Hardness Results for Intersection Non-Emptiness

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Abstract. We carefully reexamine a construction of Karakostas, Lipton, and Viglas (2003) to show that the intersection non-emptiness problem for DFA's (deterministic finite automata) characterizes the complexity class NL. In particular, if restricted to a binary work tape alphabet, then there exist constants c_1 and c_2 such that for every k intersection non-emptiness for k DFA's is solvable in $c_1 k \log(n)$ space, but is not solvable in $c_2 k \log(n)$ space. We optimize the construction to show that for an arbitrary number of DFA's intersection non-emptiness is not solvable in $o(\frac{n}{\log(n)\log(\log(n))})$ space. Furthermore, if there exists a function $f(k) = o(k)$ such that for every k intersection non-emptiness for k DFA's is solvable in $n^{f(k)}$ time, then P $\neq$ NL. If there does not exist a constant c such that for every k intersection non-emptiness for k DFA's is solvable in n^c time, then P does not contain any space complexity class larger than NL.

1 Introduction

Let $\mathcal{A}$ denote a class of machines. The intersection non-emptiness problem for $\mathcal{A}$, denoted by $\text{IE}_{\mathcal{A}}$, consists of all finite lists of machines in $\mathcal{A}$ whose underlying languages have a non-empty intersection. By fixing the number of machines in the input to k, one obtains intersection non-emptiness for k machines which we denote by k-$\text{IE}_{\mathcal{A}}$. Intersection non-emptiness problems can be motivated by the following scenario. Consider that you are trying to construct an object x for a particular application. You propose a finite list of conditions for x to satisfy such that each condition can be decided by a machine in $\mathcal{A}$. An algorithm that solves intersection non-emptiness for $\mathcal{A}$ provides a method for checking if there exists an object x satisfying the proposed conditions.

Let $\text{IE}_{\mathcal{D}}$ denote the intersection non-emptiness problem for DFA's. One can solve $\text{IE}_{\mathcal{D}}$ by checking reachability in a product machine. Given an input consisting of k machines each of size at most m, the product machine has size at most m^k. Therefore, checking reachability takes at most m^{ck} time for some constant c. $\text{IE}_{\mathcal{D}}$ is a well known PSPACE-complete problem [5]. In [6], it was shown that one can pad strings in $\text{IE}_{\mathcal{D}}$ to obtain problems hard for smaller complexity classes such as $\text{NSPACE}(g(n)\log(n))$ where g is a slow growing log-space-constructible

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function such as $\log^*(n)$. In [4], it was shown that improvements to the standard algorithm imply separation results. In particular, if there exists a function $f(k) = o(k)$ such that $\text{IE}_{\mathcal{D}}$ is solvable in $m_1 \cdot m_2^{f(k)}$ time where m_1 is the size of a designated largest machine and all other machines have size at most m_2, then NL $\neq$ P.

In this paper, we carefully reexamine and optimize the construction from [4] in order to prove new results. We show that if restricted to a binary work tape alphabet, then there exist constants c_1 and c_2 such that for every k, k-$\text{IE}_{\mathcal{D}}$ $\in \text{NSPACE}(c_1 k \log(n))$ and k-$\text{IE}_{\mathcal{D}} \notin \text{NSPACE}(c_2 k \log(n))$. Then, we introduce an optimized construction to show that $\text{IE}_{\mathcal{D}} \notin \text{NSPACE}(o(\frac{n}{\log(n)\log(\log(n))}))$. Finally, we combine these results with a diagonalization argument to show that if there exists a function $f(k) = o(k)$ such that for every k, k-$\text{IE}_{\mathcal{D}} \in \text{DTIME}(n^{f(k)})$, then P $\neq$ NL. If there does not exist a constant c such that for every k, k-$\text{IE}_{\mathcal{D}}$ $\in \text{DTIME}(n^c)$, then $\text{NSPACE}(f(n)) \not\subseteq$ P for all $f(n) = \omega(\log(n))$ such that f is space-constructible.

2 Notation and Conventions

The input for $\text{IE}_{\mathcal{D}}$ is an encoding of a finite list of DFA's. For each encoding, n will denote the length and k will denote the number of machines that are represented. For each natural number k, k-$\text{IE}_{\mathcal{D}}$ denotes a restriction of the $\text{IE}_{\mathcal{D}}$ problem such that we only accept inputs that encode at most k machines.

Whenever we use the term Turing machine, we refer to a deterministic or non-deterministic machine with a two-way read only input tape and a two-way read/write work tape. For our purposes, we will only consider Turing machines where the work tape alphabet is binary. A work tape over a binary alphabet will be referred to as a binary work tape. A cell on a binary work tape will be referred to as a bit cell.

