Chapter 5
Metrics on Normed Structures

In this chapter we consider a special class of metrics defined on some *normed structures*, as the norm of the difference between two given elements. This structure can be a group (with a *group norm*), a vector space (with a *vector norm* or, simply, a *norm*), a vector lattice (with a *Riesz norm*), a field (with a *valuation*), etc.

Any norm is *subadditive*, i.e., triangle inequality $||x + y|| \leq ||x|| + ||y||$ holds. A norm is *submultiplicative* if **multiplicative triangle inequality** $||xy|| \leq ||x||||y||$ holds.

- **Group norm metric**
  A **group norm metric** is a metric on a *group* $(G, +, 0)$ defined by

$$||x + (-y)|| = ||x - y||,$$

  where $||.||$ is a *group norm* on G, i.e., a function $||.|| : G \to \mathbb{R}$ such that, for all $x, y \in G$, we have the following properties:

  1. $||x|| \geq 0$, with $||x|| = 0$ if and only if $x = 0$;
  2. $||x|| = || - x||$;
  3. $||x + y|| \leq ||x|| + ||y||$ (triangle inequality).

  Any group norm metric d is **right-invariant**, i.e., $d(x, y) = d(x + z, y + z)$ for any $x, y, z \in G$. Conversely, any right-invariant (as well as any **left-invariant**, and, in particular, any **bi-invariant**) metric d on G is a group norm metric, since one can define a group norm on G by $||x|| = d(x, 0)$.
- ***F*-norm metric**
  A *vector space* (or *linear space*) over a *field* $\mathbb{F}$ is a set V equipped with operations of *vector addition* $+ : V \times V \to V$ and *scalar multiplication* $\cdot : \mathbb{F} \times V \to V$ such that $(V, +, 0)$ forms an *Abelian group* (where $0 \in V$ is the *zero vector*), and, for all *vectors* $x, y \in V$ and any *scalars* $a, b \in \mathbb{F}$, we have the following properties: $1 \cdot x = x$ (where 1 is the multiplicative unit of $\mathbb{F}$), $(ab) \cdot x = a \cdot (b \cdot x)$, $(a + b) \cdot x = a \cdot x + b \cdot x$, and $a \cdot (x + y) = a \cdot x + a \cdot y$.

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A vector space over the field $\mathbb{R}$ of real numbers is called a *real vector space*. A vector space over the field $\mathbb{C}$ of complex numbers is called *complex vector space*.

A **F-norm metric** is a metric on a real (complex) vector space V defined by

$$||x - y||_F,$$

where $||.||_F$ is an *F-norm* on V, i.e., a function $||.||_F : V \to \mathbb{R}$ such that, for all $x, y \in V$ and for any scalar a with $|a| = 1$, we have the following properties:

1. $||x||_F \geq 0$, with $||x||_F = 0$ if and only if $x = 0$;
2. $||ax||_F \leq ||x||_F$ if $|a| \leq 1$;
3. $\lim_{a \to 0} ||ax||_F = 0$;
4. $||x + y||_F \leq ||x||_F + ||y||_F$ (triangle inequality).

An F-norm is called *p-homogeneous* if $||ax||_F = |a|^p ||x||_F$ for any scalar a.

Any F-norm metric d is a **translation invariant metric**, i.e., $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in V$. Conversely, if d is a translation invariant metric on V, then $||x||_F = d(x, 0)$ is an F-norm on V.

- **F^*-metric**

An **F^*-metric** is an **F-norm metric** $||x - y||_F$ on a real (complex) vector space V such that the operations of scalar multiplication and vector addition are continuous with respect to $||.||_F$. Thus $||.||_F$ is a function $||.||_F : V \to \mathbb{R}$ such that, for all $x, y, x_n \in V$ and for all scalars a, a_n, we have the following properties:

1. $||x||_F \geq 0$, with $||x||_F = 0$ if and only if $x = 0$;
2. $||ax||_F = ||x||_F$ for all a with $|a| = 1$;
3. $||x + y||_F \leq ||x||_F + ||y||_F$;
4. $||a_n x||_F \to 0$ if $a_n \to 0$;
5. $||ax_n||_F \to 0$ if $x_n \to 0$;
6. $||a_n x_n||_F \to 0$ if $a_n \to 0$, $x_n \to 0$.

