Generalized Lagrange Identity for Discrete Symplectic Systems and Applications in Weyl–Titchmarsh Theory

Roman Šimon Hilscher and Petr Zemánek

Abstract In this paper we consider discrete symplectic systems with analytic dependence on the spectral parameter. We derive the Lagrange identity, which plays a fundamental role in the spectral theory of discrete symplectic and Hamiltonian systems. We compare it to several special cases well known in the literature. We also examine the applications of this identity in the theory of Weyl disks and square summable solutions for such systems. As an example we show that a symplectic system with the exponential coefficient matrix is in the limit point case.

1 Introduction

In this paper we consider a $2n$-dimensional discrete symplectic system

$$z_{k+1}(\lambda) = \mathbb{S}_k(\lambda)\, z_k(\lambda), \tag{S_λ}$$

whose coefficient matrix $\mathbb{S}_k(\lambda) \in \mathbb{C}^{2n\times 2n}$ is analytic in the spectral parameter $\lambda \in \mathbb{C}$ in a neighborhood of 0 and satisfies a symplectic-type identity, i.e.,

$$\mathbb{S}_k(\lambda) = \sum_{j=0}^{\infty} \lambda^j \mathscr{S}_k^{[j]}, \quad \mathbb{S}_k^*(\lambda)\, \mathscr{J}\, \mathbb{S}_k(\bar{\lambda}) = \mathscr{J}, \quad \mathscr{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1}$$

R. Šimon Hilscher (✉) · P. Zemánek
Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 61137 Brno, Czech Republic
e-mail: hilscher@math.muni.cz

P. Zemánek
e-mail: zemanekp@math.muni.cz

Z. AlSharawi et al. (eds.), *Theory and Applications of Difference Equations and Discrete Dynamical Systems*, Springer Proceedings in Mathematics & Statistics 102,
DOI 10.1007/978-3-662-44140-4_10

The superscript $*$ denotes the conjugate transpose and $M^*(\lambda) := [M(\lambda)]^*$. For the applications we will in addition assume that a certain weight matrix $\Psi(\lambda) \in \mathbb{C}^{2n\times 2n}$ is positive semidefinite. The terminology "symplectic system" refers to the fact that $\mathbb{S}_k(\lambda)$ and the fundamental matrix of system (S_λ) are complex symplectic (also called conjugate symplectic or $\mathcal{J}$-unitary) when λ is real, i.e., they satisfy the identity $M^*\mathcal{J}M = \mathcal{J}$.

For convenience we write system (S_λ) as two n-dimensional equations with $z_k(\lambda) = (x_k^*(\lambda),\ u_k^*(\lambda))^*$ and $\mathbb{S}_k(\lambda) = \begin{pmatrix} \mathscr{A}_k(\lambda) & \mathscr{B}_k(\lambda) \\ \mathscr{C}_k(\lambda) & \mathscr{D}_k(\lambda) \end{pmatrix}$. System ($\mathrm{S}_\lambda$) was studied in the literature in several special cases. In [2–4, 6, 9] the first equation in (S_λ) does not depend on λ and the second equation is linear in λ, which by [2, Remark 3(iii)] gives the form

$$z_{k+1}(\lambda) = \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k \\ \mathscr{C}_k - \lambda W_k \mathscr{A}_k & \mathscr{D}_k - \lambda W_k \mathscr{B}_k \end{pmatrix} z_k(\lambda), \tag{2}$$

where $\mathscr{S}_k := \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k \\ \mathscr{C}_k & \mathscr{D}_k \end{pmatrix}$ is complex symplectic, W_k is Hermitian, and $W_k \geq 0$. Note that system (2) covers also the classical second order Sturm–Liouville equation

$$-\Delta(R_k \Delta y_k(\lambda)) + Q_k y_{k+1}(\lambda) = \lambda W_k y_{k+1}(\lambda) \tag{3}$$

with Hermitian matrices $R_k, Q_k, W_k \in \mathbb{C}^{n\times n}$, invertible R_k, and $W_k > 0$, see Example 1. System (S_λ) with a general linear dependence on λ

$$z_{k+1}(\lambda) = (\mathscr{S}_k + \lambda \mathscr{V}_k)\, z_k(\lambda) \tag{4}$$

was studied in [18, 19], where the matrix $\mathscr{S}_k$ is complex symplectic, $\mathscr{V}_k^* \mathcal{J} \mathscr{S}_k$ is Hermitian and positive semidefinite, and $\mathscr{V}_k^* \mathcal{J} \mathscr{V}_k = 0$. In [15, 16] the linear Hamiltonian difference system

$$\Delta \begin{pmatrix} x_k(\lambda) \\ u_k(\lambda) \end{pmatrix} = \begin{pmatrix} A_k & B_k + \lambda W_k^{[1]} \\ C_k - \lambda W_k^{[2]} & -A_k^* \end{pmatrix} \begin{pmatrix} x_{k+1}(\lambda) \\ u_k(\lambda) \end{pmatrix} \tag{5}$$

is considered with the matrices $A_k, B_k, C_k, W_k^{[1]}, W_k^{[2]} \in \mathbb{C}^{n\times n}$, $\tilde{A}_k := (I - A_k)^{-1}$ exists, $B_k, C_k, W_k^{[1]}, W_k^{[2]}$ are Hermitian, and $W_k^{[1]} \geq 0$, $W_k^{[2]} \geq 0$. Upon expanding the forward difference in (5), we can verify that system (5) corresponds to a discrete symplectic system (S_λ) with quadratic dependence on λ. Another linear Hamiltonian system

$$\Delta \begin{pmatrix} x_k(\lambda) \\ u_k(\lambda) \end{pmatrix} = \lambda \mathcal{J} H_k \begin{pmatrix} x_{k+1}(\lambda) \\ u_k(\lambda) \end{pmatrix}, \qquad H_k := \begin{pmatrix} -C_k & A_k^* \\ A_k & B_k \end{pmatrix}, \tag{6}$$

with $H_k \in \mathbb{C}^{2n\times 2n}$ Hermitian and $\tilde{A}_k(\lambda) := (I - \lambda A_k)^{-1}$ is studied in [13, 14]. Upon expanding the latter inverse into a power series we get the analytic dependence on λ in the coefficient matrix of system (6). More generally, the linear Hamiltonian system corresponding to

$$\Delta \begin{pmatrix} x_k(\lambda) \\ u_k(\lambda) \end{pmatrix} = \mathscr{J}(H_k^{[0]} + \lambda H_k^{[1]}) \begin{pmatrix} x_{k+1}(\lambda) \\ u_k(\lambda) \end{pmatrix}$$

with Hermitian $H_k^{[0]}, H_k^{[1]} \in \mathbb{C}^{2n\times 2n}$ is considered in [5]. Finally, a discrete symplectic system (S_λ) with analytic dependence on λ and $\mathscr{S}_k^{[0]} = I$ is studied in [7]. The latter paper also motivated the present study.

All the above mentioned references are devoted to various results in the spectral theory of the corresponding system. As it is known, the Lagrange identity plays a fundamental role in these investigations. This identity connects the $\mathscr{J}$-skew-product of two solutions of system (S_λ) with the associated weight matrix $\Psi_k(\lambda)$. In this paper we prove a general form of the Lagrange identity for system (S_λ) with analytic dependence on $\lambda \in \mathbb{C}$ and calculate the corresponding weight matrix explicitly in terms of the coefficients of (S_λ). This result includes the Lagrange identities for the above mentioned special systems. As a consequence we obtain the $\mathscr{J}$-monotonicity of the fundamental matrix $\Phi_k(\lambda)$ of (S_λ), which is used in [7] for proving the Krein traffic rules for the eigenvalues of $\Phi_k(\lambda)$. We also investigate applications of the generalized Lagrange identity in the discrete Weyl–Titchmarsh theory. In particular, we show that under an appropriate Atkinson condition involving the weight matrix $\Psi_k(\lambda)$, the theory of eigenvalues, Weyl disks, and square summable solutions developed in [18, 19] for system (4) remains valid without any change also for system (S_λ) with the analytic dependence on λ.

