Some irrational numbers

Chapter 8

> *"π is irrational"*

This was already conjectured by Aristotle, when he claimed that diameter and circumference of a circle are not commensurable. The first proof of this fundamental fact was given by Johann Heinrich Lambert in 1766. In fact, Lambert even showed that $\tan r$ is irrational for rational $r \neq 0$; the irrationality of π follows from this since $\tan \frac{\pi}{4} = 1$. Our Book Proof is due to Ivan Niven, 1947: an extremely elegant one-page proof that needs only elementary calculus. Its idea is powerful, and quite a bit more can be derived from it, as was shown by Iwamoto and Koksma, respectively:

- π^2 is irrational and
- e^r is irrational for rational $r \neq 0$.

Charles Hermite

Niven's method does, however, have its roots and predecessors: It can be traced back to the classical paper by Charles Hermite from 1873 which first established that e is transcendental, that is, that e is not a zero of a polynomial with rational coefficients.

Before we treat π we will look at e and its powers, and see that these are irrational. This is much easier, and we thus also follow the historical order in the development of the results.

$$\begin{aligned} e &:= 1 + \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \cdots \\ &= 2.718281828... \end{aligned}$$

$$\begin{aligned} e^x &:= 1 + \tfrac{x}{1} + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \tfrac{x^4}{24} + \cdots \\ &= \sum_{k \geq 0} \tfrac{x^k}{k!} \end{aligned}$$

To start with, it is rather easy to see (as did Fourier in 1815) that $e = \sum_{k \geq 0} \frac{1}{k!}$ is irrational. Indeed, if we had $e = \frac{a}{b}$ for integers a and $b > 0$, then we would get

$$n!be = n!a$$

for *every* $n \geq 0$. But this cannot be true, because on the right-hand side we have an integer, while the left-hand side with

$$e = \Big(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\Big) + \Big(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\Big)$$

decomposes into an integral part

$$bn!\Big(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\Big)$$

and a second part

$$b\Big(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\Big)$$

M. Aigner, G.M. Ziegler, *Proofs from THE BOOK*, DOI 10.1007/978-3-662-44205-0_8,

Geometric series

For the infinite geometric series

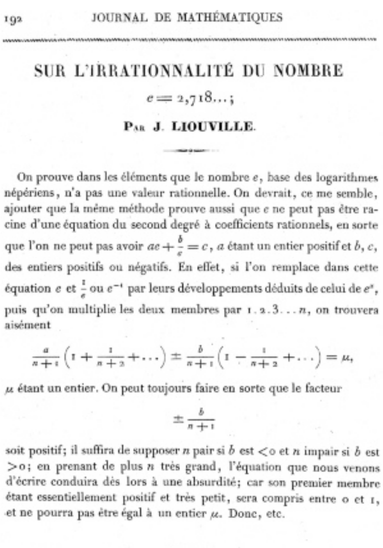
$Q = \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \cdots$

with $q > 1$ we clearly have

$qQ = 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots = 1 + Q$

and thus

$Q = \dfrac{1}{q-1}.$


192 JOURNAL DE MATHÉMATIQUES

SUR L'IRRATIONNALITÉ DU NOMBRE

$e = 2,718\ldots;$

PAR J. LIOUVILLE.

On prouve dans les éléments que le nombre e, base des logarithmes népériens, n'a pas une valeur rationnelle. On devrait, ce me semble, ajouter que la même méthode prouve aussi que e ne peut pas être racine d'une équation du second degré à coefficients rationnels, en sorte que l'on ne peut pas avoir $ae + \frac{b}{e} = c$, a étant un entier positif et b, c, des entiers positifs ou négatifs. En effet, si l'on remplace dans cette équation e et $\frac{1}{e}$ ou e^{-1} par leurs développements déduits de celui de e^x, puis qu'on multiplie les deux membres par $1.2.3\ldots n$, on trouvera aisément

$$\frac{a}{n+1}\left(1+\frac{1}{n+2}+\ldots\right) \pm \frac{b}{n+1}\left(1-\frac{1}{n+2}+\ldots\right) = \mu,$$

μ étant un entier. On peut toujours faire en sorte que le facteur

$$\pm\frac{b}{n+1}$$

soit positif; il suffira de supposer n pair si b est <0 et n impair si b est >0; en prenant de plus n très grand, l'équation que nous venons d'écrire conduira dès lors à une absurdité; car son premier membre étant essentiellement positif et très petit, sera compris entre 0 et 1, et ne pourra pas être égal à un entier μ. Donc, etc.

