Chapter 7
Riemannian and Hermitian Metrics

Riemannian Geometry is a multidimensional generalization of the intrinsic geometry of 2D surfaces in the Euclidean space $\mathbb{E}^3$. It studies *real smooth manifolds* equipped with **Riemannian metrics**, i.e., collections of positive-definite symmetric bilinear forms $((g_{ij}))$ on their tangent spaces which vary smoothly from point to point. The geometry of such (*Riemannian*) manifolds is based on the *line element* $ds^2 = \sum_{i,j} g_{ij} dx_i dx_j$. This gives, in particular, local notions of angle, length of curve, and volume.

From these notions some other global quantities can be derived, by integrating local contributions. Thus, the value ds is interpreted as the length of the vector $(dx_1, \dots, dx_n)$, and it is called the **infinitesimal distance**. The arc length of a curve γ is expressed by $\int_\gamma \sqrt{\sum_{i,j} g_{ij} dx_i dx_j}$, and then the **intrinsic metric** on a Riemannian manifold is defined as the infimum of lengths of curves joining two given points of the manifold.

Therefore, a Riemannian metric is not an ordinary metric, but it induces an ordinary metric, in fact, the intrinsic metric, called **Riemannian distance**, on any connected Riemannian manifold. A Riemannian metric is an infinitesimal form of the corresponding Riemannian distance.

As particular special cases of Riemannian Geometry, there occur *Euclidean Geometry* as well as the two standard types, *Elliptic Geometry* and *Hyperbolic Geometry*, of *non-Euclidean Geometry*. If the bilinear forms $((g_{ij}))$ are nondegenerate but indefinite, one obtains *pseudo-Riemannian Geometry*. In the case of dimension four (and *signature* $(1, 3)$) it is the main object of the General Theory of Relativity.

If $ds = F(x_1, \dots, x_n, dx_1, \dots, dx_n)$, where F is a real positive-definite convex function which cannot be given as the square root of a symmetric bilinear form (as in the Riemannian case), one obtains the *Finsler Geometry* generalizing Riemannian Geometry.

M.M. Deza, E. Deza, *Encyclopedia of Distances*,
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Hermitian Geometry studies *complex manifolds* equipped with **Hermitian metrics**, i.e., collections of positive-definite symmetric *sesquilinear forms* (or $\frac{3}{2}$-*linear forms*) since they are linear in one argument and *antilinear* in the other) on their tangent spaces, which vary smoothly from point to point. It is a complex analog of Riemannian Geometry.

A special class of Hermitian metrics form **Kähler metrics** which have a closed fundamental form w. A generalization of Hermitian metrics give **complex Finsler metrics** which cannot be written as a bilinear symmetric positive-definite sesqulinear form.

7.1 Riemannian Metrics and Generalizations

A *real* n-*manifold* M^n *with boundary* is (cf. Chap. 2) a **Hausdorff space** in which every point has an open neighborhood homeomorphic to either an open subset of $\mathbb{E}^n$, or an open subset of the closed half of $\mathbb{E}^n$. The set of points which have an open neighborhood homeomorphic to $\mathbb{E}^n$ is called the *interior* (of the manifold); it is always nonempty.

The complement of the interior is called the *boundary* (of the manifold); it is an $(n-1)$-dimensional manifold. If it is empty, one obtains a *real* n-*manifold without boundary*. Such manifold is called *closed* if it is compact, and *open*, otherwise.

An open set of M^n together with a homeomorphism between the open set and an open set of $\mathbb{E}^n$ is called a *coordinate chart*. A collection of charts which cover M^n is an *atlas* on M^n. The homeomorphisms of two overlapping charts provide a transition mapping from a subset of $\mathbb{E}^n$ to some other subset of $\mathbb{E}^n$.

If all these mappings are continuously differentiable, then M^n is a *differentiable manifold*. If they are k times (infinitely often) continuously differentiable, then the manifold is a C^k *manifold* (respectively, a *smooth manifold*, or C^∞ *manifold*).

An atlas of a manifold is called *oriented* if the Jacobians of the coordinate transformations between any two charts are positive at every point. An *orientable manifold* is a manifold admitting an oriented atlas.

Manifolds inherit many local properties of the Euclidean space: they are locally path-connected, locally compact, and locally metrizable. Every smooth Riemannian manifold embeds isometrically (Nash, 1956) in some finite-dimensional Euclidean space.

Associated with every point on a differentiable manifold is a *tangent space* and its dual, a *cotangent space*. Formally, let M^n be a C^k manifold, $k \geq 1$, and p a point of M^n. Fix a chart $\varphi : U \to \mathbb{E}^n$, where U is an open subset of M^n containing p. Suppose that two curves $\gamma^1 : (-1,1) \to M^n$ and $\gamma^2 : (-1,1) \to M^n$ with $\gamma^1(0) = \gamma^2(0) = p$ are given such that $\varphi \cdot \gamma^1$ and $\varphi \cdot \gamma^2$ are both differentiable at 0.

Then γ^1 and γ^2 are called *tangent at* 0 if $(\varphi \cdot \gamma^1)'(0) = (\varphi \cdot \gamma^2)'(0)$. If the functions $\varphi \cdot \gamma^i : (-1,1) \to \mathbb{E}^n$, $i = 1, 2$, are given by n real-valued component functions $(\varphi \cdot \gamma^i)_1(t), \dots, (\varphi \cdot \gamma^i)_n(t)$, the condition above means that their Jacobians

$\left(\frac{d(\varphi\cdot\gamma^i)_1(t)}{dt}, \dots, \frac{d(\varphi\cdot\gamma^i)_n(t)}{dt}\right)$ coincide at 0. This is an equivalence relation, and the equivalence class $\gamma'(0)$ of the curve γ is called a *tangent vector* of M^n at p.

The *tangent space* $T_p(M^n)$ of M^n at p is defined as the set of all tangent vectors at p. The function $(d\varphi)_p : T_p(M^n) \to \mathbb{E}^n$ defined by $(d\varphi)_p(\gamma'(0)) = (\varphi \cdot \gamma)'(0)$, is bijective and can be used to transfer the vector space operations from $\mathbb{E}^n$ over to $T_p(M^n)$.

All the tangent spaces $T_p(M^n)$, $p \in M^n$, when "glued together", form the *tangent bundle* $T(M^n)$ of M^n. Any element of $T(M^n)$ is a pair (p, v), where $v \in T_p(M^n)$.

If for an open neighborhood U of p the function $\varphi : U \to \mathbb{R}^n$ is a coordinate chart, then the preimage V of U in $T(M^n)$ admits a mapping $\psi : V \to \mathbb{R}^n \times \mathbb{R}^n$ defined by $\psi(p, v) = (\varphi(p), d\varphi(p))$. It defines the structure of a smooth $2n$-dimensional manifold on $T(M^n)$. The *cotangent bundle* $T^*(M^n)$ of M^n is obtained in similar manner using cotangent spaces $T_p^*(M^n)$, $p \in M^n$.

A *vector field* on a manifold M^n is a *section* of its tangent bundle $T(M^n)$, i.e., a smooth function $f : M^n \to T(M^n)$ which assigns to every point $p \in M^n$ a vector $v \in T_p(M^n)$.

A *connection* (or *covariant derivative*) is a way of specifying a derivative of a vector field along another vector field on a manifold.

Formally, the covariant derivative ∇ of a vector u (defined at a point $p \in M^n$) in the direction of the vector v (defined at the same point p) is a rule that defines a third vector at p, called $\nabla_v u$ which has the properties of a derivative. A Riemannian metric uniquely defines a special covariant derivative called the *Levi-Civita connection*. It is the torsion-free connection ∇ of the tangent bundle, preserving the given Riemannian metric.

The *Riemann curvature tensor* R is the standard way to express the *curvature* of *Riemannian manifolds*. The Riemann curvature tensor can be given in terms of the Levi-Civita connection ∇ by the following formula:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w,$$

where $R(u, v)$ is a linear transformation of the tangent space of the manifold M^n; it is linear in each argument. If $u = \frac{\partial}{\partial x_i}$ and $v = \frac{\partial}{\partial x_j}$ are coordinate vector fields, then $[u, v] = 0$, and the formula simplifies to $R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w$, i.e., the curvature tensor measures anti-commutativity of the covariant derivative. The linear transformation $w \to R(u, v)w$ is also called the *curvature transformation*.

The *Ricci curvature tensor* (or *Ricci curvature*) *Ric* is obtained as the trace of the full curvature tensor R. It can be thought of as a Laplacian of the Riemannian metric tensor in the case of Riemannian manifolds. Ricci curvature is a linear operator on the tangent space at a point. Given an orthonormal basis $(e_i)_i$ in the tangent space $T_p(M^n)$, we have

$$Ric(u) = \sum_i R(u, e_i)e_i.$$

The value of $Ric(u)$ does not depend on the choice of an orthonormal basis. Starting with dimension four, the Ricci curvature does not describe the curvature tensor completely.

The *Ricci scalar* (or *scalar curvature*) Sc of a Riemannian manifold M^n is the full trace of the curvature tensor; given an orthonormal basis $(e_i)_i$ at $p \in M^n$, we have

$$Sc = \sum_{i,j} \langle R(e_i, e_j)e_j, e_i \rangle = \sum_i \langle Ric(e_i), e_i \rangle.$$

The *sectional curvature* $K(\sigma)$ of a Riemannian manifold M^n is defined as the *Gauss curvature* of an σ*-section* at a point $p \in M^n$, where a σ*-section* is a locally-defined piece of surface which has the 2-plane σ as a tangent plane at p, obtained from geodesics which start at p in the directions of the image of σ under the exponential mapping.

- **Metric tensor**

  The **metric** (or *basic, fundamental*) **tensor** is a symmetric tensor of rank 2, that is used to measure distances and angles in a real n-dimensional differentiable manifold M^n. Once a local coordinate system $(x_i)_i$ is chosen, the metric tensor appears as a real symmetric $n \times n$ matrix $((g_{ij}))$.

  The assignment of a metric tensor on M^n introduces a *scalar product* (i.e., symmetric bilinear, but in general not positive-definite, form) $\langle , \rangle_p$ on the tangent space $T_p(M^n)$ at any $p \in M^n$ defined by

  $$\langle x, y \rangle_p = g_p(x, y) = \sum_{i,j} g_{ij}(p) x_i y_j,$$

  where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in T_p(M^n)$. The collection of all these scalar products is called the **metric** g with the metric tensor $((g_{ij}))$. The length ds of the vector $(dx_1, \dots, dx_n)$ is expressed by the quadratic differential form

  $$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j,$$

  which is called the *line element* (or *first fundamental form*) of the metric g.

  The *length* of a curve γ is expressed by the formula $\int_\gamma \sqrt{\sum_{i,j} g_{ij} dx_i dx_j}$. In general it may be real, purely imaginary, or zero (an *isotropic curve*).

  Let p, q and r be the numbers of positive, negative and zero *eigenvalues* of the matrix $((g_{ij}))$ of the metric g; so, $p + q + r = n$. The **metric signature** (or, simply, *signature*) of g is the pair (p, q). A **nondegenerated metric** (i.e., one with $r = 0$) is Riemannian or pseudo-Riemannian if its signature is *positive-definite* ($q = 0$) or *indefinite* ($pq > 0$), respectively.

  The **nonmetricity tensor** is the *covariant derivative* of a metric tensor. It is 0 for **Riemannian metrics** but can be $\neq 0$ for **pseudo-Riemannian** ones.

- **Nondegenerate metric**
  A **nondegenerate metric** is a metric g with the metric tensor $((g_{ij}))$, for which the *metric discriminant* $\det((g_{ij})) \neq 0$. All Riemannian and pseudo-Riemannian metrics are nondegenerate.
  A **degenerate metric** is a metric g with $\det((g_{ij})) = 0$ (cf. **semi-Riemannian metric** and **semi-pseudo-Riemannian metric**). A manifold with a degenerate metric is called an *isotropic manifold*.
- **Diagonal metric**
  A **diagonal metric** is a metric g with a metric tensor $((g_{ij}))$ which is zero for $i \neq j$. The Euclidean metric is a diagonal metric, as its metric tensor has the form $g_{ii} = 1, g_{ij} = 0$ for $i \neq j$.
- **Riemannian metric**
  Consider a real n-dimensional differentiable manifold M^n in which each tangent space is equipped with an *inner product* (i.e., a symmetric positive-definite bilinear form) which varies smoothly from point to point.
  A **Riemannian metric** on M^n is a collection of inner products $\langle , \rangle_p$ on the tangent spaces $T_p(M^n)$, one for each $p \in M^n$.
  Every inner product $\langle , \rangle_p$ is completely defined by inner products $\langle e_i, e_j \rangle_p = g_{ij}(p)$ of elements $e_1, \dots, e_n$ of a standard basis in $\mathbb{E}^n$, i.e., by the real symmetric and positive-definite $n \times n$ matrix $((g_{ij})) = ((g_{ij}(p)))$, called a **metric tensor**. In fact, $\langle x, y \rangle_p = \sum_{i,j} g_{ij}(p) x_i y_j$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in T_p(M^n)$. The smooth function g completely determines the Riemannian metric.
  A Riemannian metric on M^n is not an ordinary metric on M^n. However, for a connected manifold M^n, every Riemannian metric on M^n induces an ordinary metric on M^n, in fact, the **intrinsic metric** of M^n,
  For any points $p, q \in M^n$ the **Riemannian distance** between them is defined as

$$\inf_{\gamma} \int_0^1 \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle^{\frac{1}{2}} dt = \inf_{\gamma} \int_0^1 \sqrt{\sum_{i,j} g_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt}} dt,$$

  where the infimum is over all rectifiable curves $\gamma : [0, 1] \to M^n$, connecting p and q.
  A *Riemannian manifold* (or *Riemannian space*) is a real n-dimensional differentiable manifold M^n equipped with a Riemannian metric. The theory of Riemannian spaces is called *Riemannian Geometry*. The simplest examples of Riemannian spaces are Euclidean spaces, *hyperbolic spaces*, and *elliptic spaces*.
- **Conformal metric**
  A *conformal structure on a vector space V* is a class of pairwise-homothetic Euclidean metrics on V. Any Euclidean metric d_E on V defines a conformal structure $\{\lambda d_E : \lambda > 0\}$.

