

Approximating the Maximum Overlap of Polygons under Translation

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Abstract. Let P and Q be two simple polygons in the plane of total complexity n , each of which can be decomposed into at most k convex parts. We present an $(1 - \varepsilon)$ -approximation algorithm, for finding the translation of Q , which maximizes its area of overlap with P . Our algorithm runs in $O(cn)$ time, where c is a constant that depends only on k and ε .

This suggests that for polygons that are “close” to being convex, the problem can be solved (approximately), in near linear time.

1 Introduction

Shape matching is an important problem in databases, robotics, visualization and many other fields. Given two shapes, we want to find how similar (or dissimilar) they are. Typical problems include matching point sets by the Hausdorff distance metric, or matching polygons by the Hausdorff or Fréchet distance between their boundaries. See the survey by Alt and Guibas [5].

The maximum area of overlap is one possible measure for shape matching that is not significantly effected by noise. Mount *et al.* [18] studied the behavior of the area of overlap function, when one simple polygon is translated over another simple polygon. They showed that the function is continuous and piece-wise polynomial of degree at most two. If the polygons P and Q have complexity m and n , respectively, the area of overlap function can have complexity of $\Theta(m^2n^2)$. Known algorithms to find the maximum of the function work by constructing the entire overlap function. It is also known that the problem is 3SUM-Hard [8], that is, it is believed no subquadratic time algorithm is possible for the problem.

Approximating maximum overlap of general polygons. Cheong *et al.* [13] gave a $(1 - \varepsilon)$ -approximation algorithm for maximizing the area of overlap under translation of one simple polygon over the other using random sampling techniques. However, the error associated with the algorithm is additive, and the algorithm runs in near quadratic time. Specifically, the error is an ε fraction of the area of the smaller of the two polygons. Under rigid motions, the running time deteriorates to being near cubic. More recently, Cheng and Lam [12] improved the running times, and can also handle rigid motions, and present a near linear time approximation algorithm if one of the polygons is convex.

Maximum overlap in the convex case under translations. de Berg *et al.* [10] showed that finding maximum overlap translation is relatively easier in case of convex polygons. Specifically, the overlap function in this case is unimodal (as a consequence of the Brunn-Minkowski Theorem). Using this property, they gave a near linear time exact algorithm for computing the translation that maximizes the area of overlap of two convex polygons. The complexity of the graph of the overlap function is only $O(m^2 + n^2 + \min(m^2n, mn^2))$ in this case. Alt *et al.* [4] gave a constant-factor approximation for the minimum area of the symmetric difference of two convex polygons.

Approximating maximum overlap in the convex case. As for $(1-\epsilon)$ -approximation, assuming that the two polygons are provided in an appropriate form (i.e., the vertices are in an array in their order along the boundary of the polygon), then one can get a sub-linear time approximation algorithm. Specifically, Ahn *et al.* [3] show an $(1-\epsilon)$ -approximation algorithm, with running time $O((1/\epsilon) \log(n/\epsilon))$ for the case of translation, and $O((1/\epsilon) \log n + (1/\epsilon)^2 \log 1/\epsilon)$ for the case of rigid motions. (For a result using similar ideas in higher dimensions see the work by Chazelle *et al.* [11].)

Overlap of union of balls. de Berg *et al.* [9] considered the case where X and Y are disjoint unions of m and n unit disks, with $m \leq n$. They computed a $(1-\epsilon)$ approximation for the maximal area of overlap of X and Y under translations in time $O((nm/\epsilon^2) \log(n/\epsilon))$. Cheong *et al.* [13] gave an additive error ϵ -approximation algorithm for this case, with near linear running time.

Other relevant results. Avis *et al.* [6] computes the overlap of a polytope and a translated hyperplane in linear time, if the polytope is represented by a lattice of its faces. Vigneron [21] presented $(1-\epsilon)$ -approximation algorithms for maximum overlap of polyhedra (in constant dimension) that runs in polynomial time. Ahn *et al.* [1] approximates the maximum overlap of two convex polytopes in three dimensions under rigid motions. Ahn *et al.* [2] approximates the maximum overlap of two polytopes in \mathbb{R}^d under translation in $O(n^{\lfloor d/2 \rfloor + 1} \log^d n)$ time.

Our Results

As the above indicates, there is a big gap between the algorithms known for the convex and non-convex case. Our work aims to bridge this gap, showing that for “close” to convex polygons, under translation, the problem can be solved approximately in near linear time.

Specifically, assume we are given two polygons P and Q of total complexity n , such that they can be decomposed into k convex parts, we show that one can $(1-\epsilon)$ -approximate the translation of Q , which maximizes its area of overlap with P , in linear time (for k and ϵ constants). The translation returned has overlap area which is at least $(1-\epsilon)\mu_{\max}(P, Q)$, where $\mu_{\max}(P, Q)$ is the maximum area of overlap of the given polygons.