The metric space $(V, ||x - y||_F)$ with an F^*-metric is called a n**F^*-space**. Equivalently, an F^*-space is a metric space (V, d) with a **translation invariant metric** d such that the operation of scalar multiplication and vector addition are continuous with respect to this metric.

A **complete** F^*-space is called an **F-space**. A **locally convex** F-space is known as a **Fréchet space** (cf. Chap. 2) in Functional Analysis.

A **modular space** is an F^*-space $(V, ||.||_F)$ in which the F-norm $||.||_F$ is defined by

$$||x||_F = \inf\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) < \lambda\},$$

and ρ is a *metrizing modular* on V, i.e., a function $\rho : V \to [0, \infty]$ such that, for all $x, y, x_n \in V$ and for all scalars a, a_n, we have the following properties:

1. $\rho(x) = 0$ if and only if $x = 0$;
2. $\rho(ax) = \rho(x)$ implies $|a| = 1$;
3. $\rho(ax + by) \leq \rho(x) + \rho(y)$ implies $a, b \geq 0, a + b = 1$;
4. $\rho(a_n x) \to 0$ if $a_n \to 0$ and $\rho(x) < \infty$;
5. $\rho(ax_n) \to 0$ if $\rho(x_n) \to 0$ (*metrizing property*);
6. For any $x \in V$, there exists $k > 0$ such that $\rho(kx) < \infty$.

- **Norm metric**

A **norm metric** is a metric on a real (complex) vector space V defined by

$$||x - y||,$$

where $||.||$ is a *norm* on V, i.e., a function $||.|| : V \to \mathbb{R}$ such that, for all $x, y \in V$ and for any scalar a, we have the following properties:

1. $||x|| \geq 0$, with $||x|| = 0$ if and only if $x = 0$;
2. $||ax|| = |a|||x||$;
3. $||x + y|| \leq ||x|| + ||y||$ (triangle inequality).

Therefore, a norm $||.||$ is a 1-*homogeneous F-norm*. The vector space $(V, ||.||)$ is called a *normed vector space* or, simply, *normed space*.

Any metric space can be embedded isometrically in some normed vector space as a closed linearly independent subset. Every finite-dimensional normed space is **complete**, and all norms on it are equivalent.

In general, the norm $||.||$ is equivalent (Maligranda, 2008) to the norm

$$||x||_{u,p} = (||x + ||x|| \cdot u||^p + ||x - ||x|| \cdot u||^p)^{\frac{1}{p}},$$

introduced, for any $u \in V$ and $p \geq 1$, by Odell and Schlumprecht, 1998.

The **norm-angular distance** between x and y is defined (Clarkson, 1936) by

$$d(x, y) = ||\frac{x}{||x||} - \frac{y}{||y||}||.$$

The following sharpening of the triangle inequality (Maligranda, 2003) holds:

$$\frac{||x - y|| - |||x|| - ||y|||}{\min\{||x||, ||y||\}} \leq d(x, y) \leq \frac{||x - y|| + |||x|| - ||y|||}{\max\{||x||, ||y||\}}, \text{ i.e.,}$$

$$(2 - d(x, -y)) \min\{||x||, ||y||\} \leq ||x|| + ||y|| - ||x + y||$$
$$\leq (2 - d(x, -y)) \max\{||x||, ||y||\}.$$

Dragomir, 2004, call $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$ *continuous triangle inequality*.

- **Reverse triangle inequality**

The triangle inequality $||x + y|| \leq ||x|| + ||y||$ in a normed space $(V, ||.||)$ is equivalent to the following inequality, for any $x_1, \dots, x_n \in V$ with $n \geq 2$:

$$||\sum_{i=1}^{n} x_i|| \leq \sum_{i=1}^{n} ||x_i||.$$

If in the normed space $(V, ||.||)$, for some $C \geq 1$ one has

$$C||\sum_{i=1}^{n} x_i|| \geq \sum_{i=1}^{n} ||x_i||,$$

then this inequality is called the **reverse triangle inequality**.
This term is used, sometimes, also for the **inverse triangle inequality** (cf. **kinematic metric** in Chap. 26) and for the eventual inequality $Cd(x,z) \geq d(x,y) + d(y,z)$ with $C \geq 1$ in a metric space (X,d).
The triangle inequality $||x+y|| \leq ||x|| + ||y||$, for any $x, y \in V$, in a normed space $(V, ||.||)$ is, for any number $q > 1$, equivalent (Belbachir, Mirzavaziri and Moslenian, 2005) to the following inequality:

$$||x+y||^q \leq 2^{q-1}(||x||^q + ||y||^q).$$

The *parallelogram inequality* $||x+y||^2 \leq 2(||x||^2 + ||y||^2)$ is the case $q = 2$ of above.
Given a number q, $0 < q \leq 1$, the norm is called *q-subadditive* if $||x+y||^q \leq ||x||^q + ||y||^q$ holds for $x, y \in V$.