A *conformal structure on a manifold* is a field of conformal structures on the tangent spaces or, equivalently, a class of *conformally equivalent Riemannian metrics*. Two Riemannian metrics g and h on a smooth manifold M^n are called *conformally equivalent* if $g = f \cdot h$ for some positive function f on M^n, called a *conformal factor*.

A **conformal metric** is a Riemannian metric that represents the conformal structure. Cf. **conformally invariant metric** in Chap. 8.

- **Conformal space**

The **conformal space** (or *inversive space*) is the Euclidean space $\mathbb{E}^n$ extended by an ideal point (at infinity). Under *conformal* transformations, i.e., continuous transformations preserving local angles, the ideal point can be taken to be an ordinary point. Therefore, in a conformal space a sphere is indistinguishable from a plane: a plane is a sphere passing through the ideal point.

Conformal spaces are considered in *Conformal* (or *angle-preserving*, *Möbius*) *Geometry* in which properties of figures are studied that are invariant under conformal transformations. It is the set of transformations that map spheres into spheres, i.e., generated by the Euclidean transformations together with *inversions* which in coordinate form are conjugate to $x_i \to \frac{r^2 x_i}{\sum_j x_j^2}$, where r is the radius of the inversion. An inversion in a sphere becomes an everywhere well defined automorphism of period two. Any angle inverts into an equal angle.

The 2D conformal space is the *Riemann sphere*, on which the conformal transformations are given by the *Möbius transformations* $z \to \frac{az+b}{cz+d}$, $ad-bc \neq 0$.

In general, a **conformal mapping** between two Riemannian manifolds is a diffeomorphism between them such that the pulled back metric is *conformally equivalent* to the original one. A *conformal Euclidean space* is a *Riemannian space* admitting a conformal mapping onto an Euclidean space.

In the General Theory of Relativity, conformal transformations are considered on the *Minkowski space* $\mathbb{R}^{1,3}$ extended by two ideal points.

- **Space of constant curvature**

A **space of constant curvature** is a *Riemannian space* M^n for which the sectional curvature $K(\sigma)$ is constant in all 2D directions σ.

A *space form* is a connected complete space of constant curvature k. Examples of a *flat space form*, i.e., with $k = 0$, are the Euclidean space and flat torus. The sphere and hyperbolic space are space forms with $k > 0$ and $k < 0$, respectively.

- **Generalized Riemannian space**

A **generalized Riemannian space** is a metric space with the **intrinsic metric**, subject to certain restrictions on the curvature. Such spaces include *spaces of bounded curvature*, *Riemannian spaces*, etc. They are defined and investigated on the basis of their metric alone, without coordinates.

A *space of bounded curvature* ($\leq k$ and $\geq k'$) is defined by the condition: for any sequence of *geodesic triangles* T_n contracting to a point, we have

$$k \geq \overline{\lim} \frac{\overline{\delta}(T_n)}{\sigma(T_n^0)} \geq \underline{\lim} \frac{\overline{\delta}(T_n)}{\sigma(T_n^0)} \geq k',$$

where a *geodesic triangle* $T = xyz$ is the triplet of geodesic segments $[x, y]$, $[y, z]$, $[z, x]$ (the sides of T) connecting in pairs three different points x, y, z, $\bar{\delta}(T) = \alpha + \beta + \gamma - \pi$ is the *excess* of the geodesic triangle T, and $\sigma(T^0)$ is the area of a Euclidean triangle T^0 with the sides of the same lengths. The **intrinsic metric** on the space of bounded curvature is called a **metric of bounded curvature**.

Such a space turns out to be Riemannian under two additional conditions: local compactness of the space (this ensures the condition of local existence of geodesics), and local extendability of geodesics. If in this case $k = k'$, it is a Riemannian space of constant curvature k (cf. **space of geodesics** in Chap. 6).

A space of curvature $\leq k$ is defined by the condition $\overline{\lim}\frac{\bar{\delta}(T_n)}{\sigma(T_n^0)} \leq k$. In such a space any point has a neighborhood in which the sum $\alpha + \beta + \gamma$ of the angles of a geodesic triangle T does not exceed the sum $\alpha_k + \beta_k + \gamma_k$ of the angles of a triangle T^k with sides of the same lengths in a space of constant curvature k. The intrinsic metric of such space is called a k**-concave metric**.

A space of curvature $\geq k$ is defined by the condition $\underline{\lim}\frac{\bar{\delta}(T_n)}{\sigma(T_n^0)} \geq k$. In such a space any point has a neighborhood in which $\alpha + \beta + \gamma \geq \alpha_k + \beta_k + \gamma_k$ for triangles T and T^k. The intrinsic metric of such space is called a K**-concave metric**.

An *Alexandrov metric space* is a generalized Riemannian space with upper, lower or integral curvature bounds. Cf. a **CAT(κ_1) space** in Chap. 6.

- **Complete Riemannian metric**

  A Riemannian metric g on a manifold M^n is called **complete** if M^n forms a complete metric space with respect to g.

  Any Riemannian metric on a compact manifold is complete.

- **Ricci-flat metric**

  A **Ricci-flat metric** is a Riemannian metric with vanished Ricci curvature tensor. A *Ricci-flat manifold* is a Riemannian manifold equipped with a Ricci-flat metric. Ricci-flat manifolds represent vacuum solutions to the *Einstein field equation*, and are special cases of *Kähler–Einstein manifolds*. Important Ricci-flat manifolds are *Calabi–Yau manifolds*, and *hyper-Kähler manifolds*.

- **Osserman metric**

  An **Osserman metric** is a Riemannian metric for which the Riemannian curvature tensor R is *Osserman*, i.e., the eigenvalues of the *Jacobi operator* $\mathcal{J}(x) : y \to R(y, x)x$ are constant on the *unit sphere* S^{n-1} in $\mathbb{E}^n$ (they are independent of the unit vectors x).

- G**-invariant Riemannian metric**

  Given a *Lie group* $(G, \cdot, id)$ of transformations, a Riemannian metric g on a differentiable manifold M^n is called G**-invariant**, if it does not change under any $x \in G$. The group $(G, \cdot, id)$ is called the *group of motions* (or *group of isometries*) of the Riemannian space (M^n, g). Cf. G**-invariant metric** in Chap. 10.

- **Ivanov–Petrova metric**
  Let R be the Riemannian curvature tensor of a Riemannian manifold M^n, and let $\{x, y\}$ be an orthogonal basis for an oriented 2-plane π in the tangent space $T_p(M^n)$ at a point p of M^n.
  The **Ivanov–Petrova metric** is a Riemannian metric on M^n for which the eigenvalues of the antisymmetric curvature operator $\mathcal{R}(\pi) = R(x, y)$ [IvSt95] depend only on the point p of a Riemannian manifold M^n, but not upon the plane π.
- **Zoll metric**
  A **Zoll metric** is a Riemannian metric on a smooth manifold M^n whose geodesics are all simple closed curves of an equal length. A 2D sphere S^2 admits many such metrics, besides the obvious metrics of constant curvature. In terms of cylindrical coordinates (z, θ) ($z \in [-1, 1]$, $\theta \in [0, 2\pi]$), the *line element*

$$ds^2 = \frac{(1 + f(z))^2}{1 - z^2} dz^2 + (1 - z^2) d\theta^2$$

  defines a Zoll metric on S^2 for any smooth odd function $f : [-1, 1] \to (-1, 1)$ which vanishes at the endpoints of the interval.
- **Berger metric**
  The **Berger metric** is a Riemannian metric on the *Berger sphere* (i.e., the three-sphere S^3 squashed in one direction) defined by the *line element*

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\alpha (d\psi + \cos\theta d\phi)^2,$$

  where α is a constant, and θ, ϕ, ψ are *Euler angles*.
- **Cycloidal metric**
  The **cycloidal metric** is a Riemannian metric on the half-plane $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$ defined by the *line element*

$$ds^2 = \frac{dx_1^2 + dx_2^2}{2x_2}.$$

  It is called *cycloidal* because its geodesics are cycloid curves. The corresponding distance $d(x, y)$ between two points $x, y \in \mathbb{R}^2_+$ is equivalent to the distance

$$\rho(x, y) = \frac{|x_1 - y_1| + |x_2 - y_2|}{\sqrt{x_1} + \sqrt{x_2} + \sqrt{|x_2 - y_2|}}$$

  in the sense that $d \le C\rho$, and $\rho \le Cd$ for some positive constant C.

- **Klein metric**
The **Klein metric** is a Riemannian metric on the *open unit ball* $B^n = \{x \in \mathbb{R}^n : ||x||_2 < 1\}$ in $\mathbb{R}^n$ defined by

$$\frac{\sqrt{||y||_2^2 - (||x||_2^2||y||_2^2 - \langle x, y\rangle^2)}}{1 - ||x||_2^2}$$

for any $x \in B^n$ and $y \in T_x(B^n)$, where $||.||_2$ is the Euclidean norm on $\mathbb{R}^n$, and $\langle , \rangle$ is the ordinary *inner product* on $\mathbb{R}^n$.
The Klein metric is the hyperbolic case $a = -1$ of the general form

$$\frac{\sqrt{(1 + a||x||^2)||y||^2 - a\langle x, y\rangle^2}}{1 + a||x||^2},$$

while $a = 0, 1$ correspond to the Euclidean and spherical cases.
- **Carnot–Carathéodory metric**
A *distribution* (or *polarization*) on a manifold M^n is a subbundle of the tangent bundle $T(M^n)$ of M^n. Given a distribution $H(M^n)$, a vector field in $H(M^n)$ is called *horizontal*. A curve γ on M^n is called *horizontal* (or *distinguished*, *admissible*) with respect to $H(M^n)$ if $\gamma'(t) \in H_{\gamma(t)}(M^n)$ for any t.
A distribution $H(M^n)$ is called *completely nonintegrable* if the Lie brackets of $H(M^n)$, i.e., $[\cdots, [H(M^n), H(M^n)]]$, span the tangent bundle $T(M^n)$, i.e., for all $p \in M^n$ any tangent vector v from $T_p(M^n)$ can be presented as a linear combination of vectors of the following types: u, $[u, w]$, $[u, [w, t]]$, $[u, [w, [t, s]]], \dots \in T_p(M^n)$, where all vector fields $u, w, t, s, \dots$ are horizontal.
The **Carnot–Carathéodory metric** (or **CC metric**, **sub-Riemannian metric**, *control metric*) is a metric on a manifold M^n with a completely nonintegrable horizontal distribution $H(M^n)$ defined as the section g_C of positive-definite *scalar products* on $H(M^n)$. The distance $d_C(p, q)$ between any points $p, q \in M^n$ is defined as the infimum of the g_C-lengths of the horizontal curves joining p and q.
A *sub-Riemannian manifold* (or *polarized manifold*) is a manifold M^n equipped with a Carnot–Carathéodory metric. It is a generalization of a Riemannian manifold. Roughly, in order to measure distances in a sub-Riemannian manifold, one is allowed to go only along curves tangent to horizontal spaces.
- **Pseudo-Riemannian metric**
Consider a real n-dimensional differentiable manifold M^n in which every tangent space $T_p(M^n)$, $p \in M^n$, is equipped with a *scalar product* which varies smoothly from point to point and is nondegenerate, but indefinite.
A **pseudo-Riemannian metric** on M^n is a collection of scalar products $\langle , \rangle_p$ on the tangent spaces $T_p(M^n)$, $p \in M^n$, one for each $p \in M^n$.
Every scalar product $\langle , \rangle_p$ is completely defined by scalar products $\langle e_i, e_j\rangle_p = g_{ij}(p)$ of elements $e_1, \dots, e_n$ of a standard basis in $\mathbb{E}^n$, i.e., by the real symmetric indefinite $n \times n$ matrix $((g_{ij})) = ((g_{ij}(p)))$, called a **metric tensor**

(cf. **Riemannian metric** in which case this tensor is not only nondegenerate but, moreover, positive-definite).