Approach. We break the two polygons into a minimum number of convex parts. We then approximate the overlap function for each pair of pieces (everywhere). This is required as one cannot just approximate the two polygons (as done by Ahn *et al.* [3]) since the optimal solution does not realize the maximum overlap of each pair of parts separately, and the alignment of each pair of parts might be arbitrary.

To this end, if the two convex parts are of completely different sizes, we approximate the smaller part, and approximate the overlap function by taking slices (i.e., level sets) of the overlap function. In the other case, where the two parts are “large”, which is intuitively easier, we can approximate both convex parts, and then the overlap function has constant complexity. Finally, we overlap all these functions together, argue that the overlap has low complexity, and find the maximum area of overlap.

Our approach has some overlap in ideas with the work of Ahn *et al.* [3]. In particular, a similar distinction between large and small overlap, as done in Section 4.1 and Section 4.2 was already done in [3, Theorem 17].

Why the “naive” solution fails? The naive solution to our problem is to break the two polygons into k convex polygons, and then apply to each pair of them the approximation of Ahn *et al.* [3]. Now, just treat the input polygon as the union of their respective approximations, and solve problem using brute force approach. This fails miserably as the approximation of Ahn *et al.* [3] captures only the maximum overlap of the two polygons. It does not, and can not, approximate the overlap if two convex polygons are translated such that their overlap is “far” from the maximum configuration, especially if the two polygons are of different sizes. This issue is demonstrated in more detail in the beginning of Section 4.1. A more detailed counterexample is presented in the full version of the paper [16].

Paper organization. We start in Section 2 by defining formally the problem, and review some needed results. In Section 3, we build some necessary tools. Specifically, we start in Section 3.1 by observing that one can get $O(1/\varepsilon)$ approximation of a convex polygon, where the error is an ε -fraction of the width of the polygon. In Section 3.2, we show how to compute a level set of the overlap function of two convex polygons efficiently. In Section 3.3, we show that, surprisingly, the polygon formed by the maximum overlap of two convex polygons, contains (up to scaling by a small constant and translation) the intersection of any translation of these two convex polygons. Among other things this implies an easy linear time constant factor approximation for the maximum overlap (which also follows, of course, by the result of Ahn *et al.* [3]). In Section 4, we present the technical main contribution of this paper, showing how to approximate, by a compact representation that has roughly linear complexity, the area overlap function of two convex polygons. In Section 5 we put everything together and present our approximation algorithm for the non-convex case.

2 Preliminaries

For any vector $\mathbf{t} \in \mathbb{R}^2$ and a set Q , let $\mathbf{t} + Q$ denote the translation of Q by \mathbf{t} ; formally, $\mathbf{t} + Q = \left\{ \mathbf{t} + \mathbf{q} \mid \mathbf{q} \in Q \right\}$. Also let $\mu(P, Q) = \text{area}(P \cap Q)$, which is the *area of overlap* of sets P and Q . We are interested in the following problem.

Problem 1. We are given two polygons X and Y in the plane, such that each can be decomposed into at most k convex polygons. The task is to compute the translation \mathbf{t} of Y , which maximizes the area of overlap between X and $\mathbf{t} + Y$. Specifically our purpose is to approximate the quantity

$$\mu_{\max}(X, Y) = \max_{\mathbf{t} \in \mathbb{R}^2} \mu(X, \mathbf{t} + Y).$$

For a polygon P , let $|P|$ denote the number of vertices of P . For $X, Y \subseteq \mathbb{R}^d$, the set X is *contained under translation* in Y , denoted by $X \sqsubseteq Y$, if there exists \mathbf{x} such that $\mathbf{x} + X \subseteq Y$.

Unimodal. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *unimodal*, if there is a value α , such that f is monotonically increasing (formally, non-decreasing) in the range $[-\infty, \alpha]$, and f is monotonically decreasing (formally, non-increasing) in the interval $[\alpha, +\infty]$.

From width to inner radius. For a convex polygon P , the *width* of P , denoted by $\omega(P)$, is the minimum distance between two parallel lines that enclose P .

Lemma 1 ([14]). *For a convex shape X in the plane, we have that the largest disk enclosed inside X , has radius at least $\text{width}(X) / 2\sqrt{3}$.*

Convex Decomposition of Simple Polygons. A vertex of a polygon is a *notch* if the internal angle at this vertex is reflex (i.e. $> 180^\circ$). For a non-convex polygon P with n vertices and r notches, Keil and Snoeyink [17] solves the minimal convex decomposition problem in $O(n + r^2 \min(r^2, n))$ time, that is, they compute a decomposition of P into minimum number of convex polygons. Observe, that if the number of components in the minimum convex decomposition is k , the number of notches r is upper bounded by $2k$.

Scaling similarity between polygons. For two convex polygons X and Y , let us define their *scaling similarity*, denoted by $\text{ssim}(X, Y)$, as the minimum number $\alpha \geq 0$, such that $X \sqsubseteq \alpha Y$. Using low-dimensional linear programming, one can compute $\text{ssim}(X, Y)$ in linear time. In particular, the work by Sharir and Toledo [19] implies the following.

