Chapter 8
Distances on Surfaces and Knots

8.1 General Surface Metrics

A *surface* is a real 2D (two-dimensional) *manifold* M^2, i.e., a **Hausdorff space**, each point of which has a neighborhood which is homeomorphic to a plane $\mathbb{E}^2$, or a closed half-plane (cf. Chap. 7).

A compact orientable surface is called *closed* if it has no boundary, and it is called a *surface with boundary*, otherwise. There are compact nonorientable surfaces (closed or with boundary); the simplest such surface is the *Möbius strip*. Noncompact surfaces without boundary are called *open*.

Any closed connected surface is homeomorphic to either a sphere with, say, g (cylindric) handles, or a sphere with, say, g *cross-caps* (i.e., caps with a twist like Möbius strip in them). In both cases the number g is called the *genus* of the surface. In the case of handles, the surface is orientable; it is called a *torus* (doughnut), *double torus*, and *triple torus* for $g = 1, 2$ and 3, respectively. In the case of cross-caps, the surface is nonorientable; it is called the *real projective plane*, *Klein bottle*, and *Dyck's surface* for $g = 1, 2$ and 3, respectively. The genus is the maximal number of disjoint simple closed curves which can be cut from a surface without disconnecting it (the *Jordan curve theorem* for surfaces).

The *Euler–Poincaré characteristic* of a surface is (the same for all polyhedral decompositions of a given surface) the number $\chi = v - e + f$, where v, e and f are, respectively, the number of vertices, edges and faces of the decomposition. Then $\chi = 2 - 2g$ if the surface is orientable, and $\chi = 2 - g$ if not. Every surface with boundary is homeomorphic to a sphere with an appropriate number of (disjoint) *holes* (i.e., what remains if an open disk is removed) and handles or cross-caps. If h is the number of holes, then $\chi = 2 - 2g - h$ holds if the surface is orientable, and $\chi = 2 - g - h$ if not.

The *connectivity number* of a surface is the largest number of closed cuts that can be made on the surface without separating it into two or more parts. This number is equal to $3 - \chi$ for closed surfaces, and $2 - \chi$ for surfaces with boundaries. A surface

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with connectivity number 1, 2 and 3 is called, respectively, *simply, doubly* and *triply connected*. A sphere is simply connected, while a torus is triply connected.

A surface can be considered as a metric space with its own **intrinsic metric**, or as a figure in space. A surface in $\mathbb{E}^3$ is called *complete* if it is a **complete** metric space with respect to its intrinsic metric.

Useful *shape-aware* (preserved by isomorphic deformations of the surface) distances on the interior of a surface mesh can be defined by isometric embedding of the surface into a suitable high-dimensional Euclidean space; for example, **diffusion metric** (cf. Chap. 15 and **histogram diffusion distance** from Chap. 21) and Rustamov et al., 2009.

A surface is called *differentiable*, *regular*, or *analytic*, respectively, if in a neighborhood of each of its points it can be given by an expression

$$r = r(u, v) = r(x_1(u, v), x_2(u, v), x_3(u, v)),$$

where the *position vector* $r = r(u, v)$ is a differentiable, *regular* (i.e., a sufficient number of times differentiable), or *real analytic*, respectively, vector function satisfying the condition $r_u \times r_v \neq 0$.

Any regular surface has the intrinsic metric with the *line element* (or *first fundamental form*)

$$ds^2 = dr^2 = E(u, v)du^2 + 2F(u, v)du\,dv + G(u, v)dv^2,$$

where $E(u, v) = \langle r_u, r_u\rangle$, $F(u, v) = \langle r_u, r_v\rangle$, $G(u, v) = \langle r_v, r_v\rangle$. The length of a curve defined on the surface by the equations $u = u(t)$, $v = v(t)$, $t \in [0, 1]$, is computed by

$$\int_0^1 \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt,$$

and the distance between any points $p, q \in M^2$ is defined as the infimum of the lengths of all curves on M^2, connecting p and q. A **Riemannian metric** is a generalization of the first fundamental form of a surface.

For surfaces, two kinds of *curvature* are considered: *Gaussian curvature*, and *mean curvature*. To compute these curvatures at a given point of the surface, consider the intersection of the surface with a plane, containing a fixed *normal vector*, i.e., a vector which is perpendicular to the surface at this point. This intersection is a plane curve. The *curvature* k of this plane curve is called the *normal curvature* of the surface at the given point. If we vary the plane, the normal curvature k will change, and there are two extremal values, the *maximal curvature* k_1, and the *minimal curvature* k_2, called the *principal curvatures* of the surface. A curvature is taken to be *positive* if the curve turns in the same direction as the surface's chosen normal, otherwise it is taken to be *negative*.

