On the Construction of Analytic Sequent Calculi for Sub-classical Logics*

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Abstract. We study the question of when a given set of derivable rules in some basic analytic propositional sequent calculus forms itself an analytic calculus. First, a general syntactic criterion for analyticity in the family of pure sequent calculi is presented. Next, given a basic calculus admitting this criterion, we provide a method to construct weaker pure calculi by collecting simple derivable rules of the basic calculus. The obtained calculi are analytic-by-construction. While the criterion and the method are completely syntactic, our proofs are semantic, based on interpretation of sequent calculi via non-deterministic valuation functions. In particular, this method captures calculi for a wide variety of paraconsistent logics, as well as some extensions of Gurevich and Neeman's primal infon logic.

1 Introduction

Proof theory reveals a wide mosaic of possibilities for sub-classical logics. These are logics that are strictly contained (as consequence relations) in classical logic. Thus, by choosing a subset of axioms and derivation rules that are derivable in (some proof system for) classical logic, one easily obtains a (proof system for a) sub-classical logic. Various important and useful non-classical logics can be formalized in this way, with the most prominent example being intuitionistic logic. In general, the resulting logics come at first with no semantics. They might be also unusable for computational purposes, since the new calculi might not be *analytic*: it is often the case that proofs of some formula φ must contain formulas that are not subformulas of φ. This is evident within the framework of Hilbert-style calculi, that are rarely analytic. But, even for Gentzen-type sequent calculi, where the initial proof system for classical logic **LK** is analytic, there is no guarantee that an arbitrary collection of classically derivable sequent rules constitutes an analytic sequent calculus.

In this paper, we focus on a general family of relatively simple sequent calculi for propositional logics, called *pure* sequent calculi (originally studied in [2]), of which (the propositional fragment of) **LK** is the prototype example. Our contribution is twofold. First, we generalize the coherence condition from [4] to provide a decidable sufficient syntactic criterion for analyticity of a given pure

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sequent calculus. Here we employ a general concept of analyticity, based on a parametrized notion of a subformula, that shares the attractive features with the usual subformula property. This criterion is useful in many cases, e.g. for proving the analyticity of a sequent calculus for the logic of first-degree entailment [1], and of course, the analyticity of the propositional fragment of **LK**. Second, we show that calculi admitting this criterion can be utilized for constructing other analytic sequent calculi. Taking a basic calculus **B**, we present a method for obtaining other analytic-by-construction sub-calculi of **B**, by collecting derivable rules of **B** that have a certain "safe" form.

The proposed method is general enough to capture a wide variety of known sequent calculi for sub-classical logics. This includes:

- A large family of sequent calculi for propositional paraconsistent logics, originated from philosophical motivations, and obtained by replacing the usual left introduction rule of negation with weaker rules, each of which is derivable in **LK**.
- A sequent calculus for primal logic (without quotations) from [7], as well as some natural extensions of it. This calculus originated from practical computational motivations, aiming to allow efficient proof search. It is obtained by replacing the usual right introduction rule of implication with a weaker rule, and discarding the rule for introducing disjunction on the left hand-side.

Our approach is semantic: We formulate and use a semantic property of sequent calculi that is equivalent to analyticity. The semantics, however, plays a role only in our arguments, while the actual use of the proposed methods includes only syntactic considerations.

Related Work. The family of pure sequent calculi was defined in [2]. The semantics for these calculi which lies in the basis of our proofs, is similar to the one in [10] (and takes its inspiration from [6]). Nevertheless, [10] investigates translations of derivability in analytic pure calculi to the classical satisfiability problem, leaving open the tasks of constructing analytic calculi, and checking whether a given calculus is analytic. In this paper we aim to fill this gap, by providing simple sufficient conditions (that hold in various known cases) for analyticity. Furthermore, we note that the notion of analyticity employed in [10] is generalized in the current paper: (1) here we also consider derivations from assumptions (also known as "non-logical axioms"); and (2) we use a more general parametrized notion of a subformula. A particular well-behaved subfamily of pure calculi, called *canonical calculi* was studied in [4]. For these calculi, it was shown that analyticity and cut-admissibility are equivalent, and both were precisely characterized by a simple and decidable *coherence* criterion. However, various useful pure calculi (some of which are included in examples below) are not canonical, and still their analyticity can be shown using the results of the current paper. Finally, the general framework of [11] allows one to encode all pure calculi in linear logic, and use linear logic to reason about them. Among the pure calculi, it is again only the canonical ones for which a decidable criterion for cut-admissibility is given in [11].

2 Pure Sequent Calculi

In what follows, we assume a propositional language for classical logic, that consists of a countably infinite set of atomic variables $At = \{p_1, p_2, \ldots\}$, the binary connectives $\wedge$, $\vee$ and $\supset$, the unary connective $\neg$, and the nullary connectives $\top$ and $\bot$. A *sequent* is a pair $\langle \Gamma, \Delta \rangle$ (denoted by $\Gamma \Rightarrow \Delta$) where Γ and Δ are finite sets of formulas. We employ the standard sequent notations, e.g. when writing expressions like $\Gamma, \psi \Rightarrow \Delta$ or $\Rightarrow \psi$. The union of sequents is defined by $(\Gamma_1 \Rightarrow \Delta_1) \cup (\Gamma_2 \Rightarrow \Delta_2) = \Gamma_1 \cup \Gamma_2 \Rightarrow \Delta_1 \cup \Delta_2$. For a sequent $\Gamma \Rightarrow \Delta$, $frm(\Gamma \Rightarrow \Delta) = \Gamma \cup \Delta$. This notation is naturally extended to sets of sequents. Given a set $\mathcal{F}$ of formulas, we say that a sequent s is an *$\mathcal{F}$-sequent* if $frm(s) \subseteq \mathcal{F}$. A *substitution* is a function from At to the set of formulas. A substitution σ is naturally extended to compound formulas by $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for every compound formula $\diamond(\psi_1, \ldots, \psi_n)$. Substitutions are also naturally extended to sets of formulas, sequents and sets of sequents.

We focus on a general family of relatively simple sequent calculi, called *pure sequent calculi*. Roughly speaking, these are propositional fully-structural calculi (calculi that include the structural rules: exchange, contraction and weakening),[1] whose derivation rules do not enforce any limitations on the context formulas (following [2], the adjective "pure" stands for this requirement). We note that additive applications are employed (i.e., all premises share one context sequent), rather than multiplicative ones. In the context of this paper, this is just a matter of taste, since the two options are obviously equivalent when all structural rules are available.

Definition 1. A *pure rule* is a pair $\langle S, s \rangle$ (denoted by $S \,/\, s$) where S is a set of sequents and s is a sequent. The elements of S are called the *premises* of the rule and s is called the *conclusion* of the rule. The set S of premises of a pure rule is usually written without set braces, and its elements are separated by ";".

Definition 2. An *application* of a pure rule $s_1, \ldots, s_n \,/\, s$ is a pair of the form $\langle \{\sigma(s_1) \cup c, \ldots, \sigma(s_n) \cup c\}, \sigma(s) \cup c \rangle$ (denoted by $\frac{\sigma(s_1) \cup c, \ldots, \sigma(s_n) \cup c}{\sigma(s) \cup c}$) where σ is a substitution and c is a sequent (called a *context sequent*). The sequents $\sigma(s_i) \cup c$ are called the *premises* of the application and $\sigma(s) \cup c$ is called the *conclusion* of the application.

Note that every application of a pure rule is itself a pure rule. Moreover, every pure rule is an application of itself, obtained by taking the identity substitution and the empty context sequent. This duality will be exploited when we expand analytic calculi with new rules in the form of applications of other rules.

[1] Exchange and contraction are implicitly included, since sequents are taken to be pairs of *sets*.

