A Generalization of Modal Frame Definability

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Abstract. A class of Kripke frames is called modally definable if there is a set of modal formulas such that the class consists exactly of frames on which every formula from that set is valid, i.e. globally true under any valuation. Here, existential definability of Kripke frame classes is defined analogously, by demanding that each formula from a defining set is satisfiable under any valuation. The notion of modal definability is then generalized by combining these two. Model theoretic characterizations of these types of definability are given.

Keywords: modal logic, model theory, modal definability.

1 Introduction

Some questions about the power of modal logic to express properties of relational structures are addressed in this paper. One way to determine the expressive power of a language is to establish a model theoretic characterization of properties definable in that language. Such characterizations depend not only on language, but also on a choice of semantics.

Only the Kripke semantics is considered in this paper. Even so, we have several perspectives on the meaning of modal formulas: we distinguish between their truth at some designated world, global truth on a model, and validity on a frame. Because of this, model theory provides several characterizations of modal definability, which answer to the following questions: which properties of Kripke frames (Goldblatt-Thomason [4], see also [1]), Kripke models (de Rijke and Sturm [3]), and pointed models (de Rijke, see [1]), are expressible in modal logic.

Moreover, on the level of Kripke models, we can also use the notion of satisfiability, which is dual to the global truth. In [9] the notion of existential definability of Kripke model classes (or properties) is defined as follows: a class is existentially definable if there is a set of formulas such that this class consists exactly of models in which every formula from that set is satisfiable. In [8] we combine the usual (universal) and existential definability to obtain further generalizations and we prove model theoretic characterizations for these types of definability.

The aim of this paper is to provide similar generalizations for the level of Kripke frames. Since we abstract away from the effect of the valuations, frames are the most natural semantic level for one of the basic purposes of modal logic: to express properties of accessibility relation. So, it is of interest to get a broader perspective on modal frame definability.

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As it turns out, an appropriate notion of existential definability of Kripke frame classes demands that each formula from a defining set is satisfiable under any valuation. This is equivalent to the definability by the existential fragment of modal language enriched with the universal modality, similarly as it is on the level of models (see [9] and [8]). A generalized notion of modal definability, which is defined exactly like in the case of models, by combining universal and existential definability, also corresponds to a fragment of this language.

This paper provides two characterizations at the level of frames: one for the existential definability and one for the generalized definability. Characterizations are obtained by similar proof techniques as for the definability in the usual sense, which means that saturated models (ultraproducts and ultrafilter extensions) are deeply involved in the results. Characterization theorems are useful for obtaining non-definability results, some of which are given in Section 5. These examples also show that the conditions in the characterizations are necessary.

2 Preliminaries

For the sake of notational simplicity, only the basic modal language is considered in this paper, with the exception of few remarks concerning the universal modality.

Let Φ be a set of *propositional variables.* The syntax of the *basic propositional modal language* (BML) is given by

$$\varphi ::= p \mid \bot \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \Diamond\varphi,$$

where $p \in \Phi$. We define other connectives and $\Box$ as usual. Namely, $\Box\varphi := \neg\Diamond\neg\varphi$.

The basic notions and results on the Kripke semantics are only briefly recalled here (see [1] for details if needed). A *Kripke frame* for the basic modal language is a relational structure $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and $R \subseteq W \times W$. A *Kripke model* based on a frame $\mathfrak{F}$ is $\mathfrak{M} = (W, R, V)$, where $V : \Phi \to 2^W$ is a mapping called *valuation.* For $w \in W$, we call $(\mathfrak{M}, w)$ a *pointed model.*

The *truth* of a formula is defined locally and inductively as usual, and denoted $\mathfrak{M}, w \Vdash \varphi$. Namely, a formula of a form $\Diamond\varphi$ is *true* at $w \in W$ if $\mathfrak{M}, u \Vdash \varphi$ for some u such that Rwu. A valuation is naturally extended to all modal formulas by putting $V(\varphi) = \{w \in W : \mathfrak{M}, w \Vdash \varphi\}$.

We say that a formula is *globally true* on $\mathfrak{M}$ if it is true at every $w \in W$, and we denote this by $\mathfrak{M} \Vdash \varphi$. On the other hand, a formula is called *satisfiable* in $\mathfrak{M}$ if it is true at some $w \in W$.

If a formula φ is true at w under any valuation on a frame $\mathfrak{F}$, we write $\mathfrak{F}, w \Vdash \varphi$. We say that a formula is *valid* on a frame $\mathfrak{F}$ if we have $\mathfrak{M} \Vdash \varphi$ for any model $\mathfrak{M}$ based on $\mathfrak{F}$. This is denoted $\mathfrak{F} \Vdash \varphi$. For a set Σ of formulas we write $\mathfrak{F} \Vdash \Sigma$ if $\mathfrak{F} \Vdash \varphi$ for all $\varphi \in \Sigma$. A class $\mathcal{K}$ of Kripke frames is *modally definable* if there is Σ such that $\mathcal{K}$ consists exactly of frames on which every formula from Σ is valid, i.e. $\mathcal{K} = \{\mathfrak{F} : \mathfrak{F} \Vdash \Sigma\}$. If this is the case, we say that $\mathcal{K}$ is *defined* by Σ and denote $\mathcal{K} = \mathrm{Fr}(\Sigma)$.

Model theoretic closure conditions that are necessary and sufficient for an elementary class of frames (i.e. first-order definable property of relational structures) to be modally definable are given by the famous Goldblatt-Thomason Theorem.