Lemma 2 (ssim). *Given two convex polygons X and Y of total complexity n , one can compute, in linear time, $\text{ssim}(X, Y)$, and the translation that realizes it.*

3 Building Blocks

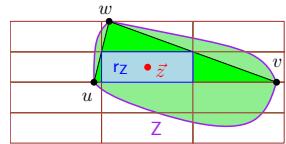
3.1 A Better Convex Approximation in the Plane

Let B be the minimum volume bounding box of some bounded convex set $K \subseteq \mathbb{R}^d$. We have that $v + c_d B \subseteq K \subseteq B$ [15], for some vector v and a constant c_d which depends only on the dimension d . This approximation can be computed in $O(n)$ time [7], where n is the number of vertices of the convex-hull of K . The more powerful result showing that a convex body can be approximated by an ellipsoid (up to a scaling factor of d), is known as John’s Theorem [15].

We need the following variant of the algorithm of Barequet and Har-Peled [7].

Lemma 3. *Given a convex polygon Z in the plane, with n vertices, one can compute, in linear time, a rectangle r_Z and a point z , such that $z + r_Z \subseteq Z \subseteq z + 5r_Z$.*

Proof. This is all well known, and we include the details for the sake of completeness. Using rotating caliper [20] compute the two vertices u and v of Z realizing its diameter. Let w be the vertex of Z furthest away from uv , Consider the rectangle r'_Z having its base on uv , having half the height of $\triangle uvw$, and contained inside this triangle. Now, let z be the center of r'_Z , and set $r_Z = r'_Z - z$, see figure on the right. It is now easy to verify that the claim holds with r_Z and z . □



Observation 1. *Given two bodies $X, Y \subseteq \mathbb{R}^2$ and a non-singular affine transformation M , we have $\frac{\text{area}(X)}{\text{area}(Y)} = \frac{\text{area}(M(X))}{\text{area}(M(Y))}$.*

Since a similar construction is described by Ahn *et al.* [3], we delegate the proof of this lemma to the full version of this paper [16].

Lemma 4 (approxPolygon). *Given a convex polygon P , and a parameter $m > 0$, we can compute, in $O(|P|)$ time, a convex polygon P' with $O(m)$ vertices, such that (i) $P' \subseteq P$, and (ii) for any point $p \in P$, its distance from P' is at most $\omega(P)/m$, where $\omega(P)$ is the width of P .*

3.2 The Level Set of the Area of Overlap Function

Definition 1. *The superlevel set of a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, for a value α is the set $L_\alpha(f) = \{p \in \mathbb{R}^d \mid f(p) \geq \alpha\}$. We will refer to it as the α -slice of f .*

Lemma 5. *Given two convex polygons X and Y , the slice $Z = L_\alpha(\mu(X, t + Y))$ is convex, and has complexity $O(m)$, where $m = |X| |Y|$. Furthermore, given a point $p \in Z$, the convex body Z can be computed in $O(m \log m)$ time.*

The proof is in the full version of the paper [16].

3.3 The Shape of the Polygon Realizing the Maximum Area Overlap

In the following, all the ellipses being considered are centered in the origin.

Lemma 6. *Given two ellipses \mathcal{E}_1 and \mathcal{E}_2 , the translation which maximizes their area of overlap is the one in which their centers are the same points.*

Proof. Translate \mathcal{E}_1 and \mathcal{E}_2 such that their centers are at the origin. Consider any unit vector \mathbf{u} , translate \mathcal{E}_2 along the direction of \mathbf{u} , and consider the behavior of the overlap function $f(x) = \mu(\mathcal{E}_1, \mathcal{E}_2 + x\mathbf{u})$, where x varies from $-\infty$ to $+\infty$. The function f is unimodal [10]. By symmetry, we have

$$f(x) = \mu(\mathcal{E}_1, \mathcal{E}_2 + x\mathbf{u}) = \mu(-\mathcal{E}_1, -(\mathcal{E}_2 + x\mathbf{u})) = \mu(\mathcal{E}_1, \mathcal{E}_2 - x\mathbf{u}) = f(-x),$$

as $\mathcal{E}_i = -\mathcal{E}_i$. If the maximum is attained at $x \neq 0$, we will get another maximum at $-x$, which implies, as f unimodal, that $f(0) = f(x) = f(-x)$, as desired. \square

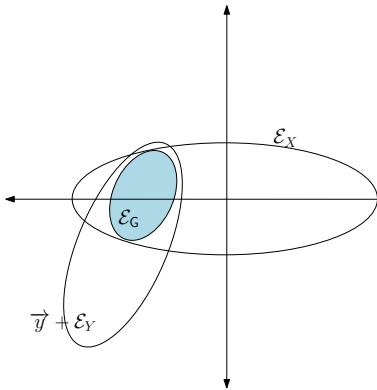


Fig. 3.1.