Definition 3. A *pure calculus* is a finite set of pure rules. A *proof* in a pure calculus $\mathbf{G}$ is defined as usual, where in addition to applications of the pure rules of $\mathbf{G}$, the following standard schemes may be used:

$$(weak)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta} \qquad (id)\ \frac{}{\Gamma, \psi \Rightarrow \psi, \Delta} \qquad (cut)\ \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Given a pure calculus $\mathbf{G}$, a set $\mathcal{F}$ of formulas, a set of $\mathcal{F}$-sequents S and an $\mathcal{F}$-sequent s, we write $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$ if there is a proof of s from S in $\mathbf{G}$ consisting only of $\mathcal{F}$-sequents. When $\mathcal{F}$ is the set of all formulas, we write $\vdash_{\mathbf{G}}$ instead of $\vdash^{\mathcal{F}}_{\mathbf{G}}$.

In what follows, all rules and calculi are pure. There are many sequent calculi for non-classical logics (admitting cut-elimination) that fall in this framework. These include calculi for three and four-valued logics, various calculi for paraconsistent logics, and all canonical sequent systems [3,4,6].

Example 1. The propositional fragment of Gentzen's fundamental calculus for classical logic can be directly presented as a pure calculus, denoted henceforth by **LK**. It consists of the following rules:

$(\bot \Rightarrow)$	$\emptyset \,/\, \bot \Rightarrow$	$(\Rightarrow \top)$	$\emptyset \,/ \Rightarrow \top$
$(\neg \Rightarrow)$	$\Rightarrow p_1 \,/\, \neg p_1 \Rightarrow$	$(\Rightarrow \neg)$	$p_1 \Rightarrow \,/ \Rightarrow \neg p_1$
$(\wedge \Rightarrow)$	$p_1, p_2 \Rightarrow \,/\, p_1 \wedge p_2 \Rightarrow$	$(\Rightarrow \wedge)$	$\Rightarrow p_1; \Rightarrow p_2 \,/ \Rightarrow p_1 \wedge p_2$
$(\vee \Rightarrow)$	$p_1 \Rightarrow; p_2 \Rightarrow \,/\, p_1 \vee p_2 \Rightarrow$	$(\Rightarrow \vee)$	$\Rightarrow p_1, p_2 \,/ \Rightarrow p_1 \vee p_2$
$(\supset\Rightarrow)$	$\Rightarrow p_1; p_2 \Rightarrow \,/\, p_1 \supset p_2 \Rightarrow$	$(\Rightarrow\supset)$	$p_1 \Rightarrow p_2 \,/ \Rightarrow p_1 \supset p_2$

Example 2. The calculus from [3] for da Costa's historical paraconsistent logic C_1 can be directly presented as a pure calculus, that we call $\mathbf{G}_{C_1}$. It consists of the rules of **LK** except for $(\neg \Rightarrow)$ that is replaced by the following rules:

$p_1 \Rightarrow \,/\, \neg\neg p_1 \Rightarrow$

$\Rightarrow p_1; \Rightarrow \neg p_1 \,/\, \neg(p_1 \wedge \neg p_1) \Rightarrow$	$\neg p_1 \Rightarrow; \neg p_2 \Rightarrow \,/\, \neg(p_1 \wedge p_2) \Rightarrow$
$\neg p_1 \Rightarrow; p_2, \neg p_2 \Rightarrow \,/\, \neg(p_1 \vee p_2) \Rightarrow$	$p_1, \neg p_1 \Rightarrow; \neg p_2 \Rightarrow \,/\, \neg(p_1 \vee p_2) \Rightarrow$
$p_1 \Rightarrow; p_2, \neg p_2 \Rightarrow \,/\, \neg(p_1 \supset p_2) \Rightarrow$	$p_1, \neg p_1 \Rightarrow; \neg p_2 \Rightarrow \,/\, \neg(p_1 \supset p_2) \Rightarrow$

The following properties of pure calculi will be particularly useful below:

Proposition 1. *Let* $\mathbf{G}$ *be a calculus,* $\mathcal{F}$ *a set of formulas,* S *a set of* $\mathcal{F}$*-sequents, and* s *an* $\mathcal{F}$*-sequent. Suppose that* $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$*. Then, the following hold:*

1. $\sigma(S) \vdash^{\sigma(\mathcal{F})}_{\mathbf{G}} \sigma(s)$ *for every substitution* σ*.*
2. $\{s' \cup c \mid s' \in S\} \vdash^{\mathcal{F}}_{\mathbf{G}} s \cup c$ *for every* $\mathcal{F}$*-sequent* c*.*

2.1 Analyticity

Analyticity is a crucial property of proof systems. In the case of fully-structural propositional sequent calculi it usually implies their decidability and consistency (the fact that the empty sequent is not derivable). Roughly speaking, a calculus is analytic if whenever a sequent s is provable in it from a set S of sequents, s can be proven using only the "syntactic material available inside s and S". Usually, this "material" is taken to consist of all subformulas occurring in s. Next, we introduce a generalized analyticity property, based on a parametrized notion of a subformula in which negation plays a special role.

Definition 4. Let $k \geq 0$. A formula φ is an *immediate k-subformula* of a formula ψ if either $\psi = \neg\varphi$, or $\psi = \varphi_1 \sharp \varphi_2$ and $\varphi = \neg^m \varphi_i$ for some formulas φ_1, φ_2, $\sharp \in \{\wedge, \vee, \supset\}$, $0 \leq m \leq k$ and $i \in \{1, 2\}$.[2] The *k-subformula relation* is the reflexive transitive closure of the immediate k-subformula relation. A k-subformula φ of a formula ψ is called *proper* if $\varphi \neq \psi$. We denote the set of k-subformulas of a formula ψ by $sub^k(\psi)$. This notation is naturally extended to sets of formulas, sequents and sets of sequents.

Definition 5. A calculus $\mathbf{G}$ is called *k-analytic* if $S \vdash_{\mathbf{G}} s$ entails $S \vdash_{\mathbf{G}}^{sub^k(S \cup \{s\})} s$ for every set S of sequents and sequent s.

0-subformulas are usual subformulas, and thus 0-analyticity amounts to the usual (global) subformula property of sequent calculi. Note that k-analyticity (for any k) ensures the decidability and consistency of a calculus. The following propositions will be useful in the sequel.

Proposition 2. *If a formula φ is a (proper) k-subformula of a formula ψ, then $\sigma(\varphi)$ is a (proper) k-subformula of $\sigma(\psi)$ for every substitution σ. Consequently, $\sigma(sub^k(\psi)) \subseteq sub^k(\sigma(\psi))$ for every formula ψ and substitution σ.*

Proposition 3. *Suppose that a calculus $\mathbf{G}'$ is obtained from a calculus $\mathbf{G}$ by one of the following:*

1. *Replacing some rule $S \,/\, \Gamma \Rightarrow \psi, \Delta$ by $S; \psi \Rightarrow \,/\, \Gamma \Rightarrow \Delta$.*
2. *Replacing some rule $S \,/\, \Gamma, \psi \Rightarrow \Delta$ by $S; \Rightarrow \psi \,/\, \Gamma \Rightarrow \Delta$.*
3. *Replacing two rules of the form $S; \Gamma \Rightarrow \Delta \,/\, s$ and $S; \Gamma' \Rightarrow \Delta' \,/\, s$ by the rule $S; \Gamma \cup \Gamma' \Rightarrow \Delta \cup \Delta' \,/\, s$, given that $\Gamma \cup \Gamma' \cup \Delta \cup \Delta' \subseteq sub^k(S \cup \{s\})$.*

Then $\vdash_{\mathbf{G}'} = \vdash_{\mathbf{G}}$ and $\mathbf{G}'$ is k-analytic iff $\mathbf{G}$ is k-analytic.