Theorem (Goldblatt-Thomason [4]). *An elementary class $\mathcal{K}$ of frames is definable by a set of modal formulas if and only if $\mathcal{K}$ is closed under surjective bounded morphisms, disjoint unions and generated subframes, and reflects ultrafilter extensions.*

All of the frame constructions used in the theorem – bounded morphisms, disjoint unions, generated subframes and ultrafilter extensions – are presented briefly in Section 4 (see [1] for more details if needed). Just to be clear, we say that a class $\mathcal{K}$ *reflects* a construction if its complement $\mathcal{K}^c$, that is the class of all Kripke frames not in $\mathcal{K}$, is closed under that construction.

Now, an alternative notion of definability is proposed here as follows.

Definition 1. A class $\mathcal{K}$ of Kripke frames is called *modally* $\exists$*-definable* if there is a set Σ of modal formulas such that for any Kripke frame $\mathfrak{F}$ we have: $\mathfrak{F} \in \mathcal{K}$ if and only if each $\varphi \in \Sigma$ is satisfiable in $\mathfrak{M}$, for every model $\mathfrak{M}$ based on $\mathfrak{F}$. If this is the case, we denote $\mathcal{K} = \mathrm{Fr}_{\exists}(\Sigma)$.

The definition does not require that all formulas of Σ are satisfied at the same point – it suffices that each formula of Σ is satisfied at some point.

In the following, a notation $\mathrm{Mod}(F)$ is used for a class of structures defined by a first-order formula F. Similarly, if $\Sigma = \{\varphi\}$ is a singleton set of modal formulas, we write $\mathrm{Fr}_{\exists}(\varphi)$ instead of $\mathrm{Fr}_{\exists}(\{\varphi\})$.

Example 1. It is well-known that the formula $p \to \Diamond p$ defines reflexivity, i.e. $\mathrm{Fr}(p \to \Diamond p) = \mathrm{Mod}(\forall x Rxx)$. Now, it is easy to see that $\mathrm{Fr}_{\exists}(p \to \Diamond p)$ is the class of all frames such that $R \neq \emptyset$, that is $\mathrm{Fr}_{\exists}(p \to \Diamond p) = \mathrm{Mod}(\exists x \exists y Rxy)$. This class is not modally definable in the usual sense, since it is clearly not closed under generated subframes. Note that the condition $R \neq \emptyset$ is $\exists$-definable also by a simpler formula $\Diamond\top$.

Next, we define a notion which generalizes both universal and existential definability.

Definition 2. A class $\mathcal{K}$ of Kripke frames is called *modally* $\forall\exists$*-definable* if there is a pair (Σ_1, Σ_2) of sets of modal formulas such that for any Kripke frame $\mathfrak{F}$ we have: $\mathfrak{F} \in \mathcal{K}$ if and only if each $\varphi \in \Sigma_1$ is valid on $\mathfrak{F}$ and each $\varphi \in \Sigma_2$ is satisfiable in $\mathfrak{M}$, for any model $\mathfrak{M}$ based on $\mathfrak{F}$, i.e. $\mathcal{K} = \mathrm{Fr}(\Sigma_1) \cap \mathrm{Fr}_{\exists}(\Sigma_2)$.

Model theoretic characterizations of these notions are given in Section 5.

3 First and Second-Order Standard Translations

The starting point of correspondence between first-order and modal logic is the *standard translation*, a mapping that translates each modal formula φ to the first-order formula $\mathrm{ST}_x(\varphi)$, as follows:

$\mathrm{ST}_x(p) = Px$, for each $p \in \Phi$,
$\mathrm{ST}_x(\bot) = \bot$,
$\mathrm{ST}_x(\neg\varphi) = \neg\,\mathrm{ST}_x(\varphi)$,
$\mathrm{ST}_x(\varphi \vee \psi) = \mathrm{ST}_x(\varphi) \vee \mathrm{ST}_x(\psi)$,
$\mathrm{ST}_x(\Diamond\varphi) = \exists y(Rxy \wedge \mathrm{ST}_y(\varphi))$.

Clearly, we have $\mathfrak{M}, w \Vdash \varphi$ if and only if $\mathfrak{M} \models \mathrm{ST}_x(\varphi)[w]$, and $\mathfrak{M} \Vdash \varphi$ if and only if $\mathfrak{M} \models \forall x\,\mathrm{ST}_x(\varphi)$. But, validity of a formula on a frame generally is not first-order expressible, since we need to quantify over valuations. We have a second-order standard translation, that is, $\mathfrak{F} \Vdash \varphi$ if and only if $\mathfrak{F} \models \forall P_1 \ldots \forall P_n \forall x\,\mathrm{ST}_x(\varphi)$, where $P_1, \ldots, P_n$ are monadic second-order variables, one for each propositional variable occurring in φ. So, the notion of modal definability is equivalent to the definability by a set of second-order formulas of the form $\forall P_1 \ldots \forall P_n \forall x\,\mathrm{ST}_x(\varphi)$. However, in many cases a formula of this type is equivalent to a first-order formula. Namely, this holds for any *Sahlqvist formula* (the definition is omitted here – see [10] or [1]), for which an equivalent first order formula is effectively computable. On the other hand, the Goldblatt-Thomason Theorem characterizes those first-order properties that are modally definable.

Now, $\exists$-definability is clearly also equivalent to the definability by a type of second-order formulas – those of the form $\forall P_1 \ldots \forall P_n \exists x\,\mathrm{ST}_x(\varphi)$. Consider another example of a modally $\exists$-definable class.

Example 2. The condition $F = \exists x \forall y(Rxy \to \exists z Ryz)$ is not modally definable, since it is not closed under generated subframes, but it is modally $\exists$-definable by the formula $\varphi = p \to \Box\Diamond p$.

