Cauchy's rigidity theorem

Chapter 14

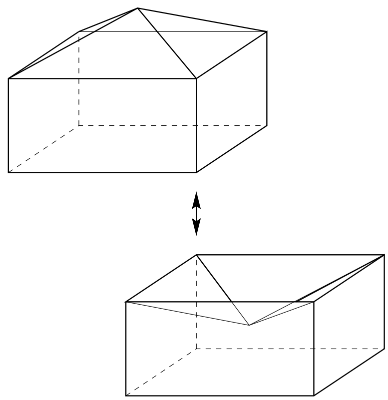
A famous result that depends on Euler's formula (specifically, on part (C) of the proposition in the previous chapter) is Cauchy's rigidity theorem for 3-dimensional polyhedra.

For the notions of congruence and of combinatorial equivalence that are used in the following we refer to the appendix on polytopes and polyhedra in the chapter on Hilbert's third problem, see page 69.

Augustin Cauchy

> **Theorem.** *If two 3-dimensional convex polyhedra P and P' are combinatorially equivalent with corresponding facets being congruent, then also the angles between corresponding pairs of adjacent facets are equal (and thus P is congruent to P').*

The illustration in the margin shows two 3-dimensional polyhedra that are combinatorially equivalent, such that the corresponding faces are congruent. But they are not congruent, and only one of them is convex. Thus the assumption of convexity is essential for Cauchy's theorem!

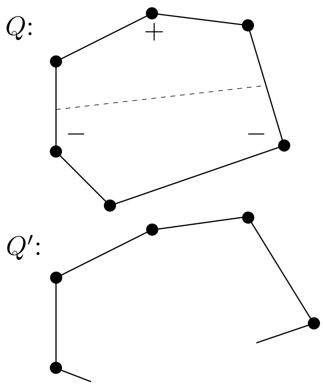


■ **Proof.** The following is essentially Cauchy's original proof. Assume that two convex polyhedra P and P' with congruent faces are given. We color the edges of P as follows: an edge is black (or "positive") if the corresponding interior angle between the two adjacent facets is larger in P' than in P; it is white (or "negative") if the corresponding angle is smaller in P' than in P.

The black and the white edges of P together form a 2-colored plane graph on the surface of P, which by radial projection, assuming that the origin is in the interior of P, we may transfer to the surface of the unit sphere. If P and P' have unequal corresponding facet-angles, then the graph is nonempty. With part (C) of the proposition in the previous chapter we find that there is a vertex p that is adjacent to at least one black or white edge, such that there are at most two changes between black and white edges (in cyclic order).

Now we intersect P with a small sphere S_ε (of radius ε) centered at the vertex p, and we intersect P' with a sphere S'_ε of the same radius ε centered at the corresponding vertex p'. In S_ε and S'_ε we find convex spherical polygons Q and Q' such that corresponding arcs have the same lengths, because of the congruence of the facets of P and P', and since we have chosen the same radius ε.

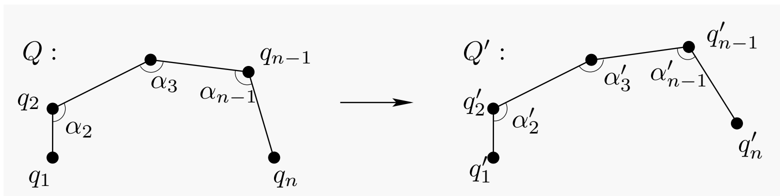
M. Aigner, G.M. Ziegler, *Proofs from THE BOOK*, DOI 10.1007/978-3-662-44205-0_14,



Now we mark by $+$ the angles of Q for which the corresponding angle in Q' is larger, and by $-$ the angles whose corresponding angle of Q' is smaller. That is, when moving from Q to Q' the $+$ angles are "opened," the $-$ angles are "closed," while all side lengths and the unmarked angles stay constant.