Liouville's paper

which is *approximately* $\frac{b}{n}$, so that for large n it certainly cannot be integral: It is larger than $\frac{b}{n+1}$ and smaller than $\frac{b}{n}$, as one can see from a comparison with a geometric series:

$$\begin{aligned}\frac{1}{n+1} &< \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1}{n}.\end{aligned}$$

Now one might be led to think that this simple multiply–by–$n!$ trick is not even sufficient to show that e^2 is irrational. This is a stronger statement: $\sqrt{2}$ is an example of a number which is irrational, but whose square is not.

From John Cosgrave we have learned that with two nice ideas/observations (let's call them "tricks") one can get two steps further nevertheless: Each of the tricks is sufficient to show that e^2 is irrational, the combination of both of them even yields the same for e^4. The first trick may be found in a one page paper by J. Liouville from 1840 — and the second one in a two page "addendum" which Liouville published on the next two journal pages.

Why is e^2 irrational? What can we derive from $e^2 = \frac{a}{b}$? According to Liouville we should write this as

$$be = ae^{-1},$$

substitute the series

$$e = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$$

and

$$e^{-1} = 1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} \pm \cdots,$$

and then multiply by $n!$, for a sufficiently large even n. Then we see that $n!be$ is nearly integral:

$$n!b\Big(1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!}\Big)$$

is an integer, and the rest

$$n!b\Big(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\Big)$$

is approximately $\frac{b}{n}$: It is larger than $\frac{b}{n+1}$ but smaller than $\frac{b}{n}$, as we have seen above.

At the same time $n!ae^{-1}$ is nearly integral as well: Again we get a large integral part, and then a rest

$$(-1)^{n+1}n!a\Big(\frac{1}{(n+1)!} - \frac{1}{(n+2)!} + \frac{1}{(n+3)!} \mp \cdots\Big),$$

and this is approximately $(-1)^{n+1}\frac{a}{n}$. More precisely: for even n the rest is larger than $-\frac{a}{n}$, but smaller than

$$-a\Big(\frac{1}{n+1} - \frac{1}{(n+1)^2} - \frac{1}{(n+1)^3} - \cdots\Big) \;=\; -\frac{a}{n+1}\Big(1-\frac{1}{n}\Big) \;<\; 0.$$

But this cannot be true, since for large even n it would imply that $n!ae^{-1}$ is just a bit smaller than an integer, while $n!be$ is a bit larger than an integer, so $n!ae^{-1} = n!be$ cannot hold. □

In order to show that e^4 is irrational, we now courageously assume that $e^4 = \frac{a}{b}$ were rational, and write this as

$$be^2 \;=\; ae^{-2}.$$

We could now try to multiply this by $n!$ for some large n, and collect the non-integral summands, but this leads to nothing useful: The sum of the remaining terms on the left-hand side will be approximately $b\frac{2^{n+1}}{n}$, on the right side $(-1)^{n+1}a\frac{2^{n+1}}{n}$, and both will be very large if n gets large.
So one has to examine the situation a bit more carefully, and make two little adjustments to the strategy: First we will not take an *arbitrary* large n, but a large power of two, $n = 2^m$; and secondly we will not multiply by $n!$, but by $\frac{n!}{2^{n-1}}$. Then we need a little lemma, a special case of Legendre's theorem (see page 10): For any $n \geq 1$ the integer $n!$ contains the prime factor 2 at most $n-1$ times — with equality if (and only if) n is a power of two, $n = 2^m$.
This lemma is not hard to show: $\lfloor\frac{n}{2}\rfloor$ of the factors of $n!$ are even, $\lfloor\frac{n}{4}\rfloor$ of them are divisible by 4, and so on. So if 2^k is the largest power of two which satisfies $2^k \leq n$, then $n!$ contains the prime factor 2 exactly