- **Seminorm semimetric**
  A **seminorm semimetric** on a real (complex) vector space V is defined by

$$||x-y||,$$

where $||.||$ is a *seminorm* (or *pseudo-norm*) on V, i.e., a function $||.|| : V \to \mathbb{R}$ such that, for all $x, y \in V$ and for any scalar a, we have the following properties:

1. $||x|| \geq 0$, with $||0|| = 0$;
2. $||ax|| = |a|||x||$;
3. $||x+y|| \leq ||x|| + ||y||$ (triangle inequality).

The vector space $(V, ||.||)$ is called a *seminormed vector space*. Many *normed vector spaces*, in particular, **Banach spaces**, are defined as the quotient space by the subspace of elements of seminorm zero.
A *quasi-normed space* is a vector space V, on which a *quasi-norm* is given. A *quasi-norm* on V is a nonnegative function $||.|| : V \to \mathbb{R}$ which satisfies the same axioms as a norm, except for the triangle inequality which is replaced by the weaker requirement: there exists a constant $C > 0$ such that, for all $x, y \in V$, the following **C-triangle inequality** (cf. **near-metric** in Chap. 1) holds:

$$||x+y|| \leq C(||x|| + ||y||)$$

An example of a quasi-normed space, that is not normed, is the *Lebesgue space* $L_p(\Omega)$ with $0 < p < 1$ in which a quasi-norm is defined by

$$||f|| = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}, f \in L_p(\Omega).$$

- **Banach space**
A **Banach space** (or *B-space*) is a **complete** metric space $(V, ||x-y||)$ on a vector space V with a norm metric $||x-y||$. Equivalently, it is the complete *normed space* $(V, ||.||)$. In this case, the norm $||.||$ on V is called the *Banach norm*. Some examples of Banach spaces are:

1. l_p^n-*spaces*, l_p^∞-*spaces*, $1 \le p \le \infty$, $n \in \mathbb{N}$;
2. The space C of convergent numerical sequences with the norm $||x|| = \sup_n |x_n|$;
3. The space C_0 of numerical sequences which converge to zero with the norm $||x|| = \max_n |x_n|$;
4. The space $C^p_{[a,b]}$, $1 \le p \le \infty$, of continuous functions on $[a,b]$ with the L_p-*norm* $||f||_p = (\int_a^b |f(t)|^p dt)^{\frac{1}{p}}$;
5. The space C_K of continuous functions on a compactum K with the norm $||f|| = \max_{t \in K} |f(t)|$;
6. The space $(C_{[a,b]})^n$ of functions on $[a,b]$ with continuous derivatives up to and including the order n with the norm $||f||_n = \sum_{k=0}^n \max_{a \le t \le b} |f^{(k)}(t)|$;
7. The space $C^n[I^m]$ of all functions defined in an m-dimensional cube that are continuously differentiable up to and including the order n with the norm of uniform boundedness in all derivatives of order at most n;
8. The space $M_{[a,b]}$ of bounded measurable functions on $[a,b]$ with the norm

$$||f|| = ess \sup_{a \le t \le b} |f(t)| = \inf_{e, \mu(e)=0} \sup_{t \in [a,b] \setminus e} |f(t)|;$$

9. The space $A(\Delta)$ of functions analytic in the open *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and continuous in the closed disk $\overline{\Delta}$ with the norm $||f|| = \max_{z \in \overline{\Delta}} |f(z)|$;
10. The **Lebesgue spaces** $L_p(\Omega)$, $1 \le p \le \infty$;
11. The *Sobolev spaces* $W^{k,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 \le p \le \infty$, of functions f on Ω such that f and its derivatives, up to some order k, have a finite L_p-*norm*, with the norm $||f||_{k,p} = \sum_{i=0}^k ||f^{(i)}||_p$;
12. The *Bohr space AP* of almost periodic functions with the norm

$$||f|| = \sup_{-\infty < t < +\infty} |f(t)|.$$

A finite-dimensional real Banach space is called a *Minkowskian space*. A norm metric of a Minkowskian space is called a **Minkowskian metric** (cf. Chap. 6). In particular, any l_p**-metric** is a Minkowskian metric.