From our choice of p we know that *some* $+$ or $-$ sign occurs, and that in cyclic order there are at most two $+/-$ changes. If only one type of signs occurs, then the lemma below directly gives a contradiction, saying that one edge must change its length. If both types of signs occur, then (since there are only two sign changes) there is a "separation line" that connects the midpoints of two edges and separates all the $+$ signs from all the $-$ signs. Again we get a contradiction from the lemma below, since the separation line cannot be both longer and shorter in Q' than in Q. □

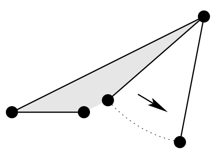
Cauchy's arm lemma.
If Q and Q' are convex (planar or spherical) n-gons, labeled as in the figure,



such that $\overline{q_iq_{i+1}} = \overline{q'_iq'_{i+1}}$ holds for the lengths of corresponding edges for $1 \le i \le n-1$, and $\alpha_i \le \alpha'_i$ holds for the sizes of corresponding angles for $2 \le i \le n-1$, then the "missing" edge length satisfies

$$\overline{q_1q_n} \;\le\; \overline{q'_1q'_n},$$

with equality if and only if $\alpha_i = \alpha'_i$ holds for all i.



It is interesting that Cauchy's original proof of the lemma was false: a continuous motion that opens angles and keeps side-lengths fixed may destroy convexity — see the figure! On the other hand, both the lemma and its proof given here, from a letter by I. J. Schoenberg to S. K. Zaremba, are valid both for planar and for spherical polygons.

■ **Proof.** We use induction on n. The case $n = 3$ is easy: If in a triangle we increase the angle γ between two sides of fixed lengths a and b, then the length c of the opposite side also increases. Analytically, this follows from the cosine theorem

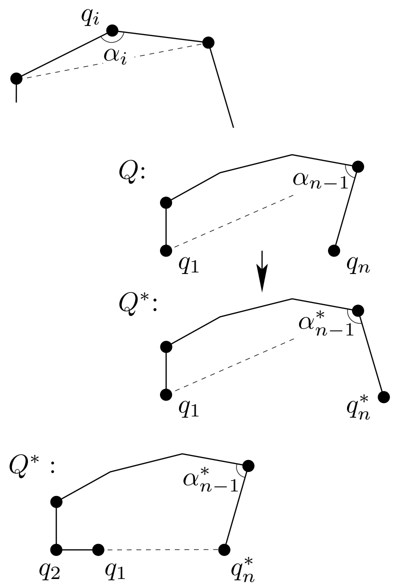
$$c^2 \;=\; a^2 + b^2 - 2ab\cos\gamma$$

in the planar case, and from the analogous result

$$\cos c \;=\; \cos a \cos b + \sin a \sin b \cos\gamma$$

in spherical trigonometry. Here the lengths a, b, c are measured on the surface of a sphere of radius 1, and thus have values in the interval $[0, \pi]$.

Now let $n \geq 4$. If for any $i \in \{2, \ldots, n-1\}$ we have $\alpha_i = \alpha'_i$, then the corresponding vertex can be cut off by introducing the diagonal from q_{i-1} to q_{i+1} resp. from q'_{i-1} to q'_{i+1}, with $\overline{q_{i-1}q_{i+1}} = \overline{q'_{i-1}q'_{i+1}}$, so we are done by induction. Thus we may assume $\alpha_i < \alpha'_i$ for $2 \leq i \leq n-1$.



Now we produce a new polygon Q^* from Q by replacing α_{n-1} by the largest possible angle $\alpha^*_{n-1} \leq \alpha'_{n-1}$ that keeps Q^* convex. For this we replace q_n by q^*_n, keeping all the other q_i, edge lengths, and angles from Q.

If indeed we can choose $\alpha^*_{n-1} = \alpha'_{n-1}$ keeping Q^* convex, then we get $\overline{q_1q_n} < \overline{q_1q^*_n} \leq \overline{q'_1q'_n}$, using the case $n = 3$ for the first step and induction as above for the second.

