Chapter 13
Distances in Functional Analysis

Functional Analysis is the branch of Mathematics concerned with the study of spaces of functions. This usage of the word *functional* goes back to the calculus of variations which studies functions whose argument is a function. In the modern view, Functional Analysis is seen as the study of complete *normed vector spaces*, i.e., **Banach spaces**.

For any real number $p \geq 1$, an example of a Banach space is given by **L_p-space** of all Lebesgue-measurable functions whose absolute value's p-th power has finite integral.

A **Hilbert space** is a Banach space in which the norm arises from an *inner product*. Also, in Functional Analysis are considered *continuous linear operators* defined on Banach and Hilbert spaces.

13.1 Metrics on Function Spaces

Let $I \subset \mathbb{R}$ be an *open interval* (i.e., a nonempty connected open set) in $\mathbb{R}$. A real function $f : I \to \mathbb{R}$ is called *real analytic* on I if it agrees with its *Taylor series* in an *open neighborhood* U_{x_0} of every point $x_0 \in I$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ for any $x \in U_{x_0}$. Let $D \subset \mathbb{C}$ be a *domain* (i.e., a *convex* open set) in $\mathbb{C}$.

A complex function $f : D \to \mathbb{C}$ is called *complex analytic* (or, simply, *analytic*) on D if it agrees with its Taylor series in an open neighborhood of every point $z_0 \in D$. A complex function f is analytic on D if and only if it is *holomorphic* on D, i.e., if it has a complex derivative $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ at every point $z_0 \in D$.

- **Integral metric**
  The **integral metric** is the L_1-*metric* on the set $C_{[a,b]}$ of all continuous real (complex) functions on a given segment $[a, b]$ defined by

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$$\int_a^b |f(x) - g(x)|dx.$$

The corresponding metric space is abbreviated by $C^1_{[a,b]}$. It is a Banach space.
In general, for any **compact** topological space X, the integral metric is defined on the set of all continuous functions $f : X \to \mathbb{R}$ ($\mathbb{C}$) by $\int_X |f(x) - g(x)|dx$.

- **Uniform metric**
The **uniform metric** (or **sup metric**) is the L_∞**-metric** on the set $C_{[a,b]}$ of all real (complex) continuous functions on a given segment $[a, b]$ defined by

$$\sup_{x \in [a,b]} |f(x) - g(x)|.$$

The corresponding metric space is abbreviated by $C^\infty_{[a,b]}$. It is a Banach space.
A generalization of $C^\infty_{[a,b]}$ is the *space of continuous functions* $C(X)$, i.e., a metric space on the set of all continuous (more generally, bounded) functions $f : X \to \mathbb{C}$ of a topological space X with the L_∞-metric $\sup_{x \in X} |f(x) - g(x)|$.
In the case of the metric space $C(X, Y)$ of continuous (more generally, bounded) functions $f : X \to Y$ from one **metric compactum** (X, d_X) to another (Y, d_Y), the sup metric between two functions $f, g \in C(X, Y)$ is defined by $\sup_{x \in X} d_Y(f(x), g(x))$.
The metric space $C^\infty_{[a,b]}$, as well as the metric space $C^1_{[a,b]}$, are two of the most important cases of the metric space $C^p_{[a,b]}$, $1 \leq p \leq \infty$, on the set $C_{[a,b]}$ with the L_p-metric $(\int_a^b |f(x) - g(x)|^p dx)^{\frac{1}{p}}$. The space $C^p_{[a,b]}$ is an example of an *L_p-space*.