Proposition 4. *Let $\mathbf{G}'$ be a calculus obtained from a calculus $\mathbf{G}$ by adding a premise s' to some rule $r = S \,/\, s$ of $\mathbf{G}$. Suppose that $S \vdash_{\mathbf{G}'}^{sub^k(frm(r))} s'$. Then, $\mathbf{G}'$ is k-analytic iff $\mathbf{G}$ is k-analytic.*[3]

Proof. Suppose that $S = \{s_1, \ldots, s_n\}$ and let $S' = S \cup \{s'\}$. To show that $\mathbf{G}'$ is k-analytic iff $\mathbf{G}$ is k-analytic, we prove that $S_0 \vdash_{\mathbf{G}'}^{\mathcal{F}} s_0$ iff $S_0 \vdash_{\mathbf{G}}^{\mathcal{F}} s_0$ for every set $\mathcal{F}$ of formulas that is closed under k-subformulas, set S_0 of $\mathcal{F}$-sequents, and $\mathcal{F}$-sequent s_0:
($\Rightarrow$): Trivially, a proof in $\mathbf{G}'$ is also a proof in $\mathbf{G}$ with some redundant sequents.
($\Leftarrow$): Suppose $S_0 \vdash_{\mathbf{G}}^{\mathcal{F}} s_0$. Let $\hat{r} = \dfrac{\sigma(s_1) \cup c, \ldots, \sigma(s_n) \cup c}{\sigma(s) \cup c}$ be an application of

[2] $\neg^m \varphi$ is inductively defined by: $\neg^0 \varphi = \varphi$, and $\neg^{m+1} \varphi = \neg\neg^m \varphi$.

[3] *frm* is extended to pure rules and their applications in the obvious way, e.g. $frm(S \,/\, s) = frm(S) \cup frm(s)$.

r in the proof of s_0 from S_0, such that $\mathit{frm}(\hat{r}) \subseteq \mathcal{F}$. Since $S \vdash_{\mathbf{G}'}^{\mathit{sub}^k(\mathit{frm}(S\,/\,s))} s'$, by Proposition 1, we have $\sigma(S) \vdash_{\mathbf{G}'}^{\sigma(\mathit{sub}^k(\mathit{frm}(S\,/\,s)))} \sigma(s')$. Since $\mathit{frm}(\hat{r}) \subseteq \mathcal{F}$ and $\mathcal{F}$ is closed under k-subformulas, $\mathit{sub}^k(\sigma(\mathit{frm}(S\,/\,s))) \subseteq \mathcal{F}$. By Proposition 2, $\sigma(\mathit{sub}^k(\mathit{frm}(S\,/\,s))) \subseteq \mathcal{F}$. Hence $\sigma(S) \vdash_{\mathbf{G}'}^{\mathcal{F}} \sigma(s')$. In addition, $\mathit{frm}(c) \subseteq \mathcal{F}$, and hence by Proposition 1, $\{\sigma(s_i) \cup c \mid 1 \leq i \leq n\} \vdash_{\mathbf{G}'}^{\mathcal{F}} \sigma(s') \cup c$. Hence we may add $\sigma(s') \cup c$ to the premises of $\hat{r}$, and obtain an application of $S'\,/\,s$ that consists only of formulas in $\mathcal{F}$. This can be done for every application of $S\,/\,s$, and hence $S_0 \vdash_{\mathbf{G}'}^{\mathcal{F}} s_0$. □

3 Sufficient Criterion for Analyticity

In this section we generalize the coherence condition from [4], and show that the generalized condition entails analyticity.

Definition 6. A rule r is called *k-closed* if its conclusion has the form $\Rightarrow \varphi$ or $\varphi \Rightarrow$, and its premises consist only of proper k-subformulas of φ. A calculus is called *k-closed* if it consists only of k-closed rules.

Notation 1. Given a k-closed rule r, we denote by φ_r the formula that appears in the conclusion of r.

The calculus **LK** (Example 1) is 0-closed (and hence it is k-closed for any k). For example, the rule $r = (\Rightarrow\supset)$ of **LK** is 0-closed and $\varphi_r = p_1 \supset p_2$. The calculus $\mathbf{G}_{C_1}$ (Example 2) is 1-closed.

Definition 7. A k-closed calculus $\mathbf{G}$ is called *(cut)-guarded* if for every two rules of $\mathbf{G}$ of the forms $S_1\,/\,\Rightarrow \varphi_1$ and $S_2\,/\,\varphi_2 \Rightarrow$, and substitutions σ_1, σ_2 such that $\sigma_1(\varphi_1) = \sigma_2(\varphi_2)$, we have that the empty sequent is derivable from $\sigma_1(S_1) \cup \sigma_2(S_2)$ using only (*cut*).

Note that it is decidable whether a given calculus is (*cut*)-guarded or not. Indeed, for each pair of rules $S_1\,/\,\Rightarrow \varphi_1$ and $S_2\,/\,\varphi_2 \Rightarrow$, one can first rename the atomic variables so that no atomic variable occurs in both rules, and then it suffices to check the above condition for the most general unifier of φ_1 and φ_2.

Theorem 1. *Every (cut)-guarded k-closed calculus is k-analytic.*

This theorem is obtained as a corollary of Theorem 2 below. Next, we present some examples of applications of it.

Example 3. **LK** is (*cut*)-guarded and 0-closed, and hence it is 0-analytic. Similarly, every canonical system (as defined in [4]) in the language of classical logic is 0-closed, and hence every (*cut*)-guarded canonical system is 0-analytic.

Example 4. The quotations-free fragment of the calculus from [5] for primal infon logic (see [7]) can be directly presented as a pure calculus, that we call **P**. It consists of the rules $(\wedge \Rightarrow)$, $(\Rightarrow \wedge)$, $(\Rightarrow \vee)$, $(\supset\Rightarrow)$, $(\Rightarrow \top)$ and $(\bot \Rightarrow)$ of **LK**, together with the rule $\Rightarrow p_2\,/\,\Rightarrow p_1 \supset p_2$. Clearly, **P** is 0-closed and (*cut*)-guarded. By Theorem 1, it is 0-analytic.

Remark 1. For the intended application of primal infon logic as a logic for access control, it is necessary to extend the language with quotations (i.e., unary connectives of the form "`q said`"), and add appropriate inference rules for them. Following [10], we note that the 0-analyticity of any given pure calculus entails the subformula property for its extension with quotations.

Example 5. The paper [8] investigates a hierarchy of weak double negations, by presenting an infinite set $\{L2^{n+2} \mid n \in \mathbb{N}\}$ of calculi. For example, the calculus $L4$, that captures the relevance logic of first-degree entailment (see [1]), can be obtained by augmenting $\mathbf{LK} \setminus \{(\neg\Rightarrow), (\Rightarrow\neg)\}$ with the following rules:

$p_1, \neg p_2 \Rightarrow \;/\; \neg(p_1 \supset p_2) \Rightarrow$ $\qquad\Rightarrow p_1; \Rightarrow \neg p_2 \;/\; \Rightarrow \neg(p_1 \supset p_2)$
$\neg p_1 \Rightarrow; \neg p_2 \Rightarrow \;/\; \neg(p_1 \wedge p_2) \Rightarrow$ $\qquad\Rightarrow \neg p_1, \neg p_2 \;/\; \Rightarrow \neg(p_1 \wedge p_2)$
$\neg p_1, \neg p_2 \Rightarrow \;/\; \neg(p_1 \vee p_2) \Rightarrow$ $\qquad\Rightarrow \neg p_1; \Rightarrow \neg p_2 \;/\; \Rightarrow \neg(p_1 \vee p_2)$
$p_1 \Rightarrow \;/\; \neg\neg p_1 \Rightarrow$ $\qquad\Rightarrow p_1 \;/\; \Rightarrow \neg\neg p_1$

This calculus is (cut)-guarded and 1-closed, and hence, by Theorem 1, it is 1-analytic. Moreover, it can be easily observed that each $L2^{n+2}$ is (cut)-guarded and $(n+1)$-closed, and thus by Theorem 1, each $L2^{n+2}$ is $(n+1)$-analytic.