In fact, $\langle x, y\rangle_p = \sum_{i,j} g_{ij}(p)x_i y_j$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in T_p(M^n)$. The smooth function g determines the pseudo-Riemannian metric.

The length ds of the vector $(dx_1, \dots, dx_n)$ is given by the quadratic differential form

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j .$$

The length of a curve $\gamma : [0, 1] \to M^n$ is expressed by the formula

$$\int_\gamma \sqrt{\sum_{i,j} g_{ij} dx_i dx_j} = \int_0^1 \sqrt{\sum_{i,j} g_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt}} dt.$$

In general it may be real, purely imaginary or zero (an *isotropic curve*).

A pseudo-Riemannian metric on M^n is a metric with a fixed, but indefinite signature (p, q), $p + q = n$. A pseudo-Riemannian metric is nondegenerate, i.e., its metric discriminant $\det((g_{ij})) \neq 0$. Therefore, it is a **nondegenerate indefinite metric**.

A *pseudo-Riemannian manifold* (or *pseudo-Riemannian space*) is a real n-dimensional differentiable manifold M^n equipped with a pseudo-Riemannian metric. The theory of pseudo-Riemannian spaces is called *Pseudo-Riemannian Geometry*.

- **Pseudo-Euclidean distance**

  The model space of a **pseudo-Riemannian space** of signature (p, q) is the *pseudo-Euclidean space* $\mathbb{R}^{p,q}$, $p + q = n$ which is a real n-dimensional vector space $\mathbb{R}^n$ equipped with the metric tensor $((g_{ij}))$ of signature (p, q) defined, for $i \neq j$, by $g_{11} = \cdots = g_{pp} = 1$, $g_{p+1,p+1} = \cdots = g_{nn} = -1$, $g_{ij} = 0$.

  The *line element* of the corresponding metric is given by

$$ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_n^2.$$

  The **pseudo-Euclidean distance** of signature $(p, q = n - p)$ on $\mathbb{R}^n$ is defined by

$$d_{pE}^2(x, y) = D(x, y) = \sum_{i=1}^{p}(x_i - y_i)^2 - \sum_{i=p+1}^{n}(x_i - y_i)^2.$$

  Such a pseudo-Euclidean space can be seen as $\mathbb{R}^p \times i\mathbb{R}^q$, where $i = \sqrt{-1}$.

  The pseudo-Euclidean space with $(p, q) = (1, 3)$ is used as flat space-time model of Special Relativity; cf. **Minkowski metric** in Chap. 26.

The points correspond to *events*; the line spanned by x and y is *space-like* if $D(x, y) > 0$ and *time-like* if $D(x, y) < 0$. If $D(x, y) > 0$, then $\sqrt{D(x, y)}$ is Euclidean distance and if $D(x, y) < 0$, then $\sqrt{|D(x, y)|}$ is the lifetime of a particle (from x to y).

The pseudo-Euclidean distance of signature $(p, q = n - p)$ is the case $A = \text{diag}(a_i)$ with $a_i = 1$ for $1 \le i \le p$ and $a_i = -1$ for $p + 1 \le i \le n$, of the **weighted Euclidean distance** $\sqrt{\sum_{1 \le i \le n} a_i (x_i - y_i)^2}$ in Chap. 17.

- **Blaschke metric**
  The **Blaschke metric** on a nondegenerate hypersurface is a pseudo-Riemannian metric, associated to the affine normal of the immersion $\phi : M^n \to \mathbb{R}^{n+1}$, where M^n is an n-dimensional manifold, and $\mathbb{R}^{n+1}$ is considered as an affine space.
- **Semi-Riemannian metric**
  A **semi-Riemannian metric** on a real n-dimensional differentiable manifold M^n is a degenerate Riemannian metric, i.e., a collection of positive-semidefinite *scalar products* $\langle x, y \rangle_p = \sum_{i,j} g_{ij}(p) x_i y_j$ on the tangent spaces $T_p(M^n)$, $p \in M^n$; the metric discriminant $det((g_{ij})) = 0$.
  A *semi-Riemannian manifold* (or *semi-Riemannian space*) is a real n-dimensional differentiable manifold M^n equipped with a semi-Riemannian metric.
  The model space of a semi-Riemannian manifold is the *semi-Euclidean space* R^n_d, $d \ge 1$ (sometimes denoted also by $\mathbb{R}^n_{n-d}$), i.e., a real n-dimensional vector space $\mathbb{R}^n$ equipped with a semi-Riemannian metric.
  It means that there exists a scalar product of vectors such that, relative to a suitably chosen basis, the scalar product $\langle x, x \rangle$ has the form $\langle x, x \rangle = \sum_{i=1}^{n-d} x_i^2$. The number $d \ge 1$ is called the *defect* (or *deficiency*) of the space.
- **Grushin metric**
  The **Grushin metric** is a semi-Riemannian metric on $\mathbb{R}^2$ defined by the *line element*

$$ds^2 = dx_1^2 + \frac{dx_2^2}{x_1^2}.$$

- **Agmon distance**
  The **Agmon metric** attached to an energy E and a potential V is defined as

$$ds^2 = \max\{0, V(x) - E_0(h)\} dx^2,$$

where dx^2 is the standard metric on $\mathbb{R}^d$. Then the **Agmon distance** on $\mathbb{R}^d$ is the corresponding Riemannian distance defined, for any $x, y \in \mathbb{R}^d$, by

$$\inf_{\gamma}\{\int_0^1 \sqrt{\max\{V(\gamma(s)) - E_0(h), 0\}} \cdot |\gamma^{'}(s)| ds : \gamma(0) = x, \gamma(1) = y, \gamma \in C^1\}.$$

- **Semi-pseudo-Riemannian metric**

  A **semi-pseudo-Riemannian metric** on a real n-dimensional differentiable manifold M^n is a degenerate pseudo-Riemannian metric, i.e., a collection of degenerate indefinite *scalar products* $\langle x, y\rangle_p = \sum_{i,j} g_{ij}(p)x_i y_j$ on the tangent spaces $T_p(M^n)$, $p \in M^n$; the metric discriminant $det((g_{ij})) = 0$. In fact, a semi-pseudo-Riemannian metric is a **degenerate indefinite metric**.

  A *semi-pseudo-Riemannian manifold* (or *semi-pseudo-Riemannian space*) is a real n-dimensional differentiable manifold M^n equipped with a semi-pseudo-Riemannian metric. The model space of such manifold is the *semi-pseudo-Euclidean space* $\mathbb{R}^n_{\substack{l_1,\dots,l_r\\ m_1,\dots,m_{r-1}}}$, i.e., a vector space $\mathbb{R}^n$ equipped with a semi-pseudo-Riemannian metric.

  It means that there exist r scalar products $\langle x, y\rangle_a = \sum \epsilon_{i_a} x_{i_a} y_{i_a}$, where $a = 1,\dots r$, $0 = m_0 < m_1 < \cdots < m_r = n$, $i_a = m_{a-1} + 1,\dots m_a$, $\epsilon_{i_a} = \pm 1$, and -1 occurs l_a times among the numbers ϵ_{i_a}. The product $\langle x, y\rangle_a$ is defined for those vectors for which all coordinates x_i, $i \le m_{a-1}$ or $i > m_a + 1$ are zero.

  The first scalar square of an arbitrary vector x is a degenerate quadratic form $\langle x, x\rangle_1 = -\sum_{i=1}^{l_1} x_i^2 + \sum_{j=l_1+1}^{n-d} x_j^2$. The number $l_1 \ge 0$ is called the *index*, and the number $d = n - m_1$ is called the *defect* of the space. If $l_1 = \cdots = l_r = 0$, we obtain a *semi-Euclidean space*. The spaces $\mathbb{R}^n_m$ and $\mathbb{R}^n_{\substack{k,l\\ m}}$ are called *quasi-Euclidean spaces*.

  The *semi-pseudo-non-Euclidean space* $\mathbb{S}^n_{\substack{l_1,\dots,l_r\\ m_1,\dots,m_{r-1}}}$ is a hypersphere in $\mathbb{R}^{n+1}_{\substack{l_1,\dots,l_r\\ m_1,\dots,m_{r-1}}}$ with identified antipodal points. It is called *semielliptic* (or *semi-non-Euclidean*) *space* if $l_1 = \cdots = l_r = 0$ and a *semihyperbolic space* if there exist $l_i \neq 0$.

- **Finsler metric**

  Consider a real n-dimensional differentiable manifold M^n in which every tangent space $T_p(M^n)$, $p \in M^n$, is equipped with a *Banach norm* $||.||$ such that the Banach norm as a function of position is smooth, and the matrix $((g_{ij}))$,

$$g_{ij} = g_{ij}(p, x) = \frac{1}{2}\frac{\partial^2 ||x||^2}{\partial x_i \partial x_j},$$

  is positive-definite for any $p \in M^n$ and any $x \in T_p(M^n)$.

  A **Finsler metric** on M^n is a collection of Banach norms $||.||$ on the tangent spaces $T_p(M^n)$, one for each $p \in M^n$. Its *line element* has the form

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j.$$

  The Finsler metric can be given by *fundamental function*, i.e., a real positive-definite convex function $F(p, x)$ of $p \in M^n$ and $x \in T_p(M^n)$ acting at the point p. $F(p, x)$ is positively homogeneous of degree one in x: $F(p, \lambda x) = \lambda F(p, x)$ for every $\lambda > 0$. Then $F(p, x)$ is the length of the vector x.

The *Finsler metric tensor* has the form $((g_{ij})) = ((\frac{1}{2}\frac{\partial^2 F^2(p,x)}{\partial x_i \partial x_j}))$. The length of a curve $\gamma : [0,1] \to M^n$ is given by $\int_0^1 F(p, \frac{dp}{dt})dt$. For each fixed p the Finsler metric tensor is Riemannian in the variables x.

The Finsler metric is a generalization of the Riemannian metric, where the general definition of the length $||x||$ of a vector $x \in T_p(M^n)$ is not necessarily given in the form of the square root of a symmetric bilinear form as in the Riemannian case.

A *Finsler manifold* (or *Finsler space*) is a real differentiable n-manifold M^n equipped with a Finsler metric. The theory of such spaces is *Finsler Geometry*.

The difference between a Riemannian space and a Finsler space is that the former behaves locally like a Euclidean space, and the latter locally like a *Minkowskian space* or, analytically, the difference is that to an ellipsoid in the Riemannian case there corresponds an arbitrary convex surface which has the origin as the center.

A **pseudo-Finsler metric** F is defined by weakening the definition of a Finsler metric): $((g_{ij}))$ should be nondegenerate and of constant signature (not necessarily positive-definite) and hence F could be negative. The pseudo-Finsler metric is a generalization of the pseudo-Riemannian metric.

- **(α, β)-metric**

  Let $\alpha(x, y) = \sqrt{\alpha_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ be a 1-form on a n-dimensional manifold M^n. Let $s = \frac{\beta}{\alpha}$ and $\phi(s)$ is an C^∞-positive function on some symmetric interval $(-r, r)$ with $r > \frac{\beta}{\alpha}$ for all (x, y) in the tangent bundle $TM = \cup_{x \in M} T_x(M^n)$ of the tangent spaces $T_x(M^n)$. Then $F = \alpha\phi(s)$ is a Finsler metric (Matsumoto, 1972) called an **(α, β)-metric**. The main examples of (α, β)-metrics follow.

  The **Kropina metric** is the case $\phi(s) = \frac{1}{s}$, i.e., $F = \frac{\alpha^2}{\beta}$.

  The **generalized Kropina metric** is the case $\phi(s) = s^m$, i.e., $F = \beta^m \alpha^{1-m}$.

  The **Randers metric** (1941) is the case $\phi(s) = 1 + s$, i.e., $F = \alpha + \beta$.

  The **Matsumoto slope metric** is the case $\phi(s) = \frac{1}{1-s}$, i.e., $F = \frac{\alpha^2}{\alpha - \beta}$.

  The **Riemann-type (α, β)-metric** is the case $\phi(s) = \sqrt{1 + s^2}$, i.e., $F = \alpha^2 + \beta^2$.

  Park and Lee, 1998, considered the case $\phi(s) = 1 + s^2$, i.e., $F = \alpha + \frac{\beta^2}{\alpha}$.

- **Shen metric**

  Given a vector $a \in \mathbb{R}^n$, $||a||_2 < 1$, the **Shen metric** (2003) is a Finsler metric on the *open unit ball* $B^n = \{x \in \mathbb{R}^n : ||x||_2 < 1\}$ in $\mathbb{R}^n$ defined by

$$\frac{\sqrt{||y||_2^2 - (||x||_2^2||y||_2^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - ||x||_2^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}$$

  for any $x \in B^n$ and $y \in T_x(B^n)$, where $||.||_2$ is the Euclidean norm on $\mathbb{R}^n$, and $\langle , \rangle$ is the ordinary *inner product* on $\mathbb{R}^n$. It is a **Randers metric** and a **projective metric**. Cf. **Klein metric** and **Berwald metric**.