The *Gaussian curvature* is $K = k_1 k_2$ (it can be given entirely in terms of the first fundamental form). The *mean curvature* is $H = \frac{1}{2}(k_1 + k_2)$.

A *minimal surface* is a surface with mean curvature zero or, equivalently, a surface of minimum area subject to constraints on the location of its boundary.

A *Riemann surface* is a one-dimensional *complex manifold*, or a 2D real manifold with a complex structure, i.e., in which the local coordinates in neighborhoods of points are related by complex analytic functions. It can be thought of as a deformed version of the complex plane. All Riemann surfaces are orientable. Closed Riemann surfaces are geometrical models of *complex algebraic curves*. Every connected Riemann surface can be turned into a *complete* 2D *Riemannian manifold* with constant curvature $-1, 0$, or 1. The Riemann surfaces with curvature -1 are called *hyperbolic*, and the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the canonical example. The Riemann surfaces with curvature 0 are called *parabolic*, and $\mathbb{C}$ is a typical example. The Riemann surfaces with curvature 1 are called *elliptic*, and the *Riemann sphere* $\mathbb{C} \cup \{\infty\}$ is a typical example.

- **Regular metric**

  The intrinsic metric of a surface is **regular** if it can be specified by the *line element*

  $$ds^2 = E du^2 + 2F dudv + G dv^2,$$

  where the coefficients of the form ds^2 are regular functions.

  Any regular surface, given by an expression $r = r(u, v)$, has a regular metric with the *line element* ds^2, where $E(u, v) = \langle r_u, r_u \rangle$, $F(u, v) = \langle r_u, r_v \rangle$, $G(u, v) = \langle r_v, r_v \rangle$.
- **Analytic metric**

  The intrinsic metric on a surface is **analytic** if it can be specified by the *line element*

  $$ds^2 = E du^2 + 2F dudv + G dv^2,$$

  where the coefficients of the form ds^2 are real analytic functions.

  Any analytic surface, given by an expression $r = r(u, v)$, has an analytic metric with the *line element* ds^2, where $E(u, v) = \langle r_u, r_u \rangle$, $F(u, v) = \langle r_u, r_v \rangle$, $G(u, v) = \langle r_v, r_v \rangle$.
- **Metric of nonpositive curvature**

  A **metric of nonpositive curvature** is the intrinsic metric on a *saddle-like surface*. A *saddle-like surface* is a generalization of a surface of negative curvature: a twice continuously-differentiable surface is a saddle-like surface if and only if at each point of the surface its Gaussian curvature is nonpositive.

  These surfaces can be seen as antipodes of *convex surfaces*, but they do not form such a natural class of surfaces as do convex surfaces.

  A **metric of negative curvature** is the intrinsic metric on a *surface of negative curvature*, i.e., a surface in $\mathbb{E}^3$ that has negative Gaussian curvature at every point.

A surface of negative curvature locally has a saddle-like structure. The intrinsic geometry of a surface of constant negative curvature (in particular, of a *pseudo-sphere*) locally coincides with the geometry of the *Lobachevsky plane*. There exists no surface in $\mathbb{E}^3$ whose intrinsic geometry coincides completely with the geometry of the Lobachevsky plane (i.e., a complete regular surface of constant negative curvature).