2 Lagrange Identity

Consider system (S_λ) with complex $2n \times 2n$ matrix $\mathbb{S}_k(\lambda)$ such that (1) holds. The parameter $\lambda \in \mathbb{C}$ is restricted to $|\lambda| < \varepsilon$ for some $\varepsilon > 0$ ($\varepsilon = \infty$ is allowed), which bounds the region of convergence of $\mathbb{S}_k(\lambda)$ in (1) for all $k \in [0,\infty)_\mathbb{Z} := [0,\infty) \cap \mathbb{Z}$. It follows that the matrices $\mathscr{S}_k^{[j]}$, $j \in [0,\infty)_\mathbb{Z}$, satisfy the identities

$$\mathscr{S}_k^{[0]*} \mathscr{J} \mathscr{S}_k^{[0]} = \mathscr{J} \tag{7}$$

$$\sum_{j=0}^{m} \mathscr{S}_k^{[j]*} \mathscr{J} \mathscr{S}_k^{[m-j]} = 0, \qquad m \in \mathbb{N} \tag{8}$$

for all $k \in [0,\infty)_\mathbb{Z}$. We note that $|\det \mathscr{S}_k^{[0]}| = 1$, as the determinant of any complex symplectic matrix is a complex unit. The second identity in (1) implies that $\mathbb{S}_k(\lambda)$ is invertible and hence

$$\mathbb{S}_k^{-1}(\lambda) = -\mathscr{J}\, \mathbb{S}_k^*(\bar{\lambda})\, \mathscr{J} = -\sum_{j=0}^{\infty} \lambda^j\, \mathscr{J}\, \mathscr{S}_k^{[j]*}\, \mathscr{J}.$$

Remark 1 From $\mathbb{S}_k(\lambda)\, \mathbb{S}_k^{-1}(\lambda) = I$ we then obtain the identity $\mathbb{S}_k(\lambda)\, \mathscr{J}\, \mathbb{S}_k^*(\bar{\lambda}) = \mathscr{J}$ or equivalently

$$\mathscr{S}_k^{[0]}\mathscr{J}\mathscr{S}_k^{[0]*} = \mathscr{J}, \qquad \sum_{j=0}^{m} \mathscr{S}_k^{[j]}\mathscr{J}\mathscr{S}_k^{[m-j]*} = 0, \qquad m \in \mathbb{N}.$$

First we study the $\mathscr{J}$-skew product of two coefficient matrices with different values of the spectral parameter. This lemma gives a main tool for the proof of the Lagrange identity.

Lemma 1 *Assume* (7)–(8). *Then for every* $\lambda, \nu \in \mathbb{C}$ *with* $|\lambda| < \varepsilon$, $|\nu| < \varepsilon$,

$$\mathbb{S}_k^*(\lambda)\, \mathscr{J}\, \mathbb{S}_k(\nu) = \mathscr{J} + (\bar{\lambda} - \nu)\, \Omega_k(\bar{\lambda}, \nu),$$

$k \in [0, \infty)_{\mathbb{Z}}$, *where the* $2n \times 2n$ *matrix* $\Omega(\bar{\lambda}, \nu)$ *is defined by*

$$\Omega_k(\bar{\lambda}, \nu) := \sum_{m=0}^{\infty} \sum_{j=0}^{m} \bar{\lambda}^{m-j} \nu^j \sum_{l=0}^{j} \mathscr{S}_k^{[m-l+1]*}\mathscr{J}\mathscr{S}_k^{[l]}. \tag{9}$$

Moreover, for $\nu = \lambda$ *the matrix* $\Omega_k(\bar{\lambda}, \lambda)$ *is Hermitian for all* $k \in [0, \infty)_{\mathbb{Z}}$.

Proof We fix $|\lambda| < \varepsilon$, $|\nu| < \varepsilon$, and $k \in [0, \infty)_{\mathbb{Z}}$. The power series for $\mathbb{S}_k^*(\lambda)$ and $\mathbb{S}_k(\nu)$ converge absolutely, so that the terms in the product $\mathbb{S}_k^*(\lambda)\, \mathscr{J}\, \mathbb{S}_k(\nu)$ can be re-arranged to the separate powers of $\bar{\lambda}^{m-j} \nu^j$, that is,

$$\mathbb{S}_k^*(\lambda)\, \mathscr{J}\, \mathbb{S}_k(\nu) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \bar{\lambda}^{m-j} \nu^j\, \mathscr{S}_k^{[m-j]*}\mathscr{J}\mathscr{S}_k^{[j]}.$$

By using identity (8) for each $m \in \mathbb{N}$, we replace the term $\nu^m \mathscr{S}_k^{[0]*}\mathscr{J}\mathscr{S}_k^{[m]}$ by

$$-\nu^m \left(\mathscr{S}_k^{[m]*}\mathscr{J}\mathscr{S}_k^{[0]} + \mathscr{S}_k^{[m-1]*}\mathscr{J}\mathscr{S}_k^{[1]} + \cdots + \mathscr{S}_k^{[1]*}\mathscr{J}\mathscr{S}_k^{[m-1]}\right).$$

Thus, with the aid of (7) we get

$$\mathbb{S}_k^*(\lambda)\, \mathscr{J}\, \mathbb{S}_k(\nu) = \mathscr{J} + \sum_{m=1}^{\infty} \sum_{j=1}^{m} (\bar{\lambda}^j - \nu^j)\, \nu^{m-j}\, \mathscr{S}_k^{[j]*}\mathscr{J}\mathscr{S}_k^{[m-j]}.$$

Upon factoring $\bar{\lambda} - \nu$ out of each term $\bar{\lambda}^j - \nu^j$ and collecting the remaining products with the same powers of $\bar{\lambda}$ and ν, we get

$$\begin{aligned}\mathbb{S}_k^*(\lambda)\,\mathscr{J}\,\mathbb{S}_k(\nu) &= \mathscr{J} + (\bar{\lambda} - \nu) \sum_{m=1}^{\infty} \sum_{j=1}^{m} \Bigg(\sum_{l=1}^{j} \bar{\lambda}^{j-l} \nu^{l-1} \Bigg) \nu^{m-j} \mathscr{S}_k^{[j]*} \mathscr{J} \mathscr{S}_k^{[m-j]} \\ &= \mathscr{J} + (\bar{\lambda} - \nu)\, \Omega_k(\bar{\lambda}, \nu),\end{aligned}$$

where $\Omega_k(\bar{\lambda}, \nu)$ is given in (9). Finally, for $\nu := \lambda$ we get by using $\mathscr{J}^* = -\mathscr{J}$ and identities (8) that the matrix $\Omega_k(\bar{\lambda}, \lambda)$ is Hermitian. This latter fact is also shown in [7, Proposition 1]. □

The following theorem provides the main result of this section. Its relationship to known discrete Lagrange identities in the literature is discussed in Sect. 3.

