarithmic in the number of players.

The efficiency of coalitionally robust solution concepts was also studied by Anshelevich et al [4] for a class of contribution games in networks, where pairwise-stable outcomes were analyzed. Most of our theorems imply social welfare bounds for strong Nash equilibria of network contribution games, that hold under much more general assumptions than the ones considered in [4].

The existence of strong Nash equilibria was examined by both game theorists and computer scientists (see e.g. [15,10,8]). [19] show that in singleton congestion games with increasing resource value functions there always exists a strong Nash equilibrium, while [11] show that for decreasing function the set of pure Nash equilibria, which is non-empty, coincides with the set of strong Nash equilibria.

2 Coalitional Smoothness

In this section we introduce the notion of coalitional smoothness and show that it captures the core of a proof on the efficiency of strong Nash equilibria in several games studied in the past, such as network cost sharing games [3,8] as well as in new classes of games that we give, which generalize the well-studied valid-utility games of Vetta [22], by dropping the assumption of submodularity.

For ease of presentation we will present the definition of coalitional smoothness for utility maximization games rather than cost minimization, but the definitions naturally extend to analogous ones for cost minimization. We will consider a standard normal form game among n players. Each player i has a strategy space S_i and a utility $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}_+$. For a subset of players $C \subseteq [n]$ we will denote with $S_C = (S_i)_{i \in C}$ the joint strategy space, with $s_C \in S_C$ a joint strategy profile and with $\Delta(S_C)$ the space of distributions over strategy profiles. We are interested in quantifying the efficiency of coalitional solution concepts with respect to the social welfare, which is defined as the sum of all player utilities: $SW(s) = \sum_{i \in [n]} u_i(s)$. For convenience, we will denote with OPT the maximal social welfare (resp. minimum social cost) achieved among all possible strategy profiles and we will try to upper bound the *price of anarchy*, which is the ratio of the optimal social welfare over the social welfare at any equilibrium in the class of solution concepts that we study (e.g. strong price of anarchy for the case of strong Nash equilibria), or equivalently to lower bound the fraction of the optimal welfare that every equilibrium in the class achieves.

The intuition behind coalitional smoothness is that it requires from the game to admit a good strategy profile such that if enough players coalitionally deviate to this strategy from any state with low social welfare then they achieve a good fraction of the optimal welfare. Specifically, it imposes that if we order the players arbitrarily and consider only the coalitional deviations of all the suffixes of this order, then the sum of utilities of the first player in each of the suffixes, after the coalitional deviation of the suffix, is at least a λ fraction of the optimal welfare or else μ times the current social welfare is at least a λ fraction of the optimal.

Definition 1 (Coalitional Smoothness). *A utility maximization game is (λ, μ)-coalitionally smooth if there exists a strategy profile s^* such that for any strategy profile s and for any permutation π of the players:*

$$\textstyle\sum_{i=1}^{n} u_i(s^*_{N_{\pi(i)}}, s_{-N_{\pi(i)}}) \geq \lambda \cdot OPT - \mu \cdot SW(s) \tag{1}$$

where $N_{\pi(i)} = \{j \in [n] : \pi(j) \geq \pi(i)\}$ is the set of all players succeeding i in the permutation and (s_{N_t}, s_{-N_t}) is the strategy profile where all players in $i \in N_t$ play s^*_i and all other players play s. [1]

We now formally define the notion of a strong Nash equilibrium introduced by Aumann [5] and show that coalitional smoothness implies high efficiency at every strong Nash equilibrium of a game.

Definition 2 (Strong Nash Equilibrium). *A strategy profile s is a strong Nash equilibrium if for any coalition $C \subseteq [n]$ and for any coalitional strategy $s_C \in S_C$, there exists a player $i \in C$ such that: $u_i(s) \geq u_i(s_C, s_{-C})$.*

Theorem 3. *If a game is (λ, μ)-coalitionally smooth for some $\lambda, \mu \geq 0$ then every strong Nash equilibrium has social welfare at least $\frac{\lambda}{1+\mu}$ of the optimal.* [2]

Proof. Let s be strong Nash equilibrium strategy profile and let s^* be the optimal strategy profile. If all players coalitionally deviate to s^* then, by the definition of a strong Nash equilibrium, there is a player i who is blocking the deviation, i.e. $u_i(s) \geq u_i(s^*)$. Without loss of generality, reorder the players such that this is player 1. Similarly, if players $\{2, \ldots, n\}$ deviate to playing their strategy in s^* then there exists some player, obviously different than 1 who is blocking the deviation. Without loss of generality, by reordering we can assume that this player is 2. Using similar reasoning we can reorder the players such that if players $\{i, \ldots, n\}$ deviate to their strategy in the optimal strategy profile x^* then player

[1] In the case of cost minimization games we would require: $\sum_{i=1}^{n} c_i(s^*_{N_{\pi(i)}}, s_{-N_{\pi(i)}}) \leq \lambda \cdot SC(s^*) + \mu \cdot SC(s)$.

[2] In cost-minimization games (λ, μ)-coalitional smoothness for $\lambda \geq 0$, $\mu \leq 1$ implies that the social cost at a strong Nash is at most $\frac{\lambda}{1-\mu}$ of the minimum cost.

i is the one blocking the deviation. That is player i's utility at the strong Nash equilibrium is at least his utility in the deviating strategy profile. Thus under this order $\forall k \in N$: $u_i(s) \geq u_i(s^*_{N_k}, s_{-N_k})$. Summing over all players and using the coalitional smoothness property for the above order we get the result:

$$SW(s) = \sum_{i=1}^{N} u_i(s) \geq \sum_{i=1}^{N} u_i(s^*_{N_i}, s_{-N_i}) \geq \lambda SW(s^*) - \mu SW(s)$$

□

Extension to Randomized Solution Concepts. Similar to smoothness, coalitional smoothness also implies efficiency bounds even for randomized coalition-proof solution concepts. Adapting randomized solution concepts such as correlated equilibria so as to make them robust to coalitional deviations is not as straightforward as in the case of unilateral stability. This is mainly due to information considerations. One such concept is that of strong correlated equilibria (see e.g. Moreno and Wooders [14]) and it's relaxation, the strong coarse correlated equilibrium (see e.g. Rozenfeld et al. [19]).

Essentially, a strong correlated equilibrium is a distribution over strategy profiles, such that if players are recommended strategies based on this distribution, then there exists no coalitional (randomized) deviation under which every member of the coalition is strictly better off. The coalitional deviation can depend on the recommendation that each member of the coalition received. Thus implicitly it is assumed that if players commit to a coalition ex-ante, then after receiving their recommendations on which strategy to play, they share it publicly among the players in the coalition and decide on a joint deviation. The strong coarse correlated equilibrium is a relaxation of the above concept where the coalitional deviation is independent of the recommendations. We defer a formal definition of the two concepts and a more elaborate discussion to the full version [6].

Non-Submodular Monotone Utility Games. Consider a utility maximization game in which every player has an s_i^{out} strategy, corresponding to the player not entering the game. Further assume that the game is monotone with respect to participation, i.e. no player can decrease the social welfare by entering the game: $\forall i \in [n], \forall s : SW(s) \geq SW(s_i^{out}, s_{-i})$. We show that the coalitional smoothness of such a game is captured exactly by the proportion of the marginal contribution to the social welfare that a player is guaranteed to get as utility.

Theorem 4. *Any monotone utility maximization game is guaranteed to be (γ, γ)-coalitionally smooth, if each player is guaranteed at least a γ fraction of his marginal contribution to the social welfare:*

$$\forall s : u_i(s) \geq \gamma \left(SW(s) - SW(s_i^{out}, s_{-i})\right) \tag{2}$$

Proof. Consider any order of the players and let s^* be the strategy profile that maximizes the social welfare. By the marginal contribution property we have:

$$\sum_{i=1}^{n} u_i(s^*_{N_i}, s_{-N_i}) \geq \gamma \cdot \sum_{i=1}^{n} \left(SW\left(s^*_{N_i}, s_{-N_i}\right) - SW\left(s_i^{out}, s_{N^*_{i+1}}, s_{-N_i}\right)\right)$$

In addition, by the monotonicity assumption the social welfare can only increase when a player enters the game with any strategy:

$$SW(s_i^{out}, s^*_{N_{i+1}}, s_{-N_i}) \leq SW(s_k, s^*_{N_{i+1}}, s_{-N_i}) = SW(s^*_{N_{i+1}}, s_{-N_{i+1}})$$

Combining the above inequalities we get a telescoping sum that yields the desired property:

$$\begin{aligned}\textstyle\sum_{i=1}^n u_i(s^*_{N_i}, s_{-N_i}) \geq\ & \gamma \cdot \textstyle\sum_{i=1}^n \left(SW(s^*_{N_i}, s_{-N_i}) - SW(s_{N^*_{i+1}}, s_{-N_{i+1}}) \right) \\ \geq\ & \gamma \cdot SW(s^*) - \gamma \cdot SW(s) = \gamma \cdot \mathrm{OPT} - \gamma \cdot SW(s)\end{aligned}$$

Which is exactly the (γ, γ)-coalitional smoothness property we wanted. □

This latter result complements Vetta's [22] results on valid-utility games. A valid-utility game is a monotone utility-maximization game with the following additional structural property: strategies of the players can be viewed as sets of some ground set of elements and the social welfare can be viewed as a monotone *submodular* set function on the union of the chosen strategies. As re-interpreted by Roughgarden [16], Vetta showed that in any monotone utility-maximization game with a submodular welfare function, if each player receives a γ fraction of their marginal contribution to the welfare, then the game is (γ, γ)-smooth implying that every Nash equilibrium achieves a $\frac{\gamma}{\gamma+1}$ fraction of the optimal welfare. In the absence of submodularity there are easy examples where the worst Nash equilibrium doesn't achieve any constant fraction of the optimal welfare, despite satisfying the marginal contribution condition. However, our result shows that even in the absence of submodularity every such game will be (γ, γ)-coalitionally smooth, implying that every strong Nash equilibrium will achieve a $\frac{\gamma}{\gamma+1}$ fraction of the welfare.

It is important to note that the approximate marginal contribution condition and the submodularity condition are very orthogonal ones. For instance, it is possible that a game satisfies the approximate marginal contribution condition for some constant, but is not submodular or even approximately submodular under existing definitions of approximate submodularity. In the full version of the paper, we give a class of welfare sharing games, where our efficiency theorem applies to give constant bounds on the string price of anarchy, whilst the price of anarchy is unbounded due to the non-submodularity of the social welfare.

Network Cost-Sharing Games. In this section we analyze the well-studied class of cost sharing games [3], using the coalitional smoothness property. The game is defined by a set of resources R each associated with a cost c_r. Each player's strategy space S_i is a set of subsets of R. The cost of each resource is shared equally among all players that use the resource and a players total cost is the sum of his cost-shares on the resources that he uses. If we denote with $n_r(s)$ the number of players using resource r under strategy profile s, then: $c_i(s) = \sum_{r \in s_i} \frac{c_r}{n_r(s)}$.

Epstein et al [8] showed that every strong Nash equilibrium of the above class of games has social cost at most H_n times the optimal, where H_n is the n-th harmonic number. Here we re-interpret that result as showing that these games are $(H_n, 0)$-coalitionally smooth. In the next section we show that the analysis of [8] can be applied to a more broad class of potential games, showing a strong connection between the price of stability and the strong price of anarchy.

Theorem 5 ([8]). *Cost sharing games are $(H_n, 0)$-coalitionally smooth.*

3 Best Nash vs. Worst Strong Nash Equilibrium

Strong Nash equilibria are a subset of Nash equilibria, so in games when strong Nash equilibria exist, the strong price of anarchy cannot be better than the price of stability (the quality of best Nash). In this section we show that in potential games these two notions are surprisingly close. We show that through the lens of coalitional smoothness there is a strong connection between the analysis of the efficiency of the worst strong Nash equilibria and the dominant analysis of the best Nash equilibria in potential games. A game admits a potential function if there exists a common function $\Phi(s)$ for all players, such that a player's difference in utility from a unilateral deviation is equal to difference in the potential:

$$u_i(s'_i, s_{-i}) - u_i(s) = \Phi(s'_i, s_{-i}) - \Phi(s) \tag{3}$$

A large amount of recent work in the algorithmic game theory literature has focused on the analysis of the efficiency of the best Nash equilibrium (price of stability). For the case of potential games the dominant way of analysing the price of stability is the Potential Method: suppose that the potential function is (λ, μ)-*close* to the social welfare, in the sense that $\lambda \cdot SW(s) \leq \Phi(s) \leq \mu \cdot SW(s)$, for some parameters $\lambda, \mu \geq 0$. Then the best Nash equilibrium achieves at least $\frac{\lambda}{\mu}$ of the optimal social welfare. The proof relies on the simple fact that the potential maximizer is always a Nash equilibrium and by the (λ, μ) property it's easy to see that the potential maximizer has social welfare that is the above fraction of the optimal social welfare.

The following theorems show that for such potential games the price of stability is very close to the strong price of anarchy, i.e. the implied quality of the best Nash equilibrium is close to the quality of the worst strong Nash equilibrium.

Theorem 6. *In a utility-maximization potential game with non-negative utilities, if the potential is (λ, μ)-close to the social welfare then the game is (λ, μ)-coalitionally smooth, implying that every strong Nash equilibrium achieves at least $\frac{\lambda}{1+\mu}$ of the optimal social welfare.*

Proof. Consider an arbitrary order of the players and some strategy profile s. By the definition of the potential function and the fact that utilities are non-negative, we have

$$\begin{aligned} u_i(s^*_{N_i}, s_{-N_i}) &= \Phi(s^*_{N_i}, s_{-N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}}) + u_i(s^*_{N_{i+1}}, s_{-N_{i+1}}) \\ &\geq \Phi(s^*_{N_i}, s_{-N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}}) \end{aligned}$$

Combining with our assumption on the relation between potential and social welfare we obtain the coalitional smoothness property:

$$\begin{aligned}\textstyle\sum_{i=1}^n u_i(s^*_{N_i}, s_{-N_i}) \geq & \textstyle\sum_{i=1}^n \left(\Phi(s^*_{N_i}, s_{-N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}})\right) \\ = & \Phi(s^*) - \Phi(s) \geq \lambda \cdot SW(s^*) - \mu \cdot SW(s)\end{aligned}$$

□

The (λ, μ)-closeness property of the potential function does not imply smoothness of the game according to the standard definition of smoothness [16] and hence a price of anarchy bound. It does so only if the potential is a submodular function and by following a similar analysis as in the case of valid utility games as we show in the full version. Such a property for instance, holds in utility congestion games with decreasing resource utilities. However, Theorem 6 does not require submodularity of the potential.

One application of Theorem 6 is in the context of network contribution games [4]. In a network contribution game each player corresponds to a node in a network. Each edge corresponds to a "friendship" between the connecting nodes or more generally some joint venture. Each player has a budget of effort that he strategically distributes among his friendships. Each friendship e between two players i and j, has a value $v_e(x_i, x_j)$ that corresponds to the value produced as a function of the efforts put into it by the two players. This value is equally split among the two players. It is easy to see that in such a game the social welfare is the total value produced, while the potential is equal to half of the social welfare. Thus, by applying Theorem 6 we get that for arbitrary "friendship" value functions $v_e(\cdot, \cdot)$ the game is $\left(\frac{1}{2}, \frac{1}{2}\right)$-coalitionally smooth and hence every strong Nash equilibrium achieves at least $\frac{1}{3}$ of the optimal welfare. In contrast, observe that Nash equilibria can have unbounded inefficiency,[3] and the game is not (λ, μ)-smooth under the unilateral notion of smoothness for any λ, μ.

For settings where a player can only have non-negative externalities on the utilities of other players by entering the game, a much stronger connection can be drawn. More concretely, a utility maximization game has non-negative externalities if for any strategy profile s and for any pair of players i, j: $u_i(s) \geq u_i(s_j^{out}, s_{-j})$.[4] The s_i^{out} strategy is not required to be a valid strategy that the player can actually pick, but rather a hypothetical strategy, requiring the property that the cost of the player in that strategy is 0, and the cost functions and the potential are extended appropriately such that the potential function property is maintained even in this augmented strategy space and the potential when all players have left the game is 0: $\Phi(s^{out}) = 0$. For instance, every congestion game has the above property if s_i^{out} is defined as the empty set of resources.

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3 Consider a line of four nodes (A, B, C, D). Each player has budget 1. Edges (A, B) and (C, D) have constant value of 1, while edge (B, C) has a huge value H, if both players put all their budget on it and 0 o.w.. Players B, C placing their budget on their alternative friendships is a Nash equilibrium, but not a strong Nash equilibrium.

4 A cost-minimization game has non-negative externalities if $c_i(s) \leq c_i(s_j^{out}, s_{-j})$.

Theorem 7. *A utility-maximization potential game with only positive externalities and such that $\Phi(s) \geq \lambda \cdot SW(s)$ is $(\lambda, 0)$-coalitionally smooth. Similarly, a cost-minimization, potential game with only positive externalities and such that $\Phi(s) \leq \lambda \cdot SC(s)$ is $(\lambda, 0)$-coalitionally smooth.*

In the context of cost-minimization, one well-studied example of such a setting is that of network cost-sharing games and the $log(n)$ strong price of anarchy result of [8] is a special instance of Theorem 7. For utility-maximization games, one example is that of network contribution games under the restriction that friendship value functions $v_e(\cdot, \cdot)$ are increasing in both coordinates. Under this restriction applying Theorem 7 we get the improved bound that every strong Nash equilibrium achieves at least 1/2 of the optimal social welfare.

4 Coalitional Best-Response Dynamics

In this section we initiate the study of efficiency of dynamic coalitional behavior. We show that if a utility game is (λ, μ)-coalitionally smooth then this implies an efficiency guarantee for out of equilibrium dynamic behavior in a certain best-response like dynamic. This is particularly interesting for games that do not admit a strong Nash equilibrium, but where coalitional deviations are bound to occur. Our approach is similar in spirit to the notion of myopic sink equilibria introduced by Goemans et al. [9]. Myopic sink equilibria correspond to steady state behavior of the Markov chain defined by iteratively doing random unilateral best respond dynamics. However, such a notion does not capture settings where players can communicate and at each step perform coalitional deviations.

We introduce a version of coalitional best-response dynamics, that allows for coalitional deviations at each time step, giving more probability to small coalitions. In our dynamic, at each step a selected group is chosen to cooperate. We assume that when a group cooperates, then they can also transfer utility, and hence will choose to optimize the total utility of all group members. Then we analyze the social welfare of the steady states arising in the long run as we perform coalitional best response dynamics for a long period. Similar to [9] we will refer to these steady states as coalitional sink equilibria. Similar to sink equilibria that are a way of studying games whose best response dynamics might not converge to a pure Nash equilibrium or even games that do not admit a pure Nash equilibrium, coalitional sink equilibria are an interesting alternative for analyzing efficiency in games that do not admit a strong Nash equilibrium, which admittedly is even more rare than the pure Nash equilibrium.

Our coalitional dynamics are as follows: At each iteration a coalition is picked at random from a distribution that favors coalitions of smaller size. Specifically, first a coalition size k is picked inversely proportional to the size and then a coalition of size k is picked uniformly at random. Next, the picked coalition deviates to the joint strategy profile that maximizes the total utility of the coalition, conditional on the current strategy of every player outside of the coalition.

Theorem 8. *If a utility maximization game with non-negative utilities is* (λ, μ)*-coalitionally smooth then the expected social welfare at every coalitional sink equilibrium is at least* $\frac{1}{H_n}\frac{\lambda}{1+\mu}$ *of the optimal.*

If we picked the coalitional size k, according to some probability distribution with a density $p(k)$ satisfying $p(k) \geq \frac{1}{c}\frac{1}{H_n \cdot k}$, then by the analysis of the above theorem we get that every coalitional sink equilibrium achieves welfare at least $\frac{1}{c \cdot H_n}\frac{\lambda}{1+\mu}$ of the optimal.

Remark 1. The Markov chain defined by the coalitional best-response dynamics might take long time to converge to a steady state. However, our analysis shows a stronger statement: at any iteration T if we take the empirical distribution defined by the best-response play up till time T, then the expected welfare of this empirical distribution is at least $\frac{T-1}{2T}\frac{\lambda}{H_n+\mu}$ of the optimal welfare.

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The Complexity of Approximating a Trembling Hand Perfect Equilibrium of a Multi-player Game in Strategic Form

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Abstract. We consider the task of computing an approximation of a trembling hand perfect equilibrium for an n-player game in strategic form, $n \geq 3$. We show that this task is complete for the complexity class FIXP_a. In particular, the task is polynomial time equivalent to the task of computing an approximation of a Nash equilibrium in strategic form games with three (or more) players.

1 Introduction

Arguably [17], the most important *refinement* of Nash equilibrium for finite games in strategic form (a.k.a. games in normal form, i.e., games given by their tables of payoffs) is Reinhard Selten's [15] notion of *trembling hand perfection.* The set of trembling hand perfect equilibria of a game is a non-empty subset of the Nash equilibria of that game. Also, many "unreasonable" Nash equilibria of many games, e.g., those relying on "empty threats" in equivalent extensive forms of those games, are not trembling hand perfect, thus motivating and justifying the notion. The importance of the notion is illustrated by the fact that Selten received the Nobel prize in economics together with Nash (and Harsanyi), "for their pioneering analysis of equilibria in the theory of non-cooperative games". In this paper, we study the computational complexity of finding trembling hand perfect equilibria of games given in strategic form.

The computational complexity of finding a *Nash* equilibrium of a game in strategic form is well-studied. When studying this computational task, we assume that the game given as input is represented as a table of integer (or rational) payoffs, with each payoff given in binary notation. The output is a strategy profile, i.e., a family of probability distributions over the strategies of each player, with each probability being a rational number with numerator and denominator given in binary notation. The computational task is therefore discrete and we are interested in the Turing machine complexity of solving it. Papadimitriou [13]

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showed that for the case of two players, the problem of computing an *exact* Nash equilibrium is in PPAD, a natural complexity class introduced in that paper, as a consequence of the Lemke-Howson algorithm [10] for solving this task. For the case of three or more players, there are games where no Nash equilibrium which uses only rational probabilities exists [12], and hence considering some relaxation of the notion of "computing" a Nash equilibrium is necessary to stay within the discrete input/output framework outlined above. In particular, Papadimitriou showed that the problem of computing an ϵ-Nash equilibrium, with $\epsilon > 0$ given as part of the input in binary notation, is also in PPAD, as a consequence of Scarf's algorithm [14] for solving this task. Here, an ϵ-Nash equilibrium is a strategy profile where no player can increase its utility by more than ϵ by deviating. In breakthrough papers, Daskalakis *et al.* [5] and Chen and Deng [4] showed that both tasks are also hard for PPAD, hence settling their complexity: Both are PPAD-complete. Subsequently, Etessami and Yannakakis [6] pointed out that for some games, ϵ-Nash equilibria can be so remote from any exact Nash equilibrium (unless ϵ is so small that its binary notation has encoding size exponential in the size of the game), that the former tells us little or nothing about the latter. For such games, the ϵ-Nash relaxation is a bad proxy for Nash equilibrium, assuming the latter is what we are actually interested in computing. Motivated by this, they suggested a different relaxation: Compute a strategy profile with ℓ_∞-distance at most δ from an exact Nash equilibrium, with $\delta > 0$ again given as part of the input in binary notation. In other words, *compute an actual Nash equilibrium to a desired number of bits of accuracy.* They showed that this problem is complete for a natural complexity class FIXP_a that they introduced in the same paper. Informally, FIXP_a is the class of discrete search problems that can be reduced to approximating (within desired ℓ_∞-distance) any one of the Brouwer fixed points of a function given by an algebraic circuit using gates: $+, -, *, /, \max, \min$. (We will formally define FIXP_a later.)