$$\Big\lfloor\frac{n}{2}\Big\rfloor+\Big\lfloor\frac{n}{4}\Big\rfloor+\cdots+\Big\lfloor\frac{n}{2^k}\Big\rfloor \;\leq\; \frac{n}{2}+\frac{n}{4}+\cdots+\frac{n}{2^k} \;=\; n\Big(1-\frac{1}{2^k}\Big) \;\leq\; n-1$$

times, with equality in both inequalities exactly if $n = 2^k$.
Let's get back to $be^2 = ae^{-2}$. We are looking at

$$b\frac{n!}{2^{n-1}}e^2 \;=\; a\frac{n!}{2^{n-1}}e^{-2} \tag{1}$$

and substitute the series

$$e^2 \;=\; 1+\frac{2}{1}+\frac{4}{2}+\frac{8}{6}+\;\cdots\;+\frac{2^r}{r!}+\cdots$$

and

$$e^{-2} \;=\; 1-\frac{2}{1}+\frac{4}{2}-\frac{8}{6}\pm\;\cdots\;+(-1)^r\frac{2^r}{r!}+\cdots$$

For $r \leq n$ we get integral summands on both sides, namely

$$b\frac{n!}{2^{n-1}}\frac{2^r}{r!} \qquad \text{resp.} \qquad (-1)^r a\frac{n!}{2^{n-1}}\frac{2^r}{r!},$$

where for $r > 0$ the denominator $r!$ contains the prime factor 2 *at most* $r-1$ times, while $n!$ contains it *exactly* $n-1$ times. (So for $r > 0$ the summands are even.)

And since n is even (we assume that $n = 2^m$), the series that we get for $r \geq n+1$ are

$$2b\Big(\frac{2}{n+1} + \frac{4}{(n+1)(n+2)} + \frac{8}{(n+1)(n+2)(n+3)} + \cdots\Big)$$

resp.

$$2a\Big(-\frac{2}{n+1} + \frac{4}{(n+1)(n+2)} - \frac{8}{(n+1)(n+2)(n+3)} \pm \cdots\Big).$$

These series will for large n be roughly $\frac{4b}{n}$ resp. $-\frac{4a}{n}$, as one sees again by comparison with geometric series. For large $n = 2^m$ this means that the left-hand side of (1) is *a bit* larger than an integer, while the right-hand side is *a bit* smaller — contradiction! □

So we know that e^4 is irrational; to show that e^3, e^5 etc. are irrational as well, we need heavier machinery (that is, a bit of calculus), and a new idea — which essentially goes back to Charles Hermite, and for which the key is hidden in the following simple lemma.

Lemma. *For some fixed $n \geq 1$, let*

$$f(x) \quad = \quad \frac{x^n(1-x)^n}{n!}.$$

(i) *The function $f(x)$ is a polynomial of the form $f(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i$, where the coefficients c_i are integers.*

(ii) *For $0 < x < 1$ we have $0 < f(x) < \frac{1}{n!}$.*

(iii) *The derivatives $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers for all $k \geq 0$.*

■ **Proof.** Parts (i) and (ii) are clear.
For (iii) note that by (i) the k-th derivative $f^{(k)}$ vanishes at $x = 0$ unless $n \leq k \leq 2n$, and in this range $f^{(k)}(0) = \frac{k!}{n!}c_k$ is an integer. From $f(x) = f(1-x)$ we get $f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$ for all x, and hence $f^{(k)}(1) = (-1)^k f^{(k)}(0)$, which is an integer. □