Theorem 1 (Lagrange identity) *Assume* (7)–(8) *and fix* $\lambda, \nu \in \mathbb{C}$ *with* $|\lambda| < \varepsilon$, $|\nu| < \varepsilon$. *For any two solutions* $z(\lambda)$ *and* $z(\nu)$ *of systems* (S_λ) *and* (S_ν), *respectively, we have for all* $k \in [0, \infty)_{\mathbb{Z}}$

$$\Delta\big(z_k^*(\lambda)\,\mathscr{J}\,z_k(\nu)\big) = (\bar{\lambda} - \nu)\; z_k^*(\lambda)\,\Omega_k(\bar{\lambda}, \nu)\,z_k(\nu), \tag{10}$$

$$z_{k+1}^*(\lambda)\,\mathscr{J}\,z_{k+1}(\nu) = z_0^*(\lambda)\,\mathscr{J}\,z_0(\nu) + (\bar{\lambda} - \nu) \sum_{j=0}^{k} z_j^*(\lambda)\,\Omega_j(\bar{\lambda}, \nu)\,z_j(\nu). \tag{11}$$

Proof Given that $z_{k+1}(\lambda) = \mathbb{S}_k(\lambda)\,z_k(\lambda)$ and $z_{k+1}(\nu) = \mathbb{S}_k(\nu)\,z_k(\nu)$ for all $k \in [0, \infty)_{\mathbb{Z}}$, we obtain from Lemma 1 that

$$\begin{aligned}\Delta\big(z_k^*(\lambda)\,\mathscr{J}\,z_k(\nu)\big) &= z_k^*(\lambda)\left[\mathbb{S}_k^*(\lambda)\,\mathscr{J}\,\mathbb{S}_k(\nu) - \mathscr{J}\right] z_k(\nu) \\ &= (\bar{\lambda} - \nu)\; z_k^*(\lambda)\,\Omega_k(\bar{\lambda}, \nu)\,z_k(\nu).\end{aligned}$$

The summation of (10) over the interval $[0, k]_{\mathbb{Z}}$ then yields (11). □

Motivated by Lemma 1, we define for $k \in [0, \infty)_{\mathbb{Z}}$ the Hermitian $2n \times 2n$ matrix

$$\Psi_k(\lambda) := \Omega_k(\bar{\lambda}, \lambda) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \bar{\lambda}^{m-j} \lambda^j \sum_{l=0}^{j} \mathscr{S}_k^{[m-l+1]*} \mathscr{J} \mathscr{S}_k^{[l]}. \tag{12}$$

The following identities show that $\Psi_k(\lambda)$ is the correct weight matrix for the spectral theory of system (S_λ), see the examples and applications in Sects. 3 and 4.

Corollary 1 *For every* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \varepsilon$ *and* $k \in [0, \infty)_{\mathbb{Z}}$ *we have*

$$\Delta\big(z_k^*(\lambda)\,\mathscr{J}\,z_k(\lambda)\big) = -2i\,\mathrm{im}(\lambda)\,z_k^*(\lambda)\,\Psi_k(\lambda)\,z_k(\lambda), \tag{13}$$

$$z_{k+1}^*(\lambda)\,\mathscr{J}\,z_{k+1}(\lambda) = z_0^*(\lambda)\,\mathscr{J}\,z_0(\lambda) - 2i\,\mathrm{im}(\lambda)\sum_{j=0}^{k} z_j^*(\lambda)\,\Psi_j(\lambda)\,z_j(\lambda), \tag{14}$$

$$z_k^*(\lambda)\,\mathscr{J}\,z_k(\bar\lambda) = z_0^*(\lambda)\,\mathscr{J}\,z_0(\bar\lambda). \tag{15}$$

Remark 2 When $|\lambda| < \varepsilon$ and $\lambda \in \mathbb{R}$, we have

$$\Psi_k(\lambda) = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\lambda^m\sum_{l=0}^{j}\mathscr{S}_k^{[m-l+1]*}\mathscr{J}\,\mathscr{S}_k^{[l]} = -\mathbb{S}_k^*(\lambda)\,\mathscr{J}\,\dot{\mathbb{S}}_k(\lambda) = \dot{\mathbb{S}}_k^*(\lambda)\,\mathscr{J}\,\mathbb{S}_k(\lambda),$$

where the dot denotes the derivative with respect to λ. The weight matrix

$$\mathscr{J}\,\dot{\mathbb{S}}_k(\lambda)\,\mathscr{J}\,\mathbb{S}_k^*(\lambda)\,\mathscr{J} = \mathbb{S}_k^{*-1}(\lambda)\,\Psi_k(\lambda)\,\mathbb{S}_k^{-1}(\lambda)$$

was used in [11, 17] in the oscillation theory of systems (S_λ) with general nonlinear dependence on λ.

3 Special Examples

In this section we show the connection of the generalized Lagrange identity in Theorem 1 to several special cases known in the literature. We also demonstrate that a positive definite weight matrix $\Psi_k(\lambda)$ can be obtained when $\mathbb{S}_k(\lambda)$ is quadratic in λ.

Example 1 In the simplest case, i.e., for the second order Sturm–Liouville difference equation (3), the Lagrange identity is

$$\Delta\big[y_k^*(\lambda)\,R_k\,\Delta y_k(\nu) - (\Delta y_k^*(\lambda))\,R_k\,y_k(\nu)\big] = (\bar\lambda - \nu)\;y_{k+1}^*(\lambda)\,W_k\,y_{k+1}(\nu),$$

see e.g. [1, Theorem 4.2.1] or [10, Theorem 2.2.3]. This can be seen from (10) and (9), in which $\varepsilon = \infty$, $x_k := y_k$, $u_k := R_k\Delta y_k$, $z_k = (x_k^*,\ u_k^*)^*$, and use the formula $x_{k+1} = x_k + R_k^{-1}u_k$. The coefficient matrix of (S_λ) is $\mathbb{S}_k(\lambda) := \mathscr{S}_k + \lambda\mathscr{V}_k$ with

$$\mathscr{S}_k := \mathscr{S}_k^{[0]} = \begin{pmatrix} I & R_k^{-1} \\ Q_k & I + Q_kR_k^{-1}\end{pmatrix},\quad \mathscr{V}_k := \mathscr{S}_k^{[1]} = -\begin{pmatrix} 0 & 0 \\ W_k & W_kR_k^{-1}\end{pmatrix},$$
$$\Omega(\bar\lambda,\nu) = \mathscr{V}_k^*\mathscr{J}\,\mathscr{S}_k = \big(I,\ R_k^{-1}\big)^*\ W_k\ \big(I,\ R_k^{-1}\big) = \Psi_k(\lambda),$$

and $\mathscr{S}_k^{[j]} := 0$ for $j \ge 2$. Note that Eqs. (7) and (8) with $m \in \{1, 2\}$ hold, since R_k, Q_k, W_k are assumed to be Hermitian.

Example 2 Consider system (4) with general linear dependence on λ. In this case $\mathscr{S}_k^{[0]} := \mathscr{S}_k$, $\mathscr{S}_k^{[1]} := \mathscr{V}_k$, $\mathscr{S}_k^{[j]} := 0$ for $j \geq 2$, $\varepsilon = \infty$, and $\Omega_k(\bar\lambda, \nu) = \mathscr{V}_k^* \mathscr{J} \mathscr{S}_k = \Psi_k(\lambda)$ is constant in λ and Hermitian. Identities (7) and (8) with $m \in \{1, 2\}$ are

$$\mathscr{S}_k^* \mathscr{J} \mathscr{S}_k = \mathscr{J}, \quad \mathscr{S}_k^* \mathscr{J} \mathscr{V}_k + \mathscr{V}_k^* \mathscr{J} \mathscr{S}_k = 0, \quad \mathscr{V}_k^* \mathscr{J} \mathscr{V}_k = 0. \tag{16}$$

The Lagrange identity in (10) is, compare with [19, Theorem 2.6],

$$\begin{aligned} \Delta\big(z_k^*(\lambda)\, \mathscr{J}\, z_k(\nu)\big) &= (\bar\lambda - \nu)\; z_k^*(\lambda)\, \mathscr{V}_k^* \mathscr{J} \mathscr{S}_k\, z_k(\nu) \\ &= (\bar\lambda - \nu)\; z_{k+1}^*(\lambda)\; \mathscr{J} \mathscr{V}_k\, \mathscr{J}\, \mathscr{S}_k^* \mathscr{J}\, z_{k+1}(\nu). \end{aligned} \tag{17}$$

Observe that by (16) the matrix $\mathscr{V}_k$ is singular. Hence, $\Omega_k(\bar\lambda, \nu)$ and $\Psi_k(\lambda)$ are in this case singular as well. Moreover, $\det \mathbb{S}_k(\lambda) = \det \mathscr{S}_k$ and thus $|\det \mathbb{S}_k(\lambda)| = 1$.