- **Metric of nonnegative curvature**
  A **metric of nonnegative curvature** is the intrinsic metric on a *convex surface*.
  A *convex surface* is a *domain* (i.e., a connected open set) on the boundary of a *convex body* in $\mathbb{E}^3$ (in some sense, it is an antipode of a saddle-like surface).
  The entire boundary of a convex body is called a *complete convex surface*. If the body is finite (bounded), the complete convex surface is called *closed*. Otherwise, it is called *infinite* (an infinite convex surface is homeomorphic to a plane or to a circular cylinder).
  Any convex surface M^2 in $\mathbb{E}^3$ is a *surface of bounded curvature*. The *total Gaussian curvature* $w(A) = \int\int_A K(x)d\sigma(x)$ of a set $A \subset M^2$ is always nonnegative (here $\sigma(.)$ is the *area*, and $K(x)$ is the *Gaussian curvature* of M^2 at a point x), i.e., a convex surface can be seen as a *surface of nonnegative curvature*.
  The intrinsic metric of a convex surface is a **convex metric** (not to be confused with **metric convexity** from Chap. 1) in the sense of Surface Theory, i.e., it displays the *convexity condition*: the sum of the angles of any triangle whose sides are shortest curves is not less that π.
  A **metric of positive curvature** is the intrinsic metric on a *surface of positive curvature*, i.e., a surface in $\mathbb{E}^3$ that has positive Gaussian curvature at every point.
- **Metric with alternating curvature**
  A **metric with alternating curvature** is the intrinsic metric on a surface with alternating (positive or negative) Gaussian curvature.
- **Flat metric**
  A **flat metric** is the intrinsic metric on a *developable surface*, i.e., a surface, on which the Gaussian curvature is everywhere zero. Cf. **flat space** in Chap. 1.
  In general, a Riemannian metric on a surface is locally Euclidean up to a third order error (distortion of metric) measured by the Gaussian curvature.
- **Metric of bounded curvature**
  A **metric of bounded curvature** is the intrinsic metric ρ on a *surface of bounded curvature*.
  A surface M^2 with an intrinsic metric ρ is called a *surface of bounded curvature* if there exists a sequence of **Riemannian metrics** ρ_n defined on M^2, such that $\rho_n \rightarrow \rho$ uniformly for any compact set $A \subset M^2$, and the sequence $|w_n|(A)$ is bounded, where $|w|_n(A) = \int\int_A |K(x)|d\sigma(x)$ is the *total absolute curvature* of the metric ρ_n (here $K(x)$ is the Gaussian curvature of M^2 at a point x, and $\sigma(.)$ is the *area*).
- **Λ-Metric**
  A **Λ-metric** (or *metric of type* Λ) is a **complete** metric on a surface with curvature bounded from above by a negative constant.

A Λ-metric does not have embeddings into $\mathbb{E}^3$. It is a generalization of the result in Hilbert, 1901: no complete regular surface of constant negative curvature (i.e., a surface whose intrinsic geometry is the geometry of the Lobachevsky plane) exists in $\mathbb{E}^3$.

- **(h, Δ)-metric**
  A **(h, Δ)-metric** is a metric on a surface with a slowly-changing negative curvature.
  A **complete** (h, Δ)-metric does not permit a regular *isometric embedding* in three-dimensional Euclidean space (cf. **Λ-metric**).
- **G-distance**
  A connected set G of points on a surface M^2 is called a *geodesic region* if, for each point $x \in G$, there exists a *disk* $B(x, r)$ with center at x, such that $B_G = G \cap B(x, r)$ has one of the following forms: $B_G = B(x, r)$ (x is a *regular interior point* of G); B_G is a semidisk of $B(x, r)$ (x is a *regular boundary point* of G); B_G is a sector of $B(x, r)$ other than a semidisk (x is an *angular point* of G); B_G consists of a finite number of sectors of $B(x, r)$ with no common points except x (a *nodal point* of G).
  The **G-distance** between any x and $y \in G$ is the greatest lower bound of the lengths of all rectifiable curves connecting x and $y \in G$ and completely contained in G.
- **Conformally invariant metric**
  Let R be a Riemann surface. A *local parameter* (or *local uniformizing parameter*, *local uniformizer*) is a complex variable z considered as a continuous function $z_{p_0} = \phi_{p_0}(p)$ of a point $p \in R$ which is defined everywhere in some neighborhood (*parametric neighborhood*) $V(p_0)$ of a point $p_0 \in R$ and which realizes a homeomorphic mapping (*parametric mapping*) of $V(p_0)$ onto the disk (*parametric disk*) $\Delta(p_0) = \{z \in \mathbb{C} : |z| < r(p_0)\}$, where $\phi_{p_0}(p_0) = 0$. Under a parametric mapping, any point function $g(p)$ defined in the parametric neighborhood $V(p_0)$, goes into a function of the local parameter z: $g(p) = g(\phi_{p_0}^{-1}(z)) = G(z)$.
  A **conformally invariant metric** is a differential $\rho(z)|dz|$ on the Riemann surface R which is invariant with respect to the choice of the local parameter z. Thus, to each local parameter z ($z : U \to \overline{\mathbb{C}}$) a function $\rho_z : z(U) \to [0, \infty]$ is associated such that, for any local parameters z_1 and z_2, we have

$$\frac{\rho_{z_2}(z_2(p))}{\rho_{z_1}(z_1(p))} = \left|\frac{dz_1(p)}{dz_2(p)}\right| \text{ for any } p \in U_1 \cap U_2.$$

  Every linear differential $\lambda(z)dz$ and every *quadratic differential* $Q(z)dz^2$ induce conformally invariant metrics $|\lambda(z)||dz|$ and $|Q(z)|^{1/2}||dz|$, respectively (cf. **Q-metric**).
- **Q-metric**
  An **Q-metric** is a **conformally invariant metric** $\rho(z)|dz| = |Q(z)|^{1/2}|dz|$ on a Riemann surface R defined by a *quadratic differential* $Q(z)dz^2$.