To prove this, we need to show $\mathrm{Fr}_\exists(\varphi) = \mathrm{Mod}(F)$. But $\mathfrak{F} = (W, R) \in \mathrm{Fr}_\exists(\varphi)$ if and only if $\mathfrak{F} \models \forall P \exists x(Px \to \forall y(Rxy \to \exists z(Ryz \wedge Pz)))$. So in particular, under the assignment which assigns the entire W to the second-order variable P, we get $\mathfrak{F} \models \exists x \forall y(Rxy \to \exists z Ryz)$, thus $\mathfrak{F} \in \mathrm{Mod}(F)$. The reverse inclusion is proved similarly.

Other changes of quantifiers or the order of first and second-order quantifiers would result in other types of definability, perhaps also worthy of exploring. In fact, this has already been done by Venema [12] and Hollenberg [7], who consider *negative definability*, which corresponds to second-order formulas of the form $\forall x \exists P_1 \ldots \exists P_n\,\mathrm{ST}_x(\neg\varphi)$. The class of frames negatively defined by Σ is denoted $\mathrm{Fr}^-(\Sigma)$. It should be noted here that the definition of $\forall\exists$-definability is inspired by the analogous notion of $\pm$-definability from [7].

A general characterization of negative definability has not been obtained, and neither has been a characterization of elementary classes which are negatively definable – it even remains unknown if all negatively definable classes are in fact elementary. But, to digress a little from the main point of this paper, we easily get the following fairly broad result.

Proposition 1. *Let φ be a modal formula which has a first-order local correspondent, i.e. there is a first-order formula $F(x)$ such that for any frame $\mathfrak{F} = (W, R)$ and any $w \in W$ we have $\mathfrak{F}, w \Vdash \varphi$ if and only if $\mathfrak{F} \models F(x)[w]$. (In particular, this holds for any Sahlqvist formula.)*

Then we have $\mathrm{Fr}^-(\varphi) = \mathrm{Mod}(\forall x \neg F(x))$.

Proof. We have $\mathfrak{F} \in \mathrm{Fr}^-(\varphi)$ if and only if $\mathfrak{F} \models \forall x \exists P_1 \ldots \exists P_n \, \mathrm{ST}_x(\neg\varphi)$ if and only if $\mathfrak{F} \not\models \exists x \forall P_1 \ldots \forall P_n \, \mathrm{ST}_x(\varphi)$. But this means that there is no $w \in W$ such that $\mathfrak{F} \models \forall P_1 \ldots \forall P_n \, \mathrm{ST}_x(\varphi)[w]$. The latter holds if and only if $\mathfrak{F}, w \Vdash \varphi$, which is by assumption equivalent to $\mathfrak{F} \models F(x)[w]$. The fact that such w does not exist, is equivalent to $\mathfrak{F} \in \mathrm{Mod}(\forall x \neg F(x))$. □

So for example, since $p \to \Diamond p$ locally corresponds to Rxx, we have that $p \to \Diamond p$ negatively defines irreflexivity, which is not modally definable property, since it is not preserved under surjective bounded morphisms.

4 Model-Theoretic Constructions

This section can be used, if needed, as a reference for the basic facts about the constructions used in the proofs of the characterizations. Otherwise it can be omitted.

A *bisimulation* between Kripke models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ is a relation $Z \subseteq W \times W'$ such that:

(at) if wZw' then we have: $w \in V(p)$ if and only if $w' \in V'(p)$, for all $p \in \Phi$,

(forth) if wZw' and Rwv, then there is a v' such that vZv' and $R'w'v'$,

(back) if wZw' and $R'w'v'$, then there is a v such that vZv' and Rwv.

The basic property of bisimulations is that (at) extends to all formulas: if wZw' then $\mathfrak{M}, w \Vdash \varphi$ if and only if $\mathfrak{M}', w' \Vdash \varphi$, i.e. $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ are modally equivalent. We get the definition of bisimulation between frames by omitting the condition (at).

A *bounded morphism* from a frame $\mathfrak{F} = (W, R)$ to $\mathfrak{F}' = (W', R')$ is a function $f : W \to W'$ such that:

(forth) Rwv implies $R'f(w)f(v)$,

(back) if $R'f(w)v'$, then there is v such that $v' = f(v)$ and Rwv.

Clearly, the graph of a bounded morphism is a bisimulation.

A *generated subframe* of $\mathfrak{F} = (W, R)$ is a frame $\mathfrak{F}' = (W', R')$ where $W' \subseteq W$ such that $w \in W'$ and Rwv implies $v \in W'$, and $R' = R \cap (W' \times W')$. A *generated submodel* of $\mathfrak{M} = (W, R, V)$ is a model based on a generated subframe, with the valuation $V'(p) = V(p) \cap W'$, for all $p \in \Phi$. It is easy to see that the global truth of a modal formula is preserved on a generated submodel.

The *disjoint union* of a family of models $\{\mathfrak{M}_i = (W_i, R_i, V_i) : i \in I\}$ is the model $\biguplus_{i \in I} \mathfrak{M}_i = (W, R, V)$ such that:

(1) $W = \bigcup_{i \in I}(W_i \times \{i\})$,

(2) $R(w, i)(v, j)$ if and only if $i = j$ and $R_i wv$,

(3) $(w, i) \in V(p)$ if and only if $w \in V_i(p)$, for all p.

It is easy to see that the disjoint union preserves the global truth of a modal formula. The definition of the disjoint union of a family of frames is obtained by omitting (3).