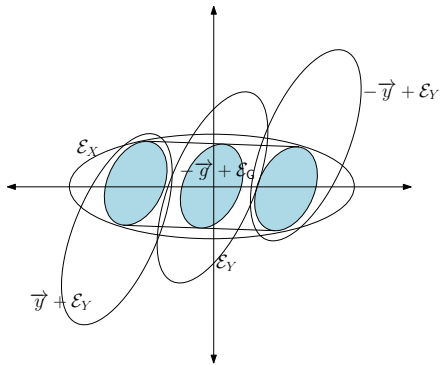


Fig. 3.2.

Lemma 7. *Consider two ellipses \mathcal{E}_X and \mathcal{E}_Y in the plane, and consider any two vectors \mathbf{x} and \mathbf{y} , then there is a vector \mathbf{u} such that $\mathbf{u} + (\mathbf{x} + \mathcal{E}_X) \cap (\mathbf{y} + \mathcal{E}_Y) \subseteq 2\mathcal{E}_X \cap 2\mathcal{E}_Y$.*

Proof. For the sake of simplicity of exposition, assume that $\mathbf{x} = 0$. Now, consider the intersection $G = \mathcal{E}_X \cap (\mathbf{y} + \mathcal{E}_Y)$, and let \mathcal{E}_G be the largest area ellipse contained inside G . John’s theorem implies that there is a translation vector \mathbf{g} , such that $\mathbf{g} + \mathcal{E}_G \subseteq G \subseteq \mathbf{g} + 2\mathcal{E}_G$, see Figure 3.1.

Observe that $\mathbf{g} + \mathcal{E}_G \subseteq \mathcal{E}_X$, and by the symmetry of \mathcal{E}_G and \mathcal{E}_X , we have that $-\mathbf{g} + \mathcal{E}_G = -\mathbf{g} - \mathcal{E}_G \subseteq -\mathcal{E}_X = \mathcal{E}_X$. This by convexity implies that $\mathcal{E}_G \subseteq \mathcal{E}_X$. A similar argument implies that $\mathcal{E}_G \subseteq \mathcal{E}_Y$. As such, $\mathcal{E}_G \subseteq \mathcal{E}_X \cap \mathcal{E}_Y$.

Thus, we have that $G \subseteq \mathbf{g} + 2\mathcal{E}_G \subseteq \mathbf{g} + 2\mathcal{E}_X \cap 2\mathcal{E}_Y$, as desired. \square

Lemma 8 ([15, Lemma 22.5]). *Any convex set $K \subseteq \mathbb{R}^d$ contained in a unit square, contains a ball of radius $\text{area}(K) / 8$*

The following lemma is one of our key insights – the maximum area of intersection of two polygons contains any intersection of translated copies of these polygons up to translation and a constant factor scaling.

Lemma 9. *Let X and Y be two convex polygons, and let M be the polygon realizing their maximum area of intersection under translation. Let \mathbf{u} be any vector in the plane, and consider the polygon $D = X \cap (\mathbf{u} + Y)$, then there exists a vector \mathbf{v} such that, $\mathbf{v} + D \subseteq c_0 M$, for some fixed constant c_0 .*

Proof. Let \mathcal{E}_X (resp., \mathcal{E}_Y) denote the maximum area ellipse (centered at the origin) contained inside X (resp. Y). By John’s Theorem, we have $\mathbf{x} + \mathcal{E}_X \subseteq X \subseteq \mathbf{x} + 2\mathcal{E}_X$ and $\mathbf{y} + \mathcal{E}_Y \subseteq Y \subseteq \mathbf{y} + 2\mathcal{E}_Y$, where \mathbf{x}, \mathbf{y} are some vector. Let $B = \mathcal{E}_X \cap \mathcal{E}_Y$, and let \mathcal{E}_B be the maximum area ellipse contained inside B . Observe that B is symmetric and centered at the origin, and by John’s theorem $\mathcal{E}_B \subseteq B \subseteq 2\mathcal{E}_B$.

By Lemma 7, there are vectors \mathbf{z} and \vec{w} , such that

$$D = X \cap (\mathbf{u} + Y) \subseteq (\mathbf{x} + 2\mathcal{E}_X) \cap (\mathbf{z} + \mathbf{y} + 2\mathcal{E}_Y) \subseteq \vec{w} + 4\mathcal{E}_X \cap 4\mathcal{E}_Y = \vec{w} + 4B \subseteq \vec{w} + 8\mathcal{E}_B.$$

Applying a similar argument, we have that $M \subseteq \vec{m} + 8\mathcal{E}_B$, for some vector \vec{m} .

Apply the linear transformation that maps \mathcal{E}_B to $\text{disk}(1/16)$, where $\text{disk}(r)$ denotes the disk of radius r centered at the origin. By Observation 1, we can continue our discussion in the transformed coordinates. This implies that $M - \vec{m} \subseteq \text{disk}(1/2)$ (which is contained inside a unit square). By Lemma 8, there is a vector \mathbf{x}_1 , such that $\mathbf{x}_1 + \text{disk}(\text{area}(M) / 8) \subseteq M$.