4 Constructing Analytic Calculi

Theorem 1 allows us to prove that many calculi are k-analytic, by observing that they are k-closed and (cut)-guarded. However, this criterion is not necessary. For example, $\mathbf{G}_{C_1}$ from Example 2 is 1-analytic (this can be shown as a consequence of cut-elimination), but it is not (cut)-guarded. Indeed, for the rules $p_1 \Rightarrow \;/\; \Rightarrow \neg p_1$ and $p_1 \Rightarrow \;/\; \neg\neg p_1 \Rightarrow$, and the substitutions σ_1, σ_2 with $\sigma_1(p_1) = \neg p_1$ and $\sigma_2(p_1) = p_1$, we have $\sigma_1(\neg p_1) = \sigma_2(\neg\neg p_1)$, but the empty sequent is not provable from $\{(\neg p_1 \Rightarrow), (p_1 \Rightarrow)\}$ only with (cut). In order to capture $\mathbf{G}_{C_1}$ and other useful calculi, we introduce a more general method to prove analyticity. More precisely, we present a method for constructing k-analytic calculi by joining applications of rules of a certain basic (cut)-guarded k-closed calculus.

In what follows **B** *denotes an arbitrary* k*-closed* (cut)*-guarded calculus, that serves as a basic calculus.*

Definition 8. An application of a rule $s_1, \ldots, s_n \;/\; s$ is called *k-safe* if it has the form $\dfrac{\sigma(s_1) \cup c, \ldots, \sigma(s_n) \cup c}{\sigma(s) \cup c}$ for some substitution σ and sequent c, such that c consists only of proper k-subformulas of formulas that occur in $\sigma(s)$.

Example 6. The following are 0-safe, 1-safe and 2-safe applications of the rule $(\supset\Rightarrow)$ of **LK** (respectively):

$$\frac{p_1 \Rightarrow p_1 \wedge p_2 \quad p_1, p_2 \Rightarrow}{p_1, (p_1 \wedge p_2) \supset p_2 \Rightarrow} \qquad \frac{\neg p_1 \Rightarrow p_1 \wedge p_2 \quad \neg p_1, p_2 \Rightarrow}{\neg p_1, (p_1 \wedge p_2) \supset p_2 \Rightarrow}$$

$$\frac{\neg\neg p_3 \Rightarrow p_1 \wedge p_2, \neg(p_1 \wedge p_2) \quad \neg\neg p_3, p_2 \supset p_3 \Rightarrow \neg(p_1 \wedge p_2)}{\neg\neg p_3, (p_1 \wedge p_2) \supset (p_2 \supset p_3) \Rightarrow \neg(p_1 \wedge p_2)}$$

Proposition 5. *Consider a k-closed rule $r = s_1, \ldots, s_n \,/\, s$, and a k-safe application of r, $\hat{r} = \dfrac{\sigma(s_1) \cup c, \ldots, \sigma(s_n) \cup c}{\sigma(s) \cup c}$. Then all formulas in $sub^k(\sigma(s_i) \cup c)$ are proper k-subformulas of $\sigma(\varphi_r)$ (and thus $sub^k(frm(\hat{r})) \subseteq sub^k(\sigma(\varphi_r))$).*

Proof. Let $\psi \in sub^k(\sigma(s_i) \cup c)$ and let $\varphi \in \sigma(frm(s_i)) \cup frm(c)$ such that ψ is a k-subformula of φ. We show that φ is a proper k-subformula of $\sigma(\varphi_r)$. Since ψ is a k-subformula of φ, it follows that ψ is also a proper k-subformula of $\sigma(\varphi_r)$. If $\varphi = \sigma(\varphi')$ for some $\varphi' \in frm(s_i)$, then since r is k-closed, φ' is a proper k-subformula of φ_r. By Proposition 2, φ is a proper k-subformula of $\sigma(\varphi_r)$. Otherwise, $\varphi \in frm(c)$, and since $\hat{r}$ is k-safe, φ is a proper k-subformula of $\sigma(\varphi_r)$. □

Theorem 2. *Every calculus that consists solely of k-safe applications of rules of* **B** *is k-analytic.*

Theorem 2 will be proved in the next section. First, observe that Theorem 1 is obtained as a corollary:

Proof (of Theorem 1). Every rule of **B** is a trivial k-safe application of itself, and hence by Theorem 2, **B** itself is k-analytic. □

Before proving Theorem 2, we present some consequences and examples of it. For these examples, we take the basic calculus **B** to be **LK** (that is (*cut*)-guarded and k-closed for every k).

Example 7. A sequent calculus $\mathbf{G}_{P_1}$ for the *atomic* paraconsistent logic P_1 from [12] can be constructed using Theorem 2.[4] Begin with $\mathbf{LK} \setminus \{(\neg \Rightarrow)\}$, and add the following 0-safe applications of $(\neg \Rightarrow)$ to allow left-introduction of negation for compound formulas:

$\Rightarrow \neg p_1 \,/\, \neg\neg p_1 \Rightarrow$ $\qquad$ $\Rightarrow p_1 \wedge p_2 \,/\, \neg(p_1 \wedge p_2) \Rightarrow$

$\Rightarrow p_1 \vee p_2 \,/\, \neg(p_1 \vee p_2) \Rightarrow$ $\qquad$ $\Rightarrow p_1 \supset p_2 \,/\, \neg(p_1 \supset p_2) \Rightarrow$

Note that the context sequent c is empty in each of these applications. By Theorem 2, this calculus is 0-analytic. In $\mathbf{G}_{P_1}$ we have $\nvdash_{\mathbf{G}_{P_1}} p_1, \neg p_1 \Rightarrow p_2$, but $\vdash_{\mathbf{G}_{P_1}} \varphi, \neg\varphi \Rightarrow \psi$ for every compound formula φ and formula ψ. Note that $\mathbf{G}_{P_1}$ is also 0-closed and (*cut*)-guarded, and hence its analyticity directly follows from Theorem 1.

In some cases, k-safe applications of rules of **B** turn out to have premises that are already derivable. For example, suppose we would like to augment the calculus **P** from Example 4 with the rule $\bot \Rightarrow p_1 \,/\, \Rightarrow \bot \supset p_1$, which is a 0-safe application of $(\Rightarrow\supset)$. Since the sequent $\bot \Rightarrow p_1$ is provable in **P**, it is a redundant premise. In this case, one can add the rule $\emptyset \,/\, \Rightarrow \bot \supset p_1$ directly. The following proposition is used for omitting redundant premises in the examples below.

[4] $\mathbf{G}_{P_1}$ is equivalent to a sequent calculus for P_1 given by Arnon Avron in an unpublished manuscript.