- **Dogkeeper distance**
Given a metric space (X, d), the **dogkeeper distance** is a metric on the set of all functions $f : [0, 1] \to X$, defined by

$$\inf_\sigma \sup_{t \in [0,1]} d(f(t), g(\sigma(t))),$$

where $\sigma : [0, 1] \to [0, 1]$ is a continuous, monotone increasing function such that $\sigma(0) = 0, \sigma(1) = 1$. This metric is a special case of the **Fréchet metric**.
For the case, when (X, d) is Euclidean space $\mathbb{R}^n$, this metric is the original (1906) **Fréchet distance** between parametric curves $f, g : [0, 1] \to \mathbb{R}^n$. This distance can be seen as the length of the shortest leash that is sufficient for the man and the dog to walk their paths f and g from start to end. For example, the Fréchet distance between two concentric circles of radius r_1 and r_2 respectively is $|r_1 - r_2|$.
The **discrete Fréchet distance** (or *coupling distance*, Eiter and Mannila, 1994) is an approximation of the Fréchet metric for polygonal curves f and g. It considers only positions of the leash where its endpoints are located at vertices of f and g. So, this distance is the minimum, over all order-preserving pairings of vertices in f and g, of the maximal Euclidean distance between paired vertices.

- **Bohr metric**
Let $\mathbb{R}$ be a metric space with a metric ρ. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *almost periodic* if, for every $\epsilon > 0$, there exists $l = l(\epsilon) > 0$ such that every interval $[t_0, t_0 + l(\epsilon)]$ contains at least one number τ for which $\rho(f(t), f(t+\tau)) < \epsilon$ for $-\infty < t < +\infty$.
The **Bohr metric** is the **norm metric** $||f-g||$ on the set AP of all almost periodic functions defined by the norm

$$||f|| = \sup_{-\infty<t<+\infty} |f(t)|.$$

It makes AP a Banach space. Some generalizations of almost periodic functions were obtained using other norms; cf. **Stepanov distance**, **Weyl distance**, **Besicovitch distance** and **Bochner metric**.
- **Stepanov distance**
The **Stepanov distance** is a distance on the set of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ with summable p-th power on each bounded integral, defined by

$$\sup_{x\in\mathbb{R}} \left(\frac{1}{l} \int_x^{x+l} |f(x) - g(x)|^p dx \right)^{1/p}.$$

The **Weyl distance** is a distance on the same set defined by

$$\lim_{l\to\infty} \sup_{x\in\mathbb{R}} \left(\frac{1}{l} \int_x^{x+l} |f(x) - g(x)|^p dx \right)^{1/p}.$$

- **Besicovitch distance**
The **Besicovitch distance** is a distance on the set of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ with summable p-th power on each bounded integral defined by

$$\left(\overline{\lim}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x) - g(x)|^p dx \right)^{1/p}.$$

The *generalized Besicovitch almost periodic functions* correspond to this distance.
- **Bochner metric**
Given a measure space $(\Omega, \mathcal{A}, \mu)$, a Banach space $(V, ||.||_V)$, and $1 \le p \le \infty$, the *Bochner space* (or *Lebesgue–Bochner space*) $L^p(\Omega, V)$ is the set of all measurable functions $f : \Omega \to V$ such that $||f||_{L^p(\Omega,V)} \le \infty$.
Here the *Bochner norm* $||f||_{L^p(\Omega,V)}$ is defined by $(\int_\Omega ||f(\omega)||_V^p d\mu(\omega))^{\frac{1}{p}}$ for $1 \le p < \infty$, and, for $p = \infty$, by $\operatorname{ess\,sup}_{\omega\in\Omega} ||f(\omega)||_V$.

- **Bergman p-metric**
  Given $1 \le p < \infty$, let $L_p(\Delta)$ be the L_p-space of Lebesgue measurable functions f on the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with $||f||_p = \left(\int_\Delta |f(z)|^p \mu(dz)\right)^{\frac{1}{p}} < \infty$.
  The *Bergman space* $L_p^a(\Delta)$ is the subspace of $L_p(\Delta)$ consisting of analytic functions, and the **Bergman p-metric** is the **L_p-metric** on $L_p^a(\Delta)$ (cf. **Bergman metric** in Chap. 7). Any Bergman space is a Banach space.
- **Bloch metric**
  The *Bloch space* B on the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the set of all analytic functions f on Δ such that $||f||_B = \sup_{z\in\Delta}(1 - |z|^2)|f'(z)| < \infty$. Using the complete *seminorm* $||.||_B$, a norm on B is defined by