A *quadratic differential* $Q(z)dz^2$ is a nonlinear differential on a Riemann surface R which is invariant with respect to the choice of the local parameter z. Thus, to each local parameter z ($z : U \to \overline{\mathbb{C}}$) a function $Q_z : z(U) \to \overline{\mathbb{C}}$ is associated such that, for any local parameters z_1 and z_2, we have

$$\frac{Q_{z_2}(z_2(p))}{Q_{z_1}(z_1(p))} = \left(\frac{dz_1(p)}{dz_2(p)}\right)^2 \text{ for any } p \in U_1 \cap U_2.$$

- **Extremal metric**
  Let Γ be a family of locally rectifiable curves on a Riemann surface R and let P be a class of **conformally invariant metrics** $\rho(z)|dz|$ on R such that $\rho(z)$ is square-integrable in the z-plane for every local parameter z, and the following Lebesgue integrals are not simultaneously equal to 0 or ∞:

  $$A_\rho(R) = \int\int_R \rho^2(z)dx\,dy \text{ and } L_\rho(\Gamma) = \inf_{\gamma\in\Gamma}\int_y \rho(z)|dz|.$$

  The *modulus of the family of curves* Γ is defined by

  $$M(\Gamma) = \inf_{\rho\in P}\frac{A_\rho(R)}{(L_\rho(\Gamma))^2}.$$

  The *extremal length of the family of curves* Γ is the reciprocal of $M(\Gamma)$.
  Let P_L be the subclass of P such that, for any $\rho(z)|dz| \in P_L$ and any $\gamma \in \Gamma$, one has $\int_\gamma \rho(z)|dz| \geq 1$. If $P_L \neq \emptyset$, then $M(\Gamma) = \inf_{\rho\in P_L} A_\rho(R)$. Every metric from P_L is called an *admissible metric* for the modulus on Γ. If there exists ρ^* for which

  $$M(\Gamma) = \inf_{\rho\in P_L} A_\rho(R) = A_{\rho^*}(R),$$

  the metric $\rho^*|dz|$ is called an **extremal metric** for the modulus on Γ. It is a **conformally invariant metric**.
- **Fréchet surface metric**
  Let (X, d) be a metric space, M^2 a compact 2D manifold, f a continuous mapping $f : M^2 \to X$, called a *parametrized surface*, and $\sigma : M^2 \to M^2$ a homeomorphism of M^2 onto itself. Two parametrized surfaces f_1 and f_2 are called *equivalent* if $\inf_\sigma \max_{p\in M^2} d(f_1(p), f_2(\sigma(p))) = 0$, where the infimum is taken over all possible homeomorphisms σ. A class f^* of parametrized surfaces, equivalent to f, is called a *Fréchet surface*. It is a generalization of the notion of a surface in Euclidean space to the case of an arbitrary metric space (X, d).
  The **Fréchet surface metric** on the set of all Fréchet surfaces is defined by

  $$\inf_\sigma \max_{p\in M^2} d(f_1(p), f_2(\sigma(p)))$$

for any Fréchet surfaces f_1^* and f_2^*, where the infimum is taken over all possible homeomorphisms σ. Cf. the **Fréchet metric** in Chap. 1.

- **Hempel metric**
  A *handlebody* of genus g is the boundary sum of g copies of a solid torus; it is homeomorphic to the closure of a regular neighborhood of some finite graph in $\mathbb{R}^3$. Given a closed orientable 3-manifold M, its *Heegaard splitting* (of genus g) is $M = A \cup_P B$ where A, B are genus g handlebodies in M such that $M = A \cup B$ and $A \cap B = \partial A = \partial B = P$. Then P is called a (genus g) *Heegaard surface* of M. In knot applications, Heegaard splitting of the *exterior of* a knot K (the complement of an open solid torus knotted like K) are considered.
  Two embedded curves are *isotopic* if there exists a continuous deformation of one embedding to another through a path of embeddings. Given a closed connected orientable surface S of genus at least two, let $C(S) = (V, E)$ denotes the graph whose vertices are isotopy classes of *essential* (not bounding disk on the surface) simple closed curves and whose edges are drawn between vertices with disjoint representative curves. This graph is connected. For any subsets of vertices $X, Y \subset V$, denote by $d_S(X, Y)$ their **set-to-set distance** $\min d_S(x, y) : x \in X, y \in Y$, where $d_S(x, y)$ is the **path metric** of $C(S)$.
  If S is the boundary of a handlebody H, let $M(H)$ denotes the set of vertices with representatives bounding *meridian disks* D of H, i.e., such that ∂D are essential simple closed curves in ∂H. The **Hempel distance** of a (genus $g \geq 2$) Heegaard splitting $M = A \cup_P B$ is defined (Hempel, 2001) to be $d_P(M(A), M(B))$.
  A Heegaard splitting $M = A \cup_P B$ is *stabilized*, if there are meridian disks D_A, D_B of A, B respectively such that ∂D_A and ∂D_B intersects transversely in a single point. The **Reidemeister–Singer distance** between two Heegaard surfaces/splittings is the minimal number of *stabilizations* (roughly, additions of a "trivial" handle) and *destabilizations* (inverse operation) relating them.