Proposition 6. *Let* $\mathbf{G}_1$ *be a* k*-analytic calculus, and* $\mathbf{G}_2$ *be a calculus that consist solely of* k*-safe applications of rules of* $\mathbf{B}$*. Suppose that* $\mathbf{G}_1 \cup \mathbf{G}_2$ *is a* k*-analytic calculus, and that* $\vdash_{\mathbf{G}_1} s$ *for every premise* s *of a rule of* $\mathbf{G}_2$*. Let* $\mathbf{G}_3 = \{\emptyset / s \mid S / s \in \mathbf{G}_2\}$*. Then* $\mathbf{G}_1 \cup \mathbf{G}_3$ *is* k*-analytic.*

Proof. Let $S / s \in \mathbf{G}_2$ and $s' \in S$. Since $\mathbf{G}_1$ is k-analytic and $\vdash_{\mathbf{G}_1} s'$, we have $\vdash_{\mathbf{G}_1}^{sub^k(s')} s'$. By Proposition 5, $sub^k(s') \subseteq sub^k(s)$. Therefore, $\vdash_{\mathbf{G}_1}^{sub^k(s)} s'$, and so $\vdash_{\mathbf{G}_1 \cup \mathbf{G}_3}^{sub^k(s)} s'$. By repeatedly applying Proposition 4, since $\mathbf{G}_1 \cup \mathbf{G}_2$ is k-analytic, we obtain that $\mathbf{G}_1 \cup \mathbf{G}_3$ is k-analytic as well. □

Example 8. The calculus $\mathbf{G}_{C_1}$ from Example 2 is 1-analytic. Using Proposition 6, we construct a 1-analytic equivalent calculus that we call $\mathbf{G}'_{C_1}$. Let $\mathbf{G}_1 = \mathbf{LK} \setminus \{(\neg \Rightarrow)\}$. $\mathbf{G}'_{C_1}$ is obtained by augmenting $\mathbf{G}_1$ with the following rules:

$\emptyset / \neg\neg p_1 \Rightarrow p_1$

$\emptyset / p_1, \neg p_1, \neg(p_1 \wedge \neg p_1) \Rightarrow$	$\emptyset / \neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2$
$\emptyset / \neg(p_1 \vee p_2) \Rightarrow \neg p_1, p_2$	$\emptyset / \neg(p_1 \vee p_2) \Rightarrow \neg p_1, \neg p_2$
$\emptyset / \neg(p_1 \vee p_2) \Rightarrow p_1, \neg p_2$	$\emptyset / \neg(p_1 \supset p_2) \Rightarrow p_1, p_2$
$\emptyset / \neg(p_1 \supset p_2) \Rightarrow p_1, \neg p_2$	$\emptyset / \neg(p_1 \supset p_2) \Rightarrow \neg p_1, \neg p_2$

Every rule in this list has the form $\emptyset / s$, where s is the conclusion of a 1-safe application of the rule $(\neg \Rightarrow)$ of $\mathbf{LK}$, whose premises are all provable in G_1. For example, the sequent $\neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2$ is the conclusion of $\frac{\Rightarrow p_1 \wedge p_2, \neg p_1, \neg p_2}{\neg(p_1 \wedge p_2) \Rightarrow \neg p_1, \neg p_2}$, which is a 1-safe application of the rule $(\neg \Rightarrow)$ of $\mathbf{LK}$, and its premise $\Rightarrow p_1 \wedge p_2, \neg p_1, \neg p_2$ is derivable in $\mathbf{G}_1$. By Theorem 2, augmenting G_1 with these applications results in a 1-analytic calculus. $\mathbf{G}'_{C_1}$ is obtained by discarding their premises, and its 1-analyticity is guaranteed by Proposition 6. Using Proposition 3, it is easy to see that $\mathbf{G}'_{C_1}$ is equivalent to $\mathbf{G}_{C_1}$, and furthermore, the 1-analyticity of $\mathbf{G}'_{C_1}$ entails the 1-analyticity of $\mathbf{G}_{C_1}$.

Example 9. The calculus $\mathbf{P}$ from Example 4 enjoys a linear time decision procedure (see, e.g., [7]). As shown in [10], it is possible to augment $\mathbf{P}$ with additional rules in order to make it somewhat closer to $\mathbf{LK}$, without compromising the linear time complexity.[5] Such extension is obtained as follows. Begin with a calculus $\mathbf{G}_0$ that consists of the rules $(\wedge \Rightarrow)$, $(\Rightarrow \wedge)$, $(\Rightarrow \vee)$, $(\supset\Rightarrow)$, $(\Rightarrow \top)$ and $(\bot \Rightarrow)$ of $\mathbf{LK}$. Add the rule $p_2, p_1 \Rightarrow p_2 / p_2 \Rightarrow p_1 \supset p_2$ to obtain a calculus that we call $\mathbf{P}'$. This rule is a 0-safe application of the rule $(\Rightarrow\supset)$ of $\mathbf{LK}$. Now, add the following set of rules to recover some natural properties of the classical connectives (none of these rules is derivable in $\mathbf{P}'$):

$\emptyset / \Rightarrow \bot \supset p_1$	$\emptyset / p_1 \vee p_1 \Rightarrow p_1$	$\emptyset / \Rightarrow p_1 \supset p_1$
$\emptyset / \bot \vee p_1 \Rightarrow p_1$	$\emptyset / p_1, \neg p_1 \Rightarrow$	$\emptyset / \Rightarrow (p_1 \wedge p_2) \supset p_1$
$\emptyset / p_1 \vee \bot \Rightarrow p_1$	$\emptyset / p_1 \vee (p_1 \wedge p_2) \Rightarrow p_1$	$\emptyset / \Rightarrow (p_1 \wedge p_2) \supset p_2$
$\emptyset / (p_1 \wedge p_2) \vee p_1 \Rightarrow p_1$	$\emptyset / \Rightarrow p_2 \supset (p_1 \supset p_2)$	

Every rule in this list has the form $\emptyset / s$, where s is the conclusion of a 0-safe

[5] A manual ad-hoc proof of analyticity of the extended calculus was needed in [10].

application of a rule of **LK**, whose premises are all derivable in $\mathbf{P}'$. For example, the sequent $\Rightarrow p_2 \supset (p_1 \supset p_2)$ is the conclusion of the 0-safe application $\frac{p_2 \Rightarrow p_1 \supset p_2}{\Rightarrow p_2 \supset (p_1 \supset p_2)}$ of $(\Rightarrow\supset)$, and its premise $p_2 \Rightarrow p_1 \supset p_2$ is derivable in $\mathbf{P}'$. By Theorem 2, augmenting $\mathbf{P}'$ with these applications results in a 0-analytic calculus. By Proposition 6, 0-analyticity is preserved when discarding their premises. Using Proposition 6 again, we may also discard the premise $p_2, p_1 \Rightarrow p_2$ of the rule $p_2, p_1 \Rightarrow p_2 \,/\, p_2 \Rightarrow p_1 \supset p_2$. Using Proposition 3, it is easy to see that we may replace the new rule $\emptyset \,/\, p_2 \Rightarrow p_1 \supset p_2$ by $\Rightarrow p_2 \,/ \Rightarrow p_1 \supset p_2$, which is the original right introduction rule of implication in **P**.

5 Proof of Theorem 2

This section is devoted to prove Theorem 2. Our proof relies on a semantic interpretation of pure calculi, that gives rise to a semantic characterization of analyticity, as was shown in [10]. Note that we have to slightly strengthen the soundness and completeness theorem given in [10] in order to cover derivations with assumptions (i.e. $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$ for non-empty set S).

Definition 9. A *bivaluation* is a function v from some set $dom(v)$ of formulas to $\{0,1\}$. A bivaluation v is extended to $dom(v)$-sequents by: $v(\Gamma \Rightarrow \Delta) = 1$ iff $v(\varphi) = 0$ for some $\varphi \in \Gamma$ or $v(\varphi) = 1$ for some $\varphi \in \Delta$. v is extended to sets of $dom(v)$-sequents by: $v(S) = min\,\{v(s) \mid s \in S\}$, where $min\,\emptyset = 1$. Given a set $\mathcal{F}$ of formulas, by an *$\mathcal{F}$-bivaluation* we refer to a bivaluation v with $dom(v) = \mathcal{F}$. A bivaluation v whose domain $dom(v)$ is the set of all formulas is called *full*.