All n-dimensional Banach spaces are pairwise isomorphic; the set of such spaces becomes compact if one introduces the **Banach–Mazur distance** by $d_{BM}(V, W) = \ln \inf_T ||T|| \cdot ||T^{-1}||$, where the infimum is taken over all operators which realize an isomorphism $T : V \to W$.

- **l_p-metric**
The **l_p-metric** d_{l_p}, $1 \leq p \leq \infty$, is a norm metric on $\mathbb{R}^n$ (or on $\mathbb{C}^n$), defined by

$$||x - y||_p,$$

where the *l_p-norm* $||.||_p$ is defined by

$$||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}.$$

For $p = \infty$, we obtain $||x||_\infty = \lim_{p\to\infty} \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} = \max_{1\leq i\leq n} |x_i|$. The metric space $(\mathbb{R}^n, d_{l_p})$ is abbreviated as l_p^n and is called *l_p^n-space*.
The **l_p-metric**, $1 \leq p \leq \infty$, on the set of all sequences $x = \{x_n\}_{n=1}^{\infty}$ of real (complex) numbers, for which the sum $\sum_{i=1}^{\infty} |x_i|^p$ (for $p = \infty$, the sum $\sum_{i=1}^{\infty} |x_i|$) is finite, is

$$(\sum_{i=1}^{\infty} |x_i - y_i|^p)^{\frac{1}{p}}.$$

For $p = \infty$, we obtain $\max_{i\geq 1} |x_i - y_i|$. This metric space is abbreviated as l_p^∞ and is called *l_p^∞-space*.
Most important are l_1-, l_2- and l_∞-metrics. Among l_p-metrics, only l_1- and l_∞-metrics are **crystalline metrics**, i.e., metrics having polygonal unit balls. On $\mathbb{R}$ all l_p-metrics coincide with the **natural metric** (cf. Chap. 12) $|x - y|$.
The l_2-norm $||(x_1, x_2)||_2 = \sqrt{x_1^2 + x_2^2}$ on $\mathbb{R}^2$ is also called *Pythagorean addition* of the numbers x_1 and x_2. Under this commutative operation, $\mathbb{R}$ form a semigroup, and $\mathbb{R}_{\geq 0}$ form a *monoid* (semigroup with identity, 0).

- **Euclidean metric**
The **Euclidean metric** (or **Pythagorean distance**, **as-the-crow-flies distance**, **beeline distance**) d_E is the metric on $\mathbb{R}^n$ defined by

$$||x - y||_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

It is the ordinary l_2-**metric** on $\mathbb{R}^n$. The metric space $(\mathbb{R}^n, d_E)$ is abbreviated as $\mathbb{E}^n$ and is called **Euclidean space** “Euclidean space” stands for the case $n = 3$, as opposed, for $n = 2$, to *Euclidean plane* and, for $n = 1$, *Euclidean* (or *real*) *line*.
In fact, $\mathbb{E}^n$ is an **inner product space** (and even a **Hilbert space**), i.e., $d_E(x, y) = ||x - y||_2 = \sqrt{\langle x - y, x - y\rangle}$, where $\langle x, y\rangle$ is the *inner product*

on $\mathbb{R}^n$ which is given in the Cartesian coordinate system by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$. In a standard coordinate system one has $\langle x, y\rangle = \sum_{i,j} g_{ij} x_i y_j$, where $g_{ij} = \langle e_i, e_j\rangle$, and the **metric tensor** $((g_{ij}))$ (cf. Chap. 7) is a positive-definite symmetric $n \times n$ matrix.
In general, a Euclidean space is defined as a space, the properties of which are described by the axioms of *Euclidean Geometry*.