$$||f|| = |f(0)| + ||f||_B.$$

  The **Bloch metric** is the **norm metric** $||f - g||$ on B. It makes B a Banach space.
- **Besov metric**
  Given $1 < p < \infty$, the *Besov space* B_p on the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the set of all analytic functions f in Δ such that $||f||_{B_p} = \left(\int_\Delta (1 - |z|^2)^p |f'(z)|^p d\lambda(z)\right)^{\frac{1}{p}} < \infty$, where $d\lambda(z) = \frac{\mu(dz)}{(1-|z|^2)^2}$ is the Möbius invariant measure on Δ. Using the complete *seminorm* $||.||_{B_p}$, the *Besov norm* on B_p is defined by

$$||f|| = |f(0)| + ||f||_{B_p}.$$

  The **Besov metric** is the **norm metric** $||f - g||$ on B_p.
  It makes B_p a Banach space. The set B_2 is the classical *Dirichlet space* of functions analytic on Δ with square integrable derivative, equipped with the **Dirichlet metric**. The *Bloch space* B can be considered as B_∞.
- **Hardy metric**
  Given $1 \le p < \infty$, the *Hardy space* $H^p(\Delta)$ is the class of functions, analytic on the *unit disk* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and satisfying the following growth condition for the *Hardy norm* $||.||_{H^p}$:

$$||f||_{H^p(\Delta)} = \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty.$$

  The **Hardy metric** is the **norm metric** $||f - g||_{H^p(\Delta)}$ on $H^p(\Delta)$. It makes $H^p(\Delta)$ a Banach space.
  In Complex Analysis, the Hardy spaces are analogs of the L_p-spaces of Functional Analysis. Such spaces are applied in Mathematical Analysis itself, and also in Scattering Theory and Control Theory (cf. Chap. 18).

- **Part metric**
  The **part metric** is a metric on a *domain* D of $\mathbb{R}^2$ defined for any $x, y \in \mathbb{R}^2$ by

$$\sup_{f \in H^+} \left| \ln \left(\frac{f(x)}{f(y)} \right) \right|,$$

  where H^+ is the set of all positive *harmonic functions* on the domain D.
  A twice-differentiable real function $f : D \to \mathbb{R}$ is called *harmonic* on D if its *Laplacian* $\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}$ vanishes on D.
- **Orlicz metric**
  Let $M(u)$ be an even convex function of a real variable which is increasing for u positive, and $\lim_{u \to 0} u^{-1}M(u) = \lim_{u \to \infty} u(M(u))^{-1} = 0$. In this case the function $p(v) = M'(v)$ does not decrease on $[0, \infty)$, $p(0) = \lim_{v \to 0} p(v) = 0$, and $p(v) > 0$ when $v > 0$. Writing $M(u) = \int_0^{|u|} p(v)dv$, and defining $N(u) = \int_0^{|u|} p^{-1}(v)dv$, one obtains a pair $(M(u), N(u))$ of *complementary functions*.
  Let $(M(u), N(u))$ be a pair of complementary functions, and let G be a bounded closed set in $\mathbb{R}^n$. The *Orlicz space* $L_M^*(G)$ is the set of Lebesgue-measurable functions f on G satisfying the following growth condition for the *Orlicz norm* $||f||_M$:

$$||f||_M = \sup \left\{ \int_G f(t)g(t)dt : \int_G N(g(t))dt \le 1 \right\} < \infty.$$