Example 3 System (2) represents a special case of Example 2, namely

$$\begin{gathered} \mathscr{S}_k := \mathscr{S}_k^{[0]} = \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k \\ \mathscr{C}_k & \mathscr{D}_k \end{pmatrix}, \quad \mathscr{V}_k := \mathscr{S}_k^{[1]} = -\begin{pmatrix} 0 & 0 \\ W_k \mathscr{A}_k & W_k \mathscr{B}_k \end{pmatrix}, \\ \Omega_k(\bar\lambda, \nu) = \mathscr{V}_k^* \mathscr{J} \mathscr{S}_k = \big(\mathscr{A}_k,\, \mathscr{B}_k\big)^*\; W_k\; \big(\mathscr{A}_k,\, \mathscr{B}_k\big) = \Psi_k(\lambda). \end{gathered}$$

In this case the Lagrange identity in (10) or (17) has the form

$$\Delta\big(z_k^*(\lambda)\, \mathscr{J}\, z_k(\nu)\big) = (\bar\lambda - \nu)\; x_{k+1}^*(\lambda)\, W_k\, x_{k+1}(\nu), \tag{18}$$

where $z(\lambda) = (x^*(\lambda),\; u^*(\lambda))^*$ and $z(\nu) = (x^*(\nu),\; u^*(\nu))^*$. Identity (18) is used in [6, Lemma 2.3] and [4, Lemma 2.6].

Example 4 In this example we discuss the symplectic analogue (or a generalization to the symplectic system) of the linear Hamiltonian system (5), which was studied in [15, 16]. We take system (S_λ) with a special quadratic dependence on λ

$$z_{k+1}(\lambda) = \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k + \lambda \mathscr{A}_k W_k^{[2]} \\ \mathscr{C}_k - \lambda W_k^{[1]} \mathscr{A}_k & \mathscr{D}_k + \lambda \mathscr{C}_k W_k^{[2]} - \lambda W_k^{[1]} (\mathscr{B}_k + \lambda \mathscr{A}_k W_k^{[2]}) \end{pmatrix} z_k(\lambda), \tag{19}$$

where $W_k^{[1]}$ and $W_k^{[2]}$ are Hermitian. That is, $\mathbb{S}_k(\lambda) = \mathscr{S}_k + \lambda \mathscr{V}_k + \lambda^2 \mathscr{W}_k$ with $\varepsilon = \infty$,

$$\begin{gathered} \mathscr{S}_k := \mathscr{S}_k^{[0]} = \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k \\ \mathscr{C}_k & \mathscr{D}_k \end{pmatrix}, \quad \mathscr{W}_k := \mathscr{S}_k^{[2]} = \mathscr{J}\, \widetilde{W}_k \mathscr{S}_k \mathscr{J}\, \widehat{W}_k = \begin{pmatrix} 0 & 0 \\ 0 & -W_k^{[1]} \mathscr{A}_k W_k^{[2]} \end{pmatrix}, \\ \mathscr{V}_k := \mathscr{S}_k^{[1]} = \mathscr{J}\, \widetilde{W}_k \mathscr{S}_k + \mathscr{S}_k \mathscr{J}\, \widehat{W}_k = \begin{pmatrix} 0 & \mathscr{A}_k W_k^{[2]} \\ -W_k^{[1]} \mathscr{A}_k & \mathscr{C}_k W_k^{[2]} - W_k^{[1]} \mathscr{B}_k \end{pmatrix}, \\ \Omega_k(\bar\lambda, \nu) = \widehat{W}_k + (I - \bar\lambda \widehat{W}_k \mathscr{J})\, \mathscr{S}_k^*\, \widetilde{W}_k\, \mathscr{S}_k\, (I + \nu \mathscr{J}\, \widehat{W}_k), \end{gathered}$$

and $\mathscr{S}_k^{[j]} := 0$ for $j \geq 3$. The Hermitian $2n \times 2n$ matrices $\widetilde{W}_k := \operatorname{diag}\{W_k^{[1]}, 0\}$ and $\widehat{W}_k := \operatorname{diag}\{0, W_k^{[2]}\}$ are block diagonal. We can see that in this case $\Psi_k(\lambda) = \Omega_k(\bar{\lambda}, \lambda)$ is Hermitian but no longer constant in λ, as was the case in Examples 1–3. The above coefficients satisfy identities (7) and (8) with $m \in \{1, 2, 3, 4\}$, i.e.,

$$\mathscr{S}_k^* \mathscr{J} \mathscr{S}_k = \mathscr{J}, \qquad \mathscr{S}_k^* \mathscr{J} \mathscr{V}_k + \mathscr{V}_k^* \mathscr{J} \mathscr{S}_k = 0, \qquad \mathscr{V}_k^* \mathscr{J} \mathscr{W}_k + \mathscr{W}_k^* \mathscr{J} \mathscr{V}_k = 0$$
$$\mathscr{S}_k^* \mathscr{J} \mathscr{W}_k + \mathscr{V}_k^* \mathscr{J} \mathscr{V}_k + \mathscr{W}_k^* \mathscr{J} \mathscr{S}_k = 0, \qquad \mathscr{W}_k^* \mathscr{J} \mathscr{W}_k = 0.$$

The Lagrange identity in (10) now reads as

$$\Delta\big(z_k^*(\lambda)\, \mathscr{J} z_k(\nu)\big) = (\bar{\lambda} - \nu)\left[x_{k+1}^*(\lambda)\, W_k^{[1]}\, x_{k+1}(\nu) + u_k^*(\lambda)\, W_k^{[2]}\, u_k(\nu)\right], \tag{20}$$

where $z(\lambda) = (x^*(\lambda),\ u^*(\lambda))^*$ and $z(\nu) = (x^*(\nu),\ u^*(\nu))^*$. Identity (20) can be found in [16, Lemma 2.2]. We note that we can factorize $\mathbb{S}_k(\lambda)$ and $\Omega_k(\bar{\lambda}, \nu)$ as

$$\mathbb{S}_k(\lambda) = \begin{pmatrix} I & 0 \\ -\lambda W_k^{[1]} & I \end{pmatrix} \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k \\ \mathscr{C}_k & \mathscr{D}_k \end{pmatrix} \begin{pmatrix} I & \lambda W_k^{[2]} \\ 0 & I \end{pmatrix},$$
$$\Omega_k(\bar{\lambda}, \nu) = \begin{pmatrix} \mathscr{A}_k^* & 0 \\ \mathscr{B}_k^* + \bar{\lambda} W_k^{[2]} \mathscr{A}_k^* & I \end{pmatrix} \begin{pmatrix} W_k^{[1]} & 0 \\ 0 & W_k^{[2]} \end{pmatrix} \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k + \nu \mathscr{A}_k W_k^{[2]} \\ 0 & I \end{pmatrix}.$$

Therefore, $\det \mathbb{S}_k(\lambda) = \det \mathscr{S}_k$ and $|\det \mathbb{S}_k(\lambda)| = 1$ as in Example 2, and

$$\det \Psi_k(\lambda) = \det \Omega_k(\bar{\lambda}, \nu) = |\det \mathscr{A}_k|^2 \times \det W_k^{[1]} \times \det W_k^{[2]}. \tag{21}$$

Equation (21) shows that the determinant of the weight matrix $\Psi_k(\lambda)$ does not depend on λ. Moreover, $\Psi_k(\lambda)$ is invertible if and only if $\mathscr{A}_k$, $W_k^{[1]}$, $W_k^{[2]}$ are invertible. And in this case the matrix $\Psi_k(\lambda)$ is positive definite if and only if $W_k^{[1]}$ and $W_k^{[2]}$ are positive definite. However, an invertible (positive definite) weight matrix $\Psi_k(\lambda)$ can occur only when system (19) corresponds to a linear Hamiltonian system (5) with invertible (positive definite) $W_k^{[1]}$ and $W_k^{[2]}$, because in this case $\mathscr{A}_k = \tilde{A}_k$ is invertible. The other coefficients of (19) are then given by $\mathscr{B}_k = \tilde{A}_k B_k$, $\mathscr{C}_k = C_k \tilde{A}_k$, and $\mathscr{D}_k = C_k \tilde{A}_k B_k + I - A_k^*$, see [16, Formula (2.3)].