- **Norm transform metric**
  A **norm transform metric** is a metric $d(x, y)$ on a vector space $(V, ||.||)$, which is a function of $||x||$ and $||y|$. Usually, $V = \mathbb{R}^n$ and, moreover, $\mathbb{E}^n = (\mathbb{R}^n, ||.||_2)$. Some examples are (p, q)**-relative metric**, M**-relative metric** and, from Chap. 19, the **British Rail metric** $||x|| + ||y||$ for $x \neq y$, (and equal to 0, otherwise), the **radar screen metric** $\min\{1, ||x - y||\}$ and $\max\{1, ||x - y||\}$ for $x \neq y$. Cf. t**-truncated** and t**-uniformly discrete** metrics in Chap. 4.
- (p, q)**-relative metric**
  Let $0 < q \leq 1$, and $p \geq \max\{1 - q, \frac{2-q}{3}\}$. Let $(V, ||.||)$ be a *Ptolemaic space*, i.e., the norm metric $||x - y||$ is a **Ptolemaic metric** (cf. Chap. 1).
  The (p, q)**-relative metric** on $(V, ||.||)$ is defined, for x or $y \neq 0$, by
  $$\frac{||x - y||}{(\frac{1}{2}(||x||^p + ||y||^p))^{\frac{q}{p}}}$$
  (and equal to 0, otherwise). In the case of $p = \infty$, it has the form
  $$\frac{||x - y||}{(\max\{||x||, ||y||\})^q}.$$
  $(p, 1)$-, $(\infty, 1)$- and the original $(1, 1)$-relative metric on $\mathbb{E}^n$ are called p**-relative** (or **Klamkin–Meir metric**), **relative metric** and **Schattschneider metric**.
- M**-relative metric**
  Let $f : [0, \infty) \to (0, \infty)$ be a convex increasing function such that $\frac{f(x)}{x}$ is decreasing for $x > 0$. Let $(V, ||.||)$ be a *Ptolemaic space*, i.e., $||x - y||$ is a **Ptolemaic metric**.
  The M**-relative metric** on $(V, ||.||)$ is defined by
  $$\frac{||x - y||}{f(||x||) \cdot f(||y||)}.$$
- **Unitary metric**
  The **unitary** (or *complex Euclidean*) **metric** is the l_2**-metric** on $\mathbb{C}^n$ defined by
  $$||x - y||_2 = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}.$$

For $n = 1$, it is the **complex modulus metric** $|x - y| = \sqrt{(x-y)\overline{(x-y)}}$ on the *Wessel–Argand plane* (cf. Chap. 12).

- L_p**-metric**

An L_p**-metric** d_{L_p}, $1 \leq p \leq \infty$, is a norm metric on $L_p(\Omega, \mathcal{A}, \mu)$ defined by

$$||f - g||_p$$

for any $f, g \in L_p(\Omega, \mathcal{A}, \mu)$. The metric space $(L_p(\Omega, \mathcal{A}, \mu), d_{L_p})$ is called the L_p**-space** (or **Lebesgue space**).

Here Ω is a set, and $\mathcal{A}$ is n σ*-algebra* of subsets of Ω, i.e., a collection of subsets of Ω satisfying the following properties:

1. $\Omega \in \mathcal{A}$;
2. If $A \in \mathcal{A}$, then $\Omega \backslash A \in \mathcal{A}$;
3. If $A = \cup_{i=1}^{\infty} A_i$ with $A_i \in \mathcal{A}$, then $A \in \mathcal{A}$.

A function $\mu : \mathcal{A} \to \mathbb{R}_{\geq 0}$ is called a *measure* on $\mathcal{A}$ if it is *additive*, i.e., $\mu(\cup_{i \geq 1} A_i) = \sum_{i \geq 1} \mu(A_i)$ for all pairwise disjoint sets $A_i \in \mathcal{A}$, and satisfies $\mu(\emptyset) = 0$. A *measure space* is a triple $(\Omega, \mathcal{A}, \mu)$.

Given a function $f : \Omega \to \mathbb{R}(\mathbb{C})$, its L_p*-norm* is defined by

$$||f||_p = \left(\int_{\Omega} |f(\omega)|^p \mu(d\omega) \right)^{\frac{1}{p}}.$$

Let $L_p(\Omega, \mathcal{A}, \mu) = L_p(\Omega)$ denote the set of all functions $f : \Omega \to \mathbb{R}$ $(\mathbb{C})$ such that $||f||_p < \infty$. Strictly speaking, $L_p(\Omega, \mathcal{A}, \mu)$ consists of equivalence classes of functions, where two functions are *equivalent* if they are equal *almost everywhere*, i.e., the set on which they differ has measure zero. The set $L_\infty(\Omega, \mathcal{A}, \mu)$ is the set of equivalence classes of measurable functions $f : \Omega \to \mathbb{R}$ $(\mathbb{C})$ whose absolute values are bounded almost everywhere.