Otherwise after a nontrivial move that yields

$$\overline{q_1q^*_n} > \overline{q_1q_n} \tag{1}$$

we "get stuck" in a situation where q_2, q_1 and q^*_n are collinear, with

$$\overline{q_2q_1} + \overline{q_1q^*_n} = \overline{q_2q^*_n}. \tag{2}$$

Now we compare this Q^* with Q' and find

$$\overline{q_2q^*_n} \leq \overline{q'_2q'_n} \tag{3}$$

by induction on n (ignoring the vertex q_1 resp. q'_1). Thus we obtain

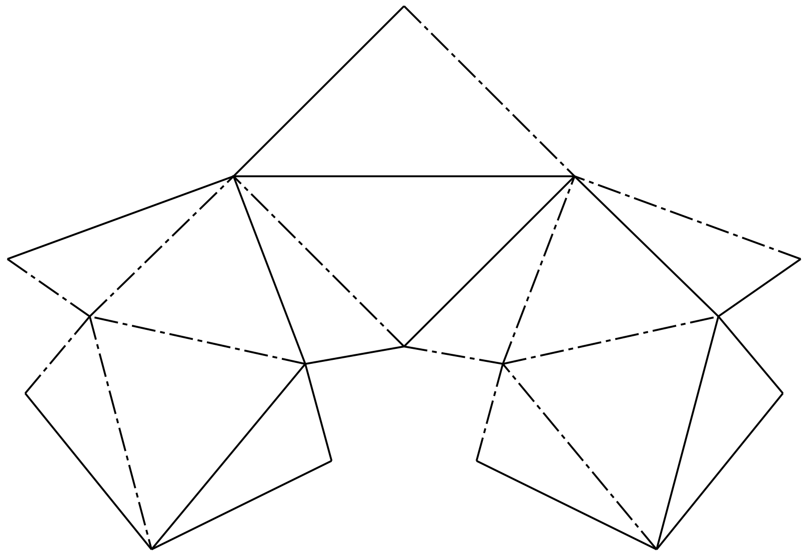
$$\overline{q'_1q'_n} \overset{(*)}{\geq} \overline{q'_2q'_n} - \overline{q'_1q'_2} \overset{(3)}{\geq} \overline{q_2q^*_n} - \overline{q_1q_2} \overset{(2)}{=} \overline{q_1q^*_n} \overset{(1)}{>} \overline{q_1q_n}\,,$$

where $(*)$ is just the triangle inequality, and all other relations have already been derived. □

We have seen an example which shows that Cauchy's theorem is not true for *nonconvex* polyhedra. The special feature of this example is, of course, that a noncontinuous "flip" takes one polyhedron to the other, keeping the facets congruent while the dihedral angles "jump." One can ask for more:

> *Could there be, for some nonconvex polyhedron, a* continuous *deformation that would keep the facets flat and congruent?*

It was conjectured that no triangulated surface, convex or not, admits such a motion. So, it was quite a surprise when in 1977 — more than 160 years after Cauchy's work — Robert Connelly presented counterexamples: closed triangulated spheres embedded in $\mathbb{R}^3$ (without self-intersections) that are flexible, with a continuous motion that keeps all the edge lengths constant, and thus keeps the triangular faces congruent.



A beautiful example of a flexible surface constructed by Klaus Steffen: The dashed lines represent the nonconvex edges in this "cut-out" paper model. Fold the normal lines as "mountains" and the dashed lines as "valleys." The edges in the model have lengths 5, 10, 11, 12 and 17 units.

The rigidity theory of surfaces has even more surprises in store: Idjad Sabitov managed to prove that when any such flexing surface moves, the *volume* it encloses must be constant. His proof is beautiful also in its use of the algebraic machinery of polynomials and determinants (outside the scope of this book).

References

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