Example 5 Consider the linear Hamiltonian difference system (6), in which

$$Sco_k^{[0]} := I, \quad \mathscr{S}_k^{[1]} := \mathscr{J} H_k, \quad \mathscr{S}_k^{[j]} := \begin{pmatrix} A_k^j & A_k^{j-1} B_k \\ C_k A_k^{j-1} & C_k A_k^{j-2} B_k \end{pmatrix}, \quad j \geq 2,$$
$$\Omega_k(\bar{\lambda}, \nu) = D_k^*(\lambda)\, H_k\, D_k(\nu), \quad D_k(\lambda) := \begin{pmatrix} \tilde{A}_k(\lambda) & \lambda \tilde{A}_k(\lambda)\, B_k \\ 0 & I \end{pmatrix},$$

where $\tilde{A}_k(\lambda) := (I - \lambda A_k)^{-1}$, see [7, p. 5]. Therefore, the dependence of $\mathbb{S}_k(\lambda)$ on λ is analytic with $\varepsilon = \inf\{1/\operatorname{sprad}(A_k),\ k \in [0, \infty)_{\mathbb{Z}}\}$, provided this infimum is positive, where $\operatorname{sprad}(M) = \max\{|\mu|,\ \mu \text{ is an eigenvalue of } M\}$ denotes the spectral radius of M. The Lagrange identity in (10) is, compare with [13, Formula (9)],

$$\Delta\big(z_k^*(\lambda)\,\mathscr{J}\,z_k(\nu)\big) = (\bar\lambda - \nu)\begin{pmatrix} x_{k+1}(\lambda) \\ u_k(\lambda) \end{pmatrix}^{*} H_k \begin{pmatrix} x_{k+1}(\nu) \\ u_k(\nu) \end{pmatrix}.$$

Example 6 In [7, 8], the discrete symplectic system (S_λ) with $\mathscr{S}_k^{[j]} := (1/j!)\, R_k^j$ for $j \in [0,\infty)_{\mathbb{Z}}$ is studied, where $R_k \in \mathbb{C}^{2n\times 2n}$ satisfies $R_k^*\mathscr{J} + \mathscr{J} R_k = 0$. This means that the coefficient matrix $\mathbb{S}_k(\lambda)$ is of exponential type, i.e., $\varepsilon = \infty$ and

$$\mathbb{S}_k(\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}\, R_k^j = \exp(\lambda R_k). \tag{22}$$

By [7, p. 6] or [8, Sect. 2], we then have

$$\Omega_k(\bar\lambda,\nu) = \sum_{j=1}^{\infty} (-1)^j\, \frac{(\bar\lambda-\nu)^{2j-1}}{(2j)!}\,(R_k^*)^j \mathscr{J} R_k^j - \sum_{j=0}^{\infty} (-1)^j\, \frac{(\bar\lambda-\nu)^{2j}}{(2j+1)!}\,(R_k^*)^j \mathscr{J} R_k^{j+1}.$$

The Lagrange identity has the same form as in (10) with the above $\Omega_k(\bar\lambda,\nu)$.

4 Weyl–Titchmarsh Theory

In this section we discuss the applications of the Lagrange identity from Theorem 1 in the Weyl–Titchmarsh theory for system (S_λ) with analytic dependence on λ. We assume that the Hermitian weight matrix $\Psi_k(\lambda)$ defined in (12) satisfies

$$\Psi_k(\lambda) \ge 0, \qquad k \in [0,\infty)_{\mathbb{Z}}. \tag{23}$$

In [19] we have recently developed the Weyl–Titchmarsh theory for system (4), i.e., for system (S_λ) with general linear dependence on λ. In this section we show that most of the results in [19] remain valid also for the analytic dependence on λ, when we modify the corresponding Atkinson-type condition to this more general setting. One of the crucial properties is that the fundamental matrix $\Phi_k(\lambda)$ of (S_λ) satisfies

$$\Phi_k^*(\lambda)\, \mathscr{J}\, \Phi_k(\bar\lambda) = \mathscr{J} \tag{24}$$

for all $k \in [0,\infty)_{\mathbb{Z}}$ whenever (24) holds at the initial point $k = 0$. Note that identity (24) now follows from (15) in Corollary 1. In the subsequent paragraphs we review the most important results, which are in particular connected to the theory of square summable solutions of (S_λ).

In [19] we identified the minimal requirements for the solutions of (S_λ) to satisfy the Atkinson condition. In this way we obtained the weak and strong Atkinson conditions, which are needed for different statements in the Weyl–Titchmarsh theory. For completeness we reformulate these conditions in the setting of this paper. Let $\Phi_k(\lambda) = (Z_k(\lambda),\ \widetilde{Z}_k(\lambda))$ be the partition of the fundamental matrix of system (S_λ) into $2n \times n$ solutions, which are given by the initial conditions $Z_0(\lambda) = \alpha^*$ and $\widetilde{Z}_0(\lambda) = -\mathscr{J}\alpha^*$ for some fixed $\alpha \in \mathbb{C}^{n\times 2n}$ with $\alpha\mathscr{J}\alpha^* = 0$ and $\alpha\alpha^* = I$. The solution $\widetilde{Z}(\lambda)$ is called the natural conjoined basis of (S_λ) and the spectral properties of the associated eigenvalue problem are formulated in terms of this natural conjoined basis. Let us fix for a moment an index $N \in [1,\infty)_{\mathbb{Z}}$.

Hypothesis 1 (*Finite weak Atkinson condition*) For all $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$ and every column $z(\lambda)$ of the natural conjoined basis $\widetilde{Z}(\lambda)$ of (S_λ) we assume that

$$\sum_{k=0}^{N} z_k^*(\lambda)\,\Psi_k(\lambda)\,z_k(\lambda) > 0. \tag{25}$$

If $\beta \in \mathbb{C}^{n\times 2n}$ with $\beta\mathscr{J}\beta^* = 0$ and $\beta\beta^* = I$ is also fixed, then we consider the symplectic eigenvalue problem

$$(\mathrm{S}_\lambda), \quad k \in [0,N]_{\mathbb{Z}}, \quad \alpha\, z_0(\lambda) = 0, \quad \beta\, z_{N+1}(\lambda) = 0. \tag{26}$$

It follows as in [19, Theorem 2.8] that under Hypothesis 1 the eigenvalues of (26) are real, isolated, and they are characterized by $\det\ \beta\,\widetilde{Z}_{N+1}(\lambda) = 0$. The corresponding eigenfunctions are then of the form $\widetilde{Z}(\lambda)\,d$ with nonzero $d \in \operatorname{Ker}\beta\,\widetilde{Z}_{N+1}(\lambda)$.

The $M(\lambda)$-function for system (S_λ) is defined by $M_k(\lambda) := -[\beta\,\widetilde{Z}_k(\lambda)]^{-1}\beta\, Z_k(\lambda)$ and it satisfies the properties in [19, Lemma 2.10 and Theorem 2.13]. In particular, $M_k^*(\lambda) = M_k(\bar{\lambda})$ and $M_k(\lambda)$ is analytic in λ. Define the Weyl solution $\chi(\lambda, M)$ of (S_λ) corresponding to $M \in \mathbb{C}^{n\times n}$ and the Hermitian matrix function $\mathscr{E}(\lambda, M)$ by

$$\chi_k(\lambda, M) := \Phi_k(\lambda)\ \left(I,\ \ M^*\right)^*, \quad \mathscr{E}_k(\lambda, M) := i\,\delta(\lambda)\,\chi_k^*(\lambda, M)\,\mathscr{J}\,\chi_k(\lambda, M), \tag{27}$$

where $\delta(\lambda) := \operatorname{sgn}\operatorname{im}(\lambda)$. The Weyl disk $D_k(\lambda)$ and the Weyl circle $C_k(\lambda)$ are then defined as the sets

$$D_k(\lambda) := \left\{M \in \mathbb{C}^{n\times n},\ \mathscr{E}_k(\lambda, M) \le 0\right\}, \quad C_k(\lambda) := \left\{M \in \mathbb{C}^{n\times n},\ \mathscr{E}_k(\lambda, M) = 0\right\}.$$

It follows that the results in [19, Sect. 3] regarding the Weyl disks and Weyl circles hold exactly in the same form, but now under the following assumption.