8.2 Intrinsic Metrics on Surfaces

In this section we list intrinsic metrics, given by their *line elements* (which, in fact, are 2D **Riemannian metrics**), for some selected surfaces.

- **Quadric metric**
  A *quadric* (or *quadratic surface*, *surface of second-order*) is a set of points in $\mathbb{E}^3$, whose coordinates in a Cartesian coordinate system satisfy an algebraic equation of degree two. There are 17 classes of such surfaces. Among them are: *ellipsoids*, *one-sheet* and *two-sheet hyperboloids*, *elliptic paraboloids*, *hyperbolic paraboloids*, *elliptic*, *hyperbolic* and *parabolic cylinders*, and *conical surfaces*.

For example, a *cylinder* can be given by the following parametric equations:

$$x_1(u,v) = a\cos v,\ \ x_2(u,v) = a\sin v,\ \ x_3(u,v) = u.$$

The intrinsic metric on it is given by the *line element*

$$ds^2 = du^2 + a^2 dv^2.$$

An *elliptic cone* (i.e., a cone with elliptical cross-section) has the following equations:

$$x_1(u,v) = a\frac{h-u}{h}\cos v,\ \ x_2(u,v) = b\frac{h-u}{h}\sin v,\ \ x_3(u,v) = u,$$

where h is the *height*, a is the *semi-major axis*, and b is the *semi-minor axis* of the cone. The intrinsic metric on it is given by the *line element*

$$ds^2 = \frac{h^2 + a^2\cos^2 v + b^2\sin^2 v}{h^2}du^2 + 2\frac{(a^2-b^2)(h-u)\cos v\sin v}{h^2}du\,dv + \\ + \frac{(h-u)^2(a^2\sin^2 v + b^2\cos^2 v)}{h^2}dv^2.$$

- **Sphere metric**
  A *sphere* is a *quadric*, given by the Cartesian equation $(x_1-a)^2 + (x_2-b)^2 + (x_3-c)^2 = r^2$, where the point (a,b,c) is the *center* of the sphere, and $r > 0$ is the *radius* of the sphere. The sphere of radius r, centered at the origin, can be given by the following parametric equations:

  $$x_1(\theta,\phi) = r\sin\theta\cos\phi,\ \ x_2(\theta,\phi) = r\sin\theta\sin\phi,\ \ x_3(\theta,\phi) = r\cos\theta,$$

  where the *azimuthal angle* $\phi \in [0,2\pi)$, and the *polar angle* $\theta \in [0,\pi]$.
  The intrinsic metric on it (in fact, the 2D **spherical metric**) is given by the *line element*

  $$ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$

  A sphere of radius r has constant positive Gaussian curvature equal to r.
- **Ellipsoid metric**
  An *ellipsoid* is a *quadric* given by the Cartesian equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$, or by the following parametric equations:

  $$x_1(\theta,\phi) = a\cos\phi\sin\theta,\ \ x_2(\theta,\phi) = b\sin\phi\sin\theta,\ \ x_3(\theta,\phi) = c\cos\theta,$$

  where the *azimuthal angle* $\phi \in [0,2\pi)$, and the *polar angle* $\theta \in [0,\pi]$.