Theorem 1. *e^r is irrational for every $r \in \mathbb{Q}\backslash\{0\}$.*

■ **Proof.** It suffices to show that e^s cannot be rational for a positive integer s (if $e^{\frac{s}{t}}$ were rational, then $\big(e^{\frac{s}{t}}\big)^t = e^s$ would be rational, too). Assume that $e^s = \frac{a}{b}$ for integers $a, b > 0$, and let n be so large that $n! > as^{2n+1}$. Put

The estimate $n! > e(\frac{n}{e})^n$ yields an explicit n that is "large enough."

$$F(x) \;:=\; s^{2n}f(x) - s^{2n-1}f'(x) + s^{2n-2}f''(x) \mp \cdots + f^{(2n)}(x),$$

where $f(x)$ is the function of the lemma.

$F(x)$ may also be written as an infinite sum

$$F(x) \;=\; s^{2n}f(x) - s^{2n-1}f'(x) + s^{2n-2}f''(x) \mp \cdots,$$

since the higher derivatives $f^{(k)}(x)$, for $k > 2n$, vanish. From this we see that the polynomial $F(x)$ satisfies the identity

$$F'(x) \;=\; -s\,F(x) + s^{2n+1}f(x).$$

Thus differentiation yields

$$\frac{d}{dx}\left[e^{sx}F(x)\right] \;=\; se^{sx}F(x) + e^{sx}F'(x) \;=\; s^{2n+1}e^{sx}f(x)$$

and hence

$$N \;:=\; b\int_0^1 s^{2n+1}e^{sx}f(x)dx \;=\; b\left[e^{sx}F(x)\right]_0^1 \;=\; aF(1) - bF(0).$$

This is an integer, since part (iii) of the lemma implies that $F(0)$ and $F(1)$ are integers. However, part (ii) of the lemma yields estimates for the size of N from below and from above,

$$0 \;<\; N \;=\; b\int_0^1 s^{2n+1}e^{sx}f(x)dx \;<\; bs^{2n+1}e^s\frac{1}{n!} \;=\; \frac{as^{2n+1}}{n!} \;<\; 1,$$

which shows that N cannot be an integer: contradiction. □

Now that this trick was so successful, we use it once more.

Theorem 2. π^2 *is irrational.*

■ **Proof.** Assume that $\pi^2 = \frac{a}{b}$ for integers $a, b > 0$. We now use the polynomial

$$F(x) \;:=\; b^n\Big(\pi^{2n}f(x) - \pi^{2n-2}f^{(2)}(x) + \pi^{2n-4}f^{(4)}(x) \mp \cdots\Big),$$

which satisfies $F''(x) = -\pi^2F(x) + b^n\pi^{2n+2}f(x)$.

From part (iii) of the lemma we get that $F(0)$ and $F(1)$ are integers. Elementary differentiation rules yield

$$\begin{aligned}\frac{d}{dx}\big[F'(x)\sin\pi x - \pi F(x)\cos\pi x\big] &= \big(F''(x) + \pi^2F(x)\big)\sin\pi x\\ &= b^n\pi^{2n+2}f(x)\sin\pi x\\ &= \pi^2a^nf(x)\sin\pi x,\end{aligned}$$

and thus we obtain

$$\begin{aligned}N := \pi\int_0^1 a^nf(x)\sin\pi x\,dx &= \Big[\frac{1}{\pi}F'(x)\sin\pi x - F(x)\cos\pi x\Big]_0^1\\ &= F(0) + F(1),\end{aligned}$$

which is an integer. Furthermore N is positive since it is defined as the

π is not rational, but it does have "good approximations" by rationals — some of these were known since antiquity:

$$\begin{aligned}\tfrac{22}{7} &= 3.142857142857...\\ \tfrac{355}{113} &= 3.141592920353...\\ \tfrac{104348}{33215} &= 3.141592653921...\\ \pi &= 3.141592653589...\end{aligned}$$

integral of a function that is positive (except on the boundary). However, if we choose n so large that $\frac{\pi a^n}{n!} < 1$, then from part (ii) of the lemma we obtain

$$0 \;<\; N \;=\; \pi \int_0^1 a^n f(x) \sin \pi x \, dx \;<\; \frac{\pi a^n}{n!} \;<\; 1,$$