- **Berwald metric**
The **Berwald metric** (1929) is a Finsler metric F_{Be} on the *open unit ball* $B^n = \{x \in \mathbb{R}^n : ||x||_2 < 1\}$ in $\mathbb{R}^n$ defined, for any $x \in B^n$ and $y \in T_x(B^n)$, by

$$\frac{\left(\sqrt{||y||_2^2 - (||x||_2^2||y||_2^2 - \langle x,y\rangle^2)} + \langle x,y\rangle\right)^2}{(1-||x||_2^2)^2\sqrt{||y||_2^2 - (||x||_2^2||y||_2^2 - \langle x,y\rangle^2)}},$$

where $||.||_2$ is the Euclidean norm on $\mathbb{R}^n$, and $\langle , \rangle$ is the *inner product* on $\mathbb{R}^n$. It is a **projective metric** and an (α,β)**-metric** with $\phi(s) = (1+s)^2$, i.e., $F = \frac{(\alpha+\beta)^2}{\alpha}$. The Riemannian metrics are special Berwald metrics. Every Berwald metric is affinely equivalent to a Riemannian metric.
In general, every Finsler metric on a manifold M^n induces a *spray* (second-order homogeneous ordinary differential equation) $y_i\frac{\partial}{\partial x_i} - 2G^i\frac{\partial}{\partial y_i}$ which determines the geodesics. A Finsler metric is a Berwald metric if the spray coefficients $G^i = G^i(x,y)$ are quadratic in $y \in T_x(M^n)$ at any point $x \in M^n$, i.e., $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$.
A Finsler metric is a more general **Landsberg metric** $\Gamma^i_{jk} = \frac{1}{2}\partial_{y^j}\partial_{y^k}(\Gamma^i_{jk}(x)y^jy^k)$. The Landsberg metric is the one for which the *Landsberg curvature* (the covariant derivative of the *Cartan torsion along a geodesic*) is zero.
- **Douglas metric**
A **Douglas metric** a Finsler metric for which the *spray coefficients* $G^i = G^i(x,y)$ have the following form:

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y_iy_k + P(x,y)y_i.$$

Every Finsler metric which is projectively equivalent to a **Berwald metric** is a Douglas metric. Every **Berwald metric** is a Douglas metric. Every known Douglas metric is (locally) projectively equivalent to a Berwald metric.
- **Bryant metric**
Let α be an angle with $|\alpha| < \frac{\pi}{2}$. Let, for any $x,y \in \mathbb{R}^n$, $A = ||y||_2^4\sin^2 2\alpha + \left(||y||_2^2\cos 2\alpha + ||x||_2^2||y||_2^2 - \langle x,y\rangle^2\right)^2$, $B = ||y||_2^2\cos 2\alpha + ||x||_2^2||y||_2^2 - \langle x,y\rangle^2$, $C = \langle x,y\rangle\sin 2\alpha$, $D = ||x||_2^4 + 2||x||_2^2\cos 2\alpha + 1$. Then we get a Finsler metric

$$\sqrt{\frac{\sqrt{A}+B}{2D} + \left(\frac{C}{D}\right)^2} + \frac{C}{D}.$$

On the 2D *unit sphere* S^2, it is the **Bryant metric** (1996).

- **m-th root pseudo-Finsler metric**
  An **m-th root pseudo-Finsler metric** is (Shimada, 1979) a **pseudo-Finsler metric** defined (with $a_{i_1 \dots i_m}$ symmetric in all its indices) by
  $$F(x, y) = (a_{i_1 \dots i_m}(x) y^{i_1 \dots i_m})^{\frac{1}{m}}.$$
  For $m = 2$, it is a pseudo-Riemannian metric. The 3rd and 4th root pseudo-Finsler metrics are called *cubic metric* and *quartic metric*, respectively.
- **Antonelli–Shimada metric**
  The **Antonelli–Shimada metric** (or *ecological Finsler metric*) is an **m-th root pseudo-Finsler metric** defined, via linearly independent 1-forms a^i, by
  $$F(x, y) = \left(\sum_{i=1}^{n} (a^i)^m \right)^{\frac{1}{m}}.$$
  The **Uchijo metric** is defined, for a constant k, by
  $$F(x, y) = \left(\sum_{i=1}^{n} (a^i)^2 \right)^{\frac{1}{2}} + k a^1.$$
- **Berwald–Moör metric**
  The **Berwald–Moör metric** is an **m-th root pseudo-Finsler metric**, defined by
  $$F(x, y) = (y^1 \dots y^n)^{\frac{1}{n}}.$$
  More general **Asanov metric** is defined, via linearly independent 1-forms a^i, by
  $$F(x, y) = (a^1 \dots a^n)^{\frac{1}{n}}.$$
  The Berwald–Moör metrics with $n = 4$ and $n = 6$ are applied in Relativity Theory and Diffusion Imaging, respectively. The pseudo-Finsler spaces which are locally isomorphic to the 4th root Berwald–Moör metric, are expected to be more general and productive space-time models than usual pseudo-Riemannian spaces, which are locally isomorphic to the Minkowski metric.
- **Kawaguchi metric**
  The **Kawaguchi metric** is a metric on a smooth n-dimensional manifold M^n, given by the arc element ds of a regular curve $x = x(t)$, $t \in [t_0, t_1]$ via the formula
  $$ds = F(x, \frac{dx}{dt}, \dots, \frac{d^k x}{dt^k}) dt,$$

where the *metric function* F satisfies Zermelo's conditions: $\sum_{s=1}^{k} sx^{(s)}F_{(s)i}= F$, $\sum_{s=r}^{k}\binom{s}{k}x^{(s-r+1)i}F_{(s)i} = 0$, $x^{(s)i} = \frac{d^s x^i}{dt^s}$, $F_{(s)i} = \frac{\partial F}{\partial x^{(s)i}}$, and $r = 2, \dots, k$. These conditions ensure that the arc element ds is independent of the parametrization of the curve $x = x(t)$.

A *Kawaguchi manifold* (or *Kawaguchi space*) is a smooth manifold equipped with a Kawaguchi metric. It is a generalization of a *Finsler manifold*.

- **Lagrange metric**

Consider a real n-dimensional manifold M^n. A set of symmetric nondegenerated matrices $((g_{ij}(p,x)))$ define a **generalized Lagrange metric** on M^n if a change of coordinates $(p,x) \to (q,y)$, such that $q_i = q_i(p_1,\dots,p_n)$, $y_i = (\partial_j q_i)x_j$ and rank $(\partial_j q_i) = n$, implies $g_{ij}(p,x) = (\partial_i q_i)(\partial_j q_j)g_{ij}(q,y)$.

A generalized Lagrange metric is called a **Lagrange metric** if there exists a *Lagrangian*, i.e., a smooth function $L(p,x)$ such that it holds

$$g_{ij}(p,x) = \frac{1}{2}\frac{\partial^2 L(p,x)}{\partial x_i \partial x_j}.$$

Every Finsler metric is a Lagrange metric with $L = F^2$.

- **DeWitt supermetric**

The **DeWitt supermetric** (or **Wheeler–DeWitt supermetric**) $G = ((G_{ijkl}))$ calculates distances between metrics on a given manifold, and it is a generalization of a Riemannian (or pseudo-Riemannian) metric $g = ((g_{ij}))$.

For example, for a given connected smooth 3-dimensional manifold M^3, consider the space $\mathcal{M}(M^3)$ of all Riemannian (or pseudo-Riemannian) metrics on M^3. Identifying points of $\mathcal{M}(M^3)$ that are related by a diffeomorphism of M^3, one obtains the space $Geom(M^3)$ of 3-*geometries* (of fixed topology), points of which are the classes of diffeomorphically equivalent metrics. The space $Geom(M^3)$ is called a *superspace*. It plays an important role in several formulations of Quantum Gravity.

A **supermetric**, i.e., a "metric on metrics", is a metric on $\mathcal{M}(M^3)$ (or on $Geom(M^3)$) which is used for measuring distances between metrics on M^3 (or between their equivalence classes). Given $g = ((g_{ij})) \in \mathcal{M}(M^3)$, we obtain

$$||\delta g||^2 = \int_{M^3} d^3x G^{ijkl}(x)\delta g_{ij}(x)\delta g_{kl}(x),$$

where G^{ijkl} is the inverse of the **DeWitt supermetric**

$$G_{ijkl} = \frac{1}{2\sqrt{det((g_{ij}))}}(g_{ik}g_{jl} + g_{il}g_{jk} - \lambda g_{ij}g_{kl}).$$

The value λ parametrizes the distance between metrics in $\mathcal{M}(M^3)$, and may take any real value except $\lambda = \frac{2}{3}$, for which the supermetric is *singular*.

- **Lund–Regge supermetric**
  The **Lund–Regge supermetric** (or **simplicial supermetric**) is an analog of the **DeWitt supermetric**, used to measure the distances between *simplicial 3-geometries* in a *simplicial configuration space*.
  More exactly, given a closed *simplicial* 3D manifold M^3 consisting of several *tetrahedra* (i.e., 3-*simplices*), a *simplicial geometry* on M^3 is fixed by an assignment of values to the squared edge lengths of M^3, and a flat Riemannian Geometry to the interior of each tetrahedron consistent with those values.
  The squared edge lengths should be positive and constrained by the triangle inequalities and their analogs for the tetrahedra, i.e., all squared measures (lengths, areas, volumes) must be nonnegative (cf. **tetrahedron inequality** in Chap. 3).
  The set $\mathcal{T}(M^3)$ of all simplicial geometries on M^3 is called a *simplicial configuration space*. The Lund–Regge supermetric $((G_{mn}))$ on $\mathcal{T}(M^3)$ is induced from the DeWitt supermetric on $\mathcal{M}(M^3)$, using for representations of points in $\mathcal{T}(M^3)$ such metrics in $\mathcal{M}(M^3)$ which are piecewise flat in the tetrahedra.
- **Space of Lorentz metrics**
  Let M^n be an n-dimensional compact manifold, and $\mathcal{L}(M^n)$ the set of all **Lorentz metrics** (i.e., the pseudo-Riemannian metrics of signature $(n-1, 1)$) on M^n.
  Given a Riemannian metric g on M^n, one can identify the vector space $S^2(M^n)$ of all symmetric 2-tensors with the vector space of endomorphisms of the tangent to M^n which are symmetric with respect to g. In fact, if $\tilde{h}$ is the endomorphism associated to a tensor h, then the distance on $S^2(M^n)$ is given by

$$d_g(h, t) = \sup_{x \in M^n} \sqrt{tr(\tilde{h}_x - \tilde{t}_x)^2}.$$

  The set $\mathcal{L}(M^n)$ taken with the distance d_g is an open subset of $S^2(M^n)$ called the **space of Lorentz metrics**. Cf. **manifold triangulation metric** in Chap. 9.
- **Perelman supermetric proof**
  The *Thurston's Geometrization Conjecture* is that, after two well-known splittings, any 3D manifold admits, as remaining components, only one of eight *Thurston model geometries*. If true, this conjecture implies the validity of the famous *Poincaré Conjecture* of 1904, that any 3-manifold, in which every simple closed curve can be deformed continuously to a point, is homeomorphic to the 3-sphere.
  In 2002, Perelman gave a gapless "sketch of an eclectic proof" of Thurston's conjecture using a kind of supermetric approach to the space of all Riemannian metrics on a given smooth 3-manifold. In a *Ricci flow* the distances decrease in directions of positive curvature since the metric is time-dependent. Perelman's modification of the standard Ricci flow permitted systematic elimination of arising singularities.

7.2 Riemannian Metrics in Information Theory

Some special Riemannian metrics are commonly used in Information Theory. A list of such metrics is given below.

- **Thermodynamic metrics**
  Given the space of all *extensive* (additive in magnitude, mechanically conserved) thermodynamic variables of a system (energy, entropy, amounts of materials), a **thermodynamic metric** is a Riemannian metric on the manifold of equilibrium states defined as the 2nd derivative of one extensive quantity, usually entropy or energy, with respect to the other extensive quantities. This information geometric approach provides a geometric description of thermodynamic systems in equilibrium.
  The **Ruppeiner metric** (Ruppeiner, 1979) is defined by the *line element* $ds_R^2 = g_{ij}^R dx^i dx^j$, where the matrix $((g_{ij}^R))$ of the symmetric metric tensor is a negative *Hessian* (the matrix of 2nd order partial derivatives) of the entropy function S:

  $$g_{ij}^R = -\partial_i \partial_j S(M, N^a).$$

  Here M is the internal energy (which is the mass in black hole applications) of the system and N^a refer to other extensive parameters such as charge, angular momentum, volume, etc. This metric is flat if and only if the statistical mechanical system is noninteracting, while curvature singularities are a signal of critical behavior, or, more precisely, of divergent **correlation lengths** (cf. Chap. 24).
  The **Weinhold metric** (Weinhold, 1975) is defined by $g_{ij}^W = \partial_i \partial_j M(S, N^a)$.
  The Ruppeiner and Weinhold metrics are *conformally equivalent* (cf. **conformal metric**) via $ds^2 = g_{ij}^R dM^i dM^j = \frac{1}{T} g_{ij}^W dS^i dS^j$, where T is the temperature.
  The **thermodynamic length** in Chap. 24 is a path function that measures the distance along a path in the state space.
- **Fisher information metric**
  In Statistics, Probability, and Information Geometry, the **Fisher information metric** is a Riemannian metric for a statistical differential manifold (see, for example, [Amar85, Frie98]). Formally, let $p_\theta = p(x, \theta)$ be a family of densities, indexed by n parameters $\theta = (\theta_1, \dots, \theta_n)$ which form the *parameter manifold* P.
  The **Fisher information metric** $g = g_\theta$ on P is a Riemannian metric, defined by the *Fisher information matrix* $((I(\theta)_{ij}))$, where

  $$I(\theta)_{ij} = \mathbb{E}_\theta \left[\frac{\partial \ln p_\theta}{\partial \theta_i} \cdot \frac{\partial \ln p_\theta}{\partial \theta_j} \right] = \int \frac{\partial \ln p(x,\theta)}{\partial \theta_i} \frac{\partial \ln p(x,\theta)}{\partial \theta_j} p(x,\theta) dx.$$

  It is a symmetric bilinear form which gives a classical measure (*Rao measure*) for the statistical distinguishability of distribution parameters.