Definition 10. A bivaluation v *respects* a rule $S \,/\, s$ if $v(\sigma(S)) \leq v(\sigma(s))$ for every substitution σ such that $\sigma(frm(S \,/\, s)) \subseteq dom(v)$. v is called **G**-*legal* for a calculus **G** if it respects all rules of **G**.

Example 10. A $\{p_1, \neg\neg p_1\}$-bivaluation v respects the rule $p_1 \Rightarrow \,/\, \neg\neg p_1 \Rightarrow$ iff either $v(p_1) = v(\neg\neg p_1) = 0$ or $v(p_1) = 1$. Note that **LK**-legal bivaluations are exactly usual classical valuation functions.

Theorem 3 (Soundness and Completeness). *Let* **G** *be a calculus,* $\mathcal{F}$ *be a set of formulas,* S *be a set of* $\mathcal{F}$*-sequents, and* s *be an* $\mathcal{F}$*-sequent. Then,* $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$ *iff* $v(S) \leq v(s)$ *for every* **G**-*legal* $\mathcal{F}$*-bivaluation* v*.*

Proof. **Soundness** Assume $S \vdash^{\mathcal{F}}_{\mathbf{G}} s$ and let v be a **G**-legal $\mathcal{F}$-bivaluation such that $v(S) = 1$. We prove that $v(s) = 1$ by induction on the length of the proof of s from S in **G**:

1. If $s \in S$ or s is a conclusion of an application of (cut), $(weak)$ or (id), then this is obvious.
2. If s is the conclusion of an application of a rule of **G**, then there exist $s_1, \ldots, s_n \,/\, s_0 \in \mathbf{G}$, an $\mathcal{F}$-sequent c and a substitution σ such that $\sigma(frm(\{s_1, \ldots, s_n, s_0\})) \subseteq \mathcal{F}$, $s = \sigma(s_0) \cup c$, and $S \vdash^{\mathcal{F}}_{\mathbf{G}} \sigma(s_i) \cup c$ for every $1 \leq i \leq n$. If $v(c) = 1$, then $v(\sigma(s_0) \cup c) = 1$. Otherwise, by the induction hypothesis, $v(\sigma(s_i)) = 1$ for every $1 \leq i \leq n$. Since v is **G**-legal, $v(\sigma(s_0)) = 1$, and hence $v(s) = v(\sigma(s_0) \cup c) = 1$.

Completeness Assume $S \nvdash_{\mathbf{G}}^{\mathcal{F}} s$. We prove that there exists a **G**-legal $\mathcal{F}$-bivaluation v such that $v(S) = 1$ and $v(s) = 0$. Define an *ω-$\mathcal{F}$-sequent* to be a pair $\langle L, R\rangle$ (denoted by $L \Rightarrow R$) such that L and R are (possibly infinite) subsets of $\mathcal{F}$. We write $S \vdash_{\mathbf{G}}^{\mathcal{F}} L \Rightarrow R$ if there exist finite $\Gamma \subseteq L$ and $\Delta \subseteq R$ such that $S \vdash_{\mathbf{G}}^{\mathcal{F}} \Gamma \Rightarrow \Delta$. All other definitions for sequents are naturally extended to ω-sequents. It is straightforward to extend s to an ω-$\mathcal{F}$-sequent $L^* \Rightarrow R^*$ that has the following properties:

- $\Gamma' \subseteq L^*$ and $\Delta' \subseteq R^*$ where $s = \Gamma' \Rightarrow \Delta'$.
- $S \nvdash_{\mathbf{G}}^{\mathcal{F}} L^* \Rightarrow R^*$.
- $S \vdash_{\mathbf{G}}^{\mathcal{F}} L^* \Rightarrow R^*, \psi$ for every $\psi \in \mathcal{F} \setminus R^*$.
- $S \vdash_{\mathbf{G}}^{\mathcal{F}} L^*, \psi \Rightarrow R^*$ for every $\psi \in \mathcal{F} \setminus L^*$.

Since the identity axiom (id) is available, we obviously have $L^* \cap R^* = \emptyset$. Similarly, using (cut), it can be shown that $\mathcal{F} = \mathit{frm}(L^* \Rightarrow R^*)$. Hence L^* and R^* partition $\mathcal{F}$. Define an $\mathcal{F}$-bivaluation v by: $v(\psi) = 1$ if $\psi \in L^*$, and $v(\psi) = 0$ if $\psi \in R^*$. Clearly, $v(L^* \Rightarrow R^*) = 0$ and therefore $v(s) = 0$. We prove that $v(S) = 1$ and that v is **G**-legal. Let $\Gamma \Rightarrow \Delta \in S$. Obviously, $S \vdash_{\mathbf{G}}^{\mathcal{F}} \Gamma \Rightarrow \Delta$. Since $S \nvdash_{\mathbf{G}}^{\mathcal{F}} L^* \Rightarrow R^*$, we have either $\Gamma \not\subseteq L^*$ or $\Delta \not\subseteq R^*$. If there exists $\varphi \in \Gamma \setminus L^*$, then $\varphi \in R^*$ and hence $v(\varphi) = 0$. Otherwise, there exists $\varphi \in \Delta \setminus R^*$, and hence $\varphi \in L^*$, which means that $v(\varphi) = 1$. Either way, $v(\Gamma \Rightarrow \Delta) = 1$.

Let $S_0 \,/\, s_0 \in \mathbf{G}$ and σ be a substitution such that $\sigma(\mathit{frm}(S_0 \,/\, s_0)) \subseteq \mathcal{F}$. We assume that $v(\sigma(S_0)) = 1$, and prove that $v(\sigma(s_0)) = 1$. Suppose that $S_0 = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$. We construct the following sequent $\Gamma \Rightarrow \Delta$: For every $1 \leq i \leq n$, there exists either $\psi_i \in \Gamma_i$ such that $v(\sigma(\psi_i)) = 0$ or $\psi_i \in \Delta_i$ such that $v(\sigma(\psi_i)) = 1$. If the first option holds, we add $\sigma(\psi_i)$ to Δ. If the second option holds, we add $\sigma(\psi_i)$ to Γ. Clearly, $v(\sigma(\Gamma \Rightarrow \Delta)) = 0$. In addition, $\Gamma \subseteq L^*$ and $\Delta \subseteq R^*$. Now, for every $1 \leq i \leq n$, using (id) and $(weak)$, we get that $S \vdash_{\mathbf{G}}^{\mathcal{F}} \sigma(\Gamma_i \Rightarrow \Delta_i) \cup (\Gamma \Rightarrow \Delta)$. Applying $S_0 \,/\, s_0$ with $\Gamma \Rightarrow \Delta$ as a context sequent, we get that $S \vdash_{\mathbf{G}}^{\mathcal{F}} \sigma(s_0) \cup (\Gamma \Rightarrow \Delta)$. Since $\Gamma \subseteq L^*$ and $\Delta \subseteq R^*$, $S \vdash_{\mathbf{G}}^{\mathcal{F}} \sigma(s_0) \cup (L^* \Rightarrow R^*)$. Let $\sigma(s_0) = \Gamma_0 \Rightarrow \Delta_0$. It follows that either $\Gamma_0 \not\subseteq L^*$ or $\Delta_0 \not\subseteq R^*$. Hence, $v(\psi) = 0$ for some $\psi \in \Gamma_0$ or $v(\psi) = 1$ for some $\psi \in \Delta_0$. Therefore, $v(\sigma(s_0)) = 1$. □

Using the theorem above, we formulate a semantic property of calculi that is equivalent to k-analyticity.