To define the ultraproducts and ultrafilter extensions, we need the notion of ultrafilters. An *ultrafilter* over a set $I \neq \emptyset$ is a family $U \subseteq \mathcal{P}(I)$ such that:

(1) $I \in U$,
(2) if $A, B \in U$, then $A \cap B \in U$,
(3) if $A \in U$ and $A \subseteq B \subseteq I$, then $B \in U$,
(4) for all $A \subseteq I$ we have: $A \in U$ if and only if $I \setminus A \notin U$.

The existence of ultrafilters is provided by a fact that any family of subsets which has the finite intersection property (that is, each finite intersection is non-empty) can be extended to an ultrafilter (see e.g. [2]).

Let $\{\mathfrak{M}_i = (W_i, R_i, V_i) : i \in I\}$ be a family of Kripke models and let U be an ultrafilter over I. The *ultraproduct* of this family over U is the model $\prod_U \mathfrak{M}_i = (W, R, V)$ such that:

(1) W is the set of equivalence classes f^U of the following relation defined on the Cartesian product of the family: $f \sim g$ if and only if $\{i \in I : f(i) = g(i)\} \in U$,
(2) $f^U R g^U$ if and only if $\{i \in I : f(i) R_i g(i)\} \in U$,
(3) $f^U \in V(p)$ if and only if $\{i \in I : f(i) \in V_i(p)\} \in U$, for all p.

The basic property of ultraproducts is that (3) extends to all formulas.

Proposition 2. *Let $\{\mathfrak{M}_i : i \in I\}$ be a family of Kripke models and let U be an ultrafilter over I.*

Then we have $\prod_U \mathfrak{M}_i, f^U \Vdash \varphi$ if and only if $\{i \in I : \mathfrak{M}_i, f(i) \Vdash \varphi\} \in U$, for any f^U. Furthermore, we have $\prod_U \mathfrak{M}_i \Vdash \varphi$ if and only if $\{i \in I : \mathfrak{M}_i \Vdash \varphi\} \in U$.

This is an analogue of Łoś's Fundamental Theorem on ultraproducts from the first-order model theory (see [2] for this, and [1] for the proof of the modal analogue). Łoś's Theorem also implies that an elementary class of models is closed under ultraproducts.

An ultraproduct such that $\mathfrak{M}_i = \mathfrak{M}$ for all $i \in I$ is called an *ultrapower* of $\mathfrak{M}$ and denoted $\prod_U \mathfrak{M}$. From Łoś's Theorem it follows that any ultrapower of a model is elementarily equivalent to the model, that is, the same first-order sentences are true on $\mathfrak{M}$ and $\prod_U \mathfrak{M}$. Definition of an ultraproduct of a family of frames is obtained by omitting the clause regarding valuation.

Another notion needed in the proofs of the characterizations is *modal saturation*, the modal analogue of *ω-saturation* from the classical model theory. The definition of saturation is omitted here (see e.g. [1]), since we only need some facts which it implies:

– While a bisimulation implies modal equivalence, the converse generally does not hold, but it does hold for modally saturated models. In fact, a modal equivalence between points of modally saturated models is a bisimulation.
– Any ω-saturated Kripke model is also modally saturated (see [1] for proofs of these facts).

Finally, the *ultrafilter extension* of a model $\mathfrak{M} = (W, R, V)$ is the model $\mathfrak{ue}\mathfrak{M} = (\mathrm{Uf}(W), R^{\mathfrak{ue}}, V^{\mathfrak{ue}})$, where $\mathrm{Uf}(W)$ is the set of all ultrafilters over W, $R^{\mathfrak{ue}}uv$ holds if and only if $A \in v$ implies $m_\Diamond(A) \in u$, where $m_\Diamond(A)$ denotes the set of all $w \in W$ such that Rwa for some $a \in A$, and $u \in V^{\mathfrak{ue}}(p)$ if and only if $V(p) \in u$. The basic property is that this extends to any modal formula, i.e. we have $u \in V^{\mathfrak{ue}}(\varphi)$ if and only if $V(\varphi) \in u$ (see [1]). From this it easily follows that the global truth of a modal formula is preserved on the ultrafilter extension. Another important fact is that the ultrafilter extension of a model is modally saturated (see [1]).

The ultrafilter extension of a frame $\mathfrak{F} = (W, R)$ is $\mathfrak{ue}\mathfrak{F} = (\mathrm{Uf}(W), R^{\mathfrak{ue}})$.

5 Characterizations

Arguments and techniques used in the proofs of the following characterizations are similar to the ones used in the proof of Goldblatt-Thomason theorem as presented in [1], so the reader might find it interesting to compare these proofs to note analogies and differences.

Theorem 1. *Let $\mathcal{K}$ be an elementary class of Kripke frames. Then $\mathcal{K}$ is modally $\exists$-definable if and only if it is closed under surjective bounded morphisms and reflects generated subframes and ultrafilter extensions.*

Proof. Let $\mathcal{K} = \mathrm{Fr}_{\exists}(\Sigma)$. Let $\mathfrak{F} = (W, R) \in \mathcal{K}$ and let f be a surjective bounded morphism from $\mathfrak{F}$ to some $\mathfrak{F}' = (W', R')$. Take any $\varphi \in \Sigma$ and any model $\mathfrak{M}' = (W', R', V')$ based on $\mathfrak{F}'$. Put $V(p) = \{w \in W : f(w) \in V'(p)\}$. Then V is a well defined valuation on $\mathfrak{F}$. Put $\mathfrak{M} = (W, R, V)$. Since $\mathfrak{F} \in \mathcal{K}$, there exists $w \in W$ such that $\mathfrak{M}, w \Vdash \varphi$. But then $\mathfrak{M}', f(w) \Vdash \varphi$. This proves that $\mathcal{K}$ is closed under surjective bounded morphisms.