Putting $i(x, \theta) = -\ln p(x, \theta)$, one obtains an equivalent formula

$$I(\theta)_{ij} = \mathbb{E}_\theta\left[\frac{\partial^2 i(x,\theta)}{\partial\theta_i \partial\theta_j}\right] = \int \frac{\partial^2 i(x,\theta)}{\partial\theta_i \partial\theta_j} p(x,\theta)dx.$$

In a coordinate-free language, we get

$$I(\theta)(u, v) = \mathbb{E}_\theta\left[u(\ln p_\theta) \cdot v(\ln p_\theta)\right],$$

where u and v are vectors tangent to the parameter manifold P, and $u(\ln p_\theta) = \frac{d}{dt} \ln p_{\theta+tu|t=0}$ is the derivative of $\ln p_\theta$ along the direction u.
A *manifold of densities* M is the image of the parameter manifold P under the mapping $\theta \to p_\theta$ with certain regularity conditions. A vector u tangent to this manifold is of the form $u = \frac{d}{dt} p_{\theta+tu|t=0}$, and the Fisher information metric $g = g_p$ on M, obtained from the metric g_θ on P, can be written as

$$g_p(u, v) = \mathbb{E}_p\left[\frac{u}{p} \cdot \frac{v}{p}\right].$$

- **Fisher–Rao metric**
Let $\mathcal{P}_n = \{p \in \mathbb{R}^n : \sum_{i=1}^n p_i = 1, p_i > 0\}$ be the simplex of strictly positive probability vectors. An element $p \in \mathcal{P}_n$ is a density of the n-point set $\{1, \dots, n\}$ with $p(i) = p_i$. An element u of the tangent space $T_p(\mathcal{P}_n) = \{u \in \mathbb{R}^n : \sum_{i=1}^n u_i = 0\}$ at a point $p \in \mathcal{P}_n$ is a function on $\{1, \dots, n\}$ with $u(i) = u_i$.
The **Fisher–Rao metric** g_p on $\mathcal{P}_n$ is a Riemannian metric defined by

$$g_p(u, v) = \sum_{i=1}^n \frac{u_i v_i}{p_i}$$

for any $u, v \in T_p(\mathcal{P}_n)$, i.e., it is the **Fisher information metric** on $\mathcal{P}_n$.
The Fisher–Rao metric is the unique (up to a constant factor) Riemannian metric on $\mathcal{P}_n$, contracting under stochastic maps [Chen72].
This metric is isometric, by $p \to 2(\sqrt{p_1}, \dots, \sqrt{p_n})$, with the standard metric on an open subset of the sphere of radius two in $\mathbb{R}^n$. This identification allows one to obtain on $\mathcal{P}_n$ the **geodesic distance**, called the **Rao distance**, by

$$2 \arccos(\sum_i p_i^{1/2} q_i^{1/2}).$$

The Fisher–Rao metric can be extended to the set $\mathcal{M}_n = \{p \in \mathbb{R}^n, p_i > 0\}$ of all finite strictly positive measures on the set $\{1, \dots, n\}$. In this case, the geodesic distance on $\mathcal{M}_n$ can be written as

$$2\left(\sum_i (\sqrt{p_i} - \sqrt{q_i})^2\right)^{1/2}$$

for any $p, q \in \mathcal{M}_n$ (cf. **Hellinger metric** in Chap. 14).

- **Monotone metrics**
Let M_n be the set of all complex $n \times n$ matrices. Let $\mathcal{M} \subset M_n$ be the manifold of all such positive-definite matrices. Let $\mathcal{D} \subset \mathcal{M}$, $\mathcal{D} = \{\rho \in \mathcal{M} : \mathrm{Tr}\rho = 1\}$, be the submanifold of all *density matrices*. It is the space of faithful states of an n-level quantum system; cf. **distances between quantum states** in Chap. 24.
The tangent space of $\mathcal{M}$ at $\rho \in \mathcal{M}$ is $T_\rho(\mathcal{M}) = \{x \in M_n : x = x^*\}$, i.e., the set of all $n \times n$ *Hermitian matrices*. The tangent space $T_\rho(\mathcal{D})$ at $\rho \in \mathcal{D}$ is the subspace of *traceless* (i.e., with trace 0) matrices in $T_\rho(\mathcal{M})$.
A Riemannian metric λ on $\mathcal{M}$ is called **monotone metric** if the inequality

$$\lambda_{h(\rho)}(h(u), h(u)) \leq \lambda_\rho(u, u)$$

holds for any $\rho \in \mathcal{M}$, any $u \in T_\rho(\mathcal{M})$, and any *stochastic*, i.e., completely positive trace preserving mapping h.
It was proved in [Petz96] that λ is monotone if and only if it can be written as

$$\lambda_\rho(u, v) = \mathrm{Tr}\ uJ_\rho(v),$$

where J_ρ is an operator of the form $J_\rho = \frac{1}{f(L_\rho/R_\rho)R_\rho}$. Here L_ρ and R_ρ are the left and the right multiplication operators, and $f : (0, \infty) \to \mathbb{R}$ is an operator monotone function which is *symmetric*, i.e., $f(t) = tf(t^{-1})$, and *normalized*, i.e., $f(1) = 1$. Then $J_\rho(v) = \rho^{-1}v$ if v and ρ are commute, i.e., any monotone metric is equal to the **Fisher information metric** on commutative submanifolds.
The **Bures metric** (or *statistical metric*) is the smallest monotone metric, obtained for $f(t) = \frac{1+t}{2}$. In this case $J_\rho(v) = g$, $\rho g + g\rho = 2v$, is the *symmetric logarithmic derivative*. For any $\rho_1, \rho_2 \in \mathcal{M}$ the **geodesic distance** defined by the Bures metric, (cf. **Bures length** in Chap. 24) can be written as

$$2\sqrt{\mathrm{Tr}(\rho_1) + \mathrm{Tr}(\rho_2) - 2\mathrm{Tr}(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}})}.$$

On the submanifold $\mathcal{D} = \{\rho \in \mathcal{M} : \mathrm{Tr}\rho = 1\}$ of density matrices it has the form

$$2 \arccos \mathrm{Tr}(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}).$$

The **right logarithmic derivative metric** (or *RLD-metric*) is the greatest monotone metric, corresponding to the function $f(t) = \frac{2t}{1+t}$. In this case $J_\rho(v) = \frac{1}{2}(\rho^{-1}v + v\rho^{-1})$ is the *right logarithmic derivative*.
The **Bogolubov–Kubo–Mori metric** (or *BKM-metric*) is obtained for $f(x) = \frac{x-1}{\ln x}$. It can be written as $\lambda_\rho(u, v) = \frac{\partial^2}{\partial s \partial t}\mathrm{Tr}(\rho + su)\ln(\rho + tv)|_{s,t=0}$.
- **Wigner–Yanase–Dyson metrics**
The **Wigner–Yanase–Dyson metrics** (or *WYD-metrics*) form a family of metrics on the manifold $\mathcal{M}$ of all complex positive-definite $n \times n$ matrices defined by

$$\lambda_\rho^\alpha(u, v) = \frac{\partial^2}{\partial t \partial s} \text{Tr} f_\alpha(\rho + tu) f_{-\alpha}(\rho + sv)|_{s,t=0},$$

where $f_\alpha(x) = \frac{2}{1-\alpha} x^{\frac{1-\alpha}{2}}$, if $\alpha \neq 1$, and is $\ln x$, if $\alpha = 1$. These metrics are monotone for $\alpha \in [-3, 3]$. For $\alpha = \pm 1$ one obtains the **Bogolubov–Kubo–Mori metric**; for $\alpha = \pm 3$ one obtains the **right logarithmic derivative metric**.
The **Wigner–Yanase metric** (or *WY-metric*) is λ_ρ^0, the smallest metric in the family. It can be written as $\lambda_\rho(u, v) = 4\text{Tr } u(\sqrt{L_\rho} + \sqrt{R_\rho})^2(v)$.

- **Connes metric**
Roughly, the **Connes metric** is a generalization (from the space of all probability measures of a set X, to the *state space* of any *unital C^*-algebra*) of the **transportation distance** (Chap. 14) defined via *Lipschitz seminorm*.
Let M^n be a smooth n-dimensional manifold. Let $A = C^\infty(M^n)$ be the (commutative) algebra of smooth complex-valued functions on M^n, represented as multiplication operators on the Hilbert space $H = L^2(M^n, S)$ of square integrable sections of the spinor bundle on M^n by $(f\xi)(p) = f(p)\xi(p)$ for all $f \in A$ and for all $\xi \in H$.
Let D be the *Dirac operator*. Let the commutator $[D, f]$ for $f \in A$ be the *Clifford multiplication* by the gradient ∇f, so that its operator norm $||.||$ in H is given by $||[D, f]|| = \sup_{p \in M^n} ||\nabla f||$.
The **Connes metric** is the **intrinsic metric** on M^n, defined by

$$\sup_{f \in A, ||\,[D,f]\,|| \leq 1} |f(p) - f(q)|.$$

This definition can also be applied to discrete spaces, and even generalized to *C^*-algebras*; cf. **Rieffel metric space**. In particular, for a labeled connected *locally finite* graph $G = (V, E)$ with the vertex-set $V = \{v_1, \dots, v_n, \dots\}$, the Connes metric on V is defined, for any $v_i, v_j \in V$, by $\sup_{||\,[D,f]\,|| = ||df|| \leq 1} |f_{v_i} - f_{v_j}|$, where $\{f = \sum f_{v_i} v_i : \sum |f_{v_i}|^2 < \infty\}$ is the set of formal sums f, forming a Hilbert space, and $||\,[D, f]\,||$ is $\sup_i (\sum_{k=1}^{deg(v_i)} (f_{v_k} - f_{v_i})^2)^{\frac{1}{2}}$.

- **Rieffel metric space**
Let V be a *normed space* (or, more generally, a **locally convex topological vector space**, cf. Chap. 2), and let V' be its **continuous dual space**, i.e., the set of all continuous linear functionals f on V. The *weak-* topology* on V' is defined as the weakest (i.e., with the fewest open sets) topology on V' such that, for every $x \in V$, the map $F_x : V' \to \mathbb{R}$ defined by $F_x(f) = f(x)$ for all $f \in V'$, remains continuous.
An *order-unit space* is a *partially ordered* real (complex) vector space $(A, \preceq)$ with a special distinguished element e (*order unit*) satisfying the following properties:

1. For any $a \in A$, there exists $r \in \mathbb{R}$ with $a \preceq re$;
2. If $a \in A$ and $a \preceq re$ for all positive $r \in \mathbb{R}$, then $a \preceq 0$ (*Archimedean property*).

The main example of an order-unit space is the vector space of all self-adjoint elements in a *unital C^*-algebra* with the identity element being the order unit. Here a *C^*-algebra* is a *Banach algebra* over $\mathbb{C}$ equipped with a special *involution*. It is called *unital* if it has a *unit* (multiplicative identity element); such C^*-algebras are also called, roughly, *compact noncommutative topological spaces*.

Main example of a unital C^*-algebra is the complex algebra of linear operators on a complex **Hilbert space** which is topologically closed in the norm topology of operators, and is closed under the operation of taking adjoints of operators.

The *state space* of an order-unit space $(A, \preceq, e)$ is the set $S(A) = \{f \in A'_+ : ||f|| = 1\}$ of *states*, i.e., continuous linear functionals f with $||f|| = f(e) = 1$.

A **Rieffel** (or *compact quantum* as in Rieffel, 1999) **metric space** is a pair $(A, ||.||_{Lip})$, where $(A, \preceq, e)$ is an order-unit space, and $||.||_{Lip}$ is a $[0, +\infty]$-valued seminorm on A (generalizing the *Lipschitz seminorm*) for which it hold:

1. For $a \in A$, $||a||_{Lip} = 0$ holds if and only if $a \in \mathbb{R}e$;
2. the metric $d_{Lip}(f, g) = \sup_{a \in A: ||a||_{Lip} \leq 1} |f(a) - g(a)|$ generates on the state space $S(A)$ its weak-* topology.