Observe that $B = \mathcal{E}_X \cap \mathcal{E}_Y \subseteq (-\mathbf{x} + X) \cap (-\mathbf{y} + Y)$. As such, the area of B must be smaller than the area of M (by the definition of M). We thus have $\text{area}(M) \geq \text{area}(B) \geq \text{area}(\mathcal{E}_B) = \text{area}(\text{disk}(1/16))$ which is a constant bounded away from zero. Therefore,

$$\begin{aligned} D &\subseteq \vec{w} + 8\mathcal{E}_B = \vec{w} + \text{disk}\left(\frac{1}{2}\right) = \vec{w} + \frac{4}{\text{area}(M)} \cdot \text{disk}\left(\frac{\text{area}(M)}{8}\right) \\ &\subseteq \vec{w} + \frac{4}{\text{area}(M)}(M - \mathbf{x}_1), \end{aligned}$$

which implies the claim. □

Constant Approximation to the Maximum Overlap

Lemma 10 (constApproxByRect). *Let X and Y be two convex polygons, and let M be the polygon realizing their maximum area intersection under translation. Then, one can compute, in $O(|X| + |Y|)$ time, a rectangle r , such that $r \subseteq \mathbf{u} + M \subseteq c_r r$, where c_r is a constant. That is, one can compute a constant factor approximation to the maximum area overlap in linear time.*

Furthermore, for any translation \mathbf{t}_Y , we have that $X \cap (Y + \mathbf{t}_Y) \subseteq c_r r$.

Proof. We are going to implement the algorithmic proof of Lemma 9. Instead of John’s ellipsoid we use the rectangle of Lemma 3. Clearly, the proof of Lemma 9 goes through with the constants being somewhat worse. Specifically, we compute, in linear time, vectors \mathbf{x}, \mathbf{y} , and rectangles r_X, r_Y , such that $\mathbf{x} + r_X \subseteq X \subseteq \mathbf{x} + 5r_X$ and $\mathbf{y} + r_Y \subseteq Y \subseteq \mathbf{y} + 5r_Y$. Again, compute a rectangle r_M , such that $r_M/5 \subseteq r_X \cap r_Y \subseteq r_M$. Arguing as in Lemma 9, and setting $r = r_M/c_3$, for some constant c_3 , is the desired rectangle. \square

4 Approximating the Overlap Function of Convex Polygons

Definition 2 Given two convex polygons X and Y in the plane, of total complexity n , and parameters $\varepsilon \in (0, 1)$, ν, ρ , a function $\psi(\mathbf{t})$ is (ε, ν, ρ) -**approximation** of $\mu(X, \mathbf{t} + Y)$, if the following conditions hold:

- (A) $\forall \mathbf{t} \in \mathbb{R}^2$, we have $|\mu(X, \mathbf{t} + Y) - \psi(\mathbf{t})| \leq \varepsilon \mu_{\max}(X, Y)$.
- (B) There are convex polygons P_1, \dots, P_ν , each of maximum complexity ρ , such that inside every face of the arrangement $\mathcal{A} = \mathcal{A}(P_1, \dots, P_\nu)$, the approximation function $\psi(\mathbf{t})$ is the same quadratic function.

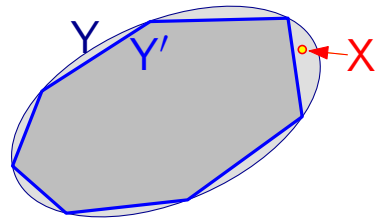
That is, the total descriptive complexity of $\psi(\cdot)$ is the complexity of the arrangement \mathcal{A} .

Algorithm 3 The input is two convex polygons X and Y in the plane, of total complexity n , and a parameter $\varepsilon \in (0, 1)$. As a first step, the algorithm is going to approximate X and Y as follows:

- (A) $r_M \leftarrow \text{constApproxByRect}(X, Y)$, see Lemma 10.
 - (B) $\mathcal{T} \leftarrow$ affine transformation that maps $2c_\tau r_M$ to $[0, 1]^2$.
 - (C) $X'_\mathcal{T} \leftarrow \text{approxPolygon}(\mathcal{T}(X), N)$ and $Y'_\mathcal{T} \leftarrow \text{approxPolygon}(\mathcal{T}(Y), N)$.
- See Lemma 4, here $N = \lceil c_4/\varepsilon \rceil$, and c_4 is a sufficiently large constant.
- (D) $X' \leftarrow \mathcal{T}^{-1}(X'_\mathcal{T})$ and $Y' \leftarrow \mathcal{T}^{-1}(Y'_\mathcal{T})$.

4.1 If One Polygon is Smaller than the Other

Assume, without loss of generality, that X is smaller than Y , that is, X can be translated so that it is entirely contained inside Y (i.e., $\text{ssim}(X, Y) \leq 1$, see Lemma 2). The maximum area of overlap is now equal to $\text{area}(X)$. The challenge is, that for any approximation of Y , we can always have a sufficiently small X which can be placed in $Y \setminus Y'$, as shown in the figure on the right. Therefore for all those translations for which X is placed inside $Y \setminus Y'$, our approximation will show zero overlap, even though the actual overlap is $\text{area}(X)$.