Definition 11. A calculus **G** is called *semantically k-analytic* if every **G**-legal bivaluation v can be extended to a **G**-legal full bivaluation, provided that $\mathit{dom}(v)$ is finite and closed under k-subformulas.

Theorem 4. *A calculus* **G** *is k-analytic iff it is semantically k-analytic.*

Proof. If **G** is not k-analytic, then there is a set S of sequents and a sequent s such that $S \vdash_{\mathbf{G}} s$ and $S \nvdash_{\mathbf{G}}^{sub^k(S \cup \{s\})} s$. Hence, there exists finite $S' \subseteq S$ such that $S' \vdash_{\mathbf{G}} s$, and $S' \nvdash_{\mathbf{G}}^{sub^k(S' \cup \{s\})} s$. According to Theorem 3, there exists a **G**-legal $sub^k(S' \cup \{s\})$-bivaluation v such that $v(S') = 1$ and $v(s) = 0$, and $u(S') \leq u(s)$

for every **G**-legal full bivaluation u. Therefore, v cannot be extended to a **G**-legal full bivaluation. In addition, $dom(v) = sub^k(S' \cup \{s\})$ is finite and closed under k-subformulas.

For the converse, suppose that v is a **G**-legal bivaluation, $dom(v)$ is finite and closed under k-subformulas, and v cannot be extended to a **G**-legal full bivaluation. Let $\Gamma = \{\psi \in dom(v) \mid v(\psi) = 1\}$, $\Delta = \{\psi \in dom(v) \mid v(\psi) = 0\}$, and $s = \Gamma \Rightarrow \Delta$. Then $dom(v) = sub^k(s)$ and $v(s) = 0$. We show that $u(s) = 1$ for every **G**-legal full bivaluation u. Indeed, every such u does not extend v. Hence there is some $\psi \in dom(v)$ such that $u(\psi) \neq v(\psi)$. Then, $u(\psi) = 0$ if $\psi \in \Gamma$, and $u(\psi) = 1$ if $\psi \in \Delta$. In either case, $u(s) = 1$. By Theorem 3, $\nvdash_{\mathbf{G}}^{sub^k(s)} s$ and $\vdash_{\mathbf{G}} s$. □

We use the semantic characterization of analyticity given in Theorem 4 to prove Theorem 2. Thus, we provide a method for extending bivaluations whose domains are finite and closed under k-subformulas.

This method is iterative: in each step we extend a given bivaluation v with a truth value for a single formula ψ, such that $dom(v) \cup \{\psi\}$ is closed under k-subformulas. We call such formulas *k-addable*:

Definition 12. A formula ψ is called *k-addable* to a bivaluation v if $dom(v)$ contains all proper k-subformulas of ψ.

The extension of partial bivaluations is determined according to the basic calculus **B**, as given in the following definition:

Definition 13. Let v be a bivaluation and ψ be a formula. The $dom(v) \cup \{\psi\}$-bivaluation $v_{\mathbf{B}}^{\psi}$ is defined as follows: *1)* $v_{\mathbf{B}}^{\psi}(\varphi) = v(\varphi)$ for every $\varphi \in dom(v)$. *2)* If $\psi \notin dom(v)$: $v_{\mathbf{B}}^{\psi}(\psi) = 1$ iff there exist a rule of the form $S \,/ \Rightarrow \varphi$ in **B** and a substitution σ such that $\sigma(frm(S)) \subseteq dom(v)$, $\sigma(\varphi) = \psi$ and $v(\sigma(S)) = 1$.

If the above extension method "works" for a given calculus **G**, we say that **G** is **B**-*k-analytic.* Formally, this is defined as follows.

Definition 14. A calculus **G** is called **B**-*k-analytic* if $v_{\mathbf{B}}^{\psi}$ is **G**-legal for every **G**-legal bivaluation v whose domain is finite and closed under k-subformulas and formula ψ that is k-addable to v.

Proposition 7. *Every* **B**-*k-analytic calculus is k-analytic.*

Proof. Let **G** be a **B**-k analytic calculus. By Theorem 4, it suffices to prove that **G** is semantically k-analytic. Let v be a **G**-legal bivaluation whose domain is finite and closed under k-subformulas. We extend v to a **G**-legal full bivaluation v'. It is a routine matter to enumerate all formulas and obtain an infinite sequence $\psi_1, \psi_2, \ldots$ such that: *a)* If $\psi_i \in dom(v)$ and $\psi_j \notin dom(v)$ then $i < j$. *b)* If ψ_i is a k-subformula of ψ_j then $i \leq j$. We define a sequence of bivaluations $v_0, v_1, \ldots$ as follows: $v_0 = v$, and $v_i = {v_{i-1}}_{\mathbf{B}}^{\psi_i}$ for every $i > 0$. $dom(v_i) = dom(v) \cup \{\psi_1, \ldots, \psi_i\}$ for every i, and therefore each ψ_i is k-addable to v_{i-1}. Since **G** is **B**-k-analytic, each v_i is $\mathbf{G}$-legal. The full bivaluation v' is defined by $v'(\psi_i) = v^i(\psi_i)$ for every $i > 0$. In order to see that v' is $\mathbf{G}$-legal, let $S \,/\, s \in \mathbf{G}$ and let σ be a substitution. Let $j = \max\{i \mid \psi_i \in \sigma(\mathit{frm}(S \,/\, s))\}$. Then $v'(\psi) = v_j(\psi)$ for every $\psi \in \sigma(\mathit{frm}(S \,/\, s))$. Recall that v_j is $\mathbf{G}$-legal, and therefore we have that $v'(\sigma(S)) = v_j(\sigma(S)) \leq v_j(\sigma(s)) = v'(\sigma(s))$. □

Next, we prove that $\mathbf{B}$-k-analyticity is preserved when a calculus is augmented with one k-safe application of a rule of $\mathbf{B}$.

Theorem 5. *Let* $\mathbf{G}$ *be a* $\mathbf{B}$-k*-analytic calculus, and* $\mathbf{G}'$ *be a calculus obtained by augmenting* $\mathbf{G}$ *with a* k*-safe application* $\hat{r}$ *of a rule* r *of* $\mathbf{B}$*. Then* $\mathbf{G}'$ *is* $\mathbf{B}$-k*-analytic.*

Proof. Suppose $r = S \,/\, s$ with $S = \{s_1, \ldots, s_n\}$, and $\hat{r} = \hat{S} \,/\, \hat{s}$. Let α be a substitution and c be a sequent such that $\hat{S} = \{\alpha(s_1) \cup c, \ldots, \alpha(s_n) \cup c\}$ and $\hat{s} = \alpha(s) \cup c$. Now, let v be a $\mathbf{G}'$-legal bivaluation whose domain is finite and closed under k-subformulas, and ψ be a formula that is k-addable to v. We prove that the bivaluation $v^{\psi}_{\mathbf{B}}$ is $\mathbf{G}'$-legal. Let $S_0 \,/\, s_0 \in \mathbf{G}'$ and σ be a substitution such that $\sigma(\mathit{frm}(S_0 \,/\, s_0)) \subseteq \mathit{dom}(v^{\psi}_{\mathbf{B}})$. We show that $v^{\psi}_{\mathbf{B}}(\sigma(S_0)) \leq v^{\psi}_{\mathbf{B}}(\sigma(s_0))$. If $S_0 \,/\, s_0 \in \mathbf{G}$ then this holds since $\mathbf{G}$ is $\mathbf{B}$-k-analytic. If $\psi \notin \sigma(\mathit{frm}(S_0 \,/\, s_0))$ or $\psi \in \mathit{dom}(v)$ then this holds since v is $\mathbf{G}'$-legal. Assume now that $S_0 \,/\, s_0 = \hat{r}$, $\psi \in \sigma(\mathit{frm}(S_0 \,/\, s_0))$ and $\psi \notin \mathit{dom}(v)$.