The intrinsic metric on it is given by the *line element*

$$ds^2 = (b^2 \cos^2 \phi + a^2 \sin^2 \phi) \sin^2 \theta d\phi^2 + (b^2 - a^2) \cos \phi \sin \phi \cos \theta \sin \theta d\theta d\phi +$$
$$+ ((a^2 \cos^2 \phi + b^2 \sin^2 \phi) \cos^2 \theta + c^2 \sin^2 \theta) d\theta^2.$$

- **Spheroid metric**
A *spheroid* is an *ellipsoid* having two axes of equal length. It is also a *rotation surface*, given by the following parametric equations:

$$x_1(u, v) = a \sin v \cos u, \ x_2(u, v) = a \sin v \sin u, \ x_3(u, v) = c \cos v,$$

where $0 \le u < 2\pi$, and $0 \le v \le \pi$.
The intrinsic metric on it is given by the *line element*

$$ds^2 = a^2 \sin^2 v\, du^2 + \frac{1}{2}(a^2 + c^2 + (a^2 - c^2) \cos(2v)) dv^2.$$

- **Hyperboloid metric**
A *hyperboloid* is a *quadric* which may be one- or two-sheeted.
The one-sheeted hyperboloid is a *surface of revolution* obtained by rotating a hyperbola about the perpendicular bisector to the line between the foci, while the two-sheeted hyperboloid is a surface of revolution obtained by rotating a hyperbola about the line joining the foci.
The one-sheeted circular hyperboloid, oriented along the x_3 axis, is given by the Cartesian equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} - \frac{x_3^2}{c^2} = 1$, or by the following parametric equations:

$$x_1(u, v) = a\sqrt{1 + u^2} \cos v, \ x_2(u, v) = a\sqrt{1 + u^2} \sin v, \ x_3(u, v) = cu,$$

where $v \in [0, 2\pi)$. The intrinsic metric on it is given by the *line element*

$$ds^2 = \left(c^2 + \frac{a^2 u^2}{u^2 + 1}\right) du^2 + a^2(u^2 + 1)\, dv^2.$$

- **Rotation surface metric**
A *rotation surface* (or *surface of revolution*) is a surface generated by rotating a 2D curve about an axis. It is given by the following parametric equations:

$$x_1(u, v) = \phi(v) \cos u, \ x_2(u, v) = \phi(v) \sin u, \ x_3(u, v) = \psi(v).$$

The intrinsic metric on it is given by the *line element*

$$ds^2 = \phi^2 du^2 + (\phi'^2 + \psi'^2)\, dv^2.$$

- **Pseudo-sphere metric**
  A *pseudo-sphere* is a half of the *rotation surface* generated by rotating a *tractrix* about its asymptote. It is given by the following parametric equations:

$$x_1(u,v) = \mathrm{sech}u\cos v,\ \ x_2(u,v) = \mathrm{sech}u\sin v,\ \ x_3(u,v) = u - \tanh u,$$

  where $u \geq 0$, and $0 \leq v < 2\pi$. The intrinsic metric on it is given by the *line element*

$$ds^2 = \tanh^2 u\, du^2 + \mathrm{sech}^2 u\, dv^2.$$

  The pseudo-sphere has constant negative Gaussian curvature equal to -1, and in this sense is an analog of the sphere which has constant positive Gaussian curvature.
- **Torus metric**
  A *torus* is a surface having genus one. A torus azimuthally symmetric about the x_3 axis is given by the Cartesian equation $(c - \sqrt{x_1^2 + x_2^2})^2 + x_3^2 = a^2$, or by the following parametric equations:

$$x_1(u,v) = (c + a\cos v)\cos u,\ \ x_2(u,v) = (c + a\cos v)\sin u,\ \ x_3(u,v) = a\sin v,$$

  where $c > a$, and $u, v \in [0, 2\pi)$.
  The intrinsic metric on it is given by the *line element*

$$ds^2 = (c + a\cos v)^2 du^2 + a^2 dv^2.$$

  For toroidally confined plasma, such as in magnetic confinement fusion, the coordinates u, v and a correspond to the directions called , respectively, *toroidal* (long, as lines of latitude, way around the torus), *poloidal* (short way around the torus) and *radial*. The **poloidal distance**, used in plasma context, is the distance in the poloidal direction.
- **Helical surface metric**
  A *helical surface* (or *surface of screw motion*) is a surface described by a plane curve γ which, while rotating around an axis at a uniform rate, also advances along that axis at a uniform rate. If γ is located in the plane of the axis of rotation x_3 and is defined by the equation $x_3 = f(u)$, the position vector of the helical surface is

$$r = (u\cos v,\ \ u\sin v,\ \ f(u) = hv),\ \ h = \mathrm{const},$$

  and the intrinsic metric on it is given by the *line element*

$$ds^2 = (1 + f'^2)du^2 + 2hf'\,du\,dv + (u^2 + h^2)dv^2.$$

If $f = const$, one has a *helicoid*; if $h = 0$, one has a *rotation surface*.