To prove that $\mathcal{K}$ reflects generated subframes and ultrafilter extensions, let $\mathfrak{F} = (W, R) \notin \mathcal{K}$. This means that there is $\varphi \in \Sigma$ and a model $\mathfrak{M} = (W, R, V)$ based on $\mathfrak{F}$ such that $\mathfrak{M} \Vdash \neg\varphi$. Let $\mathfrak{F}' = (W', R')$ be a generated subframe of $\mathfrak{F}$. Define $V'(p) = V(p) \cap W'$, for all p. Then we have $\mathfrak{M}' \Vdash \neg\varphi$, which proves $\mathfrak{F}' \notin \mathcal{K}$, as desired. Also, $\mathfrak{ue}\mathfrak{M}$ is a model based on the ultrafilter extension $\mathfrak{ue}\mathfrak{F}$ and we have $\mathfrak{ue}\mathfrak{M} \Vdash \neg\varphi$, which proves $\mathfrak{ue}\mathfrak{F} \notin \mathcal{K}$.

For the converse, let $\mathcal{K}$ be an elementary class of frames that is closed under surjective bounded morphisms and reflects generated subframes and ultrafilter extensions. Denote by Σ the set of all formulas that are satisfiable in all models based on all frames in $\mathcal{K}$. Then $\mathcal{K} \subseteq \mathrm{Fr}_{\exists}(\Sigma)$ and it remains to prove the reverse inclusion.

Let $\mathfrak{F} = (W, R) \in \mathrm{Fr}_{\exists}(\Sigma)$. Let Φ be a set of propositional variables that contains a propositional variable p_A for each $A \subseteq W$. Let $\mathfrak{M} = (W, R, V)$, where $V(p_A) = A$ for all $A \subseteq W$. Denote by Δ the set of all modal formulas over Φ which are globally true on $\mathfrak{M}$. Now, for any finite $\delta \subseteq \Delta$ there is $\mathfrak{F}_\delta \in \mathcal{K}$ and a model $\mathfrak{M}_\delta$ based on $\mathfrak{F}_\delta$ such that $\mathfrak{M}_\delta \Vdash \delta$. Otherwise, since Δ is closed under conjunctions, there is $\varphi \in \Delta$ such that $\neg\varphi \in \Sigma$, thus $\neg\varphi$ is satisfiable in $\mathfrak{M}$, which contradicts $\mathfrak{M} \Vdash \Delta$.

Now, let I be the family of all finite subsets of Δ. For each $\varphi \in \Delta$, put $\hat{\varphi} = \{\delta \in I : \varphi \in \delta\}$. The family $\{\hat{\varphi} : \varphi \in \Delta\}$ clearly has the finite intersection property, so it can be extended to an ultrafilter U over I. But for all $\varphi \in \Delta$ we have $\{\delta \in I : \mathfrak{M}_\delta \Vdash \varphi\} \supseteq \hat{\varphi}$ and $\hat{\varphi} \in U$, thus $\{\delta \in I : \mathfrak{M}_\delta \Vdash \varphi\} \in U$, so the Proposition 2 implies $\prod_U \mathfrak{M}_\delta \Vdash \varphi$. The model $\prod_U \mathfrak{M}_\delta$ is based on the frame $\prod_U \mathfrak{F}_\delta$. Since $\mathcal{K}$ is elementary, it is also closed under ultraproducts, so

$\prod_U \mathfrak{F}_\delta \in \mathcal{K}$. It remains to prove that there is a surjective bounded morphism from some ultrapower of $\prod_U \mathfrak{F}_\delta$ to a generated subframe of $\mathfrak{ue}\mathfrak{F}$. Then the assumed properties of $\mathcal{K}$ imply that $\mathfrak{F} \in \mathcal{K}$, as desired.

Classical model theory provides us with an ω-saturated ultrapower of $\prod_U \mathfrak{M}_\delta$ (cf. [2]). Let $\mathfrak{M}_\Delta$ be such an ultrapower. We have that $\mathfrak{M}_\Delta$ is modally saturated. Also, it is elementarily equivalent to $\prod_U \mathfrak{M}_\delta$, so using standard translation we obtain $\mathfrak{M}_\Delta \Vdash \Delta$. The model $\mathfrak{M}_\Delta$ is based on a frame $\mathfrak{F}_\Delta$, which is an ultrapower of $\prod_U \mathfrak{F}_\delta$. Now define a mapping from $\mathfrak{F}_\Delta$ to $\mathfrak{ue}\mathfrak{F}$ by putting $f(w) = \{A \subseteq W : \mathfrak{M}_\Delta, w \Vdash p_A\}$.

First we need to prove that f is well-defined, i.e. that $f(w)$ is indeed an ultrafilter over W.

(1) We easily obtain $W \in f(w)$, since $p_W \in \Delta$ by the definition of V.

(2) If $A, B \in f(w)$, then $\mathfrak{M}_\Delta, w \Vdash p_A \wedge p_B$. Clearly, $\mathfrak{M} \Vdash p_A \wedge p_B \leftrightarrow p_{A\cap B}$. Thus $\mathfrak{M}_\Delta \Vdash p_A \wedge p_B \leftrightarrow p_{A\cap B}$, so $\mathfrak{M}_\Delta, w \Vdash p_{A\cap B}$, i.e. $A \cap B \in f(w)$.