  The **Orlicz metric** is the norm metric $||f - g||_M$ on $L_M^*(G)$. It makes $L_M^*(G)$ a Banach space [Orli32].
  When $M(u) = u^p, 1 < p < \infty$, $L_M^*(G)$ coincides with the space $L_p(G)$, and, up to scalar factor, the L_p*-norm* $||f||_p$ coincides with $||f||_M$.
  The Orlicz norm is equivalent to the *Luxemburg norm* $||f||_{(M)} = \inf\{\lambda > 0 : \int_G M(\lambda^{-1} f(t))dt \le 1\}$; in fact, $||f||_{(M)} \le ||f||_M \le 2||f||_{(M)}$.
- **Orlicz–Lorentz metric**
  Let $w : (0, \infty) \to (0, \infty)$ be a nonincreasing function. Let $M : [0, \infty) \to [0, \infty)$ be a nondecreasing and convex function with $M(0) = 0$. Let G be a bounded closed set in $\mathbb{R}^n$.
  The *Orlicz–Lorentz space* $L_{w,M}(G)$ is the set of all Lebesgue-measurable functions f on G satisfying the following growth condition for the *Orlicz–Lorentz norm* $||f||_{w,M}$:

$$||f||_{w,M} = \inf \left\{ \lambda > 0 : \int_0^\infty w(x)M\left(\frac{f^*(x)}{\lambda}\right) dx \le 1 \right\} < \infty,$$

  where $f^*(x) = \sup\{t : \mu(|f| \ge t) \ge x\}$ is the *nonincreasing rearrangement* of f.
  The **Orlicz–Lorentz metric** is the **norm metric** $||f - g||_{w,M}$ on $L_{w,M}(G)$. It makes $L_{w,M}(G)$ a Banach space.

The Orlicz–Lorentz space is a generalization of the *Orlicz space* $L_M^*(G)$ (cf. **Orlicz metric**), and the *Lorentz space* $L_{w,q}(G)$, $1 \le q < \infty$, of all Lebesgue-measurable functions f on G satisfying the following growth condition for the *Lorentz norm*:

$$||f||_{w,q} = \left(\int_0^\infty w(x)(f^*(x))^q \right)^{\frac{1}{q}} < \infty.$$

- **Hölder metric**
Let $L^\alpha(G)$ be the set of all bounded continuous functions f defined on a subset G of $\mathbb{R}^n$, and satisfying the *Hölder condition* on G. Here, a function f satisfies the *Hölder condition* at a point $y \in G$ with *index* (or *order*) α, $0 < \alpha \le 1$, and with coefficient $A(y)$, if $|f(x) - f(y)| \le A(y)|x-y|^\alpha$ for all $x \in G$ sufficiently close to y.
If $A = \sup_{y \in G}(A(y)) < \infty$, the Hölder condition is called *uniform* on G, and A is called the *Hölder coefficient* of G. The quantity $|f|_\alpha = \sup_{x,y \in G} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$, $0 \le \alpha \le 1$, is called the *Hölder α-seminorm* of f, and the *Hölder norm* of f is defined by

$$||f||_{L^\alpha(G)} = \sup_{x \in G} |f(x)| + |f|_\alpha.$$

The **Hölder metric** is the **norm metric** $||f-g||_{L^\alpha(G)}$ on $L^\alpha(G)$. It makes $L^\alpha(G)$ a Banach space.
- **Sobolev metric**
The *Sobolev space* $W^{k,p}$ is a subset of an L_p-space such that f and its derivatives up to order k have a finite *L_p-norm*. Formally, given a subset G of $\mathbb{R}^n$, define

$$W^{k,p} = W^{k,p}(G) = \{f \in L_p(G) : f^{(i)} \in L_p(G), 1 \le i \le k\},$$

where $f^{(i)} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f$, $\alpha_1 + \dots + \alpha_n = i$, and the derivatives are taken in a weak sense. The *Sobolev norm* on $W^{k,p}$ is defined by

$$||f||_{k,p} = \sum_{i=0}^{k} ||f^{(i)}||_p.$$

In fact, it is enough to take only the first and last in the sequence, i.e., the norm defined by $||f||_{k,p} = ||f||_p + ||f^{(k)}||_p$ is equivalent to the norm above.
For $p = \infty$, the Sobolev norm is equal to the *essential supremum* of $|f|$: $||f||_{k,\infty} = ess \sup_{x \in G} |f(x)|$, i.e., it is the infimum of all numbers $a \in \mathbb{R}$ for which $|f(x)| > a$ on a set of measure zero.
The **Sobolev metric** is the **norm metric** $||f-g||_{k,p}$ on $W^{k,p}$. It makes $W^{k,p}$ a Banach space.