Hypothesis 2 (*Infinite weak Atkinson condition*) There exists $N_0 \in \mathbb{N}$ such that for all $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$ every column $z(\lambda)$ of $\widetilde{Z}(\lambda)$ satisfies (25) with $N = N_0$.

We summarize the main properties of the Weyl disks in the following.

Theorem 2 *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda| < \varepsilon$ and suppose that (23) and Hypothesis 2 hold. Then for every $k \geq N_0 + 1$ the Weyl disk and Weyl circle satisfy*

$$D_k(\lambda) = \big\{P_k(\lambda) + R_k(\lambda)\, V R_k(\bar{\lambda}),\ V \in \mathbb{C}^{n\times n},\ V^*V \leq I\big\},$$
$$C_k(\lambda) = \big\{P_k(\lambda) + R_k(\lambda)\, U R_k(\bar{\lambda}),\ U \in \mathbb{C}^{n\times n},\ U^*U = I\big\},$$

where the center $P_k(\lambda)$ and the matrix radius $R_k(\lambda)$ are defined by

$$P_k(\lambda) := -\mathscr{H}_k^{-1}(\lambda)\,\mathscr{G}_k(\lambda), \qquad R_k(\lambda) := \mathscr{H}_k^{-1/2}(\lambda) \tag{28}$$

with $\mathscr{H}_k(\lambda)$ and $\mathscr{G}_k(\lambda)$ given by $\mathscr{H}_k(\lambda) := i\,\delta(\lambda)\,\widetilde{Z}_k^*(\lambda)\,\mathscr{J}\,\widetilde{Z}_k(\lambda)$ and $\mathscr{G}_k(\lambda) := i\,\delta(\lambda)\,\widetilde{Z}_k^*(\lambda)\,\mathscr{J}\,Z_k(\lambda)$. Moreover, the Weyl disks $D_k(\lambda)$ are closed, convex, and $D_{k+1}(\lambda) \subseteq D_k(\lambda)$ for all $k \geq N_0 + 1$.

Proof The proof follows the same arguments as in [19, Theorem 3.8]. We note that by (14) in Corollary 1 we have

$$\mathscr{H}_k(\lambda) = 2\,|\mathrm{im}(\lambda)| \sum_{j=0}^{k-1} \widetilde{Z}_j^*(\lambda)\,\Psi_j(\lambda)\,\widetilde{Z}_j(\lambda). \tag{29}$$

This shows that under Hypothesis 2 the matrices $\mathscr{H}_k(\lambda)$ are Hermitian and positive definite for $k \geq N_0 + 1$, so that the center $P_k(\lambda)$ and the matrix radius $R_k(\lambda)$ are well defined. □

The properties of the Weyl disks $D_k(\lambda)$ in Theorem 2 and formula (29) imply that for $k \to \infty$ there exists the limiting Weyl disk $D_+(\lambda) := \bigcap_{k \geq N_0+1} D_k(\lambda)$, which is closed and convex and which satisfies

$$D_+(\lambda) = \big\{P_+(\lambda) + R_+(\lambda)\, V R_+(\bar{\lambda}),\ V \in \mathbb{C}^{n\times n},\ V^*V \leq I\big\},$$

where the limiting center and the limiting matrix radius are complex $n \times n$ matrices

$$P_+(\lambda) := \lim_{k\to\infty} P_k(\lambda), \qquad R_+(\lambda) := \lim_{k\to\infty} R_k(\lambda) \geq 0,$$

compare with [19, Theorem 3.9 and Corollary 3.11]. The elements of the limiting Weyl disk are characterized in the following result.

Theorem 3 *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda| < \varepsilon$ and suppose that (23) and Hypothesis 2 hold. The matrix $M \in \mathbb{C}^{n\times n}$ belongs to $D_+(\lambda)$ if and only if*

$$\sum_{k=0}^{\infty} \chi_k^*(\lambda, M)\,\Psi_k(\lambda)\,\chi_k(\lambda, M) \leq \frac{\mathrm{im}(M)}{\mathrm{im}(\lambda)}, \tag{30}$$

where $\chi(\lambda, M)$ is the Weyl solution of (S_λ) corresponding to M defined in (27).

Proof The proof follows by applying identity (14) in Corollary 1 to $\mathcal{E}_k(\lambda, M)$, see also [19, Corollary 3.12]. □

We now discuss the number of square summable solutions of (S_λ) with analytic dependence on λ. As the weight matrix $\Psi_k(\lambda)$ now depends on λ, we define for $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$ the semi-inner product and the semi-norm

$$\langle z, \widetilde{z} \rangle_{\Psi(\lambda)} := \sum_{k=0}^{\infty} z_k^* \, \Psi_k(\lambda) \, \widetilde{z}_k, \quad \|z\|_{\Psi(\lambda)} := \sqrt{\langle z, z \rangle_{\Psi(\lambda)}} = \bigg(\sum_{k=0}^{\infty} z_k^* \, \Psi_k(\lambda) \, z_k \bigg)^{1/2},$$

and the corresponding space of all square summable sequences with respect to $\Psi(\lambda)$

$$\ell^2_{\Psi(\lambda)} := \big\{ \{z_k\}_{k=0}^{\infty}, \; z_k \in \mathbb{C}^{2n}, \; \|z\|_{\Psi(\lambda)} < \infty \big\}. \tag{31}$$

Observe that the space $\ell^2_{\Psi(\lambda)}$ now also depends on λ. However, in some special cases this space can be taken independent on λ, as it is shown for systems (2), (3), (4) in Examples 1–3. Also, in view of (20) in Example 4 we may consider for systems (19) or (5) the space

$$\ell^2_{W^{[1]}, W^{[2]}} := \bigg\{ \{z_k = (x_k^*, u_k^*)^*\}_{k=0}^{\infty}, \; \sum_{k=0}^{\infty} \big(x_{k+1}^* \, W_k^{[1]} x_{k+1} + u_k^* \, W_k^{[2]} u_k \big) < \infty \bigg\},$$

which does not depend on λ. Given the space $\ell^2_{\Psi(\lambda)}$ in (31), its subspace of all square summable solutions of (S_λ) is denoted by

$$\mathcal{N}(\lambda) := \big\{ z \in \ell^2_{\Psi(\lambda)}, \; z = \{z_k\}_{k=0}^{\infty} \text{ solves} (S_\lambda) \big\}.$$

Under assumption (23) and Hypothesis 2, the result in Theorem 3 implies that $n \le \dim \mathcal{N}(\lambda) \le 2n$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda| < \varepsilon$, see also [19, Theorem 4.2]. The two extreme cases are then called as the limit point case when $\dim \mathcal{N}(\lambda) = n$, and the limit circle case when $\dim \mathcal{N}(\lambda) = 2n$. The cases when $\dim \mathcal{N}(\lambda)$ is between $n + 1$ and $2n - 1$ are called intermediate. It follows that the results in [19, Theorem 4.2, Corollary 4.15] hold for system (S_λ) with analytic dependence on λ in exactly the same form under the appropriate weak or strong Atkinson type condition. We summarize the main result regarding the number of linearly independent square summable solutions of (S_λ) in the following theorem.

Theorem 4 *Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$ with $|\lambda| < \varepsilon$ and suppose that (23) and Hypothesis 2 hold. Then system (S_λ) has exactly $n + \text{rank}\, R_+(\lambda)$ linearly independent square summable solutions, i.e.,*

$$\dim \mathcal{N}(\lambda) = n + \text{rank}\, R_+(\lambda),$$

where $R_+(\lambda)$ is the matrix radius of the limiting Weyl disk $D_+(\lambda)$.

Proof We refer to the proof of [19, Theorem 4.9] for the details. □

As a consequence of Theorem 4 we obtain the characterization of the limit point case and limit circle case for system (S_λ) with analytic dependence on λ in terms of the limiting matrix radius $R_+(\lambda)$.