In this paper, we want to similarly understand the case of trembling hand perfect equilibrium. For the case of two players, the problem of computing an exact trembling hand perfect equilibrium is PPAD-complete. This follows from a number of known exact pivoting algorithms for computing refinements of this notion [18,11,16]. For the case of three or more players, we are not aware of any natural analogue of the notion of ϵ-Nash equilibrium as an approximate proxy for a trembling hand perfect equilibrium.[1] Thus, we only discuss in this paper the approximation notion of Etessami and Yannakakis. The main result of the present paper is the following:

Theorem 1. *The following computational task is* FIXP_a*-complete for any* $n \geq 3$*: Given an integer payoff table for an* n*-player game* Γ*, and a rational* $\delta > 0$*,*

[1] The already studied notion of an ϵ-perfect equilibrium (ϵ-PE), which we discuss later, does *not* qualify as such an analogue: For some three-player games, every ϵ-PE uses irrational probabilities, and thus "computing" an (exact) ϵ-PE is just as problematic as computing an exact NE. Indeed, the notion of a ϵ-PE is used as a technical step towards the definition of trembling hand perfect equilibrium, rather than as a natural "numerical relaxation" of this notion.

with all numbers given in standard binary notation, compute (the binary representation of) a strategy profile x' with rational probabilities having ℓ_∞ distance at most δ to a trembling hand perfect equilibrium of Γ.

As an immediate corollary of our main theorem, and the results of Etessami and Yannakakis, we have that *approximating a Nash equilibrium and approximating a trembling hand perfect equilibrium are polynomial time equivalent tasks.* In particular, there is a polynomial time algorithm that finds an approximation to a trembling hand perfect equilibrium of a given game, using access to any oracle solving the corresponding approximation problem for the case of Nash equilibrium. To put this result in perspective, we note that Nash equilibrium and trembling hand perfect equilibrium are computationally quite different in other respects: if instead of *finding* an equilibrium, we want to *verify* that a given strategy profile is such an equilibrium, the case of Nash equilibrium is trivial, while the case of trembling hand perfect equilibrium for games with 3 (or more) players is NP-hard [8]. This might lead one to believe that approximating a trembling hand perfect equilibrium for games with 3 or more players is likely to be harder than approximating a Nash equilibrium, but we show that this is not the case.

1.1 About the Proof

Informally (for formal definitions, see below), FIXP (resp., FIXP_a) is defined as the complexity class of search problems that can be cast as exactly computing (resp., approximating) a *Brouwer fixed point* of functions represented by circuits over basis $\{+, *, -, /, \max, \min\}$ with rational constants. It was established in [6] that computing (resp., approximating) an actual Nash Equilibrium (NE) for a finite n-player game is FIXP-complete (resp., FIXP_a-complete), already for $n = 3$. Since trembling hand perfect equilibria constitute a refinement of Nash Equilibria, to show that approximating a trembling hand perfect equilibrium is FIXP_a-complete, we merely have to show that this task is in FIXP_a.

An ϵ-perfect equilibrium (ϵ-PE for short) is defined to be a fully mixed strategy profile, x, where every strategy j of every player i that is played with probability $x_{i,j} > \epsilon$ must be a best response to the other player's strategies x_{-i}. Then, a trembling hand perfect equilibrium (PE for short) is defined to be a limit point of a sequence of ϵ-PEs, for $\epsilon > 0$, $\epsilon \to 0$. Here, by limit point we mean, as usual, any point to which a subsequence of the sequence converges. Such a point must exist, by the Bolzano-Weierstrass theorem.

In rough outline, our proof that approximating a PE is in FIXP_a has the following structure:

1. We first define (in section 3) for any n-player game Γ, a map, F^ϵ_Γ, parameterized by a parameter $\epsilon > 0$, so that F^ϵ_Γ defines a map from D^ϵ_Γ to itself, where D^ϵ_Γ denotes the space of fully mixed strategy strategy profiles x such that every player plays each strategy with probability at least ϵ. Also, $F^\epsilon_\Gamma(x)$ is described by a $\{+, -, *, \min, \max\}$-circuit with ϵ as one of its inputs.

In particular, the Brouwer fixed point theorem applies to this map. We show that the circuit defining F_Γ^ϵ can be computed in polynomial time from the input game instance Γ, and that every Brouwer fixed point of F_Γ^ϵ is an ϵ-PE of the original game Γ, making crucial use of, and modifying, a new fixed point characterization of NEs that was defined and used in [6].

2. We then show (in section 4) that if $\epsilon^* > 0$ is made sufficiently small as a function of the encoding size $|\Gamma|$ of the game Γ, and of a parameter $\delta > 0$, specifically if $\epsilon^* \leq \delta^{2^{g(|\Gamma|)}}$, where g is some polynomial, then any ϵ^*-PE must be δ-close (in the l_∞-norm) to an actual PE. This part of the proof relies on real algebraic geometry.
3. We then observe (in section 5) that for any desired δ, we can encode such a sufficiently small $\epsilon^* > 0$ as a circuit that is polynomially large in the encoding size of Γ and δ, simply by *repeated squaring*. We think of this as constructing a *virtual infinitesimal* and believe that this technique will have many other applications in the context of proving FIXP_a membership using real algebraic geometry. Finally, plugging in the circuit for ϵ^* for the input ϵ in the circuit for F_Γ^ϵ, we obtain a Brouwer function $F_\Gamma^{\epsilon^*}(x)$, defined by a $\{+, -, *, \max, \min\}$-circuit, such that any fixed point of $F_\Gamma^{\epsilon^*}(x)$ is guaranteed to be a fully mixed strategy profile, $x^*_{\epsilon^*}$, that is also within l_∞ distance δ of a PE, x^*, of Γ. The triangle inequality completes the proof.

2 Definitions and Preliminaries

2.1 Game-Theoretic Notions

We use $\mathbb{Q}_+$ to denote the set of positive rational numbers. A finite n-player normal form game, $\Gamma = (N, \langle S_i \rangle_{i\in N}, \langle u_i \rangle_{i\in N})$, consists of a set $N = \{1, \ldots, n\}$ of n players indexed by their number, a set of n (disjoint) finite sets of *pure strategies*, S_i, one for each player $i \in N$, and n rational-valued *payoff functions* $u_i : S \to \mathbb{Q}$, from the product strategy space $S = \Pi_{i=1}^n S_i$ to $\mathbb{Q}$.

The elements of S, i.e., combinations of pure strategies, one for each player, are called *pure strategy profiles*. The assumption of rational values is for computational purposes. Each rational number r is represented as usual by its numerator and denominator in binary, and we use $size(r)$ to denote the number of bits in the representation. The size $|\Gamma|$ of the instance (game) Γ is the total number of bits needed to represent all the information in the game: the strategies of all the players and their payoffs for all $s \in S$.

A *mixed strategy*, x_i, for a player i is a probability distribution on its set S_i of pure strategies. Letting $m_i = |S_i|$, we view x_i as a real-valued vector $x_i = (x_{i,1}, \ldots, x_{i,m_i}) \in [0,1]^{m_i}$, where $x_{i,j}$ denotes the probability with which player i plays pure strategy j in the mixed strategy x_i. Note that we must have $x_i \geq 0$ and $\sum_{i=1}^{m_i} x_{i,m_i} = 1$. That is, a vector x_i is a mixed strategy of player i iff it belongs to the *unit simplex* $\Delta_{m_i} = \{y \in R^{m_i} | y \geq 0; \sum_{j=1}^{m_i} y_j = 1\}$. We use the notation $\pi_{i,j}$ to identify the pure strategy j of player i, as well as its representation as a mixed strategy that assigns probability 1 to strategy j and probability 0 to the other strategies of player i.

A *mixed strategy profile* $x = (x_1, \ldots, x_n)$ is a combination of mixed strategies for all the players. That is, vector x is a mixed strategy profile iff it belongs to the product of the n unit simplexes for the n players, $\{x \in R^m \mid x \geq 0; \sum_{j=1}^{m_i} x_{i,j} = 1$ for $i = 1, \ldots, k\}$. We let D_Γ denote the set of all mixed profiles for game Γ. The profile is *fully mixed* if all the pure strategies of all players have nonzero probability. We use the notation x_{-i} to denote the subvector of x induced by the pure strategies of all players except for player i. If y_i is a mixed strategy of player i, we use $(y_i; x_{-i})$ to denote the mixed profile where everyone plays the same strategy as x except for player i, who plays mixed strategy y_i.

The payoff function of each player can be extended from pure strategy profiles to mixed profiles, and we will use U_i to denote the expected payoff function for player i. Thus the (expected) payoff $U_i(x)$ of mixed profile x for player i is $\sum x_{1,j_1} \ldots x_{k,j_k} u_i(j_1, \ldots, j_k)$ where the sum is over all pure strategy profiles $(j_1, \ldots, j_k) \in S$.

A *Nash equilibrium* (NE) is a (mixed) strategy profile x^* such that all $i = 1, \ldots, n$ and every mixed strategy y_i for player i, $U_i(x^*) \geq U_i(y_i; x^*_{-i})$. It is sufficient to check switches to pure strategies only, i.e., x^* is a NE iff $U_i(x^*) \geq U_i(\pi_{i,j}; x^*_{-i})$ for every pure strategy $j \in S_i$, for each player $i = 1, \ldots, n$. Every finite game has at least one NE [12].

A mixed profile x is called a *ϵ-perfect equilibrium* (ϵ-PE) if it is (a) fully mixed, i.e., $x_{i,j} > 0$ for all i, and (b), for every player i and pure strategy j, if $x_{i,j} > \epsilon$, then the pure strategy $\pi_{i,j}$ is a best response for player i to x_{-i}. We call a mixed profile x^*, a *trembling hand perfect equilibrium* (PE) of Γ if it is a limit point of ϵ-PEs of the game Γ. In other words, we call x a PE if there exists a sequence $\epsilon_k > 0$, such that $\lim_{k\to\infty} \epsilon_k = 0$, and such that for all k there is a corresponding ϵ_k-PE, x^{ϵ_k} of Γ, such that $\lim_{k\to\infty} x^{\epsilon_k} = x^*$. Every finite game has at least one PE, and all PEs are NEs [15].

2.2 Complexity Theoretic Notions

A $\{+, -, *, \max, \min\}$-*circuit* is a circuit with inputs $x_1, x_2, \ldots, x_n$, as well as rational constants, and a finite number of (binary) computation gates taken from $\{+, -, *, \min, \max\}$, with a subset of the computation gates labeled $\{o_1, o_2, \ldots, o_m\}$ and called output gates.[2]

All circuits of this paper are $\{+, -, *, \min, \max\}$-circuits, so we shall often just write "circuit" for "$\{+, -, *, \min, \max\}$-circuit". A circuit computes a continuous function from $\mathbb{R}^n \to \mathbb{R}^m$ (and $\mathbb{Q}^n \to \mathbb{Q}^m$) in the natural way. Abusing notation slightly, we shall often identify the circuit with the function it computes.

By a (total) *multi-valued function*, f, with domain A and co-domain B, we mean a function that maps each $a \in A$ to a non-empty subset $f(a) \subseteq B$. We use $f : A \twoheadrightarrow B$ to denote such a function. Intuitively, when considering a multi-valued function as a computational problem, we are interested in producing just one of the elements of $f(a)$ on input a, so we refer to $f(a)$ as the set of *allowed*

[2] Note that the gates $\{+, -, *, \min, \max\}$ are of course redundant: gates $\{+, *, \max\}$ with rational constants are equally expressive.

outputs. A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \mathbb{R}^*$ is said to be in FIXP if there is a polynomial time computable map, r, that maps each instance $I \in \{0,1\}^*$ of f to $r(I) = \langle 1^{k^I}, 1^{d^I}, P^I, C^I, a^I, b^I \rangle$, where

- k^I, d^I are positive integers and $a^I, b^I \in \mathbb{Q}^{d^I}$.
- P^I is a convex polytope in $\mathbb{R}^{k^I}$, given as a set of linear inequalities with rational coefficients.
- C^I is a circuit which maps P^I to itself.
- $\phi^I : \{1,\dots,d^I\} \to \{1,\dots,k^I\}$ is a finite function given by its table.
- $f(I) = \{(a_i^I y_{\phi^I(i)} + b_i^I)_{i=1}^{d^I} \mid y \in P^I \wedge C^I(y) = y\}$. Note that $f(I) \neq \emptyset$, by Brouwer's fixed point theorem.

The above is in fact one of many equivalent characterizations of FIXP [6]. Informally, FIXP are those real vector multi-valued functions, with discrete inputs, that can be cast as Brouwer fixed point computations. A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \mathbb{R}^*$ is said to be FIXP-*complete* if:

1. $f \in$ FIXP, and
2. for all $g \in$ FIXP, there is a polynomial time computable map, mapping instances I of g to $\langle y^I, 1^{k^I}, 1^{d^I}, \phi^I, a^I, b^I \rangle$, where y^I is an instance of f, k^I and d^I are positive integers, ϕ^I maps $\{1,\dots,d^I\}$ to $\{1,\dots,k^I\}$, a^I and b^I are d^I-tuples with rational entries, so that $g(I) \supseteq \{(a_i^I z_{\phi^I(i)} + b_i^I)_{i=1}^{d^I} \mid z \in f(y^I)\}$. In other words, for any allowed output z of f on input y^I, the vector $(a_i^I z_{\phi^I(i)} + b_i^I)_{i=1}^{d^I}$ is an allowed output of g on input I.

Etessami and Yannakakis [6] showed that the multi-valued function which maps games in strategic form to their Nash equilibria is FIXP-complete.[3]

Since the output of a FIXP function consists of real-valued vectors, and as there are circuits whose fixed points are all irrational, a FIXP function is not directly computable by a Turing machine, and the class is therefore not directly comparable with standard complexity classes of total search problems (such as PPAD, PLS, or TFNP). This motivates the following definition of the discrete class FIXP_a, also from [6]. A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \{0,1\}^*$ (a.k.a. a totally defined discrete search problem) is said to be in FIXP_a if there is a function $f' \in$ FIXP, and polynomial time computable maps $\delta : \{0,1\}^* \to \mathbb{Q}_+$ and $g : \{0,1\}^* \to \{0,1\}^*$, such that for all instances I,

$$f(I) \supseteq \{\, g(\langle I, y \rangle) \mid y \in \mathbb{Q}^* \wedge \exists y' \in f'(I) : \|y - y'\|_\infty \le \delta(I) \,\}.$$

Informally, FIXP_a are those totally defined discrete search problems that reduce to approximating exact Brouwer fixed points. A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \{0,1\}^*$ is said to be FIXP_a-*complete* if:

1. $f \in \mathsf{FIXP}_a$, and

[3] To view the Nash equilibrium problem as a total multi-valued function, $f_{\text{Nash}} : \{0,1\}^* \twoheadrightarrow \mathbb{R}^*$, we can view all strings in $\{0,1\}^*$ as encoding some game, by viewing "ill-formed" input strings as encoding a fixed trivial game.

2. For all $g \in \mathsf{FIXP}_a$, there are polynomial time computable maps $r_1, r_2 : \{0,1\}^* \to \{0,1\}^*$, such that $g(I) \supseteq \{\, r_2(\langle I, z\rangle) \mid z \in f(r_1(I)) \,\}$.

Etessami and Yannakakis showed that the multi-valued function that maps pairs $\langle \Gamma, \delta\rangle$, where Γ is a strategic form game and $\delta > 0$, to the set of rational δ-approximations (in ℓ_∞-distance) of Nash equilibria of Γ, is FIXP_a-complete.

3 Computing ϵ-PEs in FIXP

Given a game Γ, let $m = \sum_{i\in N} m_i$ denote the total number of pure strategies of all players in Γ. For $\epsilon > 0$, let $D_\Gamma^\epsilon \subseteq D_\Gamma$ denote the polytope of fully mixed profiles of Γ such that furthermore every pure strategy is played with probability at least $\epsilon > 0$ (recall that D_Γ is the polytope of all strategy profiles). In this section, we show the following theorem.

Theorem 2. *There is a function, $F_\Gamma^\epsilon(x) : D_\Gamma \to D_\Gamma^\epsilon$, given by a circuit computable in polynomial time from Γ, with the circuit having both x and $\epsilon > 0$ as its inputs, such that for all fixed $0 < \epsilon < 1/m$, every Brouwer fixed point of the function $F_\Gamma^\epsilon(x)$ is an ϵ-PE of Γ. In particular, the problem of computing an ϵ-perfect equilibrium for a finite n-player normal form game is in* FIXP.

The rest of the section is devoted to the proof of Theorem 2. We will directly use, and somewhat modify, a construction developed and used in [6] (Lemmas 4.6 and 4.7, and definitions before them) which characterize the Nash Equilibria of a game as fixed points of a $\{+, -, *, \max, \min\}$-circuit. In particular, compared to Nash's original functions [12], the use of division is avoided. The construction defined in [6] that we modify amounts to a concrete algebraic realization of certain geometric characterizations of Nash Equilibria that were described by Gul, Pierce, and Stachetti in [7].

Concretely, suppose we are given $0 < \epsilon < 1/m$. For each mixed strategy profile x, let $v(x)$ be a vector which gives the expected payoff of each pure strategy of each player with respect to the profile x for the other players. That is, vector x is a vector of dimension m, whose entries are indexed by pairs $(i, j), i = 1, \ldots, n; j = 1, \ldots, m_i$, and $v(x)$ is also a vector of dimension m whose (i, j)-entry is $U_i(\pi_{i,j}; x_{-i})$. Let $h(x) = x + v(x)$. We can write $h(x)$ as $(h_1(x), \ldots, h_n(x))$ where $h_i(x)$ is the subvector corresponding to the strategies of player i. For each player i, consider the function $f_{i,x}(t) = \sum_{j\in S_i} \max(h_{ij}(x) - t, \epsilon)$. Clearly, this is a continuous, piecewise linear function of t. The function is strictly decreasing as t ranges from $-\infty$ (where $f_{i,x}(t) = +\infty$) up to $\max_j h_{ij}(x) - \epsilon$ (where $f_{i,x}(t) = m_i \cdot \epsilon$). Since we have $m_i \cdot \epsilon < 1$, there is a unique value of t, call it t_i, where $f_{i,x}(t_i) = 1$. The function F_Γ^ϵ is defined as follows:

$$F_\Gamma^\epsilon(x)_{ij} = \max(h_{ij}(x) - t_i, \epsilon)$$

for every $i = 1, \ldots, n$, and $j \in S_i$. From our choice of t_i, we have $\sum_{j\in S_i} F_\Gamma^\epsilon(x)_{ij} = 1$ for all $i = 1, \ldots, n$, thus for any mixed profile, x, we have $F_\Gamma^\epsilon(x) \in D_\Gamma^\epsilon$. So F_Γ^ϵ maps D_Γ to D_Γ^ϵ, and since it is clearly also continuous, it has fixed points, by Brouwer's theorem.

Lemma 1. *For* $0 < \epsilon < 1/m$*, every fixed point of the function* F_Γ^ϵ *is an* ϵ*-PE of* Γ*.*

Proof. If x is a fixed point of F_Γ^ϵ, then $x_{ij} = \max(x_{ij} + v(x)_{ij} - t_i, \epsilon)$ for all i, j. Recall that $v(x)_{ij} = U_i(\pi_{i,j}; x_{-i})$ is the expected payoff for player i of his j'th pure strategy $\pi_{i,j}$, with respect to strategies x_{-i} of the other players.

Note that the equation $x_{ij} = \max(x_{ij} + U_i(\pi_{i,j}; x_{-i}) - t_i, \epsilon)$ implies that $U_i(\pi_{i,j}; x_{-i}) = t_i$ for all i, j such that $x_{ij} > \epsilon$, and that $U_i(\pi_{i,j}; x_{-i}) \leq t_i$ for all i, j such that $x_{ij} = \epsilon$. Consequently, by definition, x constitutes an ϵ-PE. □

The following Lemma shows that we can implement the function $F_\Gamma^\epsilon(x)$ by a circuit which has x and ϵ as inputs. The proof exploits sorting networks.

Lemma 2. *Given* Γ*, we can construct in polynomial time a* $\{+, -, *, \max, \min\}$*-circuit that computes the function* $F_\Gamma^\epsilon(x)$*, where* x *and* ϵ *are inputs to the circuit.*

Proof. The circuit does the following.

Given a vector $x \in D_\Gamma$, first compute $y = h(x) = x + v(x)$. It is clear from the definition of $v(x)$ that y can be computed using $+, *$ gates. For each player i, let y_i be the corresponding subvector of y induced by the strategies of player i. Sort y_i in decreasing order, and let z_i be the resulting sorted vector, i.e. the components of $z_i = (z_{i1}, \ldots, z_{im_i})$ are the same as the components of y_i, but they are sorted: $z_{i1} \geq z_{i2} \geq \ldots \geq z_{im_i}$. To obtain z_i, the circuit uses a polynomial sized sorting network, W_i, for each i (see e.g. Knuth [9] for background on sorting networks). For each comparator gate of the sorting network we use a max and a min gate.

Using this, for each player i, we compute t_i as the following expression:

$$\max\{(1/l) \cdot ((\sum_{j=1}^{l} z_{ij}) + (m_i - l) \cdot \epsilon - 1) | l = 1, \cdots, m_i\}$$

We will show below that this expression does indeed give the correct value of t_i. Finally, we output $x'_{ij} = \max(y_{ij} - t_i, \epsilon)$ for each $i = 1, \ldots, d; j \in S_i$.

We now have to establish that $t_i = \max\{(1/l) \cdot ((\sum_{j=1}^{l} z_{ij}) + (m_i - l) \cdot \epsilon - 1) | l = 1, \cdots, m_i\}$. Consider the function $f_{i,x}(t) = \sum_{j \in S_i} \max(z_{ij} - t, \epsilon)$ as t decreases from $z_{i1} - \epsilon$ where the function value is at its minimum of $m_i\epsilon$, down until the function reaches the value 1. In the first interval from $z_{i1} - \epsilon$ to $z_{i2} - \epsilon$ the function is $f_{i,x}(t) = z_{i1} - t + (m_i - 1) \cdot \epsilon$; in the second interval from $z_{i2} - \epsilon$ to $z_{i3} - \epsilon$ it is $f_{i,x}(t) = z_{i1} + z_{i2} - 2t + (m_i - 2) \cdot \epsilon$, and so forth. In general, in the l-th interval, $f_{i,x}(t) = \sum_{j=1}^{l} (z_{ij} - t) + (m_i - l) \cdot \epsilon = \sum_{j=1}^{l} z_{ij} - lt + (m_i - l) \cdot \epsilon$. If the function reaches the value 1 in the l'th interval, then clearly $t_i = ((\sum_{j=1}^{l} z_{ij}) + (m_i - l) \cdot \epsilon - 1)/l$. In that case, furthermore for $k < l$, we have $\sum_{j=1}^{k} (z_{ij} - t_i) + (m_i - k) \cdot \epsilon \leq \sum_{j=1}^{l} (z_{ij} - t_i) + (m_i - l) \cdot \epsilon = 1$, because in that case we know $(z_{ij} - t_i) \geq \epsilon$ for every $j \in \{1, \ldots, l\}$. Therefore, in this case $((\sum_{j=1}^{k} z_{ij}) + (m_i - k) \cdot \epsilon - 1)/k \leq t_i$. On the other hand, if $l < m_i$, then for $k > l$ we have $t_i \geq z_{ik} - \epsilon$, i.e., $z_{ik} - t_i \leq \epsilon$, and thus for all $k > l$, $k \leq m_i$, we have $\sum_{j=1}^{k}(z_{ij} - t_i) + (m_i - k) \cdot \epsilon \leq \sum_{j=1}^{l}(z_{ij} - t_i) + (m_i - l) \cdot \epsilon = 1$. Thus again $((\sum_{j=1}^{k} z_{ij}) + (m_i - k) \cdot \epsilon - 1)/k \leq t_i$. Therefore, $t_i = \max\{(1/l) \cdot ((\sum_{j=1}^{l} z_{ij}) + (m_i - l) \cdot \epsilon - 1)|l = 1, \cdots, m_i\}$. □

Lemma 1 and Lemma 2 together immediately imply Theorem 2.