The Sobolev space $W^{k,2}$ is denoted by H^k. It is a Hilbert space for the *inner product* $\langle f, g\rangle_k = \sum_{i=1}^{k}\langle f^{(i)}, g^{(i)}\rangle_{L_2} = \sum_{i=1}^{k}\int_G f^{(i)}\overline{g}^{(i)}\mu(d\omega)$.

- **Variable exponent space metrics**
  Let G be a nonempty open subset of $\mathbb{R}^n$, and let $p : G \to [1,\infty)$ be a measurable bounded function, called a *variable exponent*. The *variable exponent Lebesgue space* $L_{p(.)}(G)$ is the set of all measurable functions $f : G \to \mathbb{R}$ for which the *modular* $\varrho_{p(.)}(f) = \int_G |f(x)|^{p(x)}dx$ is finite. The *Luxemburg norm* on this space is defined by

$$||f||_{p(.)} = \inf\{\lambda > 0 : \varrho_{p(.)}(f/\lambda) \le 1\}.$$

  The **variable exponent Lebesgue space metric** is the **norm metric** $||f-g||_{p(.)}$ on $L_{p(.)}(G)$.
  A *variable exponent Sobolev space* $W^{1,p(.)}(G)$ is a subspace of $L_{p(.)}(G)$ consisting of functions f whose distributional gradient exists almost everywhere and satisfies the condition $|\nabla f| \in L_{p(.)}(G)$. The norm

$$||f||_{1,p(.)} = ||f||_{p(.)} + ||\nabla f||_{p(.)}$$

  makes $W^{1,p(.)}(G)$ a Banach space. The **variable exponent Sobolev space metric** is the norm metric $||f-g||_{1,p(.)}$ on $W^{1,p(.)}$.
- **Schwartz metric**
  The *Schwartz space* (or *space of rapidly decreasing functions*) $S(\mathbb{R}^n)$ is the class of all *Schwartz functions*, i.e., infinitely-differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ that decrease at infinity, as do all their derivatives, faster than any inverse power of x. More precisely, f is a Schwartz function if we have the following growth condition:

$$||f||_{\alpha\beta} = \sup_{x\in\mathbb{R}^n}\left|x_1^{\beta_1}\dots x_n^{\beta_n}\frac{\partial^{\alpha_1+\dots+\alpha_n}f(x_1,\dots,x_n)}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}\right| < \infty$$

  for any nonnegative integer vectors α and β. The family of *seminorms* $||.||_{\alpha\beta}$ defines a **locally convex** topology of $S(\mathbb{R}^n)$ which is **metrizable** and complete. The **Schwartz metric** is a metric on $S(\mathbb{R}^n)$ which can be obtained using this topology (cf. **countably normed space** in Chap. 2).
  The corresponding metric space on $S(\mathbb{R}^n)$ is a *Fréchet space* in the sense of Functional Analysis, i.e., a locally convex *F-space*.
- **Bregman quasi-distance**
  Let $G \subset \mathbb{R}^n$ be a closed set with the nonempty interior G^0. Let f be a *Bregman function with zone* G.
  The **Bregman quasi-distance** $D_f : G \times G^0 \to \mathbb{R}_{\ge 0}$ is defined by

$$D_f(x,y) = f(x) - f(y) - \langle\nabla f(y), x-y\rangle,$$

where $\nabla f = (\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n})$. $D_f(x, y) = 0$ if and only if $x = y$. Also $D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle$ but, in general, D_f does not satisfy the triangle inequality, and is not symmetric.
A real-valued function f whose effective domain contains G is called a *Bregman function with zone* G if the following conditions hold:

1. f is continuously differentiable on G^0;
2. f is strictly convex and continuous on G;
3. For all $\delta \in \mathbb{R}$ the *partial level sets* $\Gamma(x, \delta) = \{y \in G^0 : D_f(x, y) \leq \delta\}$ are bounded for all $x \in G$;
4. If $\{y_n\}_n \subset G^0$ converges to y^*, then $D_f(y^*, y_n)$ converges to 0;
5. If $\{x_n\}_n \subset G$ and $\{y_n\}_n \subset G^0$ are sequences such that $\{x_n\}_n$ is bounded, $\lim_{n\to\infty} y_n = y^*$, and $\lim_{n\to\infty} D_f(x_n, y_n) = 0$, then $\lim_{n\to\infty} x_n = y^*$.

When $G = \mathbb{R}^n$, a sufficient condition for a strictly convex function to be a Bregman function has the form: $\lim_{||x||\to\infty} \frac{f(x)}{||x||} = \infty$.

13.2 Metrics on Linear Operators

A *linear operator* is a function $T : V \to W$ between two vector spaces V, W over a field $\mathbb{F}$, that is compatible with their linear structures, i.e., for any $x, y \in V$ and any scalar $k \in \mathbb{F}$, we have the following properties: $T(x + y) = T(x) + T(y)$, and $T(kx) = kT(x)$.

- **Operator norm metric**
Consider the set of all linear operators from a *normed space* $(V, ||.||_V)$ into a normed space $(W, ||.||_W)$. The *operator norm* $||T||$ of a *linear operator* $T : V \to W$ is defined as the largest value by which T stretches an element of V, i.e.,

$$||T|| = \sup_{||v||_V \neq 0} \frac{||T(v)||_W}{||v||_V} = \sup_{||v||_V = 1} ||T(v)||_W = \sup_{||v||_V \leq 1} ||T(v)||_W.$$

A linear operator $T : V \to W$ from a normed space V into a normed space W is called *bounded* if its operator norm is finite. For normed spaces, a linear operator is bounded if and only if it is *continuous*.
The **operator norm metric** is a **norm metric** on the set $B(V, W)$ of all bounded linear operators from V into W, defined by

$$||T - P||.$$

The space $(B(V, W), ||.||)$ is called the *space of bounded linear operators*. This metric space is **complete** if W is. If $V = W$ is complete, the space $B(V, V)$ is a *Banach algebra*, as the operator norm is a *submultiplicative norm*.

A linear operator $T : V \to W$ from a Banach space V into another Banach space W is called *compact* if the image of any bounded subset of V is a relatively compact subset of W. Any compact operator is bounded (and, hence, continuous). The space $(K(V, W), ||.||)$ on the set $K(V, W)$ of all compact operators from V into W with the operator norm $||.||$ is called the *space of compact operators*.

- **Nuclear norm metric**
  Let $B(V, W)$ be the space of all bounded linear operators mapping a Banach space $(V, ||.||_V)$ into another Banach space $(W, ||.||_W)$. Let the *Banach dual* of V be denoted by V', and the value of a functional $x' \in V'$ at a vector $x \in V$ by $\langle x, x' \rangle$.
  A linear operator $T \in B(V, W)$ is called a *nuclear operator* if it can be represented in the form $x \mapsto T(x) = \sum_{i=1}^{\infty} \langle x, x_i' \rangle y_i$, where $\{x_i'\}_i$ and $\{y_i\}_i$ are sequences in V' and W, respectively, such that $\sum_{i=1}^{\infty} ||x_i'||_{V'} ||y_i||_W < \infty$. This representation is called *nuclear*, and can be regarded as an expansion of T as a sum of operators of rank 1 (i.e., with one-dimensional range). The *nuclear norm* of T is defined as