To get around this problem, we will first approximate the smaller polygon X , using our approximation scheme, to get polygon X' , then we will compute level sets of the overlap function and use them to approximate it.

Lemma 11. *Given convex polygons X and Y , such that $\text{ssim}(X, Y) < 1$, and parameter $\varepsilon > 0$, and let X' be the approximation to X , as computed by Algorithm 3. Then, we have, for all translations $t \in \mathbb{R}^2$, that $\left| \mu(X', t + Y) - \mu(X, t + Y) \right| \leq \varepsilon \mu_{\max}(X, Y)$.*

Proof. Consider the overlap of $X_{\mathcal{T}} = \mathcal{T}(X)$ and $Y_{\mathcal{T}} = \mathcal{T}(Y)$. Lemma 10 implies that any intersection polygon of $X_{\mathcal{T}}$ and $Y_{\mathcal{T}}$ can be contained (via translation) in $\mathcal{T}(c_r r_M)$ (which is a translation of the square $[0, 1/2]^2$). Clearly, in this case, $X_{\mathcal{T}}$ and $X'_{\mathcal{T}}$ can both be translated to be contained in this square, both contain a disk of constant radius, the maximum distance between $X_{\mathcal{T}}$ and $X'_{\mathcal{T}}$ is $O(\varepsilon)$, and the total area of $X_{\mathcal{T}} \setminus X'_{\mathcal{T}}$ is $O(\varepsilon)$, as the perimeter of $X_{\mathcal{T}} \leq 4$. Thus, setting c_4 to be sufficiently large, implies that $\text{area}(X_{\mathcal{T}} \setminus X'_{\mathcal{T}}) \leq \varepsilon \mu_{\max}(X_{\mathcal{T}}, Y_{\mathcal{T}})$, as $\mu_{\max}(X_{\mathcal{T}}, Y_{\mathcal{T}}) = \Omega(1)$. This implies that $\left| \mu(X'_{\mathcal{T}}, t + Y_{\mathcal{T}}) - \mu(X_{\mathcal{T}}, t + Y_{\mathcal{T}}) \right| \leq \varepsilon \mu_{\max}(X_{\mathcal{T}}, Y_{\mathcal{T}})$, which implies the claims by applying \mathcal{T}^{-1} to both sides. \square

Therefore, $\mu(X', t + Y)$ is a good approximation for $\mu(X, t + Y)$. However, $\mu(X', t + Y)$ has complexity $O(|X'|^2 |Y|^2)$ [10], in the worst case, which is still too high.

Lemma 12 (approxLevelSet). *Given two convex polygons X and Y , of total complexity n , and a parameter ε , such that $\text{ssim}(X, Y) < 1$, then one can construct in $O(n/\varepsilon^2)$ time, a $(\varepsilon, O(1/\varepsilon^2), O(n/\varepsilon^2))$ -approximation $\psi(\cdot)$ to $\mu(X, t + Y)$.*

Proof. There is a translation of X such that it is contained completely in Y . Approximate X from the outside by a rectangle r , using Lemma 3. Next, spread a grid in r by partitioning each of its edges into $O(1/\varepsilon)$ equal length intervals. Let S be the set of points of the grid that are in X . It is easy to verify, that for any convex body Z and a translation t , we have

$$\left| \mu(X, t + Z) - \frac{|(t + Z) \cap S|}{|S|} \right| \leq \varepsilon \text{area}(X)$$

Namely, to approximate the overlap area for $t + Y$, we need to count the number of points of S that it covers. To this end, for each point $p \in S$, we generate a 180° rotated and translated copy of Y , denoted by Y'_p , such that $p \in t + Y$ if and only if $t \in Y'_p$.

Clearly, the generated set of polygons is the desired $(\varepsilon, O(1/\varepsilon^2), O(n/\varepsilon^2))$ -approximation $\psi(\cdot)$ to $\mu(X, t + Y)$.

The time to build this approximation is $O(n/\varepsilon^2)$.

We next describe a slightly slower algorithm that generates a slightly better approximation.

Lemma 13 (approxLevelSet). *Given two convex polygons X and Y , of total complexity n , and a parameter ε , such that $\text{ssim}(X, Y) < 1$, then one can construct in $O(\varepsilon^{-2} n \log n)$ time, a $(\varepsilon, O(1/\varepsilon), O(n/\varepsilon))$ -approximation $\psi(\cdot)$ to $\mu(X, t + Y)$.*

The proof is in the full version of the paper [16].

4.2 If the Two Polygons are Incomparable

The more interesting case, is when the maximum intersection of X and Y is significantly smaller than both polygons; that is, $\text{ssim}(X, Y) \geq 1$ and $\text{ssim}(Y, X) \geq 1$. Surprisingly, in this case, we can approximate both polygons simultaneously.

Lemma 14. *Given convex polygons X and Y , such that $\text{ssim}(X, Y) \geq 1$ and $\text{ssim}(Y, X) \geq 1$, then the widths of $X_{\mathcal{T}} = \mathcal{T}(X)$ and $Y_{\mathcal{T}} = \mathcal{T}(Y)$, as computed by Algorithm 3, are bounded by 7.*

The proof is in the full version of the paper [16].