4 Almost Implies Near

As outlined in the introduction, in this section, we want to exploit the "uniform" function $F_\Gamma^\epsilon(x)$ devised in the previous section for ϵ-PEs, and construct a "small enough" $\epsilon^* > 0$ such that any fixed point of $F_\Gamma^{\epsilon^*}(x)$ is δ-close, for a given $\delta > 0$, to an actual PE.

The following is a special case of the simple but powerful "almost implies near" paradigm of Anderson [1].

Lemma 3 (Almost implies near). *For any fixed strategic form game Γ, and any $\delta > 0$, there is an $\epsilon > 0$, so that any ϵ-PE of Γ has ℓ_∞-distance at most δ to some PE of Γ.*

Proof. Assume to the contrary that there is a game Γ and a $\delta > 0$ so that for all $\epsilon > 0$, there is an ϵ-PE x_ϵ of Γ so that there is no PE in a δ-neighborhood (with respect to l_∞ norm) of x_ϵ. Consider the sequence $(x_{1/n})_{n \in \mathbf{N}}$. Since this is a sequence in a compact space (namely, the space of mixed strategy profiles of Γ), it has a limit point, x^*, which is a PE of Γ (since x_ϵ is a ϵ-PE). But this contradicts the statement that there is no PE in a δ-neighborhood of any of the profiles $x_{1/n}$. □

A priori, we have no bound on ϵ, but we next use the machinery of real algebraic geometry [2,3] to obtain a specific bound as a "free lunch", just from the fact that Lemma 3 is true.

Lemma 4. *There is a constant c, so that for all integers $n, m, k, B \in \mathbb{N}$ and $\delta \in \mathbb{Q}_+$, the following holds. Let $\epsilon \leq \min(\delta, 1/B)^{n^{cm^3}}$. For any n-player game Γ with at most m pure strategies for all player, and with integer payoffs of absolute value at most B, any ϵ-PE of Γ has ℓ_∞-distance at most δ to some PE of Γ.*

Proof. The proof involves constructing formulas in the first order theory of real numbers, which formalize the "almost implies near" statement of Lemma 3, with δ being "hardwired" as a constant and ϵ being the only free variable. Then, we apply *quantifier elimination* to these formulas. This leads to a quantifier free statement to which we can apply standard theorems bounding the size of an instantiation of the free variable ϵ making the formula true. We shall apply and refer to theorems in the monograph of Basu, Pollack and Roy [2,3]. Note that we specifically refer to theorems and page numbers of the online edition [3]; these are in general different from the printed edition [2].

First-order formula for an ϵ-perfect equilibrium: Define $R_i(x \setminus k)$ as the polynomial expressing $U_i(\pi_{i,k}; x_{-i})$, that is, the expected payoff to player i when it uses pure strategy k, and the other players play according to their mixed strategy in the profile x. Thus,

$$R_i(x \setminus k) := \sum_{a_{-i}} u_i(k; a_{-i}) \prod_{j \neq i} x_{j,a_j} .$$

Let EPS-PE(x, ϵ) be the quantifier-free first-order formula, with free variables $x \in \mathbb{R}^m$ and $\epsilon \in \mathbb{R}$, defined by the conjunction of the following formulas that together express that x is an ϵ-perfect equilibrium:

$$x_{i,j} > 0 \quad \text{for } i = 1 \ldots, n, \text{ and } j = 1, \ldots, m_i \ ,$$
$$x_{i,1} + \cdots + x_{i,m_i} = 1 \quad \text{for } i = 1 \ldots, n \ ,$$
$$(R_i(x \setminus k) \geq R_i(x \setminus l)) \vee (x_{i,k} \leq \epsilon) \quad \text{for } i = 1 \ldots, n, \text{ and } k, l = 1, \ldots, m_i \ .$$

First-order formula for perfect equilibrium: Let PE(x) denote the following first-order formula with free variables $x \in \mathbb{R}^m$, expressing that x is a perfect equilibrium:

$$\forall \epsilon > 0 \, \exists y \in \mathbb{R}^m : \text{EPS-PE}(y, \epsilon) \wedge \|x - y\|^2 < \epsilon \ .$$

First-order formula for "almost implies near" statement: Given a *fixed* $\delta > 0$ let PE-bound$_\delta(\epsilon)$ denote the following first-order formula with free variable $\epsilon \in \mathbb{R}$, denoting that any ϵ-perfect equilibrium of G is δ-close to a perfect equilibrium (in ℓ_2-distance, and therefore also in ℓ_∞-distance):

$$\forall x \in \mathbb{R}^m \, \exists y \in \mathbb{R}^m : (\epsilon > 0) \wedge \left(\neg \, \text{EPS-PE}(x, \epsilon) \vee \left(\text{PE}(y) \wedge \|x - y\|^2 < \delta^2\right)\right) \ .$$

Suppose $\delta^2 = 2^{-k}$ and the payoffs have absolute value at most $B = 2^\tau$. Then for this formula we have

- The total degree of all involved polynomials is at most $\max(2, n-1)$.
- The bitsize of coefficients is at most $\max(k, \tau)$.
- The number of free variables is 1.
- Converting to prenex normal form, the formula has 4 blocks of quantifiers, of sizes m, m, 1, m, respectively.

We now apply quantifier elimination [3, Algorithm 14.6, page 555] to the formula PE-bound$_\delta(\epsilon)$, converting it into an equivalent quantifier free formula PE-bound$'_\delta(\epsilon)$ with a single free variable ϵ. This is simply a Boolean formula whose atoms are sign conditions on various polynomials in ϵ. The bounds given by Basu, Pollack and Roy in association to Algorithm 14.6 imply that for this formula:

- The degree of all involved polynomials (which are univariate polynomials in ϵ) is $\max(2, n-1)^{O(m^3)} = n^{O(m^3)}$.

- The bitsize of all coefficients is at most $\max(k, \tau) \max(2, n-1)^{O(m^3)} = \max(k, \tau) n^{O(m^3)}$.

By Lemma 3, we know that there exists an $\epsilon > 0$ so that the formula PE-bound$'_\delta(\epsilon)$ is true. We now apply (as the involved polynomials are univariate, simpler tools would also suffice) Theorem 13.14 of Basu, Pollack and Roy [3, Page 521] to the set of polynomials that are atoms of PE-bound$'_\delta(\epsilon)$ and conclude that PE-bound$'_\delta(\epsilon^*)$ is true for some $\epsilon^* \geq 2^{-\max(k,\tau)n^{O(m^3)}}$. By the semantics of the formula PE-bound$_\delta(\epsilon)$, we also have that PE-bound$_\delta(\epsilon')$ is true for all $\epsilon' \leq \epsilon^*$, and the statement of the lemma follows. □

5 Proof of the Main Theorem

We now prove Theorem 1. Let Γ be the n-player game given as input. Let m be the total number of pure strategies for all player. Let $B \in \mathbb{N}$ be the largest absolute value of any payoff of Γ. By the definition of FIXP_a, our task is the following. Given a parameter $\delta > 0$, we must construct a polytope P, a circuit $C : P \to P$, and a number δ', so that δ'-approximations to fixed points of C can be efficiently transformed into δ-approximations of PEs of Γ. In fact, we shall let $\delta' = \delta/2$ and ensure that δ'-approximations to fixed points of C *are* δ-approximations of PEs of Γ. The polytope P is simply the polytope D_Γ of all strategy profiles of Γ; clearly we can output the inequalities defining this polytope in polynomial time. The circuit C is the following: We construct the circuit for the function F_Γ^ϵ of Section 3. Then, we construct a circuit for the number $\epsilon^* = \min(\delta/2, B^{-1})^{2^{\lceil cm^3 \lg n \rceil}} \leq \min(\delta/2, B^{-1})^{n^{cm^3}}$, where c is the constant of Lemma 4: The circuit simply repeatedly squares the number $\min(\delta/2, B^{-1})$ (which is a rational constant) and thereby consists of exactly $\lceil cm^3 \lg n \rceil$ multiplication gates, i.e., a polynomially bounded number. We then plug in the circuit for ϵ^* for the parameter ϵ in the circuit for F_Γ^ϵ, obtaining the circuit C, which is obviously a circuit for $F_\Gamma^{\epsilon^*}$. Now, by Theorem 2, any fixed point of C on P is an ϵ^*-PE of Γ. Therefore, by Lemma 4, any fixed point of C is a $\delta/2$-approximation in ℓ_∞-distance to a PE of Γ. Finally, by the triangle inequality, any $\delta' = \delta/2$-approximation to a fixed point of C on P is a $\delta/2 + \delta/2 = \delta$ approximation of a PE of Γ. This completes the proof.

6 Conclusion

We have showed that the problem of *approximating* a trembling hand perfect equilibrium for a finite strategic form game is in FIXP_a. We do not know if *exactly* computing a trembling hand perfect equilibrium is in FIXP, and we consider this an interesting open problem, although it should be noted that if one is interested exclusively in the Turing Machine complexity of the problem, FIXP_a membership of the approximation version is arguably "the real thing". We also note that this makes our proof interesting as a case where membership in FIXP_a is *not* established as a simple corollary of the exact problem being in the "abstract class" FIXP, as was the case for all examples in the original paper of Etessami and Yannakakis.

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Approximate Well-Supported Nash Equilibria in Symmetric Bimatrix Games*

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Abstract. The ε-well-supported Nash equilibrium is a strong notion of approximation of a Nash equilibrium, where no player has an incentive greater than ε to deviate from any of the pure strategies that she uses in her mixed strategy. The smallest constant ε currently known for which there is a polynomial-time algorithm that computes an ε-well-supported Nash equilibrium in bimatrix games is slightly below 2/3. In this paper we study this problem for *symmetric* bimatrix games and we provide a polynomial-time algorithm that gives a $(1/2 + \delta)$-well-supported Nash equilibrium, for an arbitrarily small positive constant δ.

1 Introduction

The problem of computing Nash equilibria is one of the most fundamental problems in algorithmic game theory. It is now known that the complexity of computing a Nash equilibrium is PPAD-complete [4], even for two-player games [3]. Given this evidence of intractability of the problem, further research has focused on the computation of *approximate* Nash equilibria. In this context—and assuming that all payoffs are normalized to be in the interval $[0, 1]$—the standard notion of approximation is the additive approximation with a parameter $\varepsilon \in [0, 1]$. There are two different notions of additive approximation of Nash equilibria: the *ε-Nash equilibrium* and the *ε-well-supported Nash equilibrium*.

An ε-Nash equilibrium is a strategy profile—one strategy for each player—in which no player can improve her payoff by more than ε through unilateral deviation from her strategy in the strategy profile. Several polynomial-time algorithms have been proposed to find ε-Nash equilibria for $\varepsilon = 1/2$ [6], for $\varepsilon = (3 - \sqrt{5})/2 \approx 0.38$ [5], for $\varepsilon = 1/2 - 1/(3\sqrt{6}) \approx 0.36$ [2], and finally for $\varepsilon \approx 0.3393$ [13]. It is also known how to find ε-Nash equlibria in quasi-polynomial time $n^{O(\log n/\varepsilon^2)}$ for arbitrarily small $\varepsilon > 0$ [11], where n is the number of pure strategies.

The notion of an ε-well-supported Nash equilibrium requires that no player has an incentive greater than ε to deviate from any of the pure strategies she uses in her mixed strategy. It is a notion stronger than that of an ε-Nash equilibrium:

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every ε-well-supported Nash equilibrium is also an ε-Nash equilibrium, but not necessarily vice-versa. The smallest ε for which a polynomial-time algorithm is currently known that computes an ε-well-supported Nash equilibrium in an arbitrary bimatrix game is slightly above 0.6619 [9,7]. It is also known that for the class of win-lose bimatrix games one can find 1/2-well-supported Nash equilibria in polynomial time [9].

In this paper we study computation of approximate well-supported Nash equilibria in *symmetric* bimatrix games, a class of bimatrix games in which swapping the roles of the two players does not change the payoff matrices, that is if the payoff matrix of one is the transpose of the payoff matrix of the other. Symmetric games are an important class of games in game theory; their applications include auctions and congestion games. They have already been studied by Nash in his seminal paper in which he introduced the concept of a Nash equilibrium; he proved that every symmetric game has at least one symmetric Nash equilibrium, that is one in which all players use the same mixed strategy [12].

Computing Nash equilibria in symmetric bimatrix games is known to be as hard as computing Nash equilibria in arbitrary bimatrix games because there is a polynomial-time reduction from the latter to the former [8]. In contrast to arbitrary bimatrix games, it is known how to compute $(1/3 + \delta)$-Nash equilibria in symmetric bimatrix games in polynomial time, where $\delta > 0$ is arbitrarily small [10]. In this paper we improve our understanding of the approximability of Nash equilibria in symmetric bimatrix games by considering the task of computing approximate well-supported Nash equilibria. Our main result is an algorithm that computes $(1/2 + \delta)$-well-supported Nash equilibria in symmetric bimatrix games in polynomial time, where $\delta > 0$ is arbitrarily small (Theorem 3).

Our $(1/2 + \delta)$-approximation algorithm splits the analysis into two cases that are then considered independently. The first case is based on the following relaxation of the concept of a symmetric Nash equilibrium: we say that a strategy profile (x, x) *prevents exceeding* $u \in [0, 1]$ if the expected payoff of every pure strategy in the symmetric game is at most u when the other player uses strategy x. This is indeed a relaxation of the concept of the symmetric Nash equilibrium because if (x^*, x^*) is a symmetric Nash equilibrium then it prevents exceeding its value (that is, the expected payoff each player gets when they both play strategy x^*). We justify relevance of this concept by showing that a strategy profile (x, x) that prevents exceeding u is a u-well-supported Nash equilibrium, so in order to provide a latter it is sufficient to find a former. Moreover, we show that this relaxation of a symmetric Nash equilibrium is algorithmically tractable because it suffices to solve a single linear program to find a strategy profile (x, x) that prevents exceeding u, if there is one. The first case in our algorithm is to solve this linear program for $u = 1/2$ and if it succeeds then we can immediately report a 1/2-well-supported Nash equilibrium. Note that by the above, if there is indeed a symmetric Nash equilibrium with value 1/2 or smaller, then the linear program does have a solution.

If the first case in the algorithm fails to identify a 1/2-well-supported equilibrium because the game has no symmetric Nash equilibrium with value 1/2 or smaller, then we consider the other, and technically more challenging, case. We use another relaxation of the concept of a symmetric Nash equilibrium: we say that a strategy profile (x, y) *well supports* $u \in [0, 1]$ if the expected payoff of every pure strategy in the support of x is at least u when the other player uses strategy y, and the expected payoff of every pure strategy in the support of y is at least u when the other player uses strategy x. We observe that if a strategy profile (x, y) well supports u then it is a $(1 - u)$-well-supported Nash equilibrium, so in order to provide a latter it is sufficient to find a former.

Therefore, in order to obtain a $(1/2+\delta)$-well-supported Nash equilibrium, we are interested in finding a strategy profile (x, y) that well supports $u \geq 1/2 - \delta$. While it may not be easy to verify if there is such a strategy profile, let alone find one, both can be achieved in polynomial time by solving a single linear program if we happen to know the supports of strategies of each player in such a strategy profile. The obvious technical obstacle to algorithmic tractability here is that the number of all possible supports to consider is exponential in the number of pure strategies. We overcome this difficulty by proving the main technical result of the paper (Theorem 2) that for every symmetric Nash equilibrium (x^*, x^*) and for every $\delta > 0$ establishes existence of a strategy profile (x, y), with both strategies having supports of constant size, that well supports $u^* - \delta$, where u^* is the value of the Nash equilibrium. Note that by the failure of the first case every symmetric Nash equilibrium has value larger than 1/2, and hence Theorem 2 implies that there is such a strategy profile with constant-size supports that well supports $1/2-\delta$. The second case of our algorithm is to solve the linear programs mentioned above for $u = 1/2 - \delta$ and for all supports I and J of sizes at most $\kappa(\delta)$—where $\kappa(\delta)$ is a constant (which depends on δ, but does not depend on the number n of pure strategies) that is specified in Theorem 2—and to output a solution (x, y) as soon as one is found.

In order to prove our main technical result (Theorem 2) we use the probabilistic method to prove existence of constant-support strategy profiles that nearly well support the expected payoffs of a Nash equilibrium. Our construction and proof are inspired by the construction of Daskalakis et al. [5] used by them to compute $(3 - \sqrt{5})/2$-Nash equilibria in bimatrix games in polynomial time, but our analysis is different and more involved because we need to guarantee the extra condition of nearly well supporting the equilibrium values. The general idea of using sampling and Hoeffding bounds to prove existence of approximate equlibria with small supports dates back to the papers of Althofer [1] and Lipton et al. [11], who have shown that strategies with supports of size $O(\log n/\varepsilon^2)$ are sufficient for ε-Nash equilibria in games with n strategies.

2 Preliminaries

We consider bimatrix games (R, C), where $R, C \in [0, 1]^{n\times n}$ are square matrices of payoffs for the two players: the row player and the column player, respectively. If the row player uses a strategy i, $1 \leq i \leq n$ and if the column one uses a strategy j, $1 \leq j \leq n$, then the row player receives payoff R_{ij} and the column player receives payoff C_{ij}. We assume that the payoff values are in the interval $[0, 1]$; it is easy to see that equilibria in bimatrix games are invariant under additive and positive multiplicative transformations of the payoff matrices.

A *mixed strategy* $x \in [0,1]^n$ is a probability distribution on the set of *pure strategies* $\{1, 2, \ldots, n\}$. If the row player uses a mixed strategy x and the column player uses a mixed strategy y, then the row player receives payoff $x^T R y$ and the column player receives payoff $x^T C y$. A pair of strategies (x, y), the former for the row player and the latter for the column player, is often referred to as a strategy profile. We define the *support* $\text{supp}(x)$ of a mixed strategy x to be the set of pure strategies that have positive probability in x, i.e., $\text{supp}(x) = \{i : 1 \le i \le n \text{ and } x_i > 0\}$.

For every i, $1 \le i \le n$, let $R_{i\bullet}$ be the row vector of the payoffs of the payoff matrix R when the row player uses the strategy i. Note that if the row player uses a pure strategy i, $1 \le i \le n$, and if the column player uses a mixed strategy y, then the row player receives payoff $R_{i\bullet} y$. Similarly, for every j, $1 \le j \le n$, let $C_{\bullet j}$ be the column vector of the payoffs of the matrix C when the column player uses the strategy j. Note that if the column player uses a pure strategy j, $1 \le j \le n$, and if the row player uses a mixed strategy x, then the column player receives payoff $x^T C_{\bullet j}$.

Definition 1 (Nash equilibrium). *A* Nash equilibrium *is a strategy profile* (x^*, y^*) *such that*

- *for every* i, $1 \le i \le n$, *we have* $R_{i\bullet} y^* \le (x^*)^T R y^*$, *and*
- *for every* j, $1 \le j \le n$, *we have* $(x^*)^T C_{\bullet j} \le (x^*)^T C y^*$,

or, in other words, if x^* *is a best response to* y^* *and* y^* *is a best response to* x^*.

Definition 2 (Approximate Nash equilibrium). *For every* $\varepsilon > 0$, *an* ε-Nash equilibrium *is a strategy profile* (x^*, y^*) *such that*

- *for every* i, $1 \le i \le n$, *we have* $R_{i\bullet} y^* - (x^*)^T R y^* \le \varepsilon$, *and*
- *for every* j, $1 \le j \le n$, *we have* $(x^*)^T C_{\bullet j} - (x^*)^T C y^* \le \varepsilon$,

or, in other words, if x^* *is an* ε*-best response to* y^* *and* y^* *is an* ε*-best response to* x^*.

Definition 3 (Approximate well-supported Nash equilibrium). *For every* $\varepsilon > 0$, *an* ε-well-supported Nash equilibrium *is a strategy profile* (x^*, y^*) *such that*

- *for every* i, $1 \le i \le n$, *and* $i' \in \text{supp}(x^*)$, *we have* $R_{i\bullet} y^* - R_{i'\bullet} y^* \le \varepsilon$, *and*
- *for every* j, $1 \le j \le n$, *and* $j' \in \text{supp}(y^*)$, *we have* $(x^*)^T C_{\bullet j} - (x^*)^T C_{\bullet j'} \le \varepsilon$,

or, in other words, if every $i' \in \text{supp}(x^*)$ *is an* ε*-best response to* y^* *and every* $j' \in \text{supp}(y^*)$ *is an* ε*-best response to* x^*.

Definition 4 (Symmetric game, symmetric Nash equilibrium). *A bimatrix game* (R, C) *is* symmetric *if* $C = R^T$.

A symmetric Nash equilibrium *in a symmetric bimatrix game* (R, R^T) *is a strategy profile* (x^*, x^*) *such that for every* i, $1 \le i \le n$, *we have* $R_{i\bullet}x^* \le (x^*)^T Rx^*$. *Note that then it also follows that for every* j, $1 \le j \le n$, *we have:*

$$(x^*)^T R^T_{\bullet j} = R_{j\bullet}x^* \le (x^*)^T Rx^* = (Rx^*)^T x^* = (x^*)^T R^T x^*.$$

Let us recall a fundamental theorem of Nash [12] about existence of symmetric Nash equilibria in symmetric bimatrix games.

Theorem 1 ([12]). *Every symmetric bimatrix game has a symmetric Nash equilibrium.*

3 Computing Approximate Well-Supported Nash Equilibria

Fix a bimatrix game $G = (R, C)$ for the rest of the paper, where $R, C \in [0, 1]^{n \times n}$. We will use N to denote the number of bits needed to represent the matrices R and C with all their entries represented in binary. We say that a strategy x is *k-uniform*, for $k \in \mathbb{N} \setminus \{ 0 \}$, if $x_i \in \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$, for every i, $1 \le i \le n$.

3.1 Strategies that Prevent Exceeding a Payoff

Definition 5 (Preventing exceeding payoffs). *We say that a strategy* $x \in [0, 1]^n$ *for the row player* prevents exceeding $u \in [0, 1]$ *if for every* $j = 1, 2, \ldots, n$, *we have* $x^T C_{\bullet j} \le u$ *or, in other words, if the column player payoff of the best response to* x *does not exceed* u. *Similarly, we say that a strategy* $y \in [0, 1]^n$ *for the column player* prevents exceeding $v \in [0, 1]$ *if for every* $i = 1, 2, \ldots, n$, *we have* $R_{i\bullet}y \le v$ *or, in other words, if the row player payoff of the best response to* y *does not exceed* v.