So, $(S(A), d_{Lip})$ is a usual metric space. If the order-unit space $(A, \preceq, e)$ is a C^*-algebra, then d_{Lip} is the **Connes metric**, and if, moreover, the C^*-algebra is noncommutative, $(S(A), d_{Lip})$ is called a **noncommutative metric space**.

The term *quantum* is due to the belief that the Planck-scale geometry of *space-time* comes from such C^*-algebras; cf. **quantum space-time** in Chap. 24.

Kuperberg and Weaver, 2010, proposed a new definition of *quantum metric space*, in the setting of *von Neumann algebras*.

7.3 Hermitian Metrics and Generalizations

A *vector bundle* is a geometrical construct where to every point of a *topological space M* we attach a vector space so that all those vector spaces "glued together" form another topological space E. A continuous mapping $\pi : E \to M$ is called a *projection E* on M. For every $p \in M$, the vector space $\pi^{-1}(p)$ is called a *fiber* of the vector bundle.

A *real* (*complex*) *vector bundle* is a vector bundle $\pi : E \to M$ whose fibers $\pi^{-1}(p)$, $p \in M$, are real (complex) vector spaces.

In a real vector bundle, for every $p \in M$, the fiber $\pi^{-1}(p)$ locally looks like the vector space $\mathbb{R}^n$, i.e., there is an *open neighborhood U* of p, a natural number n, and a homeomorphism $\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that, for all $x \in U$ and $v \in \mathbb{R}^n$, one has $\pi(\varphi(x, v)) = v$, and the mapping $v \to \varphi(x, v)$ yields an isomorphism between $\mathbb{R}^n$ and $\pi^{-1}(x)$. The set U, together with φ, is called a *local trivialization* of the bundle.

If there exists a "global trivialization", then a real vector bundle $\pi : M \times \mathbb{R}^n \to M$ is called *trivial*. Similarly, in a complex vector bundle, for every $p \in M$, the fiber $\pi^{-1}(p)$ locally looks like the vector space $\mathbb{C}^n$. The basic example of such bundle is the trivial bundle $\pi : U \times \mathbb{C}^n \to U$, where U is an open subset of $\mathbb{R}^k$.

Important special cases of a real vector bundle are the *tangent bundle* $T(M^n)$ and the *cotangent bundle* $T^*(M^n)$ of a *real n-dimensional manifold* $M^n_{\mathbb{R}} = M^n$. Important special cases of a complex vector bundle are the tangent bundle and the cotangent bundle of a *complex n-dimensional manifold*.

Namely, a *complex n-dimensional manifold* $M^n_{\mathbb{C}}$ is a *topological space* in which every point has an open neighborhood homeomorphic to an open set of the n-dimensional complex vector space $\mathbb{C}^n$, and there is an atlas of charts such that the change of coordinates between charts is analytic. The (complex) tangent bundle $T_{\mathbb{C}}(M^n_{\mathbb{C}})$ of a complex manifold $M^n_{\mathbb{C}}$ is a vector bundle of all (complex) *tangent spaces* of $M^n_{\mathbb{C}}$ at every point $p \in M^n_{\mathbb{C}}$. It can be obtained as a *complexification* $T_{\mathbb{R}}(M^n_{\mathbb{R}}) \otimes \mathbb{C} = T(M^n) \otimes \mathbb{C}$ of the corresponding real tangent bundle, and is called the *complexified tangent bundle* of $M^n_{\mathbb{C}}$.

The *complexified cotangent bundle* of $M^n_{\mathbb{C}}$ is obtained similarly as $T^*(M^n) \otimes \mathbb{C}$. Any complex n-dimensional manifold $M^n_{\mathbb{C}} = M^n$ can be regarded as a real $2n$-dimensional manifold equipped with a *complex structure* on each tangent space.

A *complex structure* on a real vector space V is the structure of a complex vector space on V that is compatible with the original real structure. It is completely determined by the operator of multiplication by the number i, the role of which can be taken by an arbitrary linear transformation $J : V \to V$, $J^2 = -id$, where *id* is the *identity mapping*.

A *connection* (or *covariant derivative*) is a way of specifying a derivative of a *vector field* along another vector field in a vector bundle. A **metric connection** is a linear connection in a vector bundle $\pi : E \to M$, equipped with a bilinear form in the fibers, for which parallel displacement along an arbitrary piecewise-smooth curve in M preserves the form, that is, the *scalar product* of two vectors remains constant under parallel displacement.

In the case of a nondegenerate symmetric bilinear form, the metric connection is called the *Euclidean connection*. In the case of nondegenerate antisymmetric bilinear form, the metric connection is called the *symplectic connection*.

- **Bundle metric**
  A **bundle metric** is a metric on a vector bundle.
- **Hermitian metric**
  A **Hermitian metric** on a complex vector bundle $\pi : E \to M$ is a collection of *Hermitian inner products* (i.e., positive-definite symmetric sesquilinear forms) on every fiber $E_p = \pi^{-1}(p)$, $p \in M$, that varies smoothly with the point p in M. Any complex vector bundle has a Hermitian metric.
  The basic example of a vector bundle is the trivial bundle $\pi : U \times \mathbb{C}^n \to U$, where U is an open set in $\mathbb{R}^k$. In this case a Hermitian inner product on $\mathbb{C}^n$, and hence, a Hermitian metric on the bundle $\pi : U \times \mathbb{C}^n \to U$, is defined by

$$\langle u, v \rangle = u^T H \overline{v},$$

where H is a *positive-definite Hermitian matrix*, i.e., a complex $n \times n$ matrix such that $H^* = \overline{H}^T = H$, and $\overline{v}^T H v > 0$ for all $v \in \mathbb{C}^n \backslash \{0\}$. In the simplest case, one has $\langle u, v \rangle = \sum_{i=1}^{n} u_i \overline{v}_i$.
An important special case is a Hermitian metric h on a complex manifold M^n, i.e., on the complexified tangent bundle $T(M^n) \otimes \mathbb{C}$ of M^n. This is the Hermitian analog of a Riemannian metric. In this case $h = g + iw$, and its real part g is a Riemannian metric, while its imaginary part w is a nondegenerate antisymmetric bilinear form, called a *fundamental form*. Here $g(J(x), J(y)) = g(x, y)$, $w(J(x), J(y)) = w(x, y)$, and $w(x, y) = g(x, J(y))$, where the operator J is an operator of complex structure on M^n; as a rule, $J(x) = ix$. Any of the forms g, w determines h uniquely.
The term *Hermitian metric* can also refer to the corresponding Riemannian metric g, which gives M^n a Hermitian structure.
On a complex manifold, a Hermitian metric h can be expressed in local coordinates by a *Hermitian symmetric tensor* $((h_{ij}))$:

$$h = \sum_{i,j} h_{ij} dz_i \otimes d\overline{z}_j,$$

where $((h_{ij}))$ is a positive-definite Hermitian matrix. The associated fundamental form w is then written as $w = \frac{i}{2} \sum_{i,j} h_{ij} dz_i \wedge d\overline{z}_j$. A *Hermitian manifold* (or *Hermitian space*) is a complex manifold equipped with a Hermitian metric.

- **Kähler metric**
A **Kähler metric** (or *Kählerian metric*) is a Hermitian metric $h = g + iw$ on a complex manifold M^n whose fundamental form w is *closed*, i.e., $dw = 0$ holds. A *Kähler manifold* is a complex manifold equipped with a Kähler metric.
If h is expressed in local coordinates, i.e., $h = \sum_{i,j} h_{ij} dz_i \otimes d\overline{z}_j$, then the associated fundamental form w can be written as $w = \frac{i}{2} \sum_{i,j} h_{ij} dz_i \wedge d\overline{z}_j$, where $\wedge$ is the *wedge product* which is antisymmetric, i.e., $dx \wedge dy = -dy \wedge dx$ (hence, $dx \wedge dx = 0$).
In fact, w is a *differential* 2*-form* on M^n, i.e., a tensor of rank 2 that is antisymmetric under exchange of any pair of indices: $w = \sum_{i,j} f_{ij} dx^i \wedge dx^j$, where f_{ij} is a function on M^n. The *exterior derivative* dw of w is defined by $dw = \sum_{i,j} \sum_k \frac{\partial f_{ij}}{\partial x_k} dx_k \wedge dx_i \wedge dx_k$. If $dw = 0$, then w is a *symplectic* (i.e., closed nondegenerate) differential 2-form. Such differential 2-forms are called *Kähler forms*.
The metric on a Kähler manifold locally satisfies $h_{ij} = \frac{\partial^2 K}{\partial z_i \partial \overline{z}_j}$. for some function K, called the *Kähler potential*. The term *Kähler metric* can also refer to the corresponding Riemannian metric g, which gives M^n a Kähler structure. Then a Kähler manifold is defined as a complex manifold which carries a Riemannian metric and a Kähler form on the underlying real manifold.

- **Hessian metric**
  Given a smooth f on an open subset of a real vector space, the associated **Hessian metric** is defined by

$$g_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

  A Hessian metric is also called an **affine Kähler metric** since a Kähler metric on a complex manifold has an analogous description as $\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}$.
- **Calabi–Yau metric**
  The **Calabi–Yau metric** is a **Kähler metric** which is **Ricci-flat**.
  A *Calabi–Yau manifold* (or *Calabi–Yau space*) is a simply connected complex manifold equipped with a Calabi–Yau metric. It can be considered as a $2n$-dimensional (6D being particularly interesting) smooth manifold with holonomy group (i.e., the set of linear transformations of tangent vectors arising from parallel transport along closed loops) in the special unitary group.
- **Kähler–Einstein metric**
  A **Kähler–Einstein metric** is a **Kähler metric** on a complex manifold M^n whose *Ricci curvature tensor* is proportional to the metric tensor. This proportionality is an analog of the *Einstein field equation* in the General Theory of Relativity.
  A *Kähler–Einstein manifold* (or *Einstein manifold*) is a complex manifold equipped with a Kähler–Einstein metric. In this case the Ricci curvature tensor, seen as an operator on the tangent space, is just multiplication by a constant.
  Such a metric exists on any *domain* $D \subset \mathbb{C}^n$ that is bounded and *pseudo-convex*. It can be given by the *line element*

$$ds^2 = \sum_{i,j} \frac{\partial^2 u(z)}{\partial z_i \partial \overline{z}_j} dz_i d\overline{z}_j,$$

  where u is a solution to the *boundary value problem*: $det(\frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}) = e^{2u}$ on D, and $u = \infty$ on ∂D. The Kähler–Einstein metric is a complete metric. On the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ it is coincides with the **Poincaré metric**.
  Let h be the Einstein metric on a smooth compact manifold M^{n-1} without boundary, having scalar curvature $(n-1)(n-2)$. A **generalized Delaunay metric** on $\mathbb{R} \times M^{n-1}$ is (Delay, 2010) of the form $g = u^{\frac{4}{n-2}}(dy^2 + h)$, where $u = u(y) > 0$ is a periodic solution of $u'' - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0$.
  There is one parameter family of constant positive curvature conformal metrics on $\mathbb{R} \times \mathbb{S}^{n-1}$, referred to as **Delaunay metric**; cf. **Kottler metric** in Chap. 26.
- **Hodge metric**
  The **Hodge metric** is a **Kähler metric** whose *fundamental form* w defines an integral cohomology class or, equivalently, has integral periods.

A *Hodge manifold* (or *Hodge variety*) is a complex manifold equipped with a Hodge metric. A compact complex manifold is a Hodge manifold if and only if it is isomorphic to a smooth algebraic subvariety of some complex projective space.

- **Fubini–Study metric**

  The **Fubini–Study metric** (or *Cayley–Fubini–Study metric*) is a **Kähler metric** on a *complex projective space* $\mathbb{C}P^n$ defined by a *Hermitian inner product* $\langle , \rangle$ in $\mathbb{C}^{n+1}$. It is given by the *line element*

$$ds^2 = \frac{\langle x, x\rangle\langle dx, dx\rangle - \langle x, d\overline{x}\rangle\langle \overline{x}, dx\rangle}{\langle x, x\rangle^2}.$$

  The **Fubini–Study distance** between points $(x_1 : \ldots : x_{n+1})$ and $(y_1 : \ldots : y_{n+1}) \in \mathbb{C}P^n$, where $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1}) \in \mathbb{C}^{n+1} \backslash \{0\}$, is equal to

$$\arccos \frac{|\langle x, y\rangle|}{\sqrt{\langle x, x\rangle\langle y, y\rangle}}.$$

  The Fubini–Study metric is a **Hodge metric**. The space $\mathbb{C}P^n$ endowed with this metric is called a *Hermitian elliptic space* (cf. **Hermitian elliptic metric**).

- **Bergman metric**

  The **Bergman metric** is a **Kähler metric** on a bounded *domain* $D \subset \mathbb{C}^n$ defined, for the *Bergman kernel* $K(z, u)$, by the *line element*

$$ds^2 = \sum_{i,j} \frac{\partial^2 \ln K(z, z)}{\partial z_i \partial \overline{z}_j} dz_i d\overline{z}_j.$$

  It is a **biholomorhically invariant metric** on D, and it is complete if D is homogeneous. For the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ the Bergman metric coincides with the **Poincaré metric**; cf. also **Bergman** p**-metric** in Chap. 13.