(3) If $A \in f(w)$ and $A \subseteq B \subseteq W$, then from the definition of V it follows $\mathfrak{M} \Vdash p_A \to p_B$. But then also $\mathfrak{M}_\Delta \Vdash p_A \to p_B$, hence $\mathfrak{M}_\Delta, w \Vdash p_B$, so $B \in f(w)$.

(4) For all $A \subseteq W$ we have $\mathfrak{M} \Vdash p_A \leftrightarrow \neg p_{W\setminus A}$, which similarly as in the previous points implies $A \in f(w)$ if and only if $W \setminus A \notin f(w)$, as desired.

Assume for the moment that we have: $u = f(w)$ if and only if $(\mathfrak{ue}\mathfrak{M}, u)$ and $(\mathfrak{M}_\Delta, w)$ are modally equivalent. Since $\mathfrak{ue}\mathfrak{M}$ and $\mathfrak{M}_\Delta$ are modally saturated, the modal equivalence between their points is a bisimulation. So f is a bisimulation, but it is also a function, which means that it is a bounded morphism from $\mathfrak{F}_\Delta$ to $\mathfrak{ue}\mathfrak{F}$. But then the corestriction of f to its image is a surjective bounded morphism from an ultrapower of $\prod_U \mathfrak{F}_\delta$ to a generated subframe of $\mathfrak{ue}\mathfrak{F}$, which we needed.

So to conclude the proof, it remains to show that $u = f(w)$ holds if and only if $(\mathfrak{ue}\mathfrak{M}, u)$ and $(\mathfrak{M}_\Delta, w)$ are modally equivalent. Let $u = f(w)$. Then we have $\mathfrak{ue}\mathfrak{M}, u \Vdash \varphi$ if and only if $V(\varphi) \in u$, which is by the definition of f equivalent to $\mathfrak{M}_\Delta, w \Vdash p_{V(\varphi)}$. But the definition of V clearly implies $\mathfrak{M} \Vdash \varphi \leftrightarrow p_{V(\varphi)}$, so also $\mathfrak{M}_\Delta \Vdash \varphi \leftrightarrow p_{V(\varphi)}$, which provides the needed modal equivalence.

For the converse, the assumption implies that we have $\mathfrak{ue}\mathfrak{M}, u \Vdash p_A$ if and only if $\mathfrak{M}_\Delta, w \Vdash p_A$, for all $A \subseteq W$. This means that $V(p_A) = A \in u$ if and only if $A \in f(w)$, i.e. $u = f(w)$. □

In the characterization of ∀∃-definability we need the following non-standard closure condition.

Definition 3. We say that a class $\mathcal{K}$ of Kripke frames is *closed under generated interframes* if the following holds:

Let $\mathfrak{F}_1$, $\mathfrak{F}$ and $\mathfrak{F}_2$ be frames such that $\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}$ and $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}_2$. Then we have: if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are in $\mathcal{K}$, then $\mathfrak{F}$ is also in $\mathcal{K}$ (cf. [8] for the analogous notion for Kripke models).

Theorem 2. *Let $\mathcal{K}$ be an elementary class of Kripke frames. Then $\mathcal{K}$ is modally ∀∃-definable if and only if it is closed under surjective bounded morphisms, disjoint unions and generated interframes, and reflects ultrafilter extensions.*

Proof. It is easy to show that any $\forall\exists$-definable class have the desired properties, using the same arguments as in the respective directions of the proofs of Goldblatt-Thomason theorem (see [1]) and Theorem 1.

For the converse, let $\mathcal{K}$ be an elementary class of frames that is closed under surjective bounded morphisms, disjoint unions and generated interframes, and reflects ultrafilter extensions. Let Σ_1 be the set of all formulas that are valid on all frames in $\mathcal{K}$, and let Σ_2 be the set of all formulas that are satisfiable in all models based on all frames in $\mathcal{K}$. Then $\mathcal{K} \subseteq \mathrm{Fr}(\Sigma_1) \cap \mathrm{Fr}_\exists(\Sigma_2)$ and it remains to prove the reverse inclusion.

Let $\mathfrak{F} \in \mathrm{Fr}(\Sigma_1) \cap \mathrm{Fr}_\exists(\Sigma_2)$ and let Φ be a set of propositional variables that contains p_A for each $A \subseteq W$. Let $\mathfrak{M}$ be a model based on $\mathfrak{F}$ such that $V(p_A) = A$ for all $A \subseteq W$. Let $\Delta_\forall$ be the set of all formulas over Φ which are globally true on $\mathfrak{M}$ and let $\Delta_\exists$ be the set of all formulas over Φ which are satisfiable in $\mathfrak{M}$.

Denote $D_\forall = \{\forall x\, \mathrm{ST}_x(\varphi) : \varphi \in \Delta_\forall\}$, $D_\exists = \{\exists x\, \mathrm{ST}_x(\varphi) : \varphi \in \Delta_\exists\}$, and $D = D_\forall \cup D_\exists$. It is easy to see that for all $F \in D$ there is a model $\mathfrak{M}_F$ based on some $\mathfrak{F}_F \in \mathcal{K}$ such that $\mathfrak{M}_F \models F$ (the opposite assumption easily leads to a contradiction).