Corollary 2 *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda| < \varepsilon$ and suppose that (23) and Hypothesis 2 hold. System (S_λ) is in the limit point case if and only if $R_+(\lambda) = 0$, and in this case $D_+(\lambda) = \{P_+(\lambda)\}$ and $D_+(\bar{\lambda}) = \{P_+(\bar{\lambda})\}$. System (S_λ) is in the limit circle case if and only if $R_+(\lambda)$ is invertible.*

In a similar way, the results in [18] regarding the Weyl–Titchmarsh theory for system (S_λ) with jointly varying endpoints remain valid also for the analytic dependence on λ, when we assume the following finite or infinite strong Atkinson type condition.

Hypothesis 3 (*Finite strong Atkinson condition*) For all $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$, every nontrivial solution $z(\lambda)$ of (S_λ) satisfies (25).

Hypothesis 4 (*Infinite strong Atkinson condition*) There exists $N_0 \in \mathbb{N}$ such that for all $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$ every nontrivial solution $z(\lambda)$ of (S_λ) satisfies (25) with $N = N_0$.

We illustrate the Weyl–Titchmarsh theory of system (S_λ) with analytic dependence on λ by the following interesting example.

Example 7 In this example we show that the discrete symplectic system

$$z_{k+1}(\lambda) = \exp(\lambda \mathscr{J})\, z_k(\lambda). \tag{32}$$

is in the limit point case for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, we calculate the unique $2n \times n$ solution (up to an invertible multiple) of (32) whose columns lie in $\ell^2_{\Psi(\lambda)}$ and thus form a basis of $\mathcal{N}(\lambda)$. System (32) is an exponential symplectic system from Example 6, where $\varepsilon = \infty$, $\mathbb{S}_k(\lambda) := \exp(\lambda \mathscr{J})$ is given in (22) with $R_k := \mathscr{J}$. This matrix satisfies the conditions $R_k^* \mathscr{J} + \mathscr{J} R_k = 0$ in Example 6, so that by (22) we have $\mathbb{S}_k(\lambda) = \exp(\lambda \mathscr{J}) = (\cos \lambda)\, I + (\sin \lambda)\, \mathscr{J}$. Note that this matrix does not depend on k.

For simplicity we perform the calculations below in the scalar case, i.e., for $n = 1$. The general case follows with the same arguments upon multiplication by the $n \times n$ or $2n \times 2n$ identity matrices at appropriate places. The fundamental matrix $\Phi_k(\lambda)$ of (32) with $\Phi_0(\lambda) = I$ is given by

$$\Phi_k(\lambda) = \exp(k\lambda \mathscr{J}) = (\cos k\lambda)\, I + (\sin k\lambda)\, \mathscr{J} = \begin{pmatrix} \cos k\lambda & \sin k\lambda \\ -\sin k\lambda & \cos k\lambda \end{pmatrix}$$

for every $k \in [0, \infty)_\mathbb{Z}$. Since $\Phi_0(\lambda) = I$, we take $\alpha := (1,\ 0)$, so that $\alpha \mathscr{J} \alpha^* = 0$ and $\alpha\alpha^* = 1$ are satisfied. It follows that the natural conjoined basis of (32) is determined by the second column of $\Phi(\lambda)$, i.e., $\widetilde{Z}_k(\lambda) = ((\cos k\lambda)^*,\ (\sin k\lambda)^*)^*$. Since the powers of $\mathscr{J}$ repeat in a cycle of length four, the weight matrix $\Psi_k(\lambda) = \Omega(\bar{\lambda}, \lambda)$ is given in Example 6 as (we substitute $x := \mathrm{im}(\lambda)$)

$$\begin{aligned}\Psi_k(\lambda) &= \frac{1}{2x} \sum_{j=0}^{\infty} \frac{(2x)^{2j+1}}{(2j+1)!}\, I + \frac{1}{2x} \sum_{j=1}^{\infty} \frac{(2x)^{2j}}{(2j)!}\, i\mathscr{J} = \frac{\sinh 2x}{2x}\, I + \frac{\cosh 2x - 1}{2x}\, i\mathscr{J} \\ &= \frac{\sinh x}{x} \big[(\cosh x)\, I + (\sinh x)\, i\mathscr{J}\big] = \frac{\sinh x}{x} \begin{pmatrix} \cosh x & i \sinh x \\ -i \sinh x & \cosh x \end{pmatrix} > 0,\end{aligned}$$

where we used the formulas for hyperbolic functions $\sinh 2x = 2 \sinh x \cosh x$, $\cosh 2x = 2\cosh^2 x - 1$, and the identity $\cosh^2 x - \sinh^2 x = 1$. By the definition of $\mathscr{H}_k(\lambda)$ and $\mathscr{G}_k(\lambda)$ in Theorem 2,

$$\begin{aligned}\mathscr{H}_k(\lambda) &= i\,\delta(\lambda)\, (\sin k\bar{\lambda}\, \cos k\lambda - \cos k\bar{\lambda}\, \sin k\lambda) = \sinh\left(2k\, |\mathrm{im}(\lambda)|\right), \\ \mathscr{G}_k(\lambda) &= -i\,\delta(\lambda)\, (\sin k\bar{\lambda}\, \sin k\lambda + \cos k\bar{\lambda}\, \cos k\lambda) = -i\,\delta(\lambda)\, \cosh\left(2k\, \mathrm{im}(\lambda)\right),\end{aligned}$$

where we used the identities $\cosh x = \cos ix$ and $i \sinh x = \sin ix$ relating the hyperbolic and trigonometric functions. The same value for $\mathscr{H}_k(\lambda)$ is of course obtained from formula (29) after some calculations. Therefore, Hypothesis 2 is satisfied with $N_0 = 1$, and by (28)

$$P_k(\lambda) = i\, \coth\left(2k\, \mathrm{im}(\lambda)\right), \qquad R_k(\lambda) = 1/\sqrt{\sinh\left(2k\, |\mathrm{im}(\lambda)|\right)}.$$

The center and radius of the limiting disk $D_+(\lambda)$ are then

$$P_+(\lambda) = \lim_{k\to\infty} P_k(\lambda) = i\,\delta(\lambda), \qquad R_+(\lambda) = \lim_{k\to\infty} R_k(\lambda) = 0,$$

so that system (32) is in the limit point case for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. From Corollary 2 and Theorem 3 we obtain that $\dim \mathcal{N}(\lambda) = 1$, and the space $\mathcal{N}(\lambda)$ of square integrable solutions of system (32) is generated by the Weyl solution

$$\begin{aligned}\chi_k(\lambda, P_+(\lambda)) &= \Phi_k(\lambda) \begin{pmatrix} I \\ P_+(\lambda) \end{pmatrix} = \begin{pmatrix} \cos k\lambda + i\,\delta(\lambda)\, \sin k\lambda \\ -\sin k\lambda + i\,\delta(\lambda)\, \cos k\lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ i\,\delta(\lambda) \end{pmatrix} e^{i\,\delta(\lambda)\, k\lambda},\end{aligned}$$

for which (we again substitute $x := \mathrm{im}(\lambda)$)

$$\begin{aligned}\|\, \chi(\lambda, P_+(\lambda))\, \|_{\Psi(\lambda)}^2 &= \sum_{k=0}^{\infty} \chi_k^*(\lambda, P_+(\lambda))\, \Psi_k(\lambda)\, \chi_k(\lambda, P_+(\lambda)) \\ &= \frac{2 \sinh x}{x} \times [\cosh x - \delta(\lambda)\, \sinh x] \times \sum_{k=0}^{\infty} e^{-2\,|x|k}\end{aligned}$$

$$= \frac{2\sinh x}{x} \times [\cosh x - \delta(\lambda)\sinh x] \times \frac{1}{1-e^{-2|x|}} = \frac{1}{|x|}.$$