The most classical example of an L_p-metric is d_{L_p} on the set $L_p(\Omega, \mathcal{A}, \mu)$, where Ω is the open interval $(0, 1)$, $\mathcal{A}$ is the *Borel* σ*-algebra* on $(0, 1)$, and μ is the *Lebesgue measure*. This metric space is abbreviated by $L_p(0, 1)$ and is called $L_p(0, 1)$*-space*.

In the same way, one can define the L_p-metric on the set $C_{[a,b]}$ of all real (complex) continuous functions on $[a, b]$: $d_{L_p}(f, g) = (\int_a^b |f(x) - g(x)|^p dx)^{\frac{1}{p}}$. For $p = \infty$, $d_{L_\infty}(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)|$. This metric space is abbreviated by $C_{[a,b]}^p$ and is called $C_{[a,b]}^p$*-space*.

If $\Omega = \mathbb{N}$, $\mathcal{A} = 2^{\Omega}$ is the collection of all subsets of Ω, and μ is the *cardinality measure* (i.e., $\mu(A) = |A|$ if A is a finite subset of Ω, and $\mu(A) = \infty$, otherwise), then the metric space $(L_p(\Omega, 2^{\Omega}, |.|), d_{L_p})$ coincides with the space l_p^{∞}.

If $\Omega = V_n$ is a set of cardinality n, $\mathcal{A} = 2^{V_n}$, and μ is the cardinality measure, then the metric space $(L_p(V_n, 2^{V_n}, |.|), d_{L_p})$ coincides with the space l_p^n.

- **Dual metrics**
  The l_p**-metric** and the l_q**-metric**, $1 < p, q < \infty$, are called **dual** if $1/p + 1/q = 1$.
  In general, when dealing with a *normed vector space* $(V, ||.||_V)$, one is interested in the *continuous* linear functionals from V into the base field ($\mathbb{R}$ or $\mathbb{C}$). These functionals form a **Banach space** $(V', ||.||_{V'})$, called the *continuous dual* of V. The norm $||.||_{V'}$ on V' is defined by $||T||_{V'} = \sup_{||x||_V \leq 1} |T(x)|$.
  The continuous dual for the metric space l_p^n (l_p^∞) is l_q^n (l_q^∞, respectively). The continuous dual of l_1^n (l_1^∞) is l_∞^n (l_∞^∞, respectively). The continuous duals of the Banach spaces C (consisting of all convergent sequences, with l_∞**-metric**) and C_0 (consisting of the sequences converging to zero, with l_∞**-metric**) are both naturally identified with l_1^∞.
- **Inner product space**
  An **inner product space** (or *pre-Hilbert space*) is a metric space $(V, ||x-y||)$ on a real (complex) vector space V with an *inner product* $\langle x, y\rangle$ such that the norm metric $||x - y||$ is constructed using the *inner product norm* $||x|| = \sqrt{\langle x, x\rangle}$.
  An *inner product* $\langle , \rangle$ on a real (complex) vector space V is a *symmetric bilinear* (in the complex case, *sesquilinear*) form on V, i.e., a function $\langle , \rangle : V \times V \longrightarrow \mathbb{R}$ ($\mathbb{C}$) such that, for all $x, y, z \in V$ and for all scalars α, β, we have the following properties:

  1. $\langle x, x\rangle \geq 0$, with $\langle x, x\rangle = 0$ if and only if $x = 0$;
  2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$, where the bar denotes *complex conjugation*;
  3. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

  For a complex vector space, an inner product is called also a *Hermitian inner product*, and the corresponding metric space is called a *Hermitian inner product space*.
  A norm $||.||$ in a *normed space* $(V, ||.||)$ is generated by an inner product if and only if, for all $x, y \in V$, we have: $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$.
  In an inner product space, the **triangle equality** (Chap. 1) $||x-y|| = ||x||+||y||$, for $x, y \neq 0$, holds if and only if $\frac{x}{||x||} = \frac{y}{||y||}$, i.e., $x - y \in [x, y]$.
- **Hilbert space**
  A **Hilbert space** is an **inner product space** which, as a metric space, is **complete**. More precisely, a Hilbert space is a complete metric space $(H, ||x - y||)$ on a real (complex) vector space H with an *inner product* $\langle , \rangle$ such that the norm metric $||x - y||$ is constructed using the *inner product norm* $||x|| = \sqrt{\langle x, x\rangle}$. Any Hilbert space is a **Banach space**.
  An example of a Hilbert space is the set of all sequences $x = \{x_n\}_n$ of real (complex) numbers such that $\sum_{i=1}^{\infty} |x_i|^2$ converges, with the **Hilbert metric** defined by
  $$(\sum_{i=1}^{\infty}(x_i - y_i)^2)^{\frac{1}{2}}.$$