Lemma 15. *Given two convex polygons X and Y , of total complexity n , and a parameter ε , such that $\text{ssim}(X, Y) \geq 1$ and $\text{ssim}(Y, X) \geq 1$, then one can construct in $O(n + 1/\varepsilon^2)$ time, a $(\varepsilon, O(1/\varepsilon), O(1/\varepsilon))$ -approximation $\psi(\cdot)$ to $\mu(X, \mathbf{t} + Y)$.*

The proof is in the full version of the paper [16].

The result. By combining Lemma 12 and Lemma 15 (deciding which one to apply can be done by computing $\text{ssim}(X, Y)$ and $\text{ssim}(Y, X)$, which takes $O(n)$ time), we get the following.

Lemma 16. *Given two convex polygons X and Y , of total complexity n , and a parameter ε , one can construct in $O(n/\varepsilon^2)$ time, a $(\varepsilon, O(1/\varepsilon^2), O(n/\varepsilon^2))$ -approximation $\psi(\cdot)$ to $\mu(X, \mathbf{t} + Y)$.*

5 Approximating the Maximum Overlap of Polygons

The input is two polygons P and Q in the plane, of total complexity n , each of them can be decomposed into at most k convex polygons. Our purpose is to find the translation that maximizes the area of overlap.

The Algorithm. We decompose the polygons P and Q into minimum number of interior disjoint convex polygons [17], in time $O(n + k^2 \min(k^2, n))$ (some of these convex polygons can be empty). Then, for every pair P_i, Q_j , we compute an $(\varepsilon, O(1/\varepsilon^2), O(n/\varepsilon^2))$ -approximation ψ_{ij} to the overlap function of P_i and Q_j , using Lemma 16, where $\varepsilon = \varepsilon/k^2$.

Next, as each function ψ_{ij} is defined by an arrangement defined by $O(1/\varepsilon^2)$ polygons, we overlay all these arrangements together, and compute for each face of the arrangement the function $\psi = \sum_{i,j} \psi_{ij}$. Inside such a face this function is the same, and it is a quadratic function. We then find the global maximum of this function, and return it as the desired approximation.

Analysis – Quality of approximation. For any translation \mathbf{t} , we have that

$$\begin{aligned} \left| \mu(P, \mathbf{t} + Q) - \psi(\mathbf{t}) \right| &\leq \sum_{i=1}^k \sum_{j=1}^k \left| \mu(P_i, \mathbf{t} + Q_j) - \psi_{ij}(\mathbf{t}) \right| \leq \sum_{i=1}^k \sum_{j=1}^k \varepsilon \mu_{\max}(P_i, Q_j) \\ &\leq \varepsilon k^2 \mu_{\max}(P, Q) \leq \varepsilon \mu_{\max}(P, Q). \end{aligned}$$

Analysis – Running time. Computing each of the k^2 approximation function, takes $O((k/\varepsilon)^2 n)$ time. Each one of them is a $(\varepsilon/k, O(k^2/\varepsilon^2), O(k^2 n/\varepsilon^2))$ -approximation, which means that the final arrangement is the overlay of $O(k^4/\varepsilon^2)$ convex polygons, each of complexity $O(k^2 n/\varepsilon^2)$. In particular, any pair of such polygons can have at most $O(k^2 n/\varepsilon^2)$ intersection points, and thus the overall complexity of the arrangement of these polygons is $N = O\left((k^4/\varepsilon^2)^2 (k^2 n/\varepsilon^2)\right) = O(k^{10} \varepsilon^{-6} n)$. Computing this arrangement can be done by a standard sweeping algorithm. Observing that every vertical line crosses only $O(k^4/\varepsilon^2)$ segments, imply that the sweeping can be done in $O(\log(k/\varepsilon))$ time per operation, which implies that the overall running time is

$$O\left(k^2 \frac{k^2}{\varepsilon^2} n + N \log \frac{k}{\varepsilon}\right) = O\left(\frac{k^{10}}{\varepsilon^6} n \log \frac{k}{\varepsilon}\right).$$

The result.

Theorem 4. *Given two simple polygons P and Q of total complexity n, one can compute a translation which ε -approximates the maximum area of overlap of P and Q. The time required is $O(c'n)$ where $c' = \frac{k^{10}}{\varepsilon^6} \log \frac{k}{\varepsilon}$, where k is the minimum number of convex polygons in the decomposition of P and Q.*

More specifically, one gets a data-structure, such that for any query translation \mathbf{t} , one can compute, in $O(\log n)$ time, an approximation $\psi(\mathbf{t})$, such that $|\psi(\mathbf{t}) - \mu(\mathbf{P}, \mathbf{Q})| \leq \varepsilon \mu_{\max}(\mathbf{P}, \mathbf{Q})$, where $\mu_{\max}(\mathbf{P}, \mathbf{Q})$ is the maximum area of overlap between P and Q.