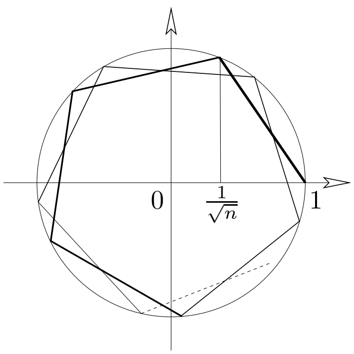
a contradiction. □

Here comes our final irrationality result.

Theorem 3. *For every odd integer $n \geq 3$, the number*

$$A(n) \;:=\; \frac{1}{\pi} \arccos\left(\frac{1}{\sqrt{n}}\right)$$

is irrational.



We will need this result for Hilbert's third problem (see Chapter 10) in the cases $n = 3$ and $n = 9$. For $n = 2$ and $n = 4$ we have $A(2) = \frac{1}{4}$ and $A(4) = \frac{1}{3}$, so the restriction to odd integers is essential. These values are easily derived by appealing to the diagram in the margin, in which the statement "$\frac{1}{\pi} \arccos\left(\frac{1}{\sqrt{n}}\right)$ is irrational" is equivalent to saying that the polygonal arc constructed from $\frac{1}{\sqrt{n}}$, all of whose chords have the same length, never closes into itself.

We leave it as an exercise for the reader to show that $A(n)$ is rational *only* for $n \in \{1, 2, 4\}$. For that, distinguish the cases when $n = 2^r$, and when n is not a power of 2.

■ **Proof.** We use the addition theorem

$$\cos \alpha + \cos \beta = 2 \cos \tfrac{\alpha+\beta}{2} \cos \tfrac{\alpha-\beta}{2}$$

from elementary trigonometry, which for $\alpha = (k+1)\varphi$ and $\beta = (k-1)\varphi$ yields

$$\cos\,(k+1)\varphi \;=\; 2\cos\varphi\, \cos\, k\varphi \;-\; \cos\,(k-1)\varphi. \tag{2}$$

For the angle $\varphi_n = \arccos\left(\frac{1}{\sqrt{n}}\right)$, which is defined by $\cos\varphi_n = \frac{1}{\sqrt{n}}$ and $0 \leq \varphi_n \leq \pi$, this yields representations of the form

$$\cos\, k\varphi_n \;=\; \frac{A_k}{\sqrt{n}^k},$$

where A_k is an integer that is not divisible by n, for all $k \geq 0$. In fact, we have such a representation for $k = 0, 1$ with $A_0 = A_1 = 1$, and by induction on k using (2) we get for $k \geq 1$

$$\cos\,(k+1)\varphi_n \;=\; 2\frac{1}{\sqrt{n}}\frac{A_k}{\sqrt{n}^k} - \frac{A_{k-1}}{\sqrt{n}^{k-1}} \;=\; \frac{2A_k - nA_{k-1}}{\sqrt{n}^{k+1}}.$$

Thus we obtain $A_{k+1} = 2A_k - nA_{k-1}$. If $n \geq 3$ is odd, and A_k is not divisible by n, then we find that A_{k+1} cannot be divisible by n, either.

Now assume that

$$A(n) \;=\; \frac{1}{\pi}\varphi_n \;=\; \frac{k}{\ell}$$

is rational (with integers $k, \ell > 0$). Then $\ell\varphi_n = k\pi$ yields

$$\pm 1 \;=\; \cos k\pi \;=\; \frac{A_\ell}{\sqrt{n}^{\ell}}.$$

Thus $\sqrt{n}^{\ell} = \pm A_\ell$ is an integer, with $\ell \geq 2$, and hence $n \,|\, \sqrt{n}^{\ell}$. With $\sqrt{n}^{\ell} \,|\, A_\ell$ we find that n divides A_ℓ, a contradiction. □

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