For brevity, we say that a strategy profile (x, y) prevents exceeding (v, u) *if* x *prevents exceeding* u *and* y *prevents exceeding* v.

Observe that the following system of linear constraints $\mathrm{PE}(v, u)$ characterizes strategy profiles (x, y) that prevent exceeding $(v, u) \in [0, 1]^2$:

$$\begin{aligned}
\sum_{i=1}^{n} x_i = 1;\ & x_i \ge 0 \text{ for all } i = 1, 2, \ldots, n; \\
\sum_{j=1}^{n} y_j = 1;\ & y_j \ge 0 \text{ for all } j = 1, 2, \ldots, n; \\
R_{i\bullet}y \le v \ & \text{ for all } i = 1, 2, \ldots, n; \\
x^T C_{\bullet j} \le u \ & \text{ for all } j = 1, 2, \ldots, n.
\end{aligned}$$

Note that if (x, y) is a Nash equilibrium then, by definition, it prevents exceeding $(x^T Ry, x^T Cy)$, which implies the following Proposition.

Proposition 1. *If (x, y) is a Nash equilibrium, $v \geq x^T Ry$, and $u \geq x^T Cy$, then* $\mathrm{PE}(v, u)$ *has a solution and it prevents exceeding* (v, u)*.*

By the following proposition, in order to find an ε-well-supported Nash equilibrium it suffices to find a strategy profile that prevents exceeding $(\varepsilon, \varepsilon)$.

Proposition 2. *If a strategy profile (x, y) prevents exceeding (v, u) then it is a* $\max(v, u)$*-well-supported Nash equilibrium.*

Proof. Let $i' \in \mathrm{supp}(x)$ and let $i \in \{ 1, 2, \ldots, n \}$. Then we have:

$$R_{i\bullet}y - R_{i'\bullet}y \leq R_{i\bullet}y \leq v,$$

where the first inequality follows from $R_{i'\bullet}y \geq 0$, and the other one holds because y prevents exceeding v. Similarly, and using the assumption that x prevents exceeding u, we can argue that for all $j' \in \mathrm{supp}(y)$ and $j \in \{ 1, 2, \ldots, n \}$, we have $x^T C_{\bullet j} - x^T C_{\bullet j'} \leq u$. It follows that (x, y) is a $\max(v, u)$-well-supported Nash equilibrium. □

3.2 Strategies that Well Support a Payoff

Definition 6 (Well supporting payoffs). *We say that a strategy $x \in [0, 1]^n$ for the row player* well supports *$v \in [0, 1]$ against a strategy $y \in [0, 1]^n$ for the column player if for every $i \in \mathrm{supp}(x)$, we have $R_{i\bullet}y \geq v$. Similarly, we say that a strategy $y \in [0, 1]^n$ for the column player* well supports *$u \in [0, 1]$ against a strategy $x \in [0, 1]^n$ for the row player if for every $j \in \mathrm{supp}(y)$, we have $x^T C_{\bullet j} \geq u$.*

For brevity, we say that a strategy profile (x, y) well supports *(v, u) if x well supports v against y and y well supports u against x.*

The following theorem states that the payoffs of every Nash equilibrium can be nearly well supported by a strategy profile with supports of constant size.

Theorem 2. *Let (x^*, y^*) be a Nash equilibrium. For every $\delta > 0$, there are $\kappa(\delta)$-uniform strategies x, y such that the strategy profile (x, y) well supports $\big((x^*)^T Ry^* - \delta, (x^*)^T Cy^* - \delta\big)$, where $\kappa(\delta) = \lceil 2 \ln(1/\delta)/\delta^2 \rceil$.*

The proof of this technical result is postponed until Section 4.

Let $v, u \in [0, 1]$, $\delta > 0$, and let $\mathcal{I}$ and $\mathcal{J}$ be multisets of pure strategies of size $\kappa(\delta)$. Consider the following system $\mathrm{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ of linear constraints:

$$\begin{aligned} x_i &= k_i/\kappa(\delta) \ \text{ for all } i = 1, 2, \ldots, n; \\ y_j &= \ell_j/\kappa(\delta) \ \text{ for all } j = 1, 2, \ldots, n; \\ R_{i\bullet}y &\geq v - \delta \ \text{ for all } i \in \mathcal{I}; \\ x^T C_{\bullet j} &\geq u - \delta \ \text{ for all } j \in \mathcal{J}; \end{aligned}$$

where k_i is the number of times i occurs in multiset $\mathcal{I}$, and ℓ_j is the number of times j occurs in multiset $\mathcal{J}$. Note that the system $\mathrm{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ of linear constraints characterizes $\kappa(\delta)$-uniform strategy profiles (x, y), such that $\mathrm{supp}(x) = \mathcal{I}$ and $\mathrm{supp}(y) = \mathcal{J}$, that well support $(v - \delta, u - \delta)$. Theorem 2 implies the following.

Corollary 1. *If (x, y) is a Nash equilibrium, $v \leq x^T Ry$, $u \leq x^T Cy$, and $\delta > 0$, then there are multisets $\mathcal{I}$ and $\mathcal{J}$ from $\{ 1, 2, \ldots, n \}$ of size $\kappa(\delta)$, such that* $\mathrm{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ *has a solution and it well supports $(v - \delta, u - \delta)$.*

By the following proposition, in order to find an ε-well-supported Nash equilibrium it suffices to find a strategy profile that well supports $(1 - \varepsilon, 1 - \varepsilon)$.

Proposition 3. *If a strategy profile (x, y) well supports (v, u) then it is a $\big(1 - \min(v, u)\big)$-well-supported Nash equilibrium.*

Proof. Let $i' \in \mathrm{supp}(x)$ and let $i \in \{ 1, 2, \ldots, n \}$. Then we have:

$$R_{i\bullet}y - R_{i'\bullet}y \leq 1 - R_{i'\bullet}y \leq 1 - v,$$

where the first inequality follows from $R_{i\bullet}y \leq 1$, and the other one holds because y well supports v. Similarly, and using the assumption that x well supports u, we can argue that for all $j' \in \mathrm{supp}(y)$ and $j \in \{ 1, 2, \ldots, n \}$, we have $x^T C_{\bullet j} - x^T C_{\bullet j'} \leq 1 - u$. It follows that (x, y) is a $\big(1 - \min(v, u)\big)$-well-supported Nash equilibrium. □

3.3 The Algorithm for Symmetric Games

Propositions 2 and 3 suggest that in order to identify a 1/2-well-supported Nash equilibrium it suffices to find either a strategy profile that prevents exceeding $(1/2, 1/2)$ or one that well supports $(1/2, 1/2)$. Moreover, verifying existence and identifying such strategy profiles can be done efficiently by solving the linear program $\mathrm{PE}(1/2, 1/2)$, and by solving linear programs $\mathrm{WS}(1/2 + \delta, 1/2 + \delta, \mathcal{I}, \mathcal{J}, \delta)$ for all multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies of size $\kappa(\delta)$, respectively.

For arbitrary bimatrix games the above scheme may fail if none of these systems of linear constraints has a solution. Note, however, that—by Proposition 1 and Corollary 1—it would indeed succeed if we could guarantee that the game had a Nash equilibrium with both payoffs at most 1/2, or with both payoffs at least $(1/2 + \delta)$. Symmetric bimatrix games nearly satisfy this requirement thanks to existence of symmetric Nash equilibria in every symmetric game [12].

If (x^*, x^*) is a symmetric Nash equilibrium in a symmetric bimatrix game (R, R^T) then—trivially—either $(x^*)^T Rx^* \leq 1/2$ or $(x^*)^T Rx^* > 1/2$. In the former case, by Proposition 1 the linear program $\mathrm{PE}(1/2, 1/2)$ has a solution, and by Proposition 2 it is a $(1/2)$-well-supported Nash equilibrium. In the latter case, by Corollary 1 there are multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies of size $\kappa(\delta)$, such that $\mathrm{WS}(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ has a solution (x, y) and it well supports $(1/2 - \delta, 1/2 - \delta)$. It then follows by Proposition 3 that (x, y) is a $(1/2 + \delta)$-well-supported Nash equilibrium.

Algorithm 1. *Let (R, R^T) be a symmetric game and let $\delta > 0$.*

1. *If* $\mathrm{PE}(1/2, 1/2)$ *has a solution x then return (x, x).*
2. *Otherwise, that is if $PE(1/2, 1/2)$ does not have a solution:*

(a) *Using exhaustive search, find multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies, both of size $\kappa(\delta)$, such that $WS(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ has a solution.*
(b) *Return a solution (x, y) of $WS(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$.* □

In order to find appropriate $\mathcal{I}$ and $\mathcal{J}$ in step 2(a), an exhaustive enumeration of all pairs of multisets $\mathcal{I}$ and $\mathcal{J}$ of size $\kappa(\delta)$ is done, and for each such pair the system of linear constraints $WS(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ is solved. Note that the number of $\kappa(\delta)$-element multisets from an n-element set is

$$\binom{n + \kappa(\delta) - 1}{\kappa(\delta)} = n^{O(\kappa(\delta))} = n^{O(\ln(1/\delta)/\delta^2)}.$$

Therefore, step 2. of the algorithm requires solving $n^{O(\ln(1/\delta)/\delta^2)}$ linear programs and hence the algorithm runs in time $N^{O(\ln(1/\delta)/\delta^2)}$.

Theorem 3. *For every $\delta > 0$, Algorithm 1 runs in time $N^{O(\ln(1/\delta)/\delta^2)}$ and it returns a strategy profile that is a $(1/2 + \delta)$-well-supported Nash equilibrium.*

4 Proof of Theorem 2

We use the probabilistic method: random $\kappa(\delta)$-uniform strategies are drawn by sampling $\kappa(\delta)$ pure strategies (with replacement) from the distributions x^* and y^*, respectively, and Hoeffding's inequality is used to show that the probability of thus selecting a strategy profile that well supports $\big(v^* - \delta, u^* - \delta\big)$ is positive if $\kappa(\delta) \geq 2\ln(1/\delta)/\delta^2$, where $v^* = (x^*)^T R y^*$ and $u^* = (x^*)^T C y^*$.

Consider $2\kappa(\delta)$ mutually independent random variables I_t and J_t, $1 \leq t \leq \kappa(\delta)$, with values in $\{\, 1, 2, \ldots, n \,\}$, the former with the same distribution as strategy x^* and the latter with the same distribution as strategy y^*, that is we have $\mathbb{P}\{I_t = i\} = x_i^*$ and $\mathbb{P}\{J_t = j\} = y_j^*$ for $i, j = 1, 2, \ldots, n$. Define the random distributions $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$, with values in $[0, 1]^n$, by setting:

$$X_i = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} [I_t = i] \qquad \text{and} \qquad Y_j = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} [J_t = j].$$

Note that every realization of Y is a $\kappa(\delta)$-uniform strategy that uses the pure strategy j, $1 \leq j \leq n$, with probability $K_j/\kappa(\delta)$, where $K_j = \sum_{t=1}^{\kappa(\delta)} [J_t = j]$ is the number of indices t, $1 \leq t \leq \kappa(\delta)$, for which $J_t = j$. A similar characterization holds for every realization of X. Observe also that $\mathrm{supp}(X) \subseteq \mathrm{supp}(x^*)$ and $\mathrm{supp}(Y) \subseteq \mathrm{supp}(y^*)$ because for all i and j, $1 \leq i, j \leq n$, the random variables X_i and Y_j are identically equal to 0 unless $x_i^* > 0$ and $y_j^* > 0$, respectively.

Since we want (a realization of) the random strategies X and Y to well support a certain pair of values, we now characterize $R_{i\bullet} Y$, for all $i \in \mathrm{supp}(x^*)$; the whole reasoning presented below for $R_{i\bullet} Y$ can be carried out analogously for $X^T C_{\bullet j}$, for all $j = 1, 2, \ldots, n$, and hence it is omitted.

First, observe that for all $i = 1, 2, \ldots, n$, we have:

$$R_{i\bullet}Y \;=\; \sum_{j=1}^{n} R_{ij}Y_j \;=\; \frac{1}{\kappa(\delta)} \cdot \sum_{j=1}^{n} R_{ij} \cdot \sum_{t=1}^{\kappa(\delta)} [J_t = j] \;=\; \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} R_{iJ_t}.$$

Therefore, the random variable $R_{i\bullet}Y$ is equal to the arithmetic average

$$\overline{Z_i} = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} Z_{it}$$

of the independent random variables $Z_{it} = R_{iJ_t}$, $1 \le t \le \kappa(\delta)$.

For every $i \in \text{supp}(x^*)$, we will apply Hoeffding's inequality to the corresponding random variable $\overline{Z_i}$. Hoeffding's inequality gives an exponential upper bound for the probability of large deviations of the arithmetic average of independent and bounded random variables from their expectation.

Lemma 1 (Hoeffding's inequality). *Let $Z_1, Z_2, \ldots, Z_k$ be independent random variables with $0 \le Z_t \le 1$ for every t, let $\overline{Z} = (1/k) \cdot \sum_{t=1}^{k} Z_t$, and let $\mathbb{E}\{\overline{Z}\}$ be its expectation. Then for all $\delta > 0$, we have $\mathbb{P}\{\overline{Z} - \mathbb{E}\{\overline{Z}\} \le -\delta\} \le e^{-2\delta^2 k}$.*

Before we apply Hoeffding's inequality to the random variables $\overline{Z_i}$ defined above, observe that for every $t = 1, 2, \ldots, \kappa(\delta)$, we have:

$$\mathbb{E}\{Z_{it}\} \;=\; \mathbb{E}\{R_{iJ_t}\} \;=\; \sum_{j=1}^{n} R_{ij} \cdot \mathbb{P}\{J_t = j\} \;=\; R_{i\bullet}y^*.$$

Note, however, that if $i \in \text{supp}(x^*)$ then $\mathbb{E}\{Z_{it}\} = R_{i\bullet}y^* = v^*$, because (x^*, y^*) is a Nash equilibrium, and hence every $i \in \text{supp}(x^*)$ is a best response to y^*. It follows that $\mathbb{E}\{\overline{Z_i}\} = (1/\kappa(\delta)) \cdot \sum_{t=1}^{\kappa(\delta)} \mathbb{E}\{Z_{it}\} = v^*$.

Applying Hoeffding's inequality, for every $i \in \text{supp}(x^*)$, we get:

$$\mathbb{P}\{R_{i\bullet}Y < v^* - \delta\} \;=\; \mathbb{P}\{\overline{Z_i} - \mathbb{E}\{\overline{Z_i}\} < -\delta\} \;\le\; e^{-2\delta^2\kappa(\delta)}. \tag{1}$$

It follows that if $I \subseteq \text{supp}(x^*)$ and $|I| \le \kappa(\delta)$, then:

$$\mathbb{P}\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I\} \;\le\; \\ \le \sum_{i \in I} \mathbb{P}\{R_{i\bullet}Y < v^* - \delta\} \;\le\; \kappa(\delta) \cdot e^{-2\delta^2\kappa(\delta)} \;=\; 2\delta^2 \ln(1/\delta) \;<\; \frac{1}{2}, \tag{2}$$

for all $\delta > 0$. The first inequality holds by the union bound, and the second follows from (1) and because $|I| \le \kappa(\delta)$. The last inequality can be verified by observing that the function $f(x) = 2x^2 \ln(1/x)$, for $x > 0$, achieves its maximum at $x = 1/\sqrt{e}$ and $f(1/\sqrt{e}) = 1/e < 1/2$.

In a similar way we can prove that if $J \subseteq \text{supp}(y^*)$ and $|J| \le \kappa(\delta)$, then:

$$\mathbb{P}\{X^T C_{\bullet j} < (x^*)^T C y^* - \delta \text{ for some } j \in J\} \;<\; \frac{1}{2}, \tag{3}$$

for all $\delta > 0$.

We are now ready to argue that

$$\mathbb{P}\big\{R_{i\bullet}Y \geq v^* - \delta \text{ for all } i \in \text{supp}(X), \\ \text{and } X^T C_{\bullet j} \geq u^* - \delta \text{ for all } j \in \text{supp}(Y)\big\} > 0,$$

and hence there must be realizations $x, y \in [0,1]^n$ of the random variables $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$, such that (x, y) well supports $\big(v^* - \delta, u^* - \delta\big)$. Indeed, we have:

$$\begin{aligned}
&\mathbb{P}\big\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in \text{supp}(X), \\
&\qquad \text{or } X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in \text{supp}(Y)\big\} \\
&\leq \sum_{I \subseteq \text{supp}(x^*)} \mathbb{P}\big\{I = \text{supp}(X) \text{ and } R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I\big\} \\
&+ \sum_{J \subseteq \text{supp}(y^*)} \mathbb{P}\big\{J = \text{supp}(Y) \text{ and } X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in J\big\} \\
&= \sum_{\substack{I \subseteq \text{supp}(x^*) \\ |I| \leq \kappa(\delta)}} \mathbb{P}\big\{I = \text{supp}(X)\big\} \cdot \mathbb{P}\big\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I \,\big|\, I = \text{supp}(X)\big\} \\
&+ \sum_{\substack{J \subseteq \text{supp}(y^*) \\ |J| \leq \kappa(\delta)}} \mathbb{P}\big\{J = \text{supp}(Y)\big\} \cdot \mathbb{P}\big\{X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in J \,\big|\, J = \text{supp}(Y)\big\} \\
&< \sum_{I \subseteq \text{supp}(x^*)} \mathbb{P}\big\{I = \text{supp}(X)\big\} \cdot \frac{1}{2} + \sum_{J \subseteq \text{supp}(y^*)} \mathbb{P}\big\{J = \text{supp}(Y)\big\} \cdot \frac{1}{2} \;=\; 1,
\end{aligned}$$

where the first inequality follows from the union bound, and from $\text{supp}(X) \subseteq \text{supp}(x^*)$ and $\text{supp}(Y) \subseteq \text{supp}(y^*)$; the equality holds because $|\text{supp}(X)| \leq \kappa(\delta)$ and $|\text{supp}(Y)| \leq \kappa(\delta)$ by the definitions of X and Y; and the latter (strict) inequality follows from (2) and (3).

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Mechanisms for Hiring a Matroid Base without Money

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Abstract. We consider the problem of designing mechanisms for hiring a matroid base without money. In our model, the elements of a given matroid correspond to agents who might misreport their actual costs that are incurred if they are hired. The goal is to hire a matroid base of minimum total cost. There are no monetary transfers involved. We assume that the reports are binding in the sense that an agent's cost is equal to the maximum of his declared and actual costs. Our model encompasses a variety of problems as special cases, such as computing a minimum cost spanning tree or finding minimum cost allocation of jobs to machines.

We derive a polynomial-time randomized mechanism that is truthful in expectation and achieves an approximation ratio of $(m - r)/2 + 1$, where m and r refer to the number of elements and the rank of the matroid, respectively. We also prove that this is best possible by showing that no mechanism that is truthful in expectation can achieve a better approximation ratio in general. If the declared costs of the agents are bounded by the cost of a socially optimal solution, we are able derive an improved approximation ratio of $3\sqrt{m}$. For example, this condition is satisfied if the costs constitute a metric in the graphical matroid.

Our mechanism iteratively extends a partial solution by adding feasible elements at random. As it turns out, this algorithm achieves the best possible approximation ratio if it is equipped with a distribution that is optimal for the allocation of a single task to multiple machines. This seems surprising given that matroids allow for much richer combinatorial structures than the assignment of a single job.

1 Introduction

The task of designing algorithms that are resilient to manipulations of strategic agents in large, distributed systems (such as the Internet) has become a major challenge in recent years. For example, in online marketplaces (such as eBay or eBid) auction formats are desired that incentivize truth revelation of the bidders' valuations for the items on auction. In online workplaces (like Elance, oDesk or Guru) that match freelance experts to clients mechanisms are sought to prevent unjustified declarations of costs.

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The classical approach to incite truth telling in strategic environments is to use *mechanism design* (see, e.g., [15,14]). Here the basic idea is to issue payments to the agents in order to convince them to behave truthfully. Typically, these payments are used to compensate for the advantage that an agent could obtain by lying. Mechanism design is a powerful approach that gave rise to several enlightening results in the past and still is a very active research area with many intriguing open questions.

However, there are many applications in which monetary transfers (as used in the traditional setting) are infeasible. As a result, researchers have more recently started to look into what is known as *mechanism design without money.* Here the basic question one asks is: Can one incite agents to behave truthfully without the use of monetary transfers? Unfortunately, classical results in voting theory show that the answer to this question is "No!" in general. In particular, the well-known Gibbard-Satterthwaite theorem [9,18] states that for unrestricted domains and at least three outcomes the only mechanism enforcing truthfulness without monetary transfers is *dictatorial*, i.e., the outcome is determined by a single agent. In particular, this also rules out the possibility of approximating any interesting objective in such a setting.

In light of this strong intractability result, there has recently been a large interest in studying more restrictive settings of mechanism design without money. A partial list of proposals that have been addressed in the literature includes the limitation of the agents' preferences [17], changing the social choice model using imposition [16] or binding reports [11,3,2].

Our Model. In this paper, we study the problem of selecting a minimum cost matroid base in a strategic environment. Here the elements of the matroid correspond to agents who might misreport their actual costs. The intuition behind our model is that a certain task can be accomplished only through the collaboration of certain groups of agents. These groups correspond to the bases of the given matroid. Each agent i declares a cost c_i for performing the task, which is not necessarily equal to his actual cost. Based on the declared costs, the mechanism designer wants to "hire" a matroid base at the cheapest possible cost. There are no monetary transfers between the mechanism designer and the agents.

As an example, suppose that the mechanism designer wants to hire a spanning tree in a given network in order to establish connectivity between all nodes at the lowest possible cost. Here the agents are the edges and each edge declares a cost that it incurs for establishing connectivity between its endpoints. This problem falls into our matroid model simply by using the graphic matroid whose bases correspond to the spanning trees of the given graph.

Another example is the problem of scheduling n jobs on m unrelated machines (possibly with restrictions). Every machine i declares for each job $j \in [n]$ it can execute a processing time p_{ij}. The goal is to determine an assignment of jobs to machines such that the total processing time is minimized. It is not hard to see that this problem is a special case of finding a minimum cost basis in a partitioning matroid and is therefore captured by our matroid model.

Binding Reports. The latter problem was studied by Koutsoupias [11] under the assumption that the reports are *binding*. This notion was first considered by Christodoulou e. al. [3] and Angel et al. [2]. Basically, this means that one can detect whether an agent overstates his actual cost. The motivation for this assumption is that in many situations costs are "observable" and thus declaring a cost that is larger than the actual one can be punished. On the other hand, if an agent understates his cost then his actual cost remains unaffected through this false declaration. For example, in the scheduling problem mentioned above binding reports means that the mechanism can enforce that the machine is busy for at least the declared processing time.

Koutsoupias [11] settles the problem of assigning one job to m machines completely. He designs a randomized algorithm that is truthful in expectation and achieves an approximation ratio of $(m+1)/2$ (which he shows is best possible). He also extends these results to the case of scheduling n jobs on m machines. The crucial insight in [11] that enables him to derive these results is a characterization of the distributions for the assignment of a single job that guarantee truthfulness in expectation. Given this characterization, he then determines a distribution that achieves the best possible approximation ratio.