Other examples of Hilbert spaces are any L_2-**space**, and any finite-dimensional inner product space. In particular, any Euclidean space is a Hilbert space.
A direct product of two **Hilbert spaces** is called a *Liouville space* (or *line space*, *extended Hilbert space*).
Given an infinite cardinal number τ and a set A of the cardinality τ, let $\mathbb{R}_a$, $a \in A$, be the copies of $\mathbb{R}$. Let $H(A) = \{\{x_a\} \in \prod_{a\in A} \mathbb{R}_a : \sum_a x_a^2 < \infty\}$; then $H(A)$ with the metric defined for $\{x_a\}, \{y_a\} \in H(A)$ as

$$(\sum_{a\in A}(x_a - y_a)^2)^{\frac{1}{2}},$$

is called the **generalized Hilbert space** of weight τ.

- **Erdös space**
The **Erdös space** (or *rational Hilbert space*) is the metric subspace of l_2 consisting of all vectors in l_2 with only rational coordinates. It has topological dimension 1 and is not complete. Erdös space is **homeomorphic** to its countable infinite power, and every nonempty open subset of it is homeomorphic to whole space.
The **complete Erdös space** (or *irrational Hilbert space*) is the complete metric subspace of l_2 consisting of all vectors in l_2 the coordinates of which are all irrational.
- **Riesz norm metric**
A *Riesz space* (or *vector lattice*) is a partially ordered vector space $(V_{Ri}, \preceq)$ in which the following conditions hold:

1. The vector space structure and the partial order structure are compatible, i.e., from $x \preceq y$ it follows that $x + z \preceq y + z$, and from $x \succ 0$, $a \in \mathbb{R}$, $a > 0$ it follows that $ax \succ 0$;
2. For any two elements $x, y \in V_{Ri}$, there exist the *join* $x \vee y \in V_{Ri}$ and *meet* $x \wedge y \in V_{Ri}$ (cf. Chap. 10).

The **Riesz norm metric** is a norm metric on V_{Ri} defined by

$$||x - y||_{Ri},$$

where $||.||_{Ri}$ is a *Riesz norm* on V_{Ri}, i.e., a norm such that, for any $x, y \in V_{Ri}$, the inequality $|x| \preceq |y|$, where $|x| = (-x) \vee (x)$, implies $||x||_{Ri} \leq ||y||_{Ri}$.
The space $(V_{Ri}, ||.||_{Ri})$ is called a *normed Riesz space*. In the case of completeness, it is called a *Banach lattice*.
- **Banach–Mazur compactum**
The **Banach–Mazur distance** d_{BM} between two n-dimensional *normed spaces* $(V, ||.||_V)$ and $(W, ||.||_W)$ is defined by

$$\ln \inf_T ||T|| \cdot ||T^{-1}||,$$

where the infimum is taken over all isomorphisms $T : V \to W$. It is a metric on the set X^n of all equivalence classes of n-dimensional normed spaces, where $V \sim W$ if and only if they are *isometric*. Then the pair (X^n, d_{BM}) is a compact metric space which is called the **Banach–Mazur compactum**.