- **Catalan surface metric**
  The *Catalan surface* is a *minimal surface*, given by the following equations:

$$x_1(u,v) = u - \sin u \cosh v,\ x_2(u,v) = 1 - \cos u \cosh v,\ x_3(u,v) = 4\sin\left(\frac{u}{2}\right)\sinh\left(\frac{v}{2}\right).$$

The intrinsic metric on it is given by the *line element*

$$ds^2 = 2\cosh^2\left(\frac{v}{2}\right)(\cosh v - \cos u)du^2 + 2\cosh^2(\frac{v}{2})\,(\cosh v - \cos u)\,dv^2.$$

- **Monkey saddle metric**
  The *monkey saddle* is a surface, given by the Cartesian equation $x_3 = x_1(x_1^2 - 3x_2^2)$, or by the following parametric equations:

$$x_1(u,v) = u,\ x_2(u,v) = v,\ x_3(u,v) = u^3 - 3uv^2.$$

This is a surface which a monkey can straddle with both legs and his tail. The intrinsic metric on it is given by the *line element*

$$ds^2 = (1 + (su^2 - 3v^2)^2)du^2 - 2(18uv(u^2 - v^2))du\,dv + (1 + 36u^2v^2)dv^2).$$

- **Distance-defined surfaces and curves**
  We give below examples of plane curves and surfaces which are the loci of points with given value of some function of their Euclidean distances to the given objects.
  A *parabola* is the locus of all points in $\mathbb{R}^2$ that are equidistant from the given point (*focus*) and given line (*directrix*) on the plane.
  A *hyperbola* is the locus of all points in $\mathbb{R}^2$ such that the ratio of their distances to the given point and line is a constant (*eccentricity*) greater than 1. It is also the locus of all points in $\mathbb{R}^2$ such that the absolute value of the difference of their distances to the two given foci is constant.
  An *ellipse* is the locus of all points in $\mathbb{R}^2$ such that the sum of their distances to the two given points (*foci*) is constant; cf. **elliptic orbit distance** in Chap. 25. A *circle* is an ellipse in which the two foci are coincident.
  A *Cassini oval* is the locus of all points in $\mathbb{R}^2$ such that the product of their distances to two given points is a constant k. If the distance between two points is $2\sqrt{k}$, then such oval is called a *lemniscate of Bernoulli*.
  A *circle of Appolonius* is the locus of points in $\mathbb{R}^2$ such that the ratio of their distances to the first and second given points is constant.
  A *Cartesian oval* is the locus of points in $\mathbb{R}^2$ such that their distances r_1, r_2 to the foci $(-1,0), (1,0)$ are related linearly by $ar_1 + br_2 = 1$. The cases $a = b, a = -b$ and $a = \frac{1}{2}$ (or $b = \frac{1}{2}$) correspond to the ellipse, hyperbola and *limaçon of Pascal*, respectively.

A *Cassinian curve* is the locus of all points in $\mathbb{R}^2$ such that the product of their distances to n given points (*poles*) is constant. If the poles form a regular n-gon, then this (algebraic of degree $2n$) curve is a *sinusoidal spiral* given also by polar equation $r^n = 2\cos(n\theta)$, and the case $n = 3$ corresponds to the *Kiepert curve*.
Farouki and Moon, 2000, considered many other multipolar generalizations of above curves. For example, their *trifocal ellipse* is the locus of all points in $\mathbb{R}^2$ (seen as the complex plane) such that the sum of their distances to the three cube roots of unity is a constant k. If $k = 2\sqrt{3}$, the curve pass through (and is singular at) the three poles.
In $\mathbb{R}^3$, a surface, rotationally symmetric about an axis, is a locus defined via Euclidean distances of its points to the two given poles belonging to this axis. For example, a *spheroid* (or *ellipsoid of revolution*) is a quadric obtained by rotating an ellipse about one of its principal axes.
It is a sphere, if this ellipse is a circle. If the ellipse is rotated about its major axis, the result is an elongated (as the rugby ball) spheroid which is the locus of all points in $\mathbb{R}^3$ such that the sum of their distances to the two given points is constant. The rotation about its minor axis results in a flattened spheroid (as the Earth) which is the locus of all points in $\mathbb{R}^3$ such that the sum of the distances to the closest and the farthest points of given circle is constant.
A *hyperboloid of revolution of two sheets* is a quadric obtained by revolving a hyperbola about its semi-major (real) axis. Such hyperboloid with axis AB is the locus of all points in $\mathbb{R}^3$ such that the absolute value of the difference of their distances to the points A and B is constant.
Any point in $\mathbb{R}^n$ is uniquely defined by its Euclidean distances to the vertices of a nondegenerated n-simplex. If a surface which is not rotationally symmetric about an axis, is a locus in $\mathbb{R}^3$ defined via distances of its points to the given poles, then three noncollinear poles is needed, and the surface is symmetric with respect to reflexion in the plane defined by the three poles.