Our Contributions. Here we continue this line of research. We consider the problem of designing mechanisms without money for the more general model of hiring a matroid base under binding reports. Our main contributions are as follows:

1. We give a randomized algorithm that is truthful in expectation and achieves an approximation ratio of $(m-r)/2+1$, where m and r refer to the number of elements and the rank of the underlying matroid, respectively.
2. We prove that this approximation ratio is best possible. More specifically, we show that no (randomized) mechanism that is truthful in expectation can achieve a better approximation ratio.
3. We then show that an improved approximation ratio of $3\sqrt{m}$ can be achieved if the declared costs of the agents are bounded by the cost of a socially optimal solution. For example, this condition is satisfied if the costs constitute a metric in the graphic matroid.

Our Techniques. Our results are based on a natural extension of the greedy algorithm for the computation of a minimum cost basis of a matroid. The algorithm iteratively extends a partial solution by adding elements that maintain feasibility. However, because of truthfulness we cannot enforce that a minimum cost element is chosen in each iteration (as in the standard greedy algorithm). Instead, we have to ensure that in each iteration each feasible addition of an element is chosen with some positive probability such that the resulting probability of picking an element meets certain monotonicity properties.

Although we have some freedom to choose these distributions, their choice impacts the resulting approximation ratio of the mechanism. Intuitively, we would like to tailor these distributions in such a way that the minimum cost element is chosen with some good probability, while the required monotonicity properties are still satisfied. Here the insights obtained by Koutsoupias [11] for assigning a single job to m machines turn out to be very useful.

Our findings show that an appropriate composition of the distribution that is proven to be optimal for the single task assignment in [11] also delivers the best possible results in the more general setting of hiring a matroid base. We find this somewhat surprising because matroids allow for combinatorially much richer structures than the assignment of a single job. In fact, the problem of optimally assigning a single job to m machines is equivalent to computing a minimum cost basis of a 1-uniform matroid (which is one of the most trivial matroids). For this special case our mechanism coincides with the one of Koutsoupias.

In order to bound the approximation ratio of our mechanism we crucially exploit properties of the matroid. However, there are many approximation algorithms that follow a similar design paradigm of iteratively extending a partial solution in a greedy manner (e.g., the greedy algorithm for the set cover problem). We conjecture that our findings might be extended to a broader context of greedy-like approximation algorithms which gives rise to some intriguing questions for follow-up research.

Additional Related Work. The design of mechanism that do not use monetary transfers has recently received considerable attention in the literature on economics and computation. Procaccia and Tennenholtz [17] initiated the study of approximate mechanism design without payments for combinatorial problems by studying facility location problems. Their studies triggered several follow-up articles on this topic (see, e.g., [1,13,12,6,7]). Dughmi and Gosh [4] derived approximate mechanisms without money for several variants of the assignment problems. Guo and Conitzer [10] studied the problem of selling items without payments for the case of two agents.

The idea of binding reports is also related to *mechanisms with verification*, whose study was first proposed by Nisan and Ronen [15]. However, the notion of verification is much stronger than the notion of binding reports that we consider here. In particular, mechanisms with verification may defer the issuing of payments to the agents until they learned the actual outcome. As a result, these mechanisms can punish misreports a posteriori by imposing very high penalties for lying.

Mechanism with binding reports are related to the notion of *imposition* proposed by Nissim et al. [16]. In the context of the facility location problem, agents might be forced to connect to the facility that is closest to their declared position instead of the one that is closest to their actual position. This approach was further pursued by Fotakis and Tzamos [8].

2 Preliminaries

In this section, we give a formal definition of the model that we consider in this paper and introduce some basic concepts.

2.1 Matroids

We first formally introduce the notion of a *matroid*:

Definition 1. *A* matroid $\mathcal{M} = (E, F)$ *is defined by a finite set E of elements and a set $F \subseteq 2^E$ of subsets of E satisfying*

1. *$\emptyset \in F$ (non-emptiness),*
2. *if $S \in F$ and $S' \subseteq S$ then $S' \in F$ (downward closure),*
3. *if $S, T \in F$ and $|S| > |T|$ then there exists some $i \in S \setminus T$ such that $T + i \in F$ (exchange property).*[1]

The sets in F are called independent sets*. An inclusion-wise maximal independent set $B \in F$ is a* basis *of $\mathcal{M}$.*[2] *The common size of all bases of $\mathcal{M}$ is called the* rank *of $\mathcal{M}$ and will be denoted by $r(\mathcal{M})$.*[3]

Throughout this paper, we assume that the matroid $\mathcal{M} = (E, F)$ is implicitly represented by an *independent set oracle*: given a set $S \subseteq E$, the oracle specifies whether S is an independent set or not. Unless specified otherwise, we identify the elements in E with the first m natural numbers, i.e., $E = [m]$. We assume that every element $i \in E$ constitutes and independent set, i.e., $i \in F$.[4] Note that this assumption is without loss of generality because we can remove all elements from E that do not occur in any independent set.

Example 1. A typical example of a matroid is the *graphic matroid*. Given a graph $G = (V, E)$, we let the edges E of G be the elements of the matroid and each subset $S \subseteq E$ of edges that does not contain a cycle in G constitutes an independent set in F. It is easy to verify that Properties 1–3 of Definition 1 are satisfied. The bases of $\mathcal{M} = (E, F)$ correspond to the spanning trees of G. The rank of $\mathcal{M}$ is $r(\mathcal{M}) = n - 1$, where n is the number of vertices in G.

2.2 Hiring a Matroid Base

Let $\mathcal{M} = (E, F)$ be a matroid. In our model, we associate an agent with each element $i \in E$ of the matroid. Each agent $i \in E$ has a non-negative cost $\bar{c}_i \in \mathbb{R}_+$. Intuitively, by choosing agent $i \in E$ a cost of $\bar{c}_i$ is incurred. The cost $\bar{c}_i$ is "private" in the sense that it is unknown to us. Our goal is to select (or hire) a base of the matroid of minimum total cost. The intuition behind our model is that the bases of the underlying matroid represent groups of agents that together can perform a certain task.

[1] For ease of notation, for a set $T \subseteq E$ and an element $i \in E$ we also use $T + i$ and $T - i$ as a short for $T \cup \{i\}$ and $T \setminus \{i\}$, respectively.

[2] Subsequently, by "maximal" we mean "inclusion-wise maximal", i.e., B is maximal if for every $i \in E \setminus B$, $B + i$ is not an independent set.

[3] Using the properties of Definition 1, it is not hard to show that all bases of a matroid $\mathcal{M}$ have the same size.

[4] We slightly abuse notation here and write $i \in F$ instead of $\{i\} \in F$ for notational convenience.

Example 2. In order to establish connectivity among all nodes in a given graph $G = (V, E)$ one may want to determine a minimum cost spanning tree of G. Here each edge $i \in E$ corresponds to an agent and selecting an edge incurs a cost of $\bar{c}_i$. Our goal then is to select a minimum cost basis of the graphic matroid.

2.3 Binding Reports

We assume that agents might misreport their costs, i.e., each agent $i \in E$ declares a cost c_i, which is possibly different from his actual cost $\bar{c}_i$. Based on the matroid $\mathcal{M}$ and the declared costs $\mathbf{c} = (c_1, \ldots, c_m)$, the mechanism selects a basis of the underlying matroid. We consider mechanisms *without money*, i.e., the mechanism does not receive/issue any payments from/to the agents.

In order to achieve truthfulness it will turn out to be crucial to allow for random selections of agents, i.e., we consider *randomized* mechanisms. Subsequently, we use $p_i(\mathbf{c})$ to refer to the probability that our (random) mechanism picks element $i \in E$, given the reported costs $\mathbf{c}$.

We assume that the reports are *binding* as proposed by Koutsoupias [11]. More precisely, if agent i's reported cost is c_i then his actual cost is $\max\{\bar{c}_i, c_i\}$. That is, if agent i overstates his actual cost by reporting $c_i > \bar{c}_i$ and agent i is selected then his actual cost becomes c_i. On the other hand, if agent i understates his actual cost $\bar{c}_i$ and is selected then his actual cost remains $\bar{c}_i$ because this is the cost incurred by i. Formally, we assume that each agent $i \in E$ strives to minimize his expected cost

$$C_i(\mathbf{c}) = \max\{\bar{c}_i, c_i\} p_i(\mathbf{c}).$$

Subsequently, we use $\bar{\mathbf{c}} = (\bar{c}_1, \ldots, \bar{c}_m) \in \mathbb{R}^m_+$ to refer to the vector of *actual* costs and $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{R}^m_+$ to refer the vector of *declared* costs.

2.4 Truthful Mechanisms

We are interested in designing mechanisms that are *truthful in expectation*, which we define next. To this aim, we first need to introduce some standard notation. Let $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{R}^m_+$ be a cost vector. Then we denote by $\mathbf{c}_{-i}$, $i \in [m]$, the $(m-1)$-dimensional vector with the ith coordinate removed, i.e.,

$$\mathbf{c}_{-i} = (c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m).$$

For a subset $T \subseteq [m]$, we will also use $\mathbf{c}_T$ to refer to the restriction of $\mathbf{c}$ to index set T, i.e., $\mathbf{c}_T = (c_{i_1}, c_{i_2}, \ldots, c_{i_{|T|}})$ with $T = \{i_1, \ldots, i_{|T|}\}$.

Definition 2 (Truthful mechanism). *A mechanism M is* truthful in expectation *if for every agent i and every vector $\boldsymbol{c}_{-i}$, the expected cost of i is minimized by declaring the actual cost truthfully, i.e., for every $i \in E$,*

$$C_i(\bar{c}_i, \boldsymbol{c}_{-i}) = \bar{c}_i p_i(\bar{c}_i, \boldsymbol{c}_{-i}) \leq \max\{\bar{c}_i, c_i\} p_i(\boldsymbol{c}) = C_i(\boldsymbol{c}).$$

There are stronger notions of truthfulness (e.g., truthfulness or universal truthfulness). However, it is easy to see that with these stronger notions of truthfulness no positive results are possible; see also [11].

2.5 Approximate Social Cost

The *social cost* function that we consider throughout this paper is the sum of the individual costs, i.e., $\mathsf{SC}(\mathbf{c}) = \sum_{i \in E} C_i(\mathbf{c})$. We use $\mathsf{OPT}(\mathbf{c})$ to refer to the cost of a socially optimal solution, i.e., the minimum cost of a base of $\mathcal{M} = (E, F)$ with respect to $\mathbf{c}$. Ideally, we would like to derive a truthful mechanism that computes a socially optimal outcome. However, this is impossible and we therefore relax the optimality condition and resort to approximate solutions.

Definition 3. *A mechanism M is α-*approximate *with $\alpha \geq 1$ if for every vector $\boldsymbol{c}$ of declared costs, the expected social cost satisfies*

$$\mathsf{SC}(\boldsymbol{c}) = \sum_{i \in E} C_i(\boldsymbol{c}) = \sum_{i \in E} \max\{\bar{c}_i, c_i\} p_i(\boldsymbol{c}) \leq \alpha \mathsf{OPT}(\boldsymbol{c}).$$

2.6 Koutsoupias' Characterization

Koutsoupias [11] considers the problem of scheduling one job on m available machines. The actual cost incurred by machine i to schedule job j is $\bar{p}_i$ and each machine wants to minimize his cost. The overall objective is to determine an assignment of minimum total cost.[5] Note that this corresponds to computing a minimum cost basis in the matroid that consists only of singletons.

Koutsoupias characterizes the set of truthful mechanisms for this problem.

Proposition 1 ([11]). *Let $p_i(\boldsymbol{c})$ be the probability that element i is chosen by mechanism M given the vector of declared costs $\mathbf{c}$. Then M is truthful in expectation if and only if for every $i \in E$:*

1. *$p_i(c_i, \boldsymbol{c}_{-i})$ is non-increasing in c_i,*
2. *$c_i p_i(c_i, \boldsymbol{c}_{-i})$ is non-decreasing in c_i.*

Based on the above characterization result, Koutsoupias then derives a distribution that satisfies the above properties and whose expected social cost is at most $(m+1)/2$ times the optimal one. He also proves that this is best possible in the sense that no other truthful in expectation mechanism (without payments) can achieve a better approximation ratio.

3 Greedy Mechanism and Truthfulness Conditions

In this section, we provide a general framework for constructing truthful mechanisms. Our framework is based on the greedy approach which iteratively extends a partial solution (i.e., independent set) by adding a least cost element. We parameterize our mechanism with a collection of distributions: for every $T \subseteq E$ we are given a distribution $d^T = \{d_i^T(\mathbf{c}_T) \mid i \in T\}$ over the elements i in T.[6]

[5] We note that in [11] also the objective of minimizing the makespan is considered.

[6] The assumption that all these distributions are given is a conceptual one. Subsequently, it will become clear that we can generate the relevant distributions considered by the algorithm efficiently.

Definition 4 (Greedy Mechanism). *Given a matroid $\mathcal{M} = (E, F)$ with a cost vector $\mathbf{c}$ and a collection of distributions $(d^T)_{T \subseteq E}$, the greedy algorithm is as follows:*

1. *Let $S \leftarrow \emptyset$*
2. *While S is not a base*
   (a) *Let $T = \{i \in E \setminus S \mid S + i \in F\}$*
   (b) *Draw $i \in T$ with probablity $d_i^T(\boldsymbol{c}_T)$*
   (c) *Set $S \leftarrow S + i$*
3. *Output S*

Note that the set T in Step 2(a) contains all elements that can be added to the independent set S without rendering it infeasible. The main difference of our algorithm to the standard greedy algorithm for matroids is that we do not require that the element $i \in T$ added to S in Step 2(c) is of minimum cost. Indeed, such a mechanism would not be truthful because it failed to satisfy Condition (b) of Proposition 1. Instead, here we choose an element i from T with probability $d_i^T(\mathbf{c}_T)$. In particular, our algorithm coincides with the standard greedy algorithm if $d_i^T(\mathbf{c}_T) > 0$ only for the minimum cost elements in T.

We next establish some sufficient conditions for the distributions used by our greedy algorithm that ensure truthfulness.

Theorem 1. *The greedy mechanism is truthful in expectation if for every $T \subseteq E$ and every $i \in T$ it holds:*

1. *$d_j^T(c_i, \boldsymbol{c}_{T-i})$ is non-decreasing in c_i for every $j \in T - i$,*
2. *$d_i^T(c_i, \boldsymbol{c}_{T-i})c_i$ is non-decreasing in c_i.*

Proof. Fix a collection of distributions $(d^T)_{T \subseteq E}$ that satisfies Properties (1) and (2). Let $p_i(\mathbf{c})$ be the probability of picking element $i \in E$ after the execution of the mechanism. We need to show that Properties (1) and (2) of Proposition 1 are satisfied, i.e.,

1. $p_i(c_i, \mathbf{c}_{-i})$ is non-increasing in c_i,
2. $p_i(c_i, \mathbf{c}_{-i})c_i$ is non-decreasing in c_i.

We prove these by induction on the number m of elements in E. If $m = 1$ then there is only one element to be picked and the properties clearly hold.

Suppose that the claim holds true for all element sets of size less than m. We show that it continues to hold for sets of size m. We use $\mathcal{M}^{(j)}$ to refer to the matroid that we obtain from $\mathcal{M}$ by contracting element j, i.e., the sub-matroid that contains only the sets that include j.

Let $p_i^{(j)}(\mathbf{c}_{-j})$ be the probability of picking element i in the matroid $\mathcal{M}^{(j)}$. Note that $p_i^{(j)}(\mathbf{c}_{-j})$ is precisely the probability of picking element i conditional on the event that player j has been picked in the first round.

First property: Using Bayes rule, we obtain

$$p_i(\mathbf{c}) = d_i^E(\mathbf{c}) + \sum_{j \in E-i} d_j^E(\mathbf{c}) p_i^{(j)}(\mathbf{c}_{-j}) = 1 - \sum_{j \in E-i} d_j^E(\mathbf{c}) + \sum_{j \in E-i} d_j^E(\mathbf{c}) p_i^{(j)}(\mathbf{c}_{-j})$$

$$= 1 + \sum_{j \in E-i} d_j^E(\mathbf{c})[p_i^{(j)}(\mathbf{c}_{-j}) - 1].$$

By assumption, $d_j^E(\mathbf{c})$ is non-decreasing in c_i for every $j \neq i$. Also, by our induction hypothesis, $p_i^{(j)}(\mathbf{c}_{-j})$ is non-increasing in c_i. Thus, the product $d_j^E(\mathbf{c})[p_i^{(j)}(\mathbf{c})-1]$ is non-increasing in c_i. We conclude that $p_i(\mathbf{c})$ is non-increasing in c_i.

Second property: Using Bayes rule, we obtain

$$p_i(\mathbf{c})c_i = d_i^E(\mathbf{c})c_i + \sum_{j \in E-i} d_j^E(\mathbf{c})[p_i^{(j)}(\mathbf{c}_{-j})c_i]$$

By assumption and our induction hypothesis, $d_j^E(\mathbf{c})$ and $p_i^{(j)}(\mathbf{c}_{-j})c_i$ are non-decreasing in c_i for every $j \neq i$. Thus, their product is non-decreasing in c_i. By assumption, also $d_i^E(\mathbf{c})c_i$ is non-decreasing in c_i. We conclude that $p_i(\mathbf{c})c_i$ is non-decreasing in c_i.

4 Optimal Distributions and Approximation Ratio

In this section we identify a distribution that satisfies Properties (1) and (2) of Theorem 1 and yields a truthful in expecation mechanism with approximation ratio $(m-r)/2+1$, where $r = r(\mathcal{M})$ is the rank of the underlying matroid $\mathcal{M}$.

4.1 Optimal Distributions

A natural choice for a collection $(d^T)_{T \subseteq E}$ of distributions to be used by the greedy algorithm is to choose each element i from a given set T with probability that is inversely proportional to its cost c_i. This distribution is also independently considered in [5]. More precisely, for every $T \subseteq E$ and every $i \in T$, we define

$$d_i^T(\mathbf{c}_T) = \frac{c_i^{-1}}{\sum_{k \in T} c_k^{-1}}. \tag{1}$$

The distribution d^T is also called the *proportional distribution*. It is not hard to show that these distributions satisfy Properties (1) and (2) of Theorem 1. However, the problem is that the greedy mechanism equipped with these distributions results in an approximation ratio which is arbitrarily close to m.

The following distribution was introduced by Koutsoupias [11] for scheduling a single job on m machines. (A similar probability distribution is considered and analyzed in the Facility Location setting in [7].)

Definition 5 (Optimal Distribution). *Let $T \subseteq E$ be a subset of elements and assume without loss of generality that $T = \{1, \ldots, |T|\}$ such that $c_1 \leq c_2 \leq \cdots \leq c_{|T|}$. Define probabilities*[7]

$$d_1^T(\boldsymbol{c}_T) = \frac{1}{c_1} \int_0^{c_1} \prod_{k \neq 1} \Big(1 - \frac{x}{c_k}\Big) dx$$

[7] This distribution corresponds to the following experiment: We select uniformly at random and independently a number $x_i \in [0, c_i]$ for each element i. The distribution d_i^T of Definition 5 corresponds to the distribution of the minimum of these x_i's.

$$d_j^T(\boldsymbol{c}_T) = \frac{1}{c_1 c_j} \int_0^{c_1} \int_0^{y} \prod_{k \neq 1,j} \Big(1 - \frac{x}{c_k}\Big) dx dy \qquad \text{for } j \neq 2.$$

This generalized distribution yields a truthful in expectation greedy mechanism that achieves the best possible approximation ratio.

Theorem 2. *The greedy mechanism equipped with the distributions of Definition 5 is truthful.*

All omitted proofs from this and future sections can be found in the full version of the paper.

4.2 Approximation Ratio

Koutsoupias [11] used the distribution d^T given in Definition 5 to handle the case of allocating a single job to m machines. He showed that the resulting mechanism achieves an approximation ratio of $(m+1)/2$. Here we prove that our greedy mechanism, equipped with the distributions in Definition 5, has an approximation ratio of $(m-r)/2+1$, where r is the rank of the matroid.

Theorem 3. *The greedy mechanism with distributions d^T as defined in Definition 5 has approximation ratio $(m-r)/2+1$.*

5 Lower Bound

In this section, we provide a general lower bound on the approximation ratio of truthful in expectation mechanisms for hiring a matroid base that matches the upper bound of our greedy algorithm established in the previous section.

We show that for any given parameters m and r, we can always construct a matroid with m elements and rank r such that no mechanism that is truthful in expectation can achieve an approximation ratio better than $(m-r)/2+1$[8].

Using the previous lemma we show that for every choice of m and r our upper bound is tight in the sense that there exists a matroid instance where any truthful mechanism has approximation ratio $(m-r)/2+1$.

Theorem 4. *Given m and r, there exists a matroid $\mathcal{M} = (E, F)$ with $|E| = m$ and $r(\mathcal{M}) = r$ for which no mechanism that is truthful in expectation can achieve an approximation ratio better than $(m-r)/2+1$.*

Finally, we show a weaker result regarding only graphical matroids. Specifically, we show that there is a family of graphs where the worst case bound of $(m-r)/2+1$ occurs.

Theorem 5. *There is no mechanism that is truthful in expectation that achieves an approximation ratio better than $(m-r)/2+1$ for graphical matroids.*

[8] Note this result does not necessarily imply that every truthful mechanism will perform poorly given any matroid set system with these parameters.

6 Improved Approximation Ratio for Metrics

In this section, we show that we can derive an improved approximation ratio of $O(\sqrt{m})$ for our greedy algorithm if each agent's declared cost is at most the cost of a socially optimal solution, i.e., for every agent $i \in E$, $c_i \leq \mathsf{OPT}(\mathbf{c})$. Said differently, this condition requires that the cost of an arbitrary base of the matroid is at least as large as $\max_{i \in E} c_i$. We call a vector $\mathbf{c} = (c_1, \ldots, c_m)$ of declared costs *opt-bounded* if it satisfies this condition.

Note that in the case of a graphical matroid this property is trivially satisfied if the declared cost vectors $\mathbf{c}$ are restricted to constitute a metric. if the declared cost vectors $\mathbf{c}$ are restricted to constitute a metric then this condition is trivially satisfied. It is interesting to note that we obtain this result for the greedy algorithm using the proportional distributions.

Theorem 6. *If the declared cost vector is opt-bounded then the greedy mechanism using the proportional distributions as defined in* (1) *is truthful in expectation and achieves an approximation ratio of* $3\sqrt{m}$.

7 Future Work

There are a lot of open problems that arise from our work. We designed an algorithm that achieves an approximation ratio based on the size of the matroid and its rank. In Section 5 we proved a lower bound that was dependent on the substitutability of elements within the matroid's bases. It could be possible to provide a more refined upper bound using this parameter.

Also there are many questions still open in the case of graphical matroids when the costs constitute a metric. We analyzed only the proportional method which generally performs worse than the distribution in Definition 5. We also have no matching lower bounds. Additionally, our iterative algorithm and generally our framework didn't depend on the matroid property of the set system to satisfy truthfulness. Thus, it will be interesting to analyze its performance in more general settings especially where the classic greedy has good approximation guarantees. Finally, we only considered social costs and not other social objectives like a minmax solution concept.

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Prediction and Welfare in Ad Auctions

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Abstract. We study how standard auction objectives in sponsored search markets are affected by refinement in the prediction of ad relevance (click-through rates). As the prediction algorithm takes more features into account, its predictions become more refined; a natural question is whether this is desirable from the perspective of auction objectives. Our focus is on mechanisms that optimize for a convex combination of efficiency and revenue, and our starting point is the observation that the objective of such a mechanism can only improve with refined prediction, making refinement in the best interest of the search engine. We demonstrate that the impact of refinement on market efficiency is not always positive; nevertheless we are able to identify natural – and to some extent necessary – conditions under which refinement is guaranteed to also improve efficiency. Our main technical contribution is in explaining how refinement changes the ranking of advertisers by value (efficiency-ranking), moving it either towards or away from their ranking by *virtual* value (revenue-ranking). These results are closely related to the literature on signaling in auctions.