- **Quotient metric**
  Given a *normed space* $(V, ||.||_V)$ with a norm $||.||_V$ and a closed subspace W of V, let $(V/W, ||.||_{V/W})$ be the normed space of cosets $x + W = \{x + w : w \in W\}$, $x \in V$, with the *quotient norm* $||x + W||_{V/W} = \inf_{w \in W} ||x + w||_V$.
  The **quotient metric** is a norm metric on V/W defined by
  $$||(x + W) - (y + W)||_{V/W}.$$
- **Tensor norm metric**
  Given *normed spaces* $(V, ||.||_V)$ and $(W, ||.||_W)$, a norm $||.||_\otimes$ on the *tensor product* $V \otimes W$ is called *tensor norm* (or *cross norm*) if $||x \otimes y||_\otimes = ||x||_V ||y||_W$ for all *decomposable* tensors $x \otimes y$.
  The **tensor product metric** is a norm metric on $V \otimes W$ defined by
  $$||z - t||_\otimes.$$
  For any $z \in V \otimes W$, $z = \sum_j x_j \otimes y_j$, $x_j \in V$, $y_j \in W$, the *projective norm* (or π*-norm*) of z is defined by $||z||_{pr} = \inf \sum_j ||x_j||_V ||y_j||_W$, where the infimum is taken over all representations of z as a sum of decomposable vectors. It is the largest tensor norm on $V \otimes W$.
- **Valuation metric**
  A **valuation metric** is a metric on a *field* $\mathbb{F}$ defined by
  $$||x - y||,$$
  where $||.||$ is a *valuation* on $\mathbb{F}$, i.e., a function $||.|| : \mathbb{F} \to \mathbb{R}$ such that, for all $x, y \in \mathbb{F}$, we have the following properties:

  1. $||x|| \geq 0$, with $||x|| = 0$ if and only if $x = 0$;
  2. $||xy|| = ||x||\,||y||$,
  3. $||x + y|| \leq ||x|| + ||y||$ (triangle inequality).

  If $||x + y|| \leq \max\{||x||, ||y||\}$, the valuation $||.||$ is called *non-Archimedean*. In this case, the valuation metric is an **ultrametric**. The simplest valuation is the *trivial valuation* $||.||_{tr}$: $||0||_{tr} = 0$, and $||x||_{tr} = 1$ for $x \in \mathbb{F} \backslash \{0\}$. It is non-Archimedean.
  There are different definitions of valuation in Mathematics. Thus, the function $\nu : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$ is called a *valuation* if $\nu(x) \geq 0$, $\nu(0) = \infty$, $\nu(xy) = \nu(x) + \nu(y)$, and $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in \mathbb{F}$. The valuation $||.||$ can be obtained from the function ν by the formula $||x|| = \alpha^{\nu(x)}$ for some fixed $0 < \alpha < 1$ (cf. p**-adic metric** in Chap. 12).

The *Kürschäk valuation* $|.|_{Krs}$ is a function $|.|_{Krs} : \mathbb{F} \to \mathbb{R}$ such that $|x|_{Krs} \geq 0$, $|x|_{Krs} = 0$ if and only if $x = 0$, $|xy|_{Krs} = |x|_{Krs}|y|_{Krs}$, and $|x + y|_{Krs} \leq C \max\{|x|_{Krs}, |y|_{Krs}\}$ for all $x, y \in \mathbb{F}$ and for some positive constant C, called the *constant of valuation*. If $C \leq 2$, one obtains the ordinary valuation $||.||$ which is non-Archimedean if $C \leq 1$. In general, any $|.|_{Krs}$ is *equivalent* to some $||.||$, i.e., $|.|^p_{Krs} = ||.||$ for some $p > 0$.

Finally, given an *ordered group* $(G, \cdot, e, \leq)$ equipped with zero, the *Krull valuation* is a function $|.| : \mathbb{F} \to G$ such that $|x| = 0$ if and only if $x = 0$, $|xy| = |x||y|$, and $|x + y| \leq \max\{|x|, |y|\}$ for any $x, y \in \mathbb{F}$. It is a generalization of the definition of non-Archimedean valuation $||.||$ (cf. **generalized metric** in Chap. 3).

- **Power series metric**

  Let $\mathbb{F}$ be an arbitrary algebraic field, and let $\mathbb{F}\langle x^{-1}\rangle$ be the field of power series of the form $w = \alpha_{-m}x^m + \cdots + \alpha_0 + \alpha_1 x^{-1} + \ldots, \alpha_i \in \mathbb{F}$. Given $l > 1$, a *non-Archimedean valuation* $||.||$ on $\mathbb{F}\langle x^{-1}\rangle$ is defined by

$$||w|| = \begin{cases} l^m, & \text{if } w \neq 0, \\ 0, & \text{if } w = 0. \end{cases}$$

  The **power series metric** is the **valuation metric** $||w - v||$ on $\mathbb{F}\langle x^{-1}\rangle$.