This shows that $\|\chi(\lambda, P_+(\lambda))\|_{\Psi(\lambda)} = 1/\sqrt{|\mathrm{im}(\lambda)|} < \infty$, and so indeed we have $\chi(\lambda, P_+(\lambda)) \in \ell^2_{\Psi(\lambda)}$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. On the other hand, we also have

$$\|\widetilde{Z}(\lambda)\|^2_{\Psi(\lambda)} = \sum_{k=0}^{\infty} \widetilde{Z}_k^*(\lambda)\,\Psi_k(\lambda)\,\widetilde{Z}_k(\lambda) \overset{(29)}{=} \frac{1}{2\,|\mathrm{im}(\lambda)|} \lim_{k\to\infty} \mathscr{H}_k(\lambda)$$
$$= \frac{1}{2\,|\mathrm{im}(\lambda)|} \lim_{k\to\infty} \sinh(2k\,|\mathrm{im}(\lambda)|) = \infty,$$

so that $\tilde{Z}(\lambda) \notin \ell^2_{\Psi(\lambda)}$. Thus, again we get that $\dim \mathcal{N}(\lambda) = 1$. Similarly, in arbitrary dimension n we get that the n columns of the Weyl solution $\chi(\lambda, P_+(\lambda))$ are linearly independent and they belong to $\ell^2_{\Psi(\lambda)}$, while the n columns of the natural conjoined basis $\widetilde{Z}(\lambda)$ are linearly independent and they do not belong to $\ell^2_{\Psi(\lambda)}$. Hence, $\dim \mathcal{N}(\lambda) = n$ and system (32) is in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Finally, as a consequence of (14) we obtain the $\mathscr{J}$-monotonicity of the fundamental matrix of (S_λ). We recall the terminology from [12, p. 7] saying that a matrix $M \in \mathbb{C}^{2n\times 2n}$ is $\mathscr{J}$-nondecreasing if $i\,M^*\mathscr{J}M \geq i\mathscr{J}$, and it is $\mathscr{J}$-nonincreasing if $i\,M^*\mathscr{J}M \leq i\mathscr{J}$. Similarly we define the corresponding notions of a $\mathscr{J}$-increasing and $\mathscr{J}$-decreasing matrix. These concepts are used in [12] to study the stability zones for continuous time periodic linear Hamiltonian systems. In a similar way, such stability zones are studied in [13, 14] for discrete linear Hamiltonian systems (6) and in [7] for discrete symplectic systems (S_λ) with $\mathscr{S}_k^{[0]} = I$.

Corollary 3 *Fix $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$ and assume (23). Let $\Phi(\lambda)$ be a fundamental matrix of system (S_λ) such that $\Phi_0(\lambda)$ is complex symplectic, i.e., $\Phi_0^*(\lambda)\,\mathscr{J}\,\Phi_0(\lambda) = \mathscr{J}$. Then for every $k \in [0,\infty)_{\mathbb{Z}}$ the matrix $\Phi_k(\lambda)$ is $\mathscr{J}$-nondecreasing or $\mathscr{J}$-nonincreasing depending on whether $\mathrm{im}(\lambda) > 0$ or $\mathrm{im}(\lambda) < 0$. Moreover, under Hypothesis 4 the $\mathscr{J}$-monotonicity of $\Phi_k(\lambda)$ is strict for $k \geq N_0 + 1$.*

Proof By applying (14) to the fundamental matrix $\Phi_k(\lambda)$ we get

$$i\,\Phi_k^*(\lambda)\,\mathscr{J}\,\Phi_k(\lambda) - i\mathscr{J} = 2\,\mathrm{im}(\lambda) \sum_{j=0}^{k-1} \Phi_j^*(\lambda)\,\Psi_j(\lambda)\,\Phi_j(\lambda). \tag{33}$$

By (23), the sum in (33) is nonnegative, so that $\Phi_k(\lambda)$ is $\mathscr{J}$-nondecreasing when $\mathrm{im}(\lambda) > 0$, and it is $\mathscr{J}$-nonincreasing when $\mathrm{im}(\lambda) < 0$. Moreover, under Hypothesis 4 the sum in (33) is positive definite for $k \geq N_0+1$, so that $\Phi_k(\lambda)$ is $\mathscr{J}$-increasing when $\mathrm{im}(\lambda) > 0$, and it is $\mathscr{J}$-decreasing when $\mathrm{im}(\lambda) < 0$. □

Acknowledgments The first author was supported by the Czech Science Foundation under grant P201/10/1032. The second author was supported by the Program of "Employment of Newly Graduated Doctors of Science for Scientific Excellence" (grant number CZ.1.07/2.3.00/30.0009) co-financed from European Social Fund and the state budget of the Czech Republic.

References

1. Atkinson, F.V.: Discrete and Continuous Boundary Problems. Academic Press, London (1964)
2. Bohner, M., Došlý, O., Kratz, W.: An oscillation theorem for discrete eigenvalue problems. Rocky Mt. J. Math. **33**(4), 1233–1260 (2003)
3. Bohner, M., Došlý, O., Kratz, W.: Sturmian and spectral theory for discrete symplectic systems. Trans. Amer. Math. Soc. **361**(6), 3109–3123 (2009)
4. Bohner, M., Sun, S.: Weyl–Titchmarsh theory for symplectic difference systems. Appl. Math. Comput. **216**(10), 2855–2864 (2010)
5. Clark, S., Gesztesy, F.: On Weyl–Titchmarsh theory for singular finite difference Hamiltonian systems. J. Comput. Appl. Math. **171**(1–2), 151–184 (2004)
6. Clark, S., Zemánek, P.: On a Weyl–Titchmarsh theory for discrete symplectic systems on a half line. Appl. Math. Comput. **217**(7), 2952–2976 (2010)
7. Došlý, O.: Symplectic difference systems with periodic coefficients: Krein's traffic rules for multipliers. Adv. Differ. Equ. **2013**(85), 13 (2013)
8. Došlý, O.: Symplectic difference systems: natural dependence on a parameter. In: Proceedings of the International Conference on Differential Equations, Difference Equations, and Special Functions (Patras, 2012), Adv. Dyn. Syst. Appl., **8**(2), 193–201 (2013)
9. Došlý, O., Kratz, W.: Oscillation theorems for symplectic difference systems. J. Differ. Equ. Appl. **13**(7), 585–605 (2007)
10. Jirari, A.: Second order Sturm-Liouville difference equations and orthogonal polynomials. Mem. Amer. Math. Soc. **113**(542), 138 (1995)
11. Kratz, W., Šimon Hilscher, R.: A generalized index theorem for monotone matrix-valued functions with applications to discrete oscillation theory. SIAM J. Matrix Anal. Appl. **34**(1), 228–243 (2013)
12. Krein, M. G.: Foundations of the theory of λ-zones of stability of a canonical system of linear differential equations with periodic coefficients. In: Leifman, L.J. (ed.) Four Papers on Ordinary Differential Equations, Amer. Math. Soc. Transl. (2), Vol. 120, pp. 1–70, AMS (1983)
13. Răsvan, V.: Stability zones for discrete time Hamiltonian systems. Arch. Math. (Brno) **36**(5), 563–573 (2000)
14. Răsvan, V.: On stability zones for discrete-time periodic linear Hamiltonian systems. Adv. Differ. Equ. **2006**, Art. 80757, 1–13 (2006)
15. Shi, Y.: Spectral theory of discrete linear Hamiltonian systems. J. Math. Anal. Appl. **289**(2), 554–570 (2004)
16. Shi, Y.: Weyl–Titchmarsh theory for a class of discrete linear Hamiltonian systems. Linear Algebra Appl. **416**, 452–519 (2006)
17. Šimon Hilscher, R.: Oscillation theorems for discrete symplectic systems with nonlinear dependence in spectral parameter. Linear Algebra Appl. **437**(12), 2922–2960 (2012)
18. Šimon Hilscher, R., Zemánek, P.: Weyl disks and square summable solutions for discrete symplectic systems with jointly varying endpoints. Adv. Differ. Equ. **2013**(232), 18 (2013)
19. Šimon Hilscher, R., Zemánek, P.: Weyl–Titchmarsh theory for discrete symplectic systems with general linear dependence on spectral parameter. J. Differ. Equ. Appl **20**(1), 84–117 (2014)