Using the same arguments as in the proof of Theorem 1, we conclude that there is an ω-saturated model $\mathfrak{M}_\forall$ based on some frame $\mathfrak{F}_\forall \in \mathcal{K}$ such that $\mathfrak{M}_\forall \models D_\forall$, i.e. $\mathfrak{M}_\forall \Vdash \Delta_\forall$. We define a mapping f from $\mathfrak{F}_\forall$ to $\mathfrak{ue}\mathfrak{F}$ by putting $f(w) = \{A \subseteq W : \mathfrak{M}_\forall, w \Vdash p_A\}$. In the same way as in the proof of Theorem 1, we show that f is a bounded morphism. Denote its image by $\mathfrak{F}'_\forall$. It is a generated subframe of $\mathfrak{ue}\mathfrak{F}$, and since $\mathcal{K}$ is closed under surjective bounded morphisms, we have $\mathfrak{F}'_\forall \in \mathcal{K}$.

On the other hand, since $\mathcal{K}$ is closed under disjoint unions, we have that $\biguplus_{F\in D_\exists} \mathfrak{F}_F \in \mathcal{K}$, while clearly $\biguplus_{F\in D_\exists} \mathfrak{M}_F \models D_\exists$. Since $\mathcal{K}$ is elementary, it is closed under ultraproducts, so an ω-saturated ultrapower $\mathfrak{M}_\exists$ of the disjoint union $\biguplus_{F\in D_\exists} \mathfrak{M}_F$ is based on some $\mathfrak{F}_\exists \in \mathcal{K}$ and it holds $\mathfrak{M}_\exists \models D_\exists$. Hence, all formulas that are satisfiable in $\mathfrak{M}$ are also satisfiable in $\mathfrak{M}_\exists$. By contraposition, all formulas that are globally true on $\mathfrak{M}_\exists$ are also globally true on $\mathfrak{M}$, thus also on $\mathfrak{ue}\mathfrak{M}$. It is not hard to show that the modal equivalence between worlds of $\mathfrak{M}_\exists$ and $\mathfrak{ue}\mathfrak{M}$ is a surjective bisimulation (this follows immediately from Lemma 1 in [8]). The domain of this bisimulation is a generated submodel $\mathfrak{M}'_\exists$ of $\mathfrak{M}_\exists$.

To prove that this bisimulation is in fact a surjective bounded morphism from $\mathfrak{M}'_\exists$ to $\mathfrak{ue}\mathfrak{M}$, it remains to prove that it is a function. Assume the opposite, i.e. that there is a world in $\mathfrak{M}_\exists$ which is modally equivalent to two different ultrafilters u, v in $\mathfrak{ue}\mathfrak{M}$. Hence, u and v are modally equivalent, i.e. for all φ we have $V(\varphi) \in u$ if and only if $V(\varphi) \in v$. In particular, for all $A \subseteq W$ we have $V(p_A) = A \in u$ if and only if $V(p_A) = A \in v$, thus $u = v$. This proves that there is a surjective bounded morphism g from $\mathfrak{F}'_\exists$ to $\mathfrak{ue}\mathfrak{F}$, where $\mathfrak{F}'_\exists$ is a generated subframe of $\mathfrak{F}_\exists$.

Let $\mathfrak{F}''_\exists$ be the frame built from $\mathfrak{ue}\mathfrak{F} \uplus (\mathfrak{F}_\exists \setminus \mathfrak{F}'_\exists)$, by extending its accessibility relation with all pairs $(w, g(v))$, for w in $\mathfrak{F}_\exists \setminus \mathfrak{F}'_\exists$ and v in $\mathfrak{F}'_\exists$ such that v is accessible from w in $\mathfrak{F}_\exists$. Now, extend g to $\mathfrak{F}_\exists$ by putting $g(w) = w$ for w in $\mathfrak{F}_\exists \setminus \mathfrak{F}'_\exists$. This makes g a surjective bounded morphism from $\mathfrak{F}_\exists$ to $\mathfrak{F}''_\exists$. Since $\mathcal{K}$ is closed under surjective bounded morphisms, we have $\mathfrak{F}''_\exists \in \mathcal{K}$. Clearly, $\mathfrak{ue}\mathfrak{F}$ is a generated subframe of $\mathfrak{F}''_\exists$. We have already proved that there is $\mathfrak{F}'_\forall \in \mathcal{K}$ which is a generated subframe of $\mathfrak{ue}\mathfrak{F}$, so the closure under generated interframes implies $\mathfrak{ue}\mathfrak{F} \in \mathcal{K}$. Since $\mathcal{K}$ reflects ultrafilter extensions, it follows $\mathfrak{F} \in \mathcal{K}$. □

The following examples show that the conditions of Theorems 1 and 2, and Goldblatt-Thomason theorem, are minimal. Each example is an elementary class which satisfies all but one of the conditions of a characterization, thus showing that this condition cannot be omitted. Almost all claims are proved routinely, so most of the details are skipped.

Example 3. Irreflexivity, i.e. the class $\mathrm{Mod}(\forall x \neg Rxx)$, is not modally definable, since it is not closed under surjective bounded morphisms. It is easy to see that this class is closed under generated subframes, generated interframes, disjoint unions, and reflects ultrafilter extensions. This shows that the closure under surjective bounded morphisms cannot be omitted in Goldblatt-Thomason theorem or Theorem 2.

To show that this condition cannot be omitted from Theorem 1 either, consider the class $\mathrm{Mod}(\exists x \neg Rxx)$, i.e. the class of frames which are not reflexive. It is easy to construct an example which shows that this class is not closed under surjective bounded morphisms, but it is also not hard to show that it reflects generated subframes and ultrafilter extensions.

Example 4. The class $\mathrm{Mod}(\forall x \forall y Rxy)$ is obviously not closed under disjoint unions, but it is closed under surjective bounded morphisms, generated subframes and generated interframes, and reflects ultrafilter extensions. This proves that the closure under disjoint unions is essential in Goldblatt-Thomason theorem and Theorem 2. It is also obvious that this class does not reflect generated subframes, which means that this condition cannot be omitted in Theorem 1.