We first prove that $\psi = \sigma(\alpha(\varphi_r))$. Otherwise, $\sigma(\alpha(\varphi_r)) \in \mathit{dom}(v)$. By Proposition 5, $\mathit{frm}(\hat{r}) \subseteq \mathit{sub}^k(\alpha(\varphi_r))$, and by Proposition 2, we also have that $\sigma(\mathit{sub}^k(\alpha(\varphi_r))) \subseteq \mathit{sub}^k(\sigma(\alpha(\varphi_r)))$, and hence $\sigma(\mathit{frm}(\hat{r})) \subseteq \mathit{sub}^k(\sigma(\alpha(\varphi_r)))$. Since $\mathit{dom}(v)$ is closed under k-subformulas and $\sigma(\alpha(\varphi_r)) \in \mathit{dom}(v)$, we have that $\mathit{sub}^k(\sigma(\alpha(\varphi_r))) \subseteq \mathit{dom}(v)$, and hence $\sigma(\mathit{frm}(\hat{r})) \subseteq \mathit{dom}(v)$. Since $\psi \in \sigma(\mathit{frm}(\hat{r}))$, it follows that $\psi \in \mathit{dom}(v)$, which is a contradiction.

Similarly, we show that $\sigma(\mathit{frm}(\hat{S})) \subseteq \mathit{dom}(v)$. Indeed, let $\varphi \in \sigma(\mathit{frm}(\hat{S}))$ and let $\varphi' \in \mathit{frm}(\hat{S})$ such that $\varphi = \sigma(\varphi')$. By Proposition 5, φ' is a proper k-subformula of $\alpha(\varphi_r)$, and hence by Proposition 2, φ is a proper k-subformula of $\psi = \sigma(\alpha(\varphi_r))$. In particular, $\varphi \neq \psi$. Since $\sigma(\mathit{frm}(\hat{S})) \subseteq \mathit{dom}(v^{\psi}_{\mathbf{B}})$, it follows that $\varphi \in \mathit{dom}(v)$.

Now suppose $v^{\psi}_{\mathbf{B}}(\sigma(\alpha(s_i) \cup c)) = 1$ for every $1 \leq i \leq n$. We prove that $v^{\psi}_{\mathbf{B}}(\sigma(\alpha(s) \cup c)) = 1$. If $v^{\psi}_{\mathbf{B}}(\sigma(c)) = 1$ then we are clearly done. Suppose otherwise. Then we have $v^{\psi}_{\mathbf{B}}(\sigma(\alpha(S))) = 1$. We prove that $v^{\psi}_{\mathbf{B}}(\sigma(\alpha(s))) = 1$ (it would then follow that $v^{\psi}_{\mathbf{B}}(\sigma(\alpha(s) \cup c)) = 1$). Since $\sigma(\mathit{frm}(\hat{S})) \subseteq \mathit{dom}(v)$, we also have $\sigma(\alpha(\mathit{frm}(S))) \subseteq \mathit{dom}(v)$. Hence, $v(\sigma(\alpha(S))) = 1$. We distinguish two cases. If s is $\Rightarrow \varphi_r$ then since $\sigma(\alpha(\mathit{frm}(S))) \subseteq \mathit{dom}(v)$, $\sigma(\alpha(\varphi_r)) = \psi$ and $v(\sigma(\alpha(S))) = 1$, by Definition 13, we have $v^{\psi}_{\mathbf{B}}(\psi) = 1$, and so $v^{\psi}_{\mathbf{B}}(\sigma(\alpha(s))) = 1$. Otherwise s is $\varphi_r \Rightarrow$. To prove that $v^{\psi}_{\mathbf{B}}(\sigma(\alpha(s))) = 1$, we show that $v^{\psi}_{\mathbf{B}}(\psi) = 0$. By Definition 13, it suffices to prove that for every rule of the form $S' \,/\, \Rightarrow \varphi'$ in $\mathbf{B}$ and substitution σ' such that $\sigma'(\mathit{frm}(S')) \subseteq \mathit{dom}(v)$ and $\sigma'(\varphi') = \psi$, we have

$v(\sigma'(S')) = 0$. Let $S' / \Rightarrow \varphi'$ and σ' as above. Since **B** is (*cut*)-guarded, the empty sequent is derivable from $\sigma(\alpha(S)) \cup \sigma'(S')$ using only (*cut*). It easily follows that $\sigma(\alpha(S)), \sigma'(S') \vdash_{\mathbf{G}'}^{dom(v)} \Rightarrow$. By Theorem 3, since v is $\mathbf{G}'$-legal and $v(\sigma(\alpha(S))) = 1$, we must have $v(\sigma'(S')) = 0$. □

Finally, we obtain Theorem 2 as a corollary:

Proof (of Theorem 2). Let **G** be a calculus that consists solely of k-safe applications of rules of **B**. Begin with the empty calculus and add the rules of **G** one by one. The empty calculus is clearly **B**-k-analytic, and by Theorem 5, in each step we obtain a **B**-k-analytic calculus. By Proposition 7, **G** is k-analytic. □

6 Further Research

While we focused on the language of classical logic for the sake of simplicity and clarity, the definitions and results of this paper can be straightforwardly adapted for arbitrary propositional languages. In addition, the following extensions and questions naturally arise and are left for a future work. First, unlike the case of canonical calculi [4], the relations between cut-elimination and analyticity in pure calculi are still unclear. We plan to apply semantic methods (see, e.g., [9]) to investigate cut-elimination in pure calculi. Second, while this paper studies only pure calculi, that have a simple semantic interpretation, we believe that a similar approach can be useful for more complicated families of sequent calculi. In particular, the family of basic sequent calculi that was studied in [9], and has a Kripke-style semantic interpretation, is an interesting subject for a similar investigation of analyticity. Lastly, it will be interesting and useful to extend the current method also for many-sided sequents (using many-valued valuation functions), as well as for calculi for first-order logics.

References

1. Anderson, A.R., Belnap, N.D.: Entailment: The Logic of Relevance and Neccessity, vol. I. Princeton University Press (1975)
2. Avron, A.: Simple consequence relations. Inf. Comput. 92(1), 105–139 (1991)
3. Avron, A., Konikowska, B., Zamansky, A.: Modular construction of cut-free sequent calculi for paraconsistent logics. In: Proceedings of the 27th Annual IEEE Symposium on Logic in Computer science (LICS), pp. 85–94 (2012)
4. Avron, A., Lev, I.: Non-deterministic multiple-valued structures. Journal of Logic and Computation 15(3), 241–261 (2005)
5. Beklemishev, L., Gurevich, Y.: Propositional primal logic with disjunction. Journal of Logic and Computation 24(1), 257–282 (2014)
6. Béziau, J.-Y.: Sequents and bivaluations. Logique et Analyse 44(176), 373–394 (2001)
7. Cotrini, C., Gurevich, Y.: Basic primal infon logic. Journal of Logic and Computation (2013)

8. Kamide, N.: A hierarchy of weak double negations. Studia Logica 101(6), 1277–1297 (2013)
9. Lahav, O., Avron, A.: A unified semantic framework for fully structural propositional sequent systems. ACM Trans. Comput. Logic 27, 1–33 (2013)
10. Lahav, O., Zohar, Y.: Sat-based decision procedure for analytic pure sequent calculi. To appear in Proceedings of the 7th International Joint Conference on Automated Reasoning, IJCAR (2014)
11. Miller, D., Pimentel, E.: A formal framework for specifying sequent calculus proof systems. Theoretical Computer Science 474(0), 98–116 (2013)
12. Sette, A.M.: On the propositional calculus P1. Mathematica Japonicae 18(13), 173–180 (1973)