  The set of all analytic functions $f \neq 0$ of class $L_2(D)$ with respect to the Lebesgue measure, forms the **Hilbert space** $L_{2,a}(D) \subset L_2(D)$ with an orthonormal basis $(\phi_i)_i$. The *Bergman kernel* is a function in the domain $D \times D \subset \mathbb{C}^{2n}$, defined by $K_D(z, u) = K(z, u) = \sum_{i=1}^{\infty} \phi_i(z)\overline{\phi_i(u)}$.
  The **Skwarczynski distance** is defined by

$$(1 \frac{|K(z, u)|}{\sqrt{K(z, z)}\sqrt{K(u, u)}})^{\frac{1}{2}}.$$

- **Hyper-Kähler metric**

  A **hyper-Kähler metric** is a Riemannian metric g on a $4n$-dimensional *Riemannian manifold* which is compatible with a quaternionic structure on the tangent bundle of the manifold.

Thus, the metric g is Kählerian with respect to 3 Kähler structures (I, w_I, g), (J, w_J, g), and (K, w_K, g), corresponding to the complex structures, as endomorphisms of the tangent bundle, which satisfy the quaternionic relationship

$$I^2 = J^2 = K^2 = IJK = -JIK = -1.$$

A *hyper-Kähler manifold* is a Riemannian manifold equipped with a hyper-Kähler metric. manifolds are Ricci-flat. Compact 4D hyper-Kähler manifolds are called K_3-*surfaces*; they are studied in Algebraic Geometry.

- **Calabi metric**
  The **Calabi metric** is a **hyper-Kähler metric** on the cotangent bundle $T^*(\mathbb{C}P^{n+1})$ of a *complex projective space* $\mathbb{C}P^{n+1}$.
  For $n = 4k + 4$, this metric can be given by the *line element*

$$ds^2 = \frac{dr^2}{1 - r^{-4}} + \frac{1}{4}r^2(1 - r^{-4})\lambda^2 + r^2(\nu_1^2 + \nu_2^2) + \frac{1}{2}(r^2 - 1)(\sigma_{1\alpha}^2 + \sigma_{2\alpha}^2) + \frac{1}{2}(r^2 + 1)(\Sigma_{1\alpha}^2 + \Sigma_{2\alpha}^2),$$

  where $(\lambda, \nu_1, \nu_2, \sigma_{1\alpha}, \sigma_{2\alpha}, \Sigma_{1\alpha}, \Sigma_{2\alpha})$, with α running over k values, are left-invariant *one-forms* (i.e., linear real-valued functions) on the coset $SU(k + 2)/U(k)$. Here $U(k)$ is the *unitary group* consisting of complex $k \times k$ *unitary matrices*, and $SU(k)$ is the *special unitary group* of complex $k \times k$ unitary matrices with determinant 1.
  For $k = 0$, the Calabi metric coincides with the **Eguchi–Hanson metric**.
- **Stenzel metric**
  The **Stenzel metric** is a **hyper-Kähler metric** on the cotangent bundle $T^*(S^{n+1})$ of a sphere S^{n+1}.
- $SO(3)$**-invariant metric**
  An $SO(3)$**-invariant metric** is a 4D 4-dimensional hyper-Kähler metric with the *line element* given, in the Bianchi type *IX* formalism (cf. **Bianchi metrics** in Chap. 26) by

$$ds^2 = f^2(t)dt^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2,$$

  where the invariant *one-forms* $\sigma_1, \sigma_2, \sigma_3$ of $SO(3)$ are expressed in terms of *Euler angles* θ, ψ, ϕ as $\sigma_1 = \frac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi)$, $\sigma_2 = -\frac{1}{2}(\cos\psi d\theta + \sin\theta\sin\psi d\phi)$, $\sigma_3 = \frac{1}{2}(d\psi + \cos\theta d\phi)$, and the normalization has been chosen so that $\sigma_i \wedge \sigma_j = \frac{1}{2}\epsilon_{ijk}d\sigma_k$. The coordinate t of the metric can always be chosen so that $f(t) = \frac{1}{2}abc$, using a suitable reparametrization.
- **Atiyah–Hitchin metric**
  The **Atiyah–Hitchin metric** is a **complete regular** $SO(3)$**-invariant metric** with the *line element*

$$ds^2 = \frac{1}{4}a^2b^2c^2\left(\frac{dk}{k(1-k^2)K^2}\right)^2 + a^2(k)\sigma_1^2 + b^2(k)\sigma_2^2 + c^2(k)\sigma_3^2,$$

where a, b, c are functions of k, $ab = -K(k)(E(k) - K(k))$, $bc = -K(k)(E(k) - (1-k^2)K(k))$, $ac = -K(k)E(k)$, and $K(k)$, $E(k)$ are the complete elliptic integrals, respectively, of the first and second kind, with $0 < k < 1$. The coordinate t is given by the change of variables $t = -\frac{2K(1-k^2)}{\pi K(k)}$ up to an additive constant.

- **Taub–NUT metric**

  The **Taub–NUT metric** (cf. also Chap. 26) is a **complete regular** $SO(3)$**-invariant metric** with the *line element*

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}dr^2 + (r^2 - m^2)(\sigma_1^2 + \sigma_2^2) + 4m^2\frac{r-m}{r+m}\sigma_3^2,$$

  where m is the relevant moduli parameter, and the coordinate r is related to t by $r = m + \frac{1}{2mt}$. NUT manifold was discovered in Ehlers, 1957, and rediscovered in Newman–Tamburino–Unti, 1963; it is closely related to the metric in Taub, 1951.

- **Eguchi–Hanson metric**

  The **Eguchi–Hanson metric** is a **complete regular** $SO(3)$**-invariant metric** with the *line element*

$$ds^2 = \frac{dr^2}{1 - \left(\frac{a}{r}\right)^4} + r^2\left(\sigma_1^2 + \sigma_2^2 + \left(1 - \left(\frac{a}{r}\right)^4\right)\sigma_3^2\right),$$

  where a is the moduli parameter, and the coordinate r is $a\sqrt{\coth(a^2t)}$.

  The Eguchi–Hanson metric coincides with the 4D **Calabi metric**.

- **Complex Finsler metric**

  A **complex Finsler metric** is an upper semicontinuous function $F : T(M^n) \to \mathbb{R}_+$ on a complex manifold M^n with the analytic tangent bundle $T(M^n)$ satisfying the following conditions:

  1. F^2 is smooth on $\check{M}^n$, where $\check{M}^n$ is the complement in $T(M^n)$ of the zero section;
  2. $F(p, x) > 0$ for all $p \in M^n$ and $x \in \check{M}^n_p$;
  3. $F(p, \lambda x) = |\lambda|F(p, x)$ for all $p \in M^n$, $x \in T_p(M^n)$, and $\lambda \in \mathbb{C}$.

  The function $G = F^2$ can be locally expressed in terms of the coordinates $(p_1, \dots, p_n, x_1, \dots, x_n)$; the *Finsler metric tensor* of the complex Finsler metric is given by the matrix$((G_{ij})) = ((\frac{1}{2}\frac{\partial^2 F^2}{\partial x_i \partial \overline{x}_j}))$, called the *Levi matrix*. If the matrix $((G_{ij}))$ is positive-definite, the complex Finsler metric F is called *strongly pseudo-convex*.

- **Distance-decreasing semimetric**
  Let d be a semimetric which can be defined on some class $\mathcal{M}$ of complex manifolds containing the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. It is called **distance-decreasing** if, for any analytic mapping $f : M_1 \to M_2$ with $M_1, M_2 \in \mathcal{M}$, the inequality $d(f(p), f(q)) \leq d(p, q)$ holds for all $p, q \in M_1$.
  The **Carathéodory semimetric** F_C, **Sibony semimetric** F_S, **Azukawa semimetric** F_A and **Kobayashi semimetric** F_K are distance-decreasing with F_C and F_K being the smallest and the greatest distance-decreasing semimetrics. They are generalizations of the **Poincaré metric** to higher-dimensional domains, since $F_C = F_K$ is the **Poincaré metric** on the unit disk Δ, and $F_C = F_K \equiv 0$ on $\mathbb{C}^n$. It holds $F_C(z, u) \leq F_S(z, u) \leq F_A(z, u) \leq F_K(z, u)$ for all $z \in D$ and $u \in \mathbb{C}^n$. If D is convex, then all these metrics coincide.
- **Biholomorphically invariant semimetric**
  A *biholomorphism* is a bijective *holomorphic* (complex differentiable in a neighborhood of every point in its domain) function whose inverse is also holomorphic.
  A semimetric $F(z, u) : D \times \mathbb{C}^n \to [0, \infty]$ on a domain D in $\mathbb{C}^n$ is called **biholomorphically invariant** if $F(z, u) = |\lambda| F(z, u)$ for all $\lambda \in \mathbb{C}$, and $F(z, u) = F(f(z), f'(z)u)$ for any biholomorphism $f : D \to D'$.
  Invariant metrics, including the **Carathéodory**, **Kobayashi**, **Sibony**, **Azukawa**, **Bergman**, and **Kähler–Einstein** metrics, play an important role in Complex Function Theory, Complex Dynamics and Convex Geometry. The first four metrics are used mostly because they are **distance-decreasing**. But they are almost never Hermitian. On the other hand, the Bergman metric and the Kähler–Einstein metric are Hermitian (in fact, Kählerian), but, in general, not distance-decreasing.
  The **Wu metric** (Cheung and Kim, 1996) is an invariant non-Kähler Hermitian metric on a complex manifold M^n which is distance-decreasing, up to a fixed constant factor, for any holomorphic mapping between two complex manifolds.
- **Kobayashi metric**
  Let D be a *domain* in $\mathbb{C}^n$. Let $\mathcal{O}(\Delta, D)$ be the set of all analytic mappings $f : \Delta \to D$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the *unit disk*.
  The **Kobayashi metric** (or **Kobayashi–Royden metric**) F_K is a **complex Finsler metric** defined, for all $z \in D$ and $u \in \mathbb{C}^n$, by

  $$F_K(z, u) = \inf\{\alpha > 0 : \exists f \in \mathcal{O}(\Delta, D), f(0) = z, \alpha f'(0) = u\}.$$

  Given a complex manifold M^n, the **Kobayashi semimetric** F_K is defined by

  $$F_K(p, u) = \inf\{\alpha > 0 : \exists f \in \mathcal{O}(\Delta, M^n), f(0) = p, \alpha f'(0) = u\}$$

  for all $p \in M^n$ and $u \in T_p(M^n)$.
  $F_K(p, u)$ is a seminorm of the tangent vector u, called the *Kobayashi seminorm*. F_K is a metric if M^n is *taut*, i.e., $\mathcal{O}(\Delta, M^n)$ is a *normal family* (every sequence has a subsequence which either converge or diverge compactly).

The Kobayashi semimetric is an infinitesimal form of the **Kobayashi semidistance** (or *Kobayashi pseudo-distance*, 1967) K_{M^n} on M^n, defined as follows. Given $p, q \in M^n$, a *chain of disks* α from p to q is a collection of points $p = p^0, p^1, \dots, p^k = q$ of M^n, pairs of points $a^1, b^1; \dots; a^k, b^k$ of the unit disk Δ, and analytic mappings $f_1, \dots f_k$ from Δ into M^n, such that $f_j(a^j) = p^{j-1}$ and $f_j(b^j) = p^j$ for all j.

The length $l(\alpha)$ of a chain α is the sum $d_P(a^1, b^1) + \dots + d_P(a^k, b^k)$, where d_P is the Poincaré metric. The Kobayashi semimetric K_{M^n} on M^n is defined by

$$K_{M^n}(p, q) = \inf_{\alpha} l(\alpha),$$

where the infimum is taken over all lengths $l(\alpha)$ of chains of disks α from p to q.

Given a complex manifold M^n, the **Kobayashi–Busemann semimetric** on M^n is the double dual of the **Kobayashi semimetric**. It is a metric if M^n is taut.

- **Carathéodory metric**

Let D be a *domain* in $\mathbb{C}^n$. Let $\mathcal{O}(D, \Delta)$ be the set of all analytic mappings $f : D \to \Delta$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the *unit disk*.

The **Carathéodory metric** F_C is a **complex Finsler metric** defined by

$$F_C(z, u) = \sup\{|f^{'}(z)u| : f \in \mathcal{O}(D, \Delta)\}$$

for any $z \in D$ and $u \in \mathbb{C}^n$.

Given a complex manifold M^n, the **Carathéodory semimetric** F_C is defined by

$$F_C(p, u) = \sup\{|f^{'}(p)u| : f \in \mathcal{O}(M^n, \Delta)\}$$

for all $p \in M^n$ and $u \in T_p(M^n)$. F_C is a metric if M^n is *taut*, i.e., every sequence in $\mathcal{O}(\Delta, M^n)$ has a subsequence which either converge or diverge compactly.

The **Carathéodory semidistance** (or *Carathéodory pseudo-distance*, 1926) C_{M^n} is a semimetric on a complex manifold M^n, defined by

$$C_{M^n}(p, q) = \sup\{d_P(f(p), f(q)) : f \in \mathcal{O}(M^n, \Delta)\},$$

where d_P is the Poincaré metric.