Note, that our analysis is far from tight. Specifically, for the sake of simplicity of exposition, it is loose in several places as far as the dependency on k and ε .

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References

1. Ahn, H.K., Cheng, S.W., Kweon, H.J., Yon, J.: Overlap of convex polytopes under rigid motion. *Comput. Geom. Theory Appl.* 47(1), 15–24 (2014), <http://dx.doi.org/10.1016/j.comgeo.2013.08.001>
2. Ahn, H.K., Cheng, S.W., Reinbacher, I.: Maximum overlap of convex polytopes under translation. *Comput. Geom. Theory Appl.* 46(5), 552–565 (2013), <http://dx.doi.org/10.1016/j.comgeo.2011.11.003>
3. Ahn, H.K., Cheong, O., Park, C.D., Shin, C.S., Vigneron, A.: Maximizing the overlap of two planar convex sets under rigid motions. *Comput. Geom. Theory Appl.* 37(1), 3–15 (2007)
4. Alt, H., Fuchs, U., Rote, G., Weber, G.: Matching convex shapes with respect to the symmetric difference. *Algorithmica* 21, 89–103 (1998), <http://citeseer.nj.nec.com/267158.html>

5. Alt, H., Guibas, L.J.: Discrete geometric shapes: Matching, interpolation, and approximation. In: Sack, J.R., Urrutia, J. (eds.) *Handbook of Computational Geometry*, pp. 121–153. Elsevier (2000)
6. Avis, D., Bose, P., Toussaint, G.T., Shermer, T.C., Zhu, B., Snoeyink, J.: On the sectional area of convex polytopes. In: *Proc. 12th Annu. Sympos. Comput. Geom.*, pp. 411–412 (1996)
7. Barequet, G., Har-Peled, S.: Efficiently approximating the minimum-volume bounding box of a point set in three dimensions. *J. Algorithms* 38, 91–109 (2001), <http://cs.uiuc.edu/~sariel/research/papers/98/bbox.html>
8. Barequet, G., Har-Peled, S.: Polygon containment and translational min-hausdorff-distance between segment sets are 3sum-hard. *Internat. J. Comput. Geom. Appl.* 11(4), 465–474 (2001)
9. de Berg, M., Cabello, S., Giannopoulos, P., Knauer, C., van Oostrum, R., Veltkamp, R.C.: Maximizing the area of overlap of two unions of disks under rigid motion. In: Hagerup, T., Katajainen, J. (eds.) *SWAT 2004*. LNCS, vol. 3111, pp. 138–149. Springer, Heidelberg (2004)
10. de Berg, M., Cheong, O., Devillers, O., van Kreveld, M., Teillaud, M.: Computing the maximum overlap of two convex polygons under translations. *Theo. Comp. Sci.* 31, 613–628 (1998), <http://link.springer-ny.com/link/service/journals/00224/bibs/31n5p613.html>
11. Chazelle, B., Liu, D., Magen, A.: Sublinear geometric algorithms. *SIAM J. Comput.* 35(3), 627–646 (2005)
12. Vigneron, A.: Geometric optimization and sums of algebraic functions. *ACM Trans. Algo.* 4, 1–4 (2014), <http://doi.acm.org/10.1145/2532647>
13. Cheong, O., Efrat, A., Har-Peled, S.: On finding a guard that sees most and a shop that sells most. *Discrete Comput.* 37(4), 545–563 (2007), <http://link.springer-ny.com/link/service/journals/00454/>
14. Gritzmann, P., Klee, V.: Inner and outer j -radii of convex bodies in finite-dimensional normed spaces. *Discrete Comput. Geom.* 7, 255–280 (1992), <http://link.springer-ny.com/link/service/journals/00454/>
15. Har-Peled, S.: *Geometric Approximation Algorithms*. *Mathematical Surveys and Monographs*, vol. 173. Amer. Math. Soc. (2011)
16. Har-Peled, S., Roy, S.: Approximating the maximum overlap of polygons under translation. *CoRR* abs/1406.5778 (2014), <http://arxiv.org/abs/1406.5778>
17. Keil, J.M., Snoeyink, J.: On the time bound for convex decomposition of simple polygons. *Internat. J. Comput. Geom. Appl.* 12(3), 181–192 (2002)
18. Mount, D.M., Silverman, R., Wu, A.Y.: On the area of overlap of translated polygons. *Computer Vision and Image Understanding: CVIU* 64(1), 53–61 (1996), <http://www.cs.umd.edu/~mount/Papers/overlap.ps>
19. Sharir, M., Toledo, S.: Extremal polygon containment problems. *Comput. Geom. Theory Appl.* 4, 99–118 (1994)
20. Toussaint, G.T.: Solving geometric problems with the rotating calipers. In: *Proc. IEEE MELECON 1983*. pp. A10.02/1–4 (1983)
21. Vigneron, A.: Geometric optimization and sums of algebraic functions. *ACM Trans. Algo.* 4, 1–4 (2014), <http://doi.acm.org/10.1145/2532647>