$$||T||_{nuc} = \inf \sum_{i=1}^{\infty} ||x_i'||_{V'} ||y_i||_W,$$

  where the infimum is taken over all possible nuclear representations of T.
  The **nuclear norm metric** is the **norm metric** $||T - P||_{nuc}$ on the set $N(V, W)$ of all nuclear operators mapping V into W. The space $(N(V, W), ||.||_{nuc})$, called the *space of nuclear operators*, is a Banach space.
  A *nuclear space* is defined as a **locally convex** space for which all continuous linear functions into an arbitrary Banach space are nuclear operators. A nuclear space is constructed as a projective limit of Hilbert spaces H_α with the property that, for each $\alpha \in I$, one can find $\beta \in I$ such that $H_\beta \subset H_\alpha$, and the embedding operator $H_\beta \ni x \to x \in H_\alpha$ is a *Hilbert–Schmidt operator*. A *normed space* is nuclear if and only if it is finite-dimensional.
- **Finite nuclear norm metric**
  Let $F(V, W)$ be the space of all *linear operators of finite rank* (i.e., with finite-dimensional range) mapping a Banach space $(V, ||.||_V)$ into another Banach space $(W, ||.||_W)$. A linear operator $T \in F(V, W)$ can be represented in the form $x \mapsto T(x) = \sum_{i=1}^{n} \langle x, x_i' \rangle y_i$, where $\{x_i'\}_i$ and $\{y_i\}_i$ are sequences in V' (*Banach dual* of V) and W, respectively, and $\langle x, x' \rangle$ is the value of a functional $x' \in V'$ at a vector $x \in V$. The *finite nuclear norm* of T is defined as

$$||T||_{fnuc} = \inf \sum_{i=1}^{n} ||x_i'||_{V'} ||y_i||_W,$$

  where the infimum is taken over all possible finite representations of T.

The **finite nuclear norm metric** is the **norm metric** $||T - P||_{fnuc}$ on $F(V, W)$. The space $(F(V, W), ||.||_{fnuc})$ is called the *space of operators of finite rank*. It is a dense linear subspace of the *space of nuclear operators* $N(V, W)$.

- **Hilbert–Schmidt norm metric**

  Consider the set of all linear operators from a Hilbert space $(H_1, ||.||_{H_1})$ into a Hilbert space $(H_2, ||.||_{H_2})$. The *Hilbert–Schmidt norm* $||T||_{HS}$ of a linear operator $T : H_1 \to H_2$ is defined by

$$||T||_{HS} = \left(\sum_{\alpha \in I} ||T(e_\alpha)||^2_{H_2}\right)^{1/2},$$

  where $(e_\alpha)_{\alpha \in I}$ is an orthonormal basis in H_1. A linear operator $T : H_1 \to H_2$ is called a *Hilbert–Schmidt operator* if $||T||^2_{HS} < \infty$.

  The **Hilbert–Schmidt norm metric** is the **norm metric** $||T - P||_{HS}$ on the set $S(H_1, H_2)$ of all Hilbert–Schmidt operators from H_1 into H_2. In Euclidean space $||.||_{HS}$ is also called *Frobenius norm*; cf. **Frobenius norm metric** in Chap. 12.

  For $H_1 = H_2 = H$, the algebra $S(H, H) = S(H)$ with the Hilbert–Schmidt norm is a *Banach algebra*. It contains operators of finite rank as a dense subset, and is contained in the space $K(H)$ of compact operators. An *inner product* $\langle , \rangle_{HS}$ on $S(H)$ is defined by $\langle T, P\rangle_{HS} = \sum_{\alpha \in I} \langle T(e_\alpha), P(e_\alpha)\rangle$, and $||T||_{HS} = \langle T, T\rangle^{1/2}_{HS}$. So, $S(H)$ is a Hilbert space (independent of the chosen basis $(e_\alpha)_{\alpha \in I}$).

- **Trace-class norm metric**

  Given a Hilbert space H, the *trace-class norm* of a linear operator $T : H \to H$ is

$$||T||_{tc} = \sum_{\alpha \in I} \langle |T|(e_\alpha), e_\alpha\rangle,$$

  where $|T|$ is the *absolute value* of T in the *Banach algebra* $B(H)$ of all bounded operators from H into itself, and $(e_\alpha)_{\alpha \in I}$ is an orthonormal basis of H.