In general, the integrated semimetric of the infinitesimal Carathéodory semimetric is **internal** for the Carathéodory semidistance, but does not equal to it.

- **Azukawa semimetric**

Let D be a *domain* in $\mathbb{C}^n$. Let $K_D(z)$ be the set of all *logarithmically plurisubharmonic* functions $f : D \to [0, 1]$ such that there exist $M, r > 0$ with $f(u) \le M||u - z||_2$ for all $u \in B(z, r) \subset D$; here $||.||_2$ is the l_2-*norm* on $\mathbb{C}^n$, and $B(z, r) = \{x \in \mathbb{C}^n : ||z - x||_2 < r\}$. Let $g_D(z, u)$ be $\sup\{f(u) : f \in K_D(z)\}$.

The **Azukawa semimetric** F_A is a **complex Finsler metric** defined by

$$F_A(z,u) = \overline{\lim}_{\lambda \to 0} \frac{1}{|\lambda|} g_D(z, z + \lambda u)$$

for all $z \in D$ and $u \in \mathbb{C}^n$.

The Azukawa metric is an infinitesimal form of the **Azukawa semidistance**.

- **Sibony semimetric**

  Let D be a *domain* in $\mathbb{C}^n$. Let $K_D(z)$ be the set of all *logarithmically plurisubharmonic* functions $f : D \to [0, 1)$ such that there exist $M, r > 0$ with $f(u) \le M||u - z||_2$ for all $u \in B(z, r) = \{x \in \mathbb{C}^n : ||z - x||_2 < r\} \subset D$. Let $C^2_{loc}(z)$ be the set of all functions of class C^2 on some open neighborhood of z.

  The **Sibony semimetric** F_S is a **complex Finsler semimetric** defined by

$$F_S(z,u) = \sup_{f \in K_D(z) \cap C^2_{loc}(z)} \sqrt{\sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(z) u_i \overline{u}_j}$$

  for all $z \in D$ and $u \in \mathbb{C}^n$.

  The Sibony semimetric is an infinitesimal form of the **Sibony semidistance**.

- **Teichmüller metric**

  A *Riemann surface* R is a one-dimensional complex manifold. Two Riemann surfaces R_1 and R_2 are called *conformally equivalent* if there exists a bijective analytic function (i.e., a conformal homeomorphism) from R_1 into R_2. More precisely, consider a fixed closed Riemann surface R_0 of a given *genus* $g \ge 2$.

  For a closed Riemann surface R of genus g, one can construct a pair (R, f), where $f : R_0 \to R$ is a homeomorphism. Two pairs (R, f) and (R_1, f_1) are called *conformally equivalent* if there exists a conformal homeomorphism $h : R \to R_1$ such that the mapping $(f_1)^{-1} \cdot h \cdot f : R_0 \to R_0$ is homotopic to the identity.

  An *abstract Riemann surface* $R^* = (R, f)^*$ is the equivalence class of all Riemann surfaces, conformally equivalent to R. The set of all equivalence classes is called the *Teichmüller space* $T(R_0)$ of the surface R_0.

  For closed surfaces R_0 of given genus g, the spaces $T(R_0)$ are isometrically isomorphic, and one can speak of the *Teichmüller space* T_g *of surfaces of genus* g. T_g is a complex manifold. If R_0 is obtained from a compact surface of genus $g \ge 2$ by removing n points, then the complex dimension of T_g is $3g - 3 + n$.

  The **Teichmüller metric** is a metric on T_g defined by

$$\frac{1}{2} \inf_h \ln K(h)$$

  for any $R_1^*, R_2^* \in T_g$, where $h : R_1 \to R_2$ is a quasi-conformal homeomorphism, homotopic to the identity, and $K(h)$ is the *maximal dilation* of h. In fact, there exists a unique extremal mapping, called the *Teichmüller mapping* which minimizes the maximal dilation of all such h, and the distance between R_1^* and R_2^* is equal to $\frac{1}{2}\ln K$, where the constant K is the dilation of the Teichmüller mapping.

In terms of the *extremal length* $ext_{R^*}(\gamma)$, the distance between R_1^* and R_2^* is

$$\frac{1}{2}\ln\sup_{\gamma}\frac{ext_{R_1^*}(\gamma)}{ext_{R_2^*}(\gamma)},$$

where the supremum is taken over all simple closed curves on R_0.

The Teichmüller space T_g, with the Teichmüller metric on it, is a **geodesic** metric space (moreover, a **straight G-space**) but it is neither **Gromov hyperbolic**, nor a **Busemann convex** metric space.

The **Thurston quasi-metric** on the *Teichmüller space* T_g is defined by

$$\frac{1}{2}\inf_h \ln||h||_{Lip}$$

for any $R_1^*, R_2^* \in T_g$, where $h : R_1 \rightarrow R_2$ is a quasi-conformal homeomorphism, homotopic to the identity, and $||.||_{Lip}$ is the *Lipschitz norm* on the set of all injective functions $f : X \rightarrow Y$ defined by $||f||_{Lip} = \sup_{x,y\in X, x\neq y}\frac{d_Y(f(x),f(y))}{d_X(x,y)}$.

The *moduli space* R_g of conformal classes of Riemann surfaces of genus g is obtained by factorization of T_g by some countable group of automorphisms of it, called the *modular group*. The **Zamolodchikov metric**, defined (1986) in terms of *exactly marginal operators*, is a natural metric on the conformal moduli spaces.

Liu, Sun and Yau, 2005, showed that all known complete metrics on the Teichmüller space and moduli space (including **Teichmüller metric**, **Bergman metric**, *Cheng–Yau–Mok* **Kähler–Einstein metric**, **Carathéodory metric**, *McMullen metric*) are equivalent since they are **quasi-isometric** (cf. Chap. 1) to the *Ricci metric* and the *perturbed Ricci metric* introduced by them.

- **Weil–Petersson metric**

  The **Weil–Petersson metric** is a **Kähler metric** on the Teichmüller space $T_{g,n}$ of abstract Riemann surfaces of genus g with n punctures and negative Euler characteristic. This metric has negative Ricci curvature; it is **geodesically convex** (cf. Chap. 1) and not complete.

  The **Weil–Peterson metric** is **Gromov hyperbolic** if and only if (Brock and Farb, 2006) the complex dimension $3g-3+n$ of $T_{g,n}$ is at most two.

- **Gibbons–Manton metric**

  The **Gibbons–Manton metric** is a $4n$-dimensional **hyper-Kähler metric** on the moduli space of *n-monopoles* which admits an isometric action of the n-dimensional torus T^n. It is a hyper-Kähler quotient of a flat quaternionic vector space.

- **Metrics on determinant lines**
  Let M^n be an n-dimensional compact smooth manifold, and let F be a flat vector bundle over M^n. Let $H^\bullet(M^n, F) = \oplus_{i=0}^n H^i(M^n, F)$ be the *de Rham cohomology* of M^n with coefficients in F. Given an n-dimensional vector space V, the *determinant line det* V of V is defined as the top exterior power of V, i.e., *det* $V = \wedge^n V$. Given a finite-dimensional graded vector space $V = \oplus_{i=0}^n V_i$, the determinant line of V is defined as the tensor product $det\ V = \otimes_{i=0}^n (detV_i)^{(-1)^i}$. Thus, the determinant line $detH^\bullet(M^n, F)$ of the cogomology $H^\bullet(M^n, F)$ can be written as $detH^\bullet(M^n, F) = \otimes_{i=0}^n (detH^i(M^n, F))^{(-1)^i}$.
  The **Reidemeister metric** is a metric on $detH^\bullet(M^n, F)$, defined by a given smooth triangulation of M^n, and the classical *Reidemeister–Franz torsion*.
  Let g^F and $g^{T(M^n)}$ be smooth metrics on the vector bundle F and tangent bundle $T(M^n)$, respectively. These metrics induce a canonical L_2**-metric** $h^{H^\bullet(M^n,F)}$ on $H^\bullet(M^n, F)$. The **Ray–Singler metric** on $detH^\bullet(M^n, F)$ is defined as the product of the metric induced on $detH^\bullet(M^n, F)$ by $h^{H^\bullet(M^n,F)}$ with the *Ray–Singler analytic torsion*. The **Milnor metric** on $detH^\bullet(M^n, F)$ can be defined in a similar manner using the *Milnor analytic torsion*. If g^F is flat, the above two metrics coincide with the Reidemeister metric. Using a co-Euler structure, one can define a *modified Ray–Singler metric* on $detH^\bullet(M^n, F)$.
  The **Poincaré–Reidemeister metric** is a metric on the cohomological determinant line $detH^\bullet(M^n, F)$ of a closed connected oriented odd-dimensional manifold M^n. It can be constructed using a combination of the Reidemeister torsion with the Poincaré duality. Equivalently, one can define the *Poincaré–Reidemeister scalar product* on $detH^\bullet(M^n, F)$ which completely determines the Poincaré–Reidemeister metric but contains an additional sign or phase information.
  The **Quillen metric** is a metric on the inverse of the cohomological determinant line of a compact Hermitian one-dimensional complex manifold. It can be defined as the product of the L_2-metric with the Ray-Singler analytic torsion.
- **Kähler supermetric**
  The **Kähler supermetric** is a generalization of the **Kähler metric** for the case of a *supermanifold*. A *supermanifold* is a generalization of the usual manifold with *fermonic* as well as *bosonic* coordinates. The bosonic coordinates are ordinary numbers, whereas the fermonic coordinates are *Grassmann numbers*.
  Here the term *supermetric* differs from the one used in this chapter.
- **Hofer metric**
  A *symplectic manifold* (M^n, w), $n = 2k$, is a smooth even-dimensional manifold M^n equipped with a *symplectic form*, i.e., a closed nondegenerate 2-form, w.
  A *Lagrangian manifold* is a k-dimensional smooth submanifold L^k of a symplectic manifold (M^n, w), $n = 2k$, such that the form w vanishes identically on L^k, i.e., for any $p \in L^k$ and any $x, y \in T_p(L^k)$, one has $w(x, y) = 0$.

Let $L(M^n, \Delta)$ be the set of all Lagrangian submanifolds of a closed symplectic manifold (M^n, w), diffeomorphic to a given Lagrangian submanifold Δ. A smooth family $\alpha = \{L_t\}_t$, $t \in [0,1]$, of Lagrangian submanifolds $L_t \in L(M^n, \Delta)$ is called an *exact path* connecting L_0 and L_1, if there exists a smooth mapping $\Psi : \Delta \times [0,1] \to M^n$ such that, for every $t \in [0,1]$, one has $\Psi(\Delta \times \{t\}) = L_t$, and $\Psi * w = dH_t \wedge dt$ for some smooth function $H : \Delta \times [0,1] \to \mathbb{R}$. The *Hofer length* $l(\alpha)$ of an exact path α is defined by $l(\alpha) = \int_0^1 \{\max_{p \in \Delta} H(p,t) - \min_{p \in \Delta} H(p,t)\} dt$.
The **Hofer metric** on the set $L(M^n, \Delta)$ is defined by

$$\inf_{\alpha} l(\alpha)$$

for any $L_0, L_1 \in L(M^n, \Delta)$, where the infimum is taken over all exact paths on $L(M^n, \Delta)$, that connect L_0 and L_1.
The Hofer metric can be defined similarly on the group $Ham(M^n, w)$ of *Hamiltonian diffeomorphisms* of a closed symplectic manifold (M^n, w), whose elements are *time-one mappings* of *Hamiltonian flows* ϕ_t^H: it is $\inf_\alpha l(\alpha)$, where the infimum is taken over all smooth paths $\alpha = \{\phi_t^H\}$, $t \in [0,1]$, connecting ϕ and ψ.

- **Sasakian metric**
  A **Sasakian metric** is a metric on a *contact manifold*, naturally adapted to the *contact structure*.
  A contact manifold equipped with a Sasakian metric is called a *Sasakian space*, and it is an odd-dimensional analog of a *Kähler manifold*. The scalar curvature of a Sasakian metric which is also **Einstein metric**, is positive.
- **Cartan metric**
  A *Killing form* (or *Cartan–Killing form*) on a finite-dimensional *Lie algebra* Ω over a field $\mathbb{F}$ is a symmetric bilinear form

$$B(x, y) = \mathrm{Tr}(ad_x \cdot ad_y),$$

where Tr denotes the trace of a linear operator, and ad_x is the image of x under *the adjoint representation* of Ω, i.e., the linear operator on the vector space Ω defined by the rule $z \to [x, z]$, where $[,]$ is the Lie bracket.
Let $e_1, \dots e_n$ be a basis for the Lie algebra Ω, and $[e_i, e_j] = \sum_{k=1}^n \gamma_{ij}^k e_k$, where γ_{ij}^k are corresponding *structure constants*. Then the Killing form is given by

$$B(x_i, x_j) = g_{ij} = \sum_{k,l=1}^{n} \gamma_{il}^k \gamma_{ik}^l.$$

In Theoretical Physics, the **metric tensor** $((g_{ij}))$ is called a **Cartan metric**.