Example 5. The class $\mathrm{Mod}(\exists x \exists y Rxy)$ is not closed under generated subframes, but it satisfies all other conditions of Goldblatt-Thomason theorem.

Example 6. Let $\mathcal{K} = \mathrm{Mod}(\forall x Rxx \vee \exists x \forall y \neg Rxy)$. This is the class of all frames that are either reflexive or have a world with no access to any world. It is easy to see that $\mathcal{K}$ is closed under disjoint unions and surjective bounded morphisms, and reflects ultrafilter extensions. But, $\mathcal{K}$ is not closed under generated interframes, thus this condition cannot be omitted in Theorem 2.

To see this, let $\mathfrak{F}_1 = (\{w\}, \{(w,w)\})$, and $\mathfrak{F}_2 = (\{w,v,u\}, \{(w,w),(v,w)\})$. Obviously $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{K}$. Let $\mathfrak{F} = (\{w,v\}, \{(w,w),(v,w)\})$. Clearly, $\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}$, and $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}_2$, but $\mathfrak{F} \notin \mathcal{K}$.

Example 7. Finally, the class $\mathcal{K} = \mathrm{Mod}(\forall x \exists y (Rxy \wedge Ryy))$, i.e. the property that every world has a reflexive R-successor, is closed under disjoint unions, generated subframes, generated interframes and surjective bounded morphisms, but does not reflect ultrafilter extensions. To prove the last claim, consider the frame $\mathfrak{F} = (\mathbb{N}, <)$, i.e. the set of natural numbers with the standard strict ordering. Obviously $\mathfrak{F} \in \mathcal{K}^c$. But, $\mathfrak{ue}\mathfrak{F} \in \mathcal{K}$. This follows from the fact that for each ultrafilter u over $\mathbb{N}$ and for each non-principal ultrafilter v over $\mathbb{N}$ we have $u <^{\mathfrak{ue}} v$ (see [1], p. 95).

The same frame shows in a similar way that the class $\mathrm{Mod}(\exists x Rxx)$ does not reflect ultrafilter extensions, and it is easy to see that it is closed under surjective bounded morphisms and reflects generated subframes. This shows that the reflection of ultrafilter extensions cannot be omitted in any of the characterizations.

6 Link to the Universal Modality

Although the approach of this paper is to define $\exists$-definability as a metalingual notion, it should be noted that it can be included in the language itself. That is, the satisfiability of a modal formula under any valuation on a frame can be expressed by a formula of the modal language enriched with the universal modality (BMLU). The syntax is an extension of the basic modal language by a new modal operator $A\varphi$, and we can also define its dual $E\varphi := \neg A \neg \varphi$. We call A the *universal modality*, and E the *existential modality*. The semantics of the new operators is standard modal semantics, with respect to the universal binary relation $W \times W$ on a frame $\mathfrak{F} = (W, R)$. This means that the standard translation of universal and existential operators is as follows (cf. [5] and [11]):

$$\mathrm{ST}_x(E\varphi) = \exists y\, \mathrm{ST}_y(\varphi),$$
$$\mathrm{ST}_x(A\varphi) = \forall y\, \mathrm{ST}_y(\varphi).$$

Now, let $\mathcal{K}$ be a class of Kripke frames. Clearly, $\mathcal{K}$ is modally $\exists$-definable if and only if it is definable by a set of formulas of the existential fragment of BMLU, i.e. by a set of formulas of the form $E\varphi$, where φ is a formula of BML. This immediately follows from the clear fact that for any frame $\mathfrak{F}$ and any φ we have $\mathfrak{F} \Vdash E\varphi$ if and only if $\mathfrak{F} \models \forall P_1 \ldots \forall P_n \exists y\, \mathrm{ST}_y(\varphi)$, where $P_1, \ldots, P_n$ correspond to propositional variables that occur in φ, and the latter holds if and only if φ is satisfiable under any valuation on $\mathfrak{F}$.

Goranko and Passy [5] gave a characterization that an elementary class is modally definable in BMLU if and only if it is closed under surjective bounded morphisms and reflects ultrafilter extension. So, from Theorem 1 we conclude that reflecting generated subframes, not surprisingly, is what distinguishes existential fragment within this language, at least with respect to elementary classes. Also, the usual notion of modal definability clearly coincides with the universal fragment of BMLU, hence the Goldblatt-Thomason Theorem tells us that closure under generated subframes and disjoint unions is essential for this fragment. Furthermore, from Theorem 2 it follows that closure under generated interframes and disjoint unions characterizes the union of universal and existential fragment of BMLU, i.e. definability by sets of formulas of the form $A\varphi$ or $E\varphi$, where φ is in BML.

On the other hand, a question is which modally $\exists$-definable classes are elementary, and whether there is an effective procedure analogous to the one for Sahlqvist formulas, to obtain a first-order formula equivalent to a second-order translation $\forall P_1 \ldots \forall P_n \exists x\, \mathrm{ST}_x(\varphi)$ for some sufficiently large and interesting class of modal formulas. Goranko and Vakarelov [6] answer this, and more: they provide a generalization of Sahlqvist formulas to languages with hybrid operators, including universal modal operator.

As for some further questions that might be worth exploring, we may be able to obtain general characterization theorems, without the assumption of the first-order definability. Furthermore, the results of this paper are easily generalized to the multi-modal framework, but more work is needed to obtain similar results for particular modal logics, for example temporal, with some restrictions on accessibility relations, e.g. transitivity.

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