1 Introduction

Sponsored search is a multi-billion dollar market; it enables contextual advertising, and generates revenue that supports innovation in search algorithms. Sponsored search markets are also technically interesting and have been investigated theoretically from several perspectives [12], including auction theory [1,4], game theory [21,3], and bipartite matching theory [15].

Sponsored search markets exhibit an interesting interplay between auctions and machine learning. Value is realized by the combination of two processes. First, the search engine displays *relevant* ads to the user, i.e., ones that maximize the odds of the user clicking on an ad. Second, users conduct a transaction with

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some probability and of some value on the advertiser's website, resulting in some (expected) value per click to the advertiser. To facilitate the first process, the search engine uses a combination of machine learning and historical data to estimate the relevance of an ad to the user [9]. The second process is not directly observable by the search engine, and so it uses an auction to elicit the value per click from the advertisers [1,4]. It then combines this with ad relevance to determine which ads to show to the user.

The explosion of data available to search engines makes it possible to improve relevance prediction by seemingly endless *refinements*, taking into account more and more features of the ad and the user. For example, consider a search query 'pizza' emanating from an unspecified location in the Bay Area. By adding a feature that pinpoints the user's location within this region, the relevance of advertisements showcasing pizza merchants based in San Francisco relative to those in nearby San Jose becomes clear, and this helps in deciding between these ads.

Refinement is often perceived as a positive, win-win opportunity making everyone better off – the users view more relevant ads and engage more with them, increasing overall value.[1] However, to our knowledge this has not been rigorously studied. *The focus of this paper is to explore how standard objectives of truthful auctions, specifically welfare, behave with refinement of relevance prediction.* We apply theory tools in order to understand the high-stake effects of refinement decisions carried out by sponsored search practitioners. We view this as a first step in better understanding the interaction between machine learning and market design objectives. We also discuss the connection to the signaling literature.

As out first contribution, we formalize the conventional wisdom that refinement aids optimization. While it holds generally that refinement only improves the efficiency of the optimally-efficient mechanism, or the revenue of the revenue-optimal mechanism [7], we build upon the latter to establish this result for all truthful mechanisms that optimize some fixed convex combination of revenue and efficiency (*trade-off optimal* mechanisms). We discuss such mechanisms and justify why the search engine would be interested in a mechanism from this class in Section 3.[2] Thus, performing refinements always benefits the search engine.

What about the impact of refinement on social welfare? Our main contribution (Section 4) is to study conditions under which refinement is simultaneously favorable for the auctioneer and for market efficiency. Indeed, this is not always

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[1] Obviously, refinement should not be at the cost of using features that violate user privacy; in this work we leave aside issues of privacy to focus on welfare considerations of refinement.

[2] The mechanisms used in practice, though not truthful, have equilibria that are allocation- and revenue-equivalent to the corresponding truthful mechanisms [3,4]. Thus, we expect the gist our results to apply to practically used mechanisms in equilibrium. This raises an interesting open problem: As we show, refinement changes advertiser ranking in non-trivial ways; how do the equilibrium bids of the advertisers change in response? Will their level of granularity mirror that of the refinement? In other words, how does personalization affect the analysis of [3,4]? The answer will depend on the informational assumptions of the model.

the case – the twin objectives of revenue and efficiency are not necessarily aligned in the context of refinement. We identify two assumptions under which refinement improves the efficiency of every trade-off optimal mechanism. The first assumption is fairly standard, and requires that the value-per-click distributions are i.i.d. and satisfy the monotone hazard rate condition. The second assumption is arguably more restrictive, requiring that refinements *distinguish* among advertisers, by causing the relevances of every pair of advertisers for every query to either grow further apart or switch order. We demonstrate the need for both assumptions by two examples (see Section 2.2 and full version [20]).

From a technical perspective, a main challenge is in understanding the mathematical effect of refinement. The revenue-optimal auction and the efficient auction both rank advertisers by a monotone function of their bids and then use this ranking to allocate them to available ad slots. The key difference is that the two mechanisms employ different ranking functions to the bids. Refinement harms efficiency precisely when it causes the revenue-optimal ranking to drift apart from the efficient ranking. The allocation ranking of every trade-off optimal mechanism is guaranteed to draw closer to the efficient ranking with refinement under the assumptions mentioned above.

2 Model

In this section we present our model, which encompasses the standard model for position auctions [12] while capturing the effect of prediction refinement.

A search engine sells m ad slots to $n \leq m$ advertisers.[3] The slots appear alongside search results for a search query q. Advertiser i has a private value $v_i \in \mathbb{R}_+$ for a click on his ad, and his value for an *impression* (appearance) of the ad is v_i multiplied by the corresponding *click-through rate*. This multiplicative relation is an important feature of the model. If the click-through rates are 1 we get a standard m-unit auction.

A standard assumption is that click-through rates are *separable*, i.e., can be separated into the advertiser's relevance to q and the slot position on the webpage. Formally, the click-through rate for advertiser i's ad in slot j is $p_{q,i}s_j$, where $1 \geq p_{q,i} > 0$ is the query-advertiser *relevance*, and $1 \geq s_1 \geq \cdots \geq s_m \geq 0$ are the *slot effects*.[4] The relevance is essentially the slot-independent click-through rate. We omit q from the notation where clear from context. We denote the value per impression in slot j by $v_{i,j} = p_i s_j v_i$, and the *realized value* without the slot effect by $r_i = p_i v_i$.

The advertisers' private values are assumed to be independently distributed according to a publicly-known distribution F with positive smooth density f (note that the realized values are not i.i.d. and so the setting is not symmetric).

[3] The assumption that $m \geq n$ is without loss of generality.

[4] The assumption that $p_{q,i} \neq 0$ is without loss, to simplify the exposition.

2.1 Prediction Schemes and Refinements

The machine learning system that predicts query-advertiser relevance has access to a set of *features*: keywords, geographic location, time, user demographics, search history, ad text, etc. As is standard we assume features are discretized [9]. The system partitions the set of query-advertiser pairs according to the features and produces a relevance estimate for each part. For example, a part can consist of pizzeria advertisers and queries for "pizza" by users located in the Bay Area. We refer to the output of the machine learning system as a *prediction scheme*:

Definition 1. *[Prediction scheme] A partition T of all query-advertiser pairs with a relevance prediction p_t for every part $t \in T$.*[5]

Overloading notation we also denote the prediction scheme itself by T. The prediction given a search query q is *according to* T if for every advertiser i, $p_{q,i} = p_t$ where t is the part in T containing the query-advertiser pair (q, i).

Refinements A prediction scheme can be *refined* by refining its partition, i.e., dividing coarse parts into finer *subparts*. This can be achieved by taking into account additional features, such as more precise user location. For example, a subpart may consist of pizzeria advertisers and queries for "pizza" by users located in a *specific city* within the Bay Area. We use the notational convention that $\bar{T}$ is a coarse partition and $\bar{t}$ a coarse part, whereas T is a refined partition and t is a subpart.

The relevence of a subpart can be very different from that of the original coarse part, and for this reason refinement can completely alter the outcome of the ad auction. However, the coarse and refined relevance predictions must maintain the following relation. Given a query-advertiser pair belonging to a coarse part $\bar{t}$, there is a certain distribution with which it falls within its different subparts. We require that in expectation over this distribution, the refined relevance prediction equals the coarse one. To summarize:

Definition 2. *[Refinement] A prediction scheme T is a* refinement *of $\bar{T}$ if its partition is a refinement of $\bar{T}$'s partition, and the relevance of every coarse part $\bar{t}$ equals the expected relevance over $\bar{t}$'s subparts: $p_{\bar{t}} = \mathbb{E}_{t \subseteq \bar{t}}[p_t]$.*

(If the subpart t and its coarse counterpart $\bar{t}$ are clear from context, we use p and $\bar{p}$ to denote their relevance predictions.)

Distinguishing Refinements A natural subclass of refinements is those which distinguish among advertisers, thus enabling a better matching between them and the search queries.

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[5] This definition matches that of Ghosh et al.'s deterministic clustering scheme [8]. In general a prediction scheme can be randomized, by including a distribution over relevance predictions for each part (cf. [5,17]). Our results hold for randomized prediction schemes as well.

Definition 3. *[Spread or flipped pairs] A pair of numbers a, b is* spread or flipped *with respect to another pair c, d if*

$$\frac{a}{b} \geq \frac{c}{d} \geq 1, \ \textit{or}\ 1 \geq \frac{c}{d} \geq \frac{a}{b}, \ \textit{or}\ \frac{a}{b} \geq 1 \geq \frac{c}{d}, \ \textit{or}\ \frac{c}{d} \geq 1 \geq \frac{a}{b}.$$

Definition 4. *[Distinguishing refinement] A prediction scheme T is a* distinguishing refinement *of $\bar{T}$ if T is a refinement of $\bar{T}$, and for every query q and pair of advertisers, their relevance p_1, p_2 according to T is spread or flipped with respect to their relevance $\bar{p}_1, \bar{p}_2$ according to $\bar{T}$.*

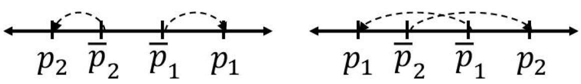

Fig. 1. An example of relevance pair p_1, p_2 spread (left) or flipped (right) with respect to $\bar{p}_1, \bar{p}_2$.

Remark 1. Our model is compatible with the standard assumption in theoretical study of sponsored search auctions, by which click-through rates are estimated accurately. It does not take into account that very fine prediction schemes may be inaccurate due to the emergence of over-thin submarkets with insufficient data. This simplifying assumption helps our goal of studying how finer prediction schemes affect auction objectives, by distilling this aspect of prediction refinement from various machine learning and other considerations.

2.2 Examples

Example 1. [Every refinement is distinguishing] If all advertisers competing for a query q belong to the same part in $\bar{T}$ and so appear equally relevant, then every refinement of $\bar{T}$ is distinguishing.

As a concrete example, consider the auction described in the introduction in which two pizzerias – the first located in San Francisco (SF) and the second in San Jose (SJ) – compete for a single advertisement slot next to search results for a query 'pizza' by a Bay Area user. Let $\bar{T}$ be a coarse prediction scheme in which both pizzerias appear equally relevant, i.e., both query-advertiser pairs belong to the same part $\bar{t} \in \bar{T}$. Let the corresponding relevance be $\bar{p}_{\bar{t}} = 3/4$. Now assume the search engine has access to a more precise location feature of the query q, indicating whether the user is in SF or in SJ, and each occurs with equal probability 1/2. When the prediction scheme is refined by including this feature, the relevances according to the refined scheme T behave antisymmetrically, and the realized values are:

	User from SF	User from SJ	City unknown
Advertiser 1 (from SF)	$p_{\mathrm{SF},1}v_1 = v_1$	$p_{\mathrm{SJ},1}v_1 = v_1/2$	$\bar{p}_1 v_1 = 3v_1/4$
Advertiser 2 (from SJ)	$p_{\mathrm{SF},2}v_2 = v_2/2$	$p_{\mathrm{SJ},2}v_2 = v_2$	$\bar{p}_2 v_2 = 3v_2/4$

In both cases it can be observed that the refined relevances are either spread or flipped with respect to $\bar{p}_1, \bar{p}_2$.

Example 2. [A non-distinguishing refinement] Consider again a single-slot position auction for a query 'pizza'. Assume now that advertiser 1 is a nationwide chain of pizzerias whose relevance does not depend on user location, while advertiser 2 is a local artisan pizzeria in SF. Consider a coarse prediction scheme $\bar{T}$ as above, and a refinement T where this time the refining feature indicates whether $q = \text{SF}$ (happens with probability $1/4 - \delta$ for $\delta = 15\epsilon/(8 - 20\epsilon)$ and some small ϵ) or $q = \neg\,\text{SF}$ (happens with probability $3/4 + \delta$). The realized values are:

	User from SF	User not from SF	City unknown
Advertiser 1 (chain)	$p_{\text{SF},1}v_1 = 4v_1/5$	$p_{\neg\,\text{SF},1}v_1 = 4v_1/5$	$\bar{p}_1 v_1 = 4v_1/5$
Advertiser 2 (from SF)	$p_{\text{SF},2}v_2 = 2v_2/5$	$p_{\neg\,\text{SF},2}v_2 = \epsilon v_2$	$\bar{p}_2 v_2 = v_2/10$

Refinement T is not distinguishing, since the relevances for $q = \text{SF}$ are neither spread nor flipped with respect to $\bar{p}_1, \bar{p}_2$.

3 Trade-off Optimal Mechanisms

3.1 Bayesian Mechanism Design

This section contains mechanism design preliminaries in the ad auction context; the expert reader may wish to skip to Section 3.2.

Given a private value $v_i \sim F$, the *inverse hazard rate* is $\lambda^F(v_i) = (1 - F(v_i))/f(v_i)$ and the *virtual value* is $\varphi^F(v_i) = v_i - \lambda^F(v_i)$. Similarly, given a realized value r_i, let G with density g be the distribution from which r_i is drawn; the *realized virtual value* is then $\varphi^G(r_i) = r_i - \lambda^G(r_i)$. Since $r_i = p_i v_i$,

$$G(r_i) = F(v_i),\ g(r_i) = \tfrac{1}{p_i} f(v_i),\ \varphi^G(r_i) = p_i \varphi^F(v_i). \tag{1}$$

From now on, we omit the distribution and value from the notation where clear from context, and follow the convention that $\varphi(v_i)$ or φ_i is the virtual value, and $\varphi(r_i)$ is the realized virtual value.

A distribution F is *MHR* (*montone hazard rate*)[6] if its inverse hazard rate function $\lambda(\cdot)$ is non-increasing, and *regular* if its virtual value function $\varphi(\cdot)$ is non-decreasing. In other words, F is MHR if for every pair of values $v_1, v_2 \sim F$ such that $v_1 > v_2$, their inverse hazard rates λ_1, λ_2 are flipped: $v_1/v_2 > 1 \geq \lambda_1/\lambda_2$. It immediately follows that their virtual values φ_1, φ_2 are spread with respect to v_1, v_2:

$$1 < v_1/v_2 < \varphi_1/\varphi_2. \tag{2}$$

We say that values are MHR (resp., regular) if they're drawn from an MHR distribution, and that a position auction is MHR if its advertisers' values are MHR. By Equation 1, MHR (resp., regular) values imply MHR realized values.

[6] The assumption of MHR values is standard in the mechanism design literature (e.g., [14]). Many commonly studied distributions are MHR, including the uniform, exponential and normal distributions, and those with log-concave densities [6].

Efficiency-Optimal and Revenue-Optimal Mechanisms The *VCG* auction maximizes efficiency while maintaining truthfulness and individual rationality (IR) [22,2,10]. In the context of position auctions, VCG allocates the slots to the m advertisers with highest realized values r_i, in high to low order [1]. We assume throughout that ties are broken lexicographically. Every bidder is charged his externality – the difference in social efficiency due to his participation.

The Myerson mechanism maximizes expected revenue among all truthful and IR mechanisms ([18], cf. [4,11]). Consider an allocation rule $x_{i,j}(\mathbf{v})$, which indicates whether bidder i wins slot j given a reported value profile $\mathbf{v}$; we say it is *monotone* if as v_i increases, i is only allocated higher slots. The following lemma is an adaptation of Myerson to the sponsored search context.

Lemma 1 (Myerson [18]).

1. *Every monotone allocation rule can be coupled with a unique pricing rule such that the resulting mechanism is truthful and IR.*[7]
2. *The expected revenue of every truthful and IR mechanism is equal to its expected* realized virtual surplus*, i.e.,* $\mathbb{E}_{\mathbf{v}}[\sum_{i,j} s_j \varphi_i(r_i) x_{i,j}(\mathbf{v})]$.

When values are regular, the Myerson mechanism allocates slots to the $\leq m$ advertisers with highest *non-negative realized virtual values*, in high to low order. By regularity this allocation rule is monotone, and so by the first part of Lemma 1, with appropriate payments we get a truthful and IR mechanism. This allocation rule maximizes the expected revenue by the second part of Lemma 1 coupled with the standard rearrangement inequality, which is applied to realized virtual values and slot effects. More formally, we say that π is a *partial ranking* of an n-element vector $\mathbf{x}$ if it ranks a subset of $n' \leq n$ *acceptable* elements. We denote by $\mathbf{x}(\pi)$ a vector of length n in which the first n' entries are the acceptable elements ranked by π, and the rest are zero entries.

Lemma 2 (Rearrangement inequality). *Let* $\mathbf{s} \geq 0, \mathbf{x}$ *be two non-increasing vectors, and let* π *be a partial ranking of* $\mathbf{x}$ *such that* $\mathbf{x}(\pi) \geq 0$*. Then* $\mathbf{s} \cdot \mathbf{x}(\pi) \leq \mathbf{s} \cdot \mathbf{x}$.

3.2 Trade-off Optimality

We now define a class of virtual value based mechanisms, of which the VCG auction and the Myerson mechanism are extremal members. We apply a result of Myerson and Satterthwaite to the sponsored search context, showing that mechanisms in this class optimize any efficiency-revenue trade-off [19,13]. Such mechanisms are termed *trade-off optimal*, and their outcomes lie on the Pareto frontier of efficiency and revenue.[8]

[7] For our purpose we need not specify the pricing rule, because the second part of this lemma gives us a handle on revenue even without knowing the precise price form.

[8] Mechanisms on the efficiency-revenue Pareto frontier are not to be confused with mechanisms that generate Pareto optimal outcomes, in which no bidder's utility can be increased without decreasing another's. Diakonikolas et al. study computational complexity aspects of the Pareto frontier; the difference between their work and ours is that we focus on trade-off optimal mechanisms, which are not required to realize *every* point on the Pareto optimal curve.

Our interest in trade-off optimal mechanisms stems from our belief that search engines aim to optimize some convex combination of efficiency and revenue. While commercial search engines are "revenue maximizers", the "revenue" they aim to maximize is not just the short-term revenue referred to in this paper; rather, they care about a combination of revenue and efficiency, due to their interest in the long-term health and efficiency of sponsored search markets.

Definition 5. *[α-virtual value] For $\alpha \geq 0$, the* α-virtual value *of $v \sim F$ is $\varphi^\alpha(v) = v - \alpha\lambda^F(v)$ (where λ^F is the inverse hazard rate).*

The α-virtual value of v can be rewritten as a combination of v and its corresponding virtual value: $\varphi^\alpha(v) = (1-\alpha)v + \alpha\varphi(v)$. The following definition encompasses the VCG auction ($\alpha = 0$) as well as the Myerson mechanism ($\alpha = 1$).

Definition 6. *[α-virtual value based mechanism] For $\alpha \geq 0$, the* α-virtual value based mechanism *is a deterministic mechanism which asks the advertisers to report their values v_i, ranks them according to their realized α-virtual values $p_i\varphi_i^\alpha$, and allocates the slots to those ranked highest with non-negative $p_i\varphi_i^\alpha$.*

Lemma 3 (Truthfulness). *For $0 \leq \alpha \leq 1$ and regular values, the α-virtual value based mechanism is truthful.*

Proof. By regularity, $\varphi^\alpha = (1-\alpha)v + \alpha\varphi(v)$ is non-decreasing in v when $0 \leq \alpha \leq 1$, thus the allocation rule is monotone; truthfulness follows from Lemma 1.

Lemma 4 (Trade-off optimal mechanisms). *Consider a regular position auction.*[9] *For $0 \leq \alpha \leq 1$, the optimal mechanism for the objective $(1-\alpha)\mathbb{E}$ [efficiency] $+ \alpha\mathbb{E}$[revenue] among all truthful and IR mechanisms is the α-virtual value based mechanism.*

Proof. From Myerson's results applied to sponsored search (Lemma 1) it follows that the optimal mechanism for the objective maximizes the realized α-virtual surplus. The standard rearrangement inequality (Lemma 2) ensures that the optimal mechanism is the α-virtual value based mechanism.

4 Refinement Effects on Auction Objectives

Our starting point is an observation regarding the search engine's incentive to perform refinement. Recall from Section 3.2 that we assume the search engine aims to optimize a fixed trade-off between revenue and efficiency. We observe that in expectation, refinement helps this objective, thus generalizing a previous result of Fu et al. beyond revenue maximization [7].[10] This indicates that up to practical limitations, the search engine would prefer as refined a prediction scheme as possible.

[9] A similar result holds for irregular position auctions, by replacing realized α-virtual values with their ironed counterparts.

[10] Note however that the result of Fu et al. [7] applies to completely general signals whereas we focus on the linear form standard in the context of sponsored search.

Lemma 5 (Refinement improves trade-off). *Let prediction scheme T be a refinement of scheme $\bar{T}$, and let q be a query belonging to a coarse part $\bar{t} \in \bar{T}$. Then with respect to its objective, a trade-off optimal mechanism M performs as well for q with scheme T as with $\bar{T}$, in expectation over value profiles and over the refined part $t \in T$ to which q belongs.*

Proof. Follows from the fact that the trade-off optimal mechanism M is α-virtual value based (Lemma 4), combined with the proof of Proposition 3 of Fu et al. [7] in which, *mutatis mutandis*, the notion of value is replaced with that of α-virtual value.

We now turn to the effect of refined relevance prediction on the efficiency guarantees of trade-off optimal mechanisms. In our main technical result, we identify natural conditions under which refining the prediction improves the efficiency of any trade-off optimal mechanism. The proof appears in Section 4.1.

Theorem 1 (Refinement improves efficiency). *Let prediction scheme T be a distinguishing refinement of scheme $\bar{T}$. Consider an i.i.d., MHR position auction for a query q. Then with respect to social efficiency, a trade-off optimal mechanism M performs as well for q with scheme T as with $\bar{T}$, for every value profile of the advertisers.*

It is instructive to compare the two above results. Lemma 5 is less conditional, that is, the conditions of i.i.d., MHR values and distinguishing refinement are not required. On the other hand, Theorem 1 holds entirely *pointwise*, that is, it does not require averaging over the value profiles or query types. The fact that Theorem 1 requires more conditions, whose necessity is discussed in the full version [20] by analyzing Examples 1 and 2, indicates a non-trivial trade-off between efficiency and revenue: When the search engine is optimizing for a combination of efficiency and revenue, refining "ad infinitum" will not always be the right thing to do in terms of social efficiency. This can be the case, for example, if the refinement is indistinguishing. On the flip side, when the conditions of Theorem 1 hold, the social interest is aligned with that of the search engine; prediction refinement is in both their best interest since it simultaneously increases social efficiency and its combination with revenue. This is formalized in Corollary 1, which is a direct consequence of Lemma 5 and Theorem 1.

Corollary 1. *Let prediction scheme T be a distinguishing refinement of scheme $\bar{T}$, and let q be a query belonging to part $\bar{t} \in \bar{T}$. Consider an i.i.d., MHR position auction for q. Then with respect to both its objective and social efficiency, a trade-off optimal mechanism M performs as well for q with scheme T as with $\bar{T}$, in expectation over value profiles and the part $t \in T$ to which q belongs.*

In particular, mechanism M in Corollary 1 can be the revenue-optimal Myerson mechanism, for which a distinguishing refinement improves both efficiency and revenue. It is an interesting question whether there are additional mechanisms for which this desirable property of simultaneous improvement holds.