  An operator $T : H \to H$ is called a *trace-class operator* if $||T||_{tc} < \infty$. Any such operator is the product of two *Hilbert–Schmidt operators*.

  The **trace-class norm metric** is the **norm metric** $||T - P||_{tc}$ on the set $L(H)$ of all trace-class operators from H into itself.

  The set $L(H)$ with the norm $||.||_{tc}$ forms a Banach algebra which is contained in the algebra $K(H)$ (of all compact operators from H into itself), and contains the algebra $S(H)$ of all Hilbert–Schmidt operators from H into itself.

- **Schatten p-class norm metric**

  Let $1 \leq p < \infty$. Given a separable Hilbert space H, the *Schatten p-class norm* of a compact linear operator $T : H \to H$ is defined by

$$||T||^p_{Sch} = \left(\sum_n |s_n|^p\right)^{\frac{1}{p}},$$

where $\{s_n\}_n$ is the sequence of *singular values* of T. A compact operator $T : H \rightarrow H$ is called a *Schatten p-class operator* if $||T||^p_{Sch} < \infty$.
The **Schatten p-class norm metric** is the **norm metric** $||T - P||^p_{Sch}$ on the set $S_p(H)$ of all Schatten p-class operators from H onto itself. The set $S_p(H)$ with the norm $||.||^p_{Sch}$ forms a Banach space. $S_1(H)$ is the *trace-class* of H, and $S_2(H)$ is the *Hilbert–Schmidt class* of H. Cf. **Schatten norm metric** (in Chap. 12) for which **trace** and **Frobenius** norm metrics are cases $p = 1$ and $p = 2$, respectively.

- **Continuous dual space**
  For any vector space V over some field, its *algebraic dual space* is the set of all linear functionals on V.
  Let $(V, ||.||)$ be a *normed vector space*. Let V' be the set of all *continuous* linear functionals T from V into the base field ($\mathbb{R}$ or $\mathbb{C}$). Let $||.||'$ be the *operator norm* on V' defined by

$$||T||' = \sup_{||x|| \leq 1} |T(x)|.$$

  The space $(V', ||.||')$ is a Banach space which is called the **continuous dual** (or *Banach dual*) of $(V, ||.||)$.
  The continuous dual of the metric space l^n_p (l^∞_p) is l^n_q (l^∞_q, respectively), where q is defined by $\frac{1}{p} + \frac{1}{q} = 1$. The continuous dual of l^n_1 (l^∞_1) is l^n_∞ (l^∞_∞, respectively).
- **Distance constant of operator algebra**
  Let $\mathcal{A}$ be an subalgebra of $B(H)$, the algebra of all bounded operators on a Hilbert space H. For any operator $T \in B(H)$, let P be a projection, $P^\perp$ be its orthogonal complement and $\beta(T, \mathcal{A}) = \sup\{||P^\perp TP|| : P^\perp \mathcal{A} P = (0)\}$.
  Let $dist(T, \mathcal{A}) = \inf_{A \in \mathcal{A}} ||T - A||$ be the *distance of T to algebra* $\mathcal{A}$; cf. **matrix nearness problems** in Chap. 12. It holds $dist(T, \mathcal{A}) \geq \beta(T, \mathcal{A})$.
  The algebra $\mathcal{A}$ is *reflexive* if $\beta(T, \mathcal{A}) = 0$ implies $T \in \mathcal{A}$; it is *hyperreflexive* if there exists a constant $C \geq 1$ such that, for any operator $T \in B(H)$, it holds

$$dist(T, \mathcal{A}) \leq C\beta(T, \mathcal{A}).$$

  The smallest such C is called the **distance constant** of the algebra $\mathcal{A}$.
  In the case of a reflexive algebra of matrices with nonzero entries specified by a given pattern, the problem of finding the distance constant can be formulated as a matrix-filling problem: given a partially completed matrix, fill in the remaining entries so that the operator norm of the resulting complete matrix is minimized.