8.3 Distances on Knots

A *knot* is a closed, self-nonintersecting curve that is embedded in S^3. The *trivial knot* (or *unknot*) O is a closed loop that is not knotted. A knot can be generalized to a link which is a set of disjoint knots. Every link has its *Seifert surface*, i.e., a compact oriented surface with the given link as boundary.

Two knots (links) are called *equivalent* if one can be smoothly deformed into another. Formally, a link is defined as a smooth one-dimensional *submanifold* of the 3-sphere S^3; a knot is a link consisting of one component; two links L_1 and L_2 are called *equivalent* if there exists an orientation-preserving homeomorphism $f : S^3 \to S^3$ such that $f(L_1) = L_2$.

All the information about a knot can be described using a *knot diagram*. It is a projection of a knot onto a plane such that no more than two points of the knot are projected to the same point on the plane, and at each such point it is indicated which strand is closest to the plane, usually by erasing part of the lower strand. Two different knot diagrams may both represent the same knot. Much of Knot Theory is devoted to telling when two knot diagrams represent the same knot.

An *unknotting operation* is an operation which changes the overcrossing and the undercrossing at a double point of a given knot diagram. The *unknotting number* of a knot K is the minimum number of unknotting operations needed to deform a diagram of K into that of the trivial knot, where the minimum is taken over all diagrams of K. Roughly, the unknotting number is the smallest number of times a knot K must be passed through itself to untie it. An $\sharp$-*unknotting operation* in a diagram of a knot K is an analog of the unknotting operation for a $\sharp$-*part* of the diagram consisting of two pairs of parallel strands with one of the pair overcrossing another. Thus, an $\sharp$-unknotting operation changes the overcrossing and the undercrossing at each vertex of obtained quadrangle.

- **Gordian distance**
  The **Gordian distance** is a metric on the set of all knots defined, for given knots K and K', as the minimum number of unknotting operations needed to deform a diagram of K into that of K', where the minimum is taken over all diagrams of K from which one can obtain diagrams of K'. The unknotting number of K is equal to the Gordian distance between K and the trivial knot O.
  Let rK be the knot obtained from K by taking its mirror image, and let $-K$ be the knot with the reversed orientation. The **positive reflection distance** $Ref_+(K)$ is the Gordian distance between K and rK. The **negative reflection distance** $Ref_-(K)$ is the Gordian distance between K and $-rK$. The **inversion distance** $Inv(K)$ is the Gordian distance between K and $-K$.
  The Gordian distance is the case $k = 1$ of the C_k**-distance** which is the minimum number of C_k-*moves* needed to transform K into K'; Habiro, 1994 and Goussarov, 1995, independently proved that, for $k > 1$, it is finite if and only if both knots have the same *Vassiliev invariants of order less than* k. A C_1-move is a single crossing change, a C_2-move (or *delta-move*) is a simultaneous crossing change for 3 arcs forming a triangle. C_2- and C_3-distances are called **delta distance** and **clasp-pass distance**, respectively.
- **$\sharp$-Gordian distance**
  The **$\sharp$-Gordian distance** (see, for example, [Mura85]) is a metric on the set of all knots defined, for given knots K and K', as the minimum number of $\sharp$-unknotting operations needed to deform a diagram of K into that of K', where the minimum is taken over all diagrams of K from which one can obtain diagrams of K'.
  Let rK be the knot obtained from K by taking its mirror image, and let $-K$ be the knot with the reversed orientation. The **positive $\sharp$-reflection distance** $Ref^{\sharp}_+(K)$ is the $\sharp$-Gordian distance between K and rK. The **negative $\sharp$-reflection distance** $Ref^{\sharp}_-(K)$ is the $\sharp$-Gordian distance between K and $-rK$. The **$\sharp$-inversion distance** $Inv^{\sharp}(K)$ is the $\sharp$-Gordian distance between K and $-K$.

- **Knot complement hyperbolic metric**
  The *complement* of a knot K (or a link L) is $S^3 \backslash K$ (or $S^3 \backslash L$, respectively).
  A knot (or, in general, a link) is called *hyperbolic* if its complement supports a complete Riemannian metric of constant curvature -1. In this case, the metric is called a **knot** (or **link**) **complement hyperbolic metric**, and it is unique.
  A knot is hyperbolic if and only if (Thurston, 1978) it is not a *satellite knot* (then it supports a complete locally homogeneous Riemannian metric) and not a *torus knot* (does not lie on a trivially embedded torus in S^3). The complement of any nontrivial knot supports a complete nonpositively curved Riemannian metric.