Application to Signaling The above results are closely related to *signaling* of seller information in auctions, studied in the economic literature since the seminal work of Milgrom and Weber [16], and more recently in the computer science literature starting with [8,5,17]. The seller can adopt a *signaling scheme* by which he communicates his information to the bidders, who adjust their realized values accordingly. In the sponsored search context, the features which determine advertiser relevance can be viewed as the seller's information, making it a special case in which the effect of the information on values is multiplicative, and refinement is equivalent to revealing more of the seller's information. To our knowledge, this mathematical equivalence between prediction and signaling schemes has not been previously observed. Our results apply to settings to which the fundamental *Linkage Principle* does not, due to the inherent asymmetry of advertiser relevance (indeed, it is not hard to see that refinement may decrease the expected revenue of mechanisms such as the second-price auction). Our main result stated in the context of signaling is that if releasing information distinguishes among i.i.d. MHR bidders, then it improves both the expected outcome of a Pareto optimal mechanism and its efficiency.

4.1 Proof of Theorem 1

Refinement has a delicate effect in the context of ad auctions. The transformation of values to realized values using different relevance terms causes the revenue-optimal ranking to differ from the efficieny-optimal one, even under assumptions of i.i.d. and MHR. This is in contrast to simple single-item multi-unit settings, where the revenue- and efficiency-optimal rankings both order bidders in the same way, and differ only in that the former excludes bidders with negative virtual values. In this section we show that the difference between the two rankings in sponsored search diminishes with refinement, as long as the conditions stated in Theorem 1 hold. In fact we show this for any trade-off optimal ranking according to α-virtual values, in addition to the revenue-optimal one where $\alpha = 1$.

Throughout, fix a query q and let prediction scheme T be a distinguishing refinement of a scheme $\bar{T}$. We begin with two lemmas, whose proofs appear in the full version [20]. The first lemma shows that if according to T, advertiser 1 has lower realized value but higher realized α-virtual value in comparison to advertiser 2, then the same holds according to $\bar{T}$. This indicates that any inefficiency due to the trade-off optimal ranking according to T occurs according to $\bar{T}$ as well, and so refinement can only increase efficiency.

Lemma 6 (Inefficient allocation). *Consider two advertisers with i.i.d. MHR values $v_1 \neq v_2$, and α-virtual values $\varphi_1^\alpha, \varphi_2^\alpha$. Let p_1, p_2 be their relevance predictions according to T, and $\bar{p}_1, \bar{p}_2$ their predictions according to $\bar{T}$. Then*

$$p_1 v_1 < p_2 v_2 \text{ and } p_1 \varphi_1^\alpha \geq p_2 \varphi_2^\alpha > 0 \implies \bar{p}_1 \varphi_1^\alpha \geq \bar{p}_2 \varphi_2^\alpha.$$

We now state a generalization of the standard rearrangement inequality in Lemma 2. Let π_1, π_2 be two partial rankings of an n-element vector $\mathbf{x}$. We say

π_1 is *more ordered* than π_2 if the same elements are acceptable in both, and for every pair of acceptable elements $x_i > x_j$ such that x_i appears before x_j in $\mathbf{x}(\pi_2)$, this pair also appears in the correct order in $\mathbf{x}(\pi_1)$.

Lemma 7 (Generalized rearrangement inequality). *Let* $\mathbf{s} \geq 0, \mathbf{x}$ *be two non-increasing vectors, and let* π_1, π_2 *be two partial rankings of* $\mathbf{x}$ *such that* π_1 *is more ordered than* π_2*, and* $\mathbf{x}(\pi_1), \mathbf{x}(\pi_2) \geq 0$*. Then* $\mathbf{s} \cdot \mathbf{x}(\pi_1) \geq \mathbf{s} \cdot \mathbf{x}(\pi_2)$*.*

Proof (Theorem 1). Consider n advertisers with i.i.d. MHR values $\mathbf{v}$, competing for $m \geq n$ ad slots to appear along search results for a query q. For every advertiser i, let $\bar{p}_i$ be the relevance prediction according to $\bar{T}$, and let p_i be the prediction according to the distinguishing refinement T. We want to show that with respect to social efficiency, the trade-off optimal mechanism M performs better with T than with $\bar{T}$.

For simplicity, rename the advertisers such that their true realized values, i.e., those according to the refined scheme T, are in decreasing order $p_1 v_1 \geq \cdots \geq p_n v_n$. (These are the true realized values since they are based on an accurate prediction of the click-through rates, and so reflect the true added efficiency from allocating a slot to each advertiser). The advertisers are now ordered according to the efficiency-optimal ranking.

We know that M is α-virtual value based for some α (Lemma 4). It thus ranks the advertisers by their realized α-virtual values – either $\bar{p}_i \varphi_i^\alpha$ if using scheme $\bar{T}$, or $p_i \varphi_i^\alpha$ if using scheme T. Let π_1 be the partial ranking of advertisers with non-negative α-virtual values according to T, and let π_2 be the same according to $\bar{T}$.

We first claim it is sufficient to show that, as partial rankings of the advertisers and their realized values $\mathbf{r} = (p_1 v_1, \ldots, p_n v_n)$, π_1 is more ordered than π_2. The sufficiency follows from the generalized rearrangement inequality (Lemma 7): Both $\mathbf{r}$ and the vector of slot effects $\mathbf{s}$ are decreasing, so if π_1 is more ordered than π_2 it holds that $\mathbf{s} \cdot \mathbf{r}(\pi_1) \geq \mathbf{s} \cdot \mathbf{r}(\pi_2)$, i.e., the efficiency of M with T is at least its efficiency with $\bar{T}$.

It's left to show that π_1 is more ordered than π_2. First observe that advertiser i is acceptable according to either partial ranking if and only if his (non-realized) α-virtual value φ_i^α is non-negative, and so π_1 and π_2 rank the same subset of advertisers as acceptable. Now consider a pair of acceptable advertisers i, j ($\varphi_i^\alpha, \varphi_j^\alpha \geq 0$), whose realized values are $r_i = p_i v_i < p_j v_j = r_j$. We claim that if their ranking according to π_1 is reversed (i appears before j even though his realized value is lower), then this will also be the case according to π_2, and so π_1 is indeed more ordered.

If the ranking according to π_1 is reversed then we know that $p_i \varphi_i^\alpha \geq p_j \varphi_j^\alpha$. We can now invoke Lemma 6 to get $\bar{p}_i \varphi_i^\alpha \geq \bar{p}_j \varphi_j^\alpha$ (note that while Lemma 6 requires that $\varphi_i^\alpha, \varphi_j^\alpha$ are both positive, if at least one of these is zero then the inequality holds trivially). We have shown that advertiser i is ranked before j in π_2, completing the proof.

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On the Impossibility of Black-Box Transformations in Mechanism Design

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Abstract. A fundamental question in algorithmic mechanism design is whether any approximation algorithm for a single-parameter social-welfare maximization problem can be turned into a dominant-strategy truthful mechanism for the same problem (while preserving the approximation ratio up to a constant factor). A particularly desirable type of transformations—called *black-box transformations*—achieve the above goal by only accessing the approximation algorithm as a black box.

A recent work by Chawla, Immorlica and Lucier (STOC 2012) demonstrates (unconditionally) the impossibility of certain *restricted* classes of black-box transformations—where the tranformation is oblivious to the feasibility constrain of the optimization problem. In this work, we remove these restrictions under standard complexity-theoretic assumptions: Assuming the existence of one-way functions, we show the impossibility of *all* black-box transformations.

1 Introduction

A central area in mechanism design focuses on designing *dominant-strategy truthful mechanisms* that 1) maximize some global objective function (e.g., social welfare) in some feasible outcome space, while 2) incentivizing agents to truthfully report their private values, no matter what everyone else does (that is, being truthful is a dominant strategy).

Our focus is on computational aspects of this task; we restrict to NP-optimization problem (that is, problems for which there exists an efficient procedure to check whether an outcome is feasible,) and are interested in computationally-efficient mechanisms that satisfy the above properties. A fundamental question in algorithmic mechanism design is whether any algorithm for solving some single-parameter social-welfare optimization problem (either exactly or approximately) can be transformed into a dominant-strategy truthful mechanism for the same problem. Ideally, we would like these transformations to be *black-box*—that is, given a description of optimization problem f (i.e., the agents'

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utility functions, the global objective function, and the feasibility constraints), they only require black-box access to the algorithm $\mathcal{A}$.

The celebrated Vickrey-Clarke-Groves mechanism gives a strong positive result of this kind for any single-parameter social welfare maximization problem with quasi-linear utilities. It shows how to efficiently convert any *exact* algorithm for a social welfare maximization problem into a dominant-strategy truthful mechanism for the same problem, by appropriately charging agents prices.

However, in many cases, we do not have access to efficient exact algorithms, but only an *approximation algorithm*, which finds an outcome that is within some approximation factor of the optimal outcome. It is well-known that the VCG transformation fails when applied to approximation algorithms [NR00], leaving open the question of whether some other black-box transformation can be used to achieve the above goal:

> *Does there exists an efficient* black-box transformation $\mathcal{T}$ *such that for any (single-parameter) class of social welfare maximization problems* $F \in \mathsf{NPO}$[1]*, any instance* $\mathsf{f} \in F$*, any approximation algorithm* $\mathcal{A}$ *for* f*,* $\mathcal{T}^{\mathcal{A}}(\mathsf{f}, F)(\cdot)$ *is a dominant-strategy truthful mechanism for f which preserves the approximation ratio of* $\mathcal{A}$ *(up to a constant factor)?*

For a relaxed version of this problem, considering a Bayesian setting where players' values are sampled from some publicly known distribution and we only require a weaker notion of Bayes Nash truthfulness, we can essentially replicate the success of VCG: work by [HL09, BH11] shows that any approximation algorithm can be efficiently converted into a mechanism that is "Bayes-Nash truthful" for social welfare maximization problems, while preserving the approximation ratio of the original algorithm to within an arbitrarily small factor. ([HKM11] show how to achieve a similar result even in a multi-parameter setting, such as multi-item auctions). But the Bayesian setting makes strong assumptions in terms of the knowledge of both the mechanism designer and the players—in many settings it is unreasonable to assume there is a publicly known distribution over player values.

For the case of dominant-strategy truthfulness, Chawla, Immorlica and Lucier [CIL12] gave an elegant partial negative result. That is, they presented a family of problem instances, together with a corresponding family of approximation algorithms for these instances, and showed that every polynomial time transformation must fail to yield a worst-case approximation-preserving dominant-strategy truthful mechanism[2] on *some* member of this family, given black-box access to the corresponding approximation algorithm. However, their result only applies to a *restricted* class of transformations that on top of having black-box access to the algorithm, have *no* access to feasibility constraint of the problem statement f (they only give the transfomation access to the utility

[1] Recall that NPO is the class of NP-optimization problems.

[2] In fact their negative result also encompassed a slightly weaker notion of truthfulness that arises in the case of randomized mechanisms, namely *truthfulness in expectation* (TIE). A mechanism is TIE if each agent wants to be truthful given knowledge of the other players' values, but not the random coin flips of the mechanism. This notion can be contrasted with full ex-post incentive compatibility (EPIC), where the agents want to be truthful even given the coin flips of the mechanism.

functions of the players). In fact, the [CIL12] analysis actually fails if the transformation knows the feasible space.

In this work, we address the above question with respect to *arbitrary* black-box transformations.

1.1 Our Results

Our central theorem shows how to remove the restrictions on the transformation from [CIL12] by using standard complexity-theoretic hardness assumptions: Assuming the existence of one-way functions, we show the impossibility of *all* black-box transformations.[3] Comparing our results to [CIL12], we rule out all black-box transformation (whereas [CIL12] only rules out restricted classes of transformation that are oblivious to the feasibility constraints). On the other hand, the impossibility result of [CIL12] is *unconditional* (whereas we assume one-way functions). We finally observe that we cannot hope to obtain our result without making complexity-theoretic assumptions (unless we prove strong complexity-theoretic hardness results): ruling our arbitrary black-box transformation requires assuming $\mathsf{NPO} \neq \mathsf{P}$[4]; roughly, this follows from the observation that if $\mathsf{NPO} = \mathsf{P}$, then there exists an exact algorithm for every social-welfare optimization problem, and we can then rely on the VCG mechanism. We defer a full proof of this observation to the full version of the paper.

At a high level, the approach in our impossibility result is to consider a family of problem instances that have their feasibility constraints expressed in an "obfuscated" form. This obfuscated description fully determines the set of feasible allocations, and enables efficiently checking whether a particular allocation is feasible, but the obfuscation makes it computationally hard to find the explicit set of feasible allocations for the problem instance given just this description.

1.2 Overview of the Construction

Before describing our results, let us briefly review the construction of [CIL12]. At a high level, their problem instances consist of choosing two privileged subsets of players, U and $V \subseteq [n]$, with $V \subseteq U$. There are three feasible allocations: a high allocation to U, a low allocation to U, and a "dummy" allocation that gives a tiny amount to all n players. When the players in V have high value, the approximation algorithm $\mathcal{A}$ returns the high allocation on U. However, when the players in U have high value, $\mathcal{A}$ returns the low allocation on U. Since $V \subseteq U$, this arrangement creates an essential non-monotonicity[5] in the output of $\mathcal{A}$: increasing the valuations of players in $U \setminus V$ causes their allocations to decrease.

[3] Our results also apply to black-box TIE transformations.

[4] Our results assume the existence of one-way functions, which imply that $\mathsf{NPO} \neq \mathsf{P}$, but it is a major open question whether $\mathsf{NPO} \neq \mathsf{P}$ implies one-way functions.

[5] It is well known that, in the single parameter setting, if an algorithm produces allocations that are monotone in the valuations of the individual players, then there is a simple pricing rule that converts it into a truthful mechanism. Hence their impossibility result must necessarily use an approximation algorithm that is non-monotone.

Additionally, $\mathcal{A}$ has the property that given an input that produces one of the high or low allocation on U, it is hard to find an input that produces the other kind of allocation. This is the crucial property that causes any transformation $\mathcal{T}$ to fail: given an input that produces one kind of allocation on U, $\mathcal{T}$ must proceed as if the other allocation didn't exist, since it can never encounter it. [CIL12] show that this leads every transformation $\mathcal{T}$ to unfairly punish the players in the set V, leading to a low approximation ratio in the transformed mechanism.

Hiding the feasible allocation set from the transformation is essential to their analysis: the argument for not being able to find the second allocation given the first depends on it. In our construction, we would like the a description of the feasible set to be accessible to the transformation.

One attempt could be to release a cryptographic "commitment" to the feasible sets. Roughly speaking, a commitment scheme can be viewed as the cryptographic analog of a "sealed enveloped". It enables a party to "commit" to a value v in a way that hides it with respect to computationally-bounded (i.e., polynomial-time) parties—this is referred to as the *hiding property* of the commitment. Yet at a later stage the party may reveal the value v by releasing some "decommitment string" and it is garanteed that it can only decommit to the actual value v—this is referred to as the *binding property* of the commitment scheme. A bit more formally, a non-interactive commitment scheme Com is a polynomial-time computable function that given a message m and a random string r outputs a commitment $c = \mathsf{Com}(m, r)$ that determines m, yet hides it with respect to polynomial-time parties; the string r is the decommitment information.

The binding property of the commitment scheme would now ensure that the released description fully determines the set of feasible allocations, while the hiding property would make it computationally infeasible for the transformation to recover the actual feasible set without the decommitment information. However, this solution is unsatisfactory, because it makes it impossible to efficiently check whether a particular allocation is feasible or not (and thus the class of optimization problems wouldn't be in NPO). Given just a set of commitments, there is no efficient way to verify if a particular allocation is the value committed to in one of them.

A natural second approach would be to release an "obfuscated" version of the functionality that checks whether a given allocation is one in the feasible set. In theory, this would allow us to release a description of the feasible allocations, together with an efficient functionality to check whether a given allocation is feasible, and still make it hard for the transformation to explicitly find the feasible sets. However, we do not know how to obfuscate general functionalities [BGI$^+$12]. Although there are results for specific functionalities [Wee05, HRV$^+$07, CRV10, BR13], and also some recent progress for weakened notions of obfuscation [GGH$^+$13], it is not clear how to use these weakened version for our purposes. Finally, even if such obfuscation were possible, we could not longer rely on the result of [CIL12]—this result no longer holds when the black-box transformation can just check whether an allocation is feasible or not.

Our key idea for overcoming these problems is by embedding the [CIL12] instance in larger instance such that the impossibility results still applies even if one can check whether an allocation to the "larger" instance is feasible (but this does not permit checking whether an allocation to the "smaller" embedded instance is feasible). More precisely, we release commitments to each of the feasible allocations, and additionally modify the problem to add in some "dummy" agents with no values, that can be allocated either 1 or 0. An allocation to these players can be interpreted as a binary decommitment string r for one of the released commitments. Thus a feasible allocation now consists of an allocation to the "real" agents, together with an allocation to the dummy agents that shows that the allocation to the real agents is in fact a correct decommitment to one of the released commitments. This scheme now satisfies our requirements: it allows for a public description of the feasibility constraints that enables efficiently checking whether a particular allocation is feasible.

To ensure that the approximation algorithm $\mathcal{A}$ can efficiently find a solution, we provide it with the decommitment information r for all commitments. This will allow $\mathcal{A}$ to compute the feasible allocations efficiently, and thus provide a good approximation. This decommitment information—which we refer to as $\mathcal{A}$'s "trapdoor" (or "secret-sauce")—however, is not publicly released and in particular, cannot be directly accessed by the transformation—since we only consider *black-box* transformations $\mathcal{T}$, $\mathcal{T}$ only gets black-box access to $\mathcal{A}$ and can only access the "secret-sauce" indirectly through the output of $\mathcal{A}$. (As is always the case with black-box separation results, we need to provide the algorithm that the reduction/transformation operates on with some extra "power" (e.g., a trapdoor, or the ability to perform super-polynomial-time computation) that the reduction/transformation itself does not get; see e.g., [IR88]. If we did not do this, we would rule out not only black-box transformations, but also *non-black-box* ones.)

1.3 Paper Layout

The layout of our paper is as follows: In Section 2 we provide preliminaries. Section 3 contains the main results of the paper. In Section 3.2 we give both a description of the family of problem instances and approximation algorithms for those instances. Section 3.3 gives a very high-level idea of the analysis of the construction; the actual proof is deferred to the full version of the paper.

2 Preliminaries

2.1 Notation

Let $\mathbb{N}$ denote the set of positive integers, and $[n]$ denote the set $\{1, 2, \ldots, n\}$. We call a function negligible if it grows more slowly than the inverse of any polynomial. By a probabilistic algorithm we mean a Turing machine that receives an auxiliary random tape as input. If M is a probabilistic algorithm, then for any input x, $M(x)$ represents the distribution of outputs of $M(x)$ when the random tape is chosen uniformly. By $x \leftarrow S$, we denote an element x is sampled from a distribution S. If F is a finite set, then $x \leftarrow F$ means x is sampled uniformly from the set F. To denote the ordered sequence in which the experiments happen we use semicolon, e.g. $(x \leftarrow S; (y, z) \leftarrow A(x))$. Using this notation we can describe probability of events. For example, if $p(\cdot, \cdot)$ denotes a predicate, then $\Pr[x \leftarrow S; (y, z) \leftarrow A(x) : p(y, z)]$ is the probability that the predicate

$p(y, z)$ is true in the ordered sequence of experiments $(x \leftarrow S; (y, z) \leftarrow A(x))$. The notation $\{(x \leftarrow S; (y, z) \leftarrow A(x) : (y, z))\}$ denotes the resulting probability distribution $\{(y, z)\}$ generated by the ordered sequence of experiments $(x \leftarrow S; (y, z) \leftarrow A(x))$.

2.2 Optimization Problems

In this work, we restrict ourselves to single-parameter social welfare maximization problems with quasilinear utilities, which is a central class of problems in the mechanism design literature.

In this class of problems we are given an input vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, representing the valuations of each of n players. Each v_i is assumed to be drawn from a known set $V_i \subseteq \mathbb{R}$ corresponding to the possible valuations for player i, and $V = V_1 \times V_2 \times \cdots \times V_n$ is the set of possible input vectors. Each problem also defines allocations $x \in \mathsf{f} \subseteq \mathbb{R}^n$, where f is called the set of feasible allocations. In allocation x, player i is allocated x_i, and that player's utility is $v_i \cdot x_i - p_i$, where p_i is the price charged to player i. The goal in a social welfare maximization problem is to choose $x \in \mathsf{f}$ such that the total social welfare of all players, $\sum_i v_i \cdot x_i$, is maximized. Different sets f and V can be thought of as defining different instances of the problem.

An algorithm $\mathcal{A}$ for the problem defines a mapping from input vectors $\mathbf{v}$ to outcomes $\mathbf{x} \in \mathbb{R}^n$. We use $\mathcal{A}(v)$ to represent the output of $\mathcal{A}$ on input v. If $\mathcal{A}$ is randomized, then $\mathcal{A}(v)$ is a random variable. We may also allow $\mathcal{A}$ to take additional inputs as "hints".

We use OPT_{f} to refer to the optimal algorithm for a social welfare problem f and the social welfare $\phi(\mathbf{x}, \mathbf{v})$, that is, for all $\mathbf{v} \in V$, $OPT_{\mathsf{f}}(\mathbf{v}) = \arg\max_{\mathbf{x} \in \mathsf{f}} \phi(\mathbf{x}, \mathbf{v})$ (for a maximization problem, as in the case of social welfare). For an arbitrary algorithm $\mathcal{A}$, we let $approx_{\mathsf{f}}(\mathcal{A})$ denote the worst-case approximation ratio of $\mathcal{A}$ for f. For a maximization problem, $approx_{\mathsf{f}}(\mathcal{A}) = \min_{\mathbf{v} \in V} \frac{\mathbb{E}[\mathcal{A}(\mathbf{v})]}{OPT_{\mathsf{f}}(\mathbf{v})}$.

An family F of optimization problems is in the class NPO if the following conditions hold:

- For every $\mathsf{f} \in F$, f's feasible allocations are each of size at most polynomial in $|\mathsf{f}|$.
- There exists a verifier Ver such that, for every $\mathsf{f} \in F$, $\mathsf{Ver}(\mathsf{f}, \mathbf{x}) = 1 \iff \mathbf{x}$ is in the set of feasible allocations for f. Additionally, Ver is polynomial-time computable.
- Given the valuations of players, the utility Φ of an allocation is polynomial-time computable (this is the case when Φ is the social-welfare maximization objective).

2.3 Mechanisms

We consider our optimization problems in a mechanism design setting with n rational agents, where each agent receives one of the n input values as private information. We think of allocation $\mathbf{x}$ as an allocation to agents, where x_i is the allocation to agent i. A (direct-revelation) mechanism then proceeds by eliciting declared values $\mathbf{b} \in \mathbb{R}^n$ from the agents, then applying an allocation algorithm $\mathcal{A} : \mathbb{R}^n \to \mathsf{f}$ that maps $\mathbf{b}$ to an allocation $\mathbf{x}$ and a payment rule that maps $\mathbf{b}$ to a payment vector $\mathbf{p}$. We write $\mathbf{x}(\mathbf{b})$ and $\mathbf{p}(\mathbf{b})$ for the allocations and payments generated on input $\mathbf{b}$. The utility of agent i, given that agents declare values $\mathbf{b}$, is taken to be $u_i(\mathbf{b}) = v_i x_i(\mathbf{b}) - p_i(\mathbf{b})$.

A (possibly randomized) mechanism $\mathcal{M}$ is called truthful in expectation (TIE) if each agent maximizes it's expected utility by reporting its value truthfully, regardless of the reports of the other agents (the expectation is over the randomness of the mechanism. That is, $\mathbb{E}[u_i(v_i, \mathbf{v}_{-i})] \geq \mathbb{E}[u_i(b_i, \mathbf{v}_{-i})]$ for all i, for all $b_i \in V_i$ and for all $\mathbf{b}_{-i} \in V_{-i}$. An algorithm is TIE if there exists a payment rule such that the resulting mechanism is TIE. It is known that an algorithm is TIE if and only if, for all i and $\mathbf{v}_{-i}$, $\mathbb{E}[x_i(v_i, \mathbf{v}_{-i})]$ is a monotone non-decreasing function of v_i, where again the expectation is over the randomness of the mechanism.

A mechanism $\mathcal{M}$ is ε-TIE if an agent cannot gain more than ε by lying, regardless of the reports of the other agents (the expectation is over the randomness of the mechanism. That is, $\mathbb{E}[u_i(v_i, \mathbf{v}_{-i})] \geq \mathbb{E}[u_i(b_i, \mathbf{v}_{-i})] - \varepsilon$ for all i, for all $b_i \in V_i$ and for all $\mathbf{b}_{-i} \in V_{-i}$.

2.4 Transformations

A polytime transformation $\mathcal{T}$ is an algorithm that is given black-box access to an algorithm $\mathcal{A}$. We will write $\mathcal{T}^{\mathcal{A}}(\mathbf{v})$ for the allocation returned by $\mathcal{T}$ on input $\mathbf{v}$ given black-box access to algorithm $\mathcal{A}$. We can think of $\mathcal{T}^{\mathcal{A}}(\cdot)$ as the algorithm $\mathcal{T}$ that runs with multiple black-box accesses to $\mathcal{A}$. We say $\mathcal{T}$ is TIE (resp. ε- TIE) if for every algorithm $\mathcal{A}$, $\mathcal{T}^{\mathcal{A}}(\cdot)$ is TIE (resp. ε-TIE).

We write $\mathcal{T}^{\mathcal{A}}$ for the allocation rule that results when $\mathcal{T}$ runs with black-box access to $\mathcal{A}$. We assume $\mathcal{T}$ is aware of the objective function ϕ and the domain V_i of values for each agent i.

A transformation $\mathcal{T}$ for a family $F \in \mathsf{NPO}$ additionally takes as input an instance f $\in F$, the utility function Φ and the verifier Ver for the family F. We say $\mathcal{T}$ is TIE for F (resp. ε- TIE) if for every instance f $\in F$, for every algorithm $\mathcal{A}$ for f, $\mathcal{T}^{\mathcal{A}}(\mathsf{f}, \Phi, \mathsf{Ver})(\cdot)$ is TIE (resp. ε-TIE)

2.5 Commitment Schemes

Recall that, roughly speaking, a non-interactive commitment scheme Com is a polynomial-time computable function that given a message m and a random string r outputs a commitment $c = \mathsf{Com}(m, r)$ that determines m, yet hides it with respect to polynomial-time parties. More formally,

Definition 1 (Commitment Schemes[Gol01]) *A non-interactive commitment scheme* Com *is a polynomial-time computable function satisfying the following two properties:*

- ***computational hiding****: for every pair of message sequences* $\{m_{0,n}\}_{n\in\mathbb{N}}$, $\{m_{1,n}\}_{n\in\mathbb{N}}$ *such that* $m_{0,n}, m_{1,n} \in \{0,1\}^n$ *for all* $n \in \mathbb{N}$*, for every non-uniform probabilistic polynomial time adversary A, there exists a negligible function* $\mu(\cdot)$ *such that for all* $n \in \mathbb{N}$*:*

$$|\Pr[r \leftarrow \{0,1\}^n : A(\mathsf{Com}(m_{0,n}, r)) = 0] - \Pr[r \leftarrow \{0,1\}^n : A(\mathsf{Com}(m_{1,n}, r)) = 0]| \leq \mu(n)$$

(In other words, A should not be able to distinguish between commitments to any pair of messages $m_{0,n}$ *and* $m_{1,n}$ *except with negligible probability.)*

– ***perfect binding****: there do not exist* $m_0, m_1, r_0, r_1 \in \{0,1\}^n$ *such that* $m_0 \neq m_1$ *but* $\mathsf{Com}(m_0, r_0) = \mathsf{Com}(m_1, r_1)$. *(In other words, no commitment c can be opened to two different values* m_0 *and* m_1.*)*

Non-interactive commitment schemes can be constructed based on any one-to-one one-way function (see Section 4.4.1 of [Gol01]).

In many applications—and in particular ours—it, however, suffices to consider a *family* of non-interactive commitments that are parametrized by some initialization string k, and the security properties of the commitment (i.e., hiding and binding) need only hold with all but negligible probability over the choice of k. Such commitment schemes can be constructed based on any "plain" one-way function [Nao91, HILL99]. For ease of notation, in this extented abstract, we here simply assume the existence of non-interactive commitment schemes and note that our construction can be easily modified to work also with any family of non-interactive commitments (and thus be based on any one-way function).

3 Main Result

3.1 Problem Definition and Main Theorem

In this section, we will give the main result of this paper:

Theorem 1 (Main Theorem) *Assuming the existence of one-way functions, there exist constants* $c, d > 0$*, a family* $F \in \mathsf{NPO}$ *of single-parameter social-welfare optimization problems with verifier* Ver*, and a sequence of distributions* $\{D_n\}_{n\in\mathbb{N}}$ *such that* D_n *is a distribution over pairs* $(\mathsf{f}, \mathcal{A})$ *where each* $\mathsf{f} \in F$*, and* $\mathcal{A}$ *is a polynomial size approximation algorithm for* f*, such that for any poly-time* ε*-TIE transformation* $\mathcal{T}$ *with* $\varepsilon \leq 1/n^d$*, for all* n*, with all but negligible probability over* $(\mathsf{f}, \mathcal{A}) \leftarrow D_n$*,* $\frac{approx_{\mathsf{f}}(\mathcal{A})}{approx_{\mathsf{f}}(\mathcal{T}^{\mathcal{A}}(\mathsf{f},\mathsf{Ver}))} \geq n^c$.

As mentioned above, our overall approach will be to appropriately modify the construction of [CIL12]—roughly speaking, embedding each instance into a larger instance, which uses cryptographic commitments to release the feasibility constraints to the transformation.

3.2 Construction

We consider instances where there are $3n$ agents. The first n agents have private values v_i drawn from $\{v, 1\}$, where $0 < v < 1$ is a parameter we set below. The remaining $2n$ agents are dummy agents, that all have value 0, however allocating to these agents is necessary in order to satisfy the feasibility constraints f, which we describe below. We refer to the set of the first n agents as Y_1, the second n agents as Y_2, and the third n agents as Y_3. We can therefore interpret an input vector as a subset $y \subseteq Y_1$, corresponding to those of the first n agents that have value 1 (the remaining agents in Y_1 have value v, and all agents in Y_2 and Y_3 have value 0).

Since every player in sets Y_2 and Y_3 always has value 0, we will sometimes omit these players from the input to $\mathcal{A}$, since their valuations are known by default. Accordingly, for every $y \subseteq Y_1$, we define $\mathcal{A}(y)$, $OPT_{\mathsf{f}}(y)$, and so on, as the output of the algorithms on the set of values corresponding to the subset y, with the valuations of the players in Y_2 and Y_3 set to 0.

Also, for $a_1, a_2, a_3 \geq 0$, and $y_1 \subseteq Y_1, y_2 \subseteq Y_2, y_3 \subseteq Y_3$, we will let $(\mathbf{x}_{y_1}^{a_1}, \mathbf{x}_{y_2}^{a_2}, \mathbf{x}_{y_3}^{a_3})$ denote the allocation where each agent $i \in y_1$ is allocated a_1, each agent $j \in y_2$ is allocated a_2, each agent $k \in y_3$ is allocated a_3 and all remaining agents are allocated 0. We will also sometimes refer to allocations in the form (x_y^a, r, r'), where $a \geq 0, y \subseteq Y$, and $r, r' \in \{0,1\}^n$. This corresponds to allocating a to every $i \in y$, allocating 1 to every $j \in Y_2$ such that r_j, the jth bit of r, is 1, and also allocating 1 to every $k \in Y_3$ such that r'_k, the kth bit of r', is 1, while allocating 0 to all remaining agents in $Y_1 \cup Y_2 \cup Y_3$.

Problem Instances: We now define the family F of problems corresponding to the problem instances. Each instance f will have a trapdoor trap. Intuitively, the set of feasible allocations for f will be computationally difficult to find without knowledge of trap.

Let Com be a non-interactive commitment scheme.[6] Then each instance f in our family will consist of a choice of $\alpha, \gamma \in (0,1)$ with $\gamma < \alpha$, sets $S, T \subseteq Y_1$ of agents, and commitments to the preceding parameters $c_1 = \mathsf{Com}(S, r_S)$, $c_2 = \mathsf{Com}(T, r_T)$, $c_3 = \mathsf{Com}(\alpha, r_\alpha)$, for $r_S, r_T, r_\alpha \in \{0,1\}^n$. We let $\boldsymbol{c} = \langle c_1, c_2, c_3 \rangle$, $\boldsymbol{r} = \langle r_S, r_T, r_\alpha \rangle$. The feasible allocations are:

- $(\mathbf{x}_{[n]}^{\gamma}, 0^n, 0^n)$
- $(\mathbf{x}_{S'}^{1}, r, 0^n)$ such that $c_1 = \mathsf{Com}(S', r)$
- $(\mathbf{x}_{T'}^{\alpha'}, r, r')$ such that $c_2 = \mathsf{Com}(T', r)$ and $c_3 = \mathsf{Com}(\alpha', r')$.

We also require that S and T satisfy the same properties as in [CIL12], namely that they are sufficiently large, and have a sufficiently large intersection. Formally, we define parameters $r \geq 1$, and $t \geq 1$ (which we fix below to functions of n), such that $t \gg r \gg \gamma^{-1} \gg \alpha^{-1}$, and $\frac{t}{\gamma n} \ll 1$. We think of t as a bound on "small" sets, and r as a ratio between "small" and "large" sets. We also use a third set $V \subseteq [n]$ as part of our description. Then, if V, S and T are subsets of $[n]$, we say that the triple V, S, T is *admissible* if:

1. $|S| = |T| = r^3 t$
2. $|S \cap T| = r^2 t$
3. $V \subset S \cap T$ and
4. $|V| = rt$

Additionally, for an admissible V, S, T, we let $U = S \cap T$. Then, each problem instance corresponds to a choice of V, S and T and a value α, together with $\boldsymbol{c}$ corresponding to commitments to S, T and α under randomness given by $\boldsymbol{r}$. The description of a problem instance is thus given by

$$\mathsf{f}_{V,S,T,\alpha,\boldsymbol{r}} = (\boldsymbol{c}, \gamma)$$

[6] As mentioned above, such schemes can be constructed based on any one-to-one one-way function. But, by slightly modifying our construction, we can also rely on a family of non-interactive commitment schemes which can be based on any "plain" one-way function. We defer the details to the full version.

and the trapdoor for that instance is given by

$$\mathsf{trap}_{V,S,T,\alpha,\boldsymbol{r}} = \langle S, T, \alpha, \boldsymbol{r} \rangle$$

The utility function Φ for each instance is simply the social welfare maximization objective. The verifier Ver, given an instance $\mathsf{f}_{V,S,T,\alpha,\boldsymbol{r}}$ and an allocation $\mathbf{x}$ simply checks whether $\mathbf{x}$ falls into one of the three feasible types of allocations described above. This corresponds to performing at most three decommitments, implying Ver is efficient.

Note that, given the trapdoor, it is easy to come up with feasible allocations corresponding to $\mathsf{f}_{V,S,T,\alpha,\boldsymbol{r}}$: simply use the decommitments in the trapdoor together with α to create any of the three different allocations. However, as we will argue later that, without the trapdoor, it is *infeasible* to construct any allocation other than the first, and this argument will be the heart of the impossibility result. Also, notice that neither f nor trap depend on V. Note also that each of the three feasible allocations is independent of V, and thus it is not strictly necessary to include V in our description of a problem instance. However, we include it for notational convenience, as it is an important component of the approximation algorithm, as we will see shortly.

Algorithm. We now give an approximation algorithm for the family of problem instances described above. Our algorithm closely follows the approximation algorithm of [CIL12], and is designed to resist black-box transformation into a mechanism. We note that the algorithm depends on f, and in particular needs to have trap for f baked into it.

Algorithm 1. Allocation algorithm $\mathcal{A}_{V,S,T,\alpha,\boldsymbol{r}}$

Input: A subset $y \in Y_1$ or agents with value 1
Output: An allocation valid with respect to $\mathsf{f}_{V,S,T,\alpha,\boldsymbol{r}}$

if $n_S(y) \geq t$, $n_S(y) \geq \gamma|n|$, and $n_S(y) \geq n_T(y)$ **then**
  return $(\mathbf{x}_S^1, r_S, 0^n)$
else if $n_T(y) \geq t$, $n_T(y) \geq \gamma|n|$, and $n_T(y) \geq n_S(y)$ **then**
  return $(\mathbf{x}_T^\alpha, r_T, r_\alpha)$
else
  return $(\mathbf{x}_{[n]}^\gamma, 0^n, 0^n)$
end if

As in [CIL12], we define the following functions used by our algorithm given an input $y \in Y_1$:

$$n_T(y) = |y \cap T| + |y \cap U|$$

and

$$n_S(y) = |y \cap S| + 2|y \cap V|$$

We also inherit Lemma 3.2 from [CIL12], which guarantees the approximation ratio of the above algorithm. We restate this lemma here:

Lemma 2 $\mathit{approx}_{\mathsf{f}_{V,S,T,\alpha,\boldsymbol{r}}}(\mathcal{A}_{V,S,T,\alpha,\boldsymbol{r}}) \geq \alpha/6$

Finally, we also have the following version of Claim 3.3 from [CIL12], closely following their proof, but modified to handle ε-TIE transformations. The lemma guarantees that any mechanism solving one of the problem instances we described must allocate similar amounts to agents in U on inputs $y = U$ and $y = V$.

Lemma 3 *Suppose $\mathcal{A}'$ is a ε-TIE algorithm for $\mathsf{f}_{V,S,T,\alpha,\boldsymbol{r}}$. Let a_U and a_V be the expected allocation to each agent in U in $\mathcal{A}'(U)$ and $\mathcal{A}'(V)$ respectively. Then $a_V - a_U \leq \varepsilon \cdot (|U| - |V|)$*

Proof. Consider any set W with $V \subseteq W \subseteq U$ and $|W| = |V| + 1$. Then, on input W, the expected allocation to the agent in $W \setminus V$ must not decrease by more than ϵ. Since all allocations are constant on U, this means that the expected allocation to each agent in U must not decrease by more than ε. By the same argument, for each W such that $V \subseteq W \subseteq U$, $\mathcal{A}'$ must allocate at least $a_V - \varepsilon \cdot (|W| - |V|)$ to each agent in U, and in particular, this holds for $W = U$. □

3.3 Analysis of Construction

The crux of the analysis of [CIL12] is showing that when $\mathcal{T}^{\mathcal{A}}(\cdot)$ is given input $y = V$, it has difficulty finding the feasible output $\mathbf{x}_T^{\alpha}$, while on input $y = U$, it cannot find the feasible output $\mathbf{x}_S^1$. Being unable to find these allocations, and simultaneously needing to maintain the TIE property, the transformed algorithm is forced to make poor choices. The proof of these result in [CIL12], however, crucially rely on the fact that the the feasible allocations are not revealed to the transformation. In the full version of the paper, we show how to extend their proof to also work when commitments to the feasible allocations are given to the transformation. The key idea is that to analyze the probability of a "bad event" (i.e., that the transformation finds a feasible output in a situation when it shouldn't), we can consider a mental experiment where the actual commitments are replaced with commitments to 0. In this mental experiment we can then rely on the [CIL12] analysis to bound the probability of the bad event, and finally, we rely on the hiding propery of the commitment scheme to argue that the probability of the bad even in the real experiment is also small. Due to lack of space, the actual analysis is omitted.

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Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis (Extended Abstract)*

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In this paper, we consider two-sided, many-to-one matching problems where agents on one side of the market (hospitals) impose some distributional constraints. The *regional maximum quotas* provide one such example, where a hospital belongs to a region, and each region has an upper bound on the number of assigned agents on the other side (doctors). Furthermore, *minimum quotas* are relevant in many markets, e.g., school districts may need at least a certain number of students in each school in order for the school to operate. Yet another type of constraints takes the form of *diversity constraints*, e.g., public schools are often required to satisfy balance on the composition of students, typically in terms of socioeconomic status. Several mechanisms have been proposed for each of these various constraints, but previous studies have focused on tailoring mechanisms to specific settings, rather than providing a general framework.

We show that when the preference of the hospitals is represented as an $M^\natural$-concave function, the following desirable properties hold: (i) the *time complexity* of the generalized Deferred Acceptance (DA) mechanism is $O(|X|^3)$, where $|X|$ is the number of possible contracts, (ii) the generalized DA mechanism is *strategyproof*, (iii) the obtained matching is *stable*, and (iv) the obtained matching is *optimal* for doctors within all stable matchings. Equipped with this general result, we study conditions under which the hospitals' preferences can be represented by an $M^\natural$-concave function. We start by separating the preference of hospitals into two parts, i.e., hard distributional constraints for the contracts to be feasible, and soft preferences over a family of feasible contracts. We show that if a family of hospital-feasible contracts forms a *matroid*, and the soft preferences satisfy certain easy-to-verify conditions (e.g., it can be represented as a sum of weights associated with individual contracts), then hospital preferences can be represented by an $M^\natural$-concave function. These conditions are general enough to cover most of existing works, i.e., stability notions under constraints in existing works can be mapped to stability for preferences that satisfy $M^\natural$-concavity, and our generalized DA mechanism corresponds to proposed solutions. These conditions provide a *recipe* for non-experts in matching theory or discrete convex analysis to develop desirable mechanisms in such settings.

* A draft full version is available at http://mpra.ub.uni-muenchen.de/56189/

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Monotonicity, Revenue Equivalence and Budgets*

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Abstract. Budget constraints are central to big business auctions. In Google's GSP keyword auction and other search engine advertising platforms, the bidders are required to specify their bids as well as their budget limits. We study multidimensional mechanism design in a common scenario where players have private information about their willingness to pay and their ability to pay. We provide a necessary and sufficient conditions for the dominant-strategy incentive-compatible implementability of direct mechanisms. Immediate applications of these results include simple characterizations for auctions with publicly-known budgets and for auctions without monetary transfers.

The celebrated revenue equivalence theorem states that the seller's revenue for a broad class of standard auction formats and settings will be the same in equilibrium. Our main application is a revenue equivalence theorem for financially constrained multidimensional bidders.

* A significant part of this research was done while the author was a Post-Doctoral Fellow at the Social and Information Sciences Laboratory, Caltech.

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The Price of Spite in Spot-Checking Games
(Brief Announcement)

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We introduce the class of spot-checking games (SC games). These games can be seen as the graphical counterpart of *security games*, and can be used to model problems where the goal is to distribute fare inspectors over a toll network. In an SC game, the pure strategies of network users correspond to paths in a graph, and those of the fare inspectors are subset of edges to be controlled. Mixed strategies of the network users entail a non-atomic traffic model without congestion, and can be represented using multicommodity flows. Similarly, mixed strategies of the inspector can be represented by flows in a time-extended duty graph and yield Markovian patrolling policies. With this model, best responses of the network users to a given inspector's strategy correspond to shortest paths for some weights that depends on the control intensities.

Although SC games are not zero-sum, we show that a mixed Nash equilibrium can be computed by linear programming. However, the computation of a strong Stackelberg equilibrium is more relevant – because the inspector can credibly commit to a strategy– and we give a mixed integer programming (MIP) formulation for this problem. We show that the computation of such an equilibrium is NP-hard, even in the simplest case, in which the game has a "zero-sum plus costs" structure; Consequently, it is NP-hard to compute a strong Stackelberg equilibrium in a polymatrix game, *even if the game is pairwise zero-sum.*

Then, we study the *price of spite*, which measures how the payoff of the inspector degrades when committing to a Nash equilibrium, that is, when he looses his ability to credibly commit. In fact, the Nash equilibrium is easy to compute and we regard it as an efficient heuristic for the inspector, but this is also the most harmful strategy for the network users (thus the name *spite*). We give two upper bounds for this measure. The first one depends on the detour done by the flow of users' best responses to the Nash controlling strategy, with respect to a particular metric. The second one is valid for networks with a distance-based toll: in this situation the price of spite is bounded from above by a constant which depends only on the ratio of the toll rate per kilometer to the average penalty to pay per evaded kilometer for a uniform control.

Finally, we report computational experiments on instances constructed from real data, for an application to the enforcement of a truck toll in Germany. These numerical results show the efficiency of the proposed methods, as well as the quality of the bounds derived in this article.

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Brief Announcement: A Model for Multilevel Network Games*

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Abstract Today's networks, like the Internet, do not consist of one but a mixture of several interconnected networks. Each has individual qualities and hence the performance of a network node results from the networks' interplay.

We introduce a new game theoretic model capturing the interplay between a high-speed backbone network and a low-speed general purpose network. In our model, n nodes are connected by a static network and each node can decide individually to become a gateway node. A gateway node pays a fixed price for its connection to the high-speed network, but can utilize the high-speed network to gain communication distance 0 to all other gateways. Communication distances in the low-speed network are given by the hop distances. The effective communication distance between any two nodes then is given by the shortest path, which is possibly improved by using gateways as shortcuts.

Every node v has the objective to minimize its communication costs, given by the sum (SUM-game) or maximum (MAX-game) of the effective communication distances from v to all other nodes plus a fixed price $\alpha > 0$, if it decides to be a gateway. For both games and different ranges of α, we study the existence of equilibria, the price of anarchy, and convergence properties of best-response dynamics.

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Complexity of Optimal Lobbying in Threshold Aggregation
(Brief Announcement)

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Optimal Lobbying is the problem a lobbyist or a campaign manager faces in a full-information voting scenario of a multi-issue referendum when trying to influence the result. The Lobby is faced with a profile that specifies for each voter and each issue whether the voter approves or rejects the issue, and seeks to find the smallest set of voters it must influence to change their vote, for a desired outcome to be obtained. This computational problem also describes problems arising in other scenarios of aggregating complex opinions, such as principal-agents incentives scheme in a complex combinatorial problem, and bribery and manipulation in Truth-Functional Judgement Aggregation. We study the computational complexity of Optimal Lobbying when the issues are aggregated using an anonymous monotone function and the family of desired outcomes is an upward-closed family. We analyze this problem with regard to two parameters: the minimal number of supporters needed to pass an issue, and the size of the maximal minterm of the desired set (the maximal issues set that is desired s.t. each subset of it is not desired). We show that for the extreme values of the parameters, the problem is tractable, and provide algorithms. On the other hand, we prove intractability of the problem for the non-extremal values, which are common values for the parameters.

Previous versions of this work were presented at AAMAS 2013 (Extended abstract) and at The 5th Israeli Game Theory conference.

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