For each k, there are acceptance problems for space and time bounded Turing machines denoted by $N^S_{k\log}$ and $D^T_{n^k}$, respectively. $N^S_{k\log}$ refers to the problem where we are given an encoding of a non-deterministic Turing machine M with a binary work tape and an input s. We accept (M, s) if and only if M accepts s using at most $k\log(n)$ work tape bit cells where n denotes the length of s. $D^T_{n^k}$ is defined similarly for n^k deterministic time. We denote by $\text{NSPACE}^2(h(n))$ the set of problems solvable by a non-deterministic Turing machine using at most $h(n)$ work tape bit cells. Such classes are used to measure the binary space complexity of problems [2]. We associate $N^S_{k\log}$ with $\text{NSPACE}^2(k\log(n))$ and $D^T_{n^k}$ with $\text{DTIME}(n^k)$.

3 Binary Space Complexity

We introduce a function $\text{S}_{\text{NL}}(k)$ that measures the actual space complexities of the $N^S_{k\log}$ problems. In particular, $\text{S}_{\text{NL}}(k)$ is defined as follows:

$$\text{S}_{\text{NL}}(k) := \min\{\, d \in \mathbb{N} \mid N^S_{k\log} \in \text{NSPACE}^2(d\log(n)) \,\}. \tag{1}$$

In this section, we sketch how one could apply standard techniques from the space hierarchy theorem to prove that there exist constants c_1 and c_2 such that for every k sufficiently large, $N^S_{k\log} \in \mathrm{NSPACE}^2(c_1 k\log(n))$ and $N^S_{k\log} \notin \mathrm{NSPACE}^2(c_2 k\log(n))$. Using the function $\mathrm{S_{NL}}(k)$, we express this result as $\mathrm{S_{NL}}(k) = \Theta(k)$.

Proposition 1. $\mathrm{S_{NL}}(k) = O(k)$.

Sketch of proof. Using the simulation found in any common proof of the space hierarchy theorem, one shows that $N^S_{\log} \in \mathrm{NL}$. Further, one shows $\mathrm{S_{NL}}(k) = O(k)$ by using padding to reduce $N^S_{k\log}$ to $N^S_{\log}$ for every k. □

Proposition 2. $\mathrm{S_{NL}}(k) = \Omega(k)$.

Sketch of proof. Using the standard diagonalization argument found in any common proof of the non-deterministic space hierarchy theorem, one shows $\mathrm{S_{NL}}(k) = \Omega(k)$. Notice that in order to carry out the diagonalization one needs to show there exists c such that for all k,

$$\mathrm{NSPACE}^2(k\log(n)) \subseteq co\text{-}\mathrm{NSPACE}^2(ck\log(n)). \tag{2}$$

First, one applies the result $\mathrm{NL} = co\text{-}\mathrm{NL}$ to show that there exists c such that $N^S_{\log} \in co\text{-}\mathrm{NSPACE}^2(c\log(n))$. Further, one shows (2) by using padding to reduce $N^S_{k\log}$ to $N^S_{\log}$ for every k. □

Corollary 3. $\mathrm{S_{NL}}(k) = \Theta(k)$.

4 Reductions

We introduce a function $\mathrm{S_{IE}}(k)$ that measures the actual space complexities of the k-$\mathrm{IE}_{\mathcal{D}}$ problems. In particular, $\mathrm{S_{IE}}(k)$ is defined as follows:

$$\mathrm{S_{IE}}(k) := \min\{\, d \in \mathbb{N} \mid k\text{-}\mathrm{IE}_{\mathcal{D}} \in \mathrm{NSPACE}^2(d\log(n))\,\}. \tag{3}$$

In this section, we carefully reexamine the construction from [4] to show that there exist constants c_1 and c_2 such that for every k sufficiently large, k-$\mathrm{IE}_{\mathcal{D}} \in \mathrm{NSPACE}^2(c_1 k\log(n))$ and k-$\mathrm{IE}_{\mathcal{D}} \notin \mathrm{NSPACE}^2(c_2 k\log(n))$. Using the function $\mathrm{S_{IE}}(k)$, we can express this result as $\mathrm{S_{IE}}(k) = \Theta(\mathrm{S_{NL}}(k)) = \Theta(k)$.

Proposition 4. $\mathrm{S_{IE}}(k) = O(k)$.

Sketch of proof. As was previously discussed, one can solve $\mathrm{IE}_{\mathcal{D}}$ by checking reachability in a product machine. A state of the product machine can be stored as a string of $k\log(n)$ bits. Given such a state, we can non-deterministically guess which state comes next. There exists a path from an initial state to a final state if and only if there exists a path from an initial state to a final state of length at most n^k. Therefore, k-$\mathrm{IE}_{\mathcal{D}}$ is solvable using at most $ck\log(n)$ bits for some constant c. □

Theorem 5. $S_{IE}(k) = \Omega(S_{NL}(k))$.

Proof. We will describe a reduction from $N^S_{k\log}$ to k-$IE_{\mathcal{D}}$. Then, we will discuss encoding details to show that this is a log-space reduction.

Let a $k\log(n)$ space bounded non-deterministic Turing machine M and an input string s of length n be given. Our first task is to construct k DFA's, denoted by $< D_i >_{i\in[k]}$, each of size at most $p(n)$ for some fixed polynomial p such that M accepts s if and only if $\bigcap_{i\in[k]} L(D_i)$ is non-empty. The DFA's will read in a string that represents a computation of M on s and verify that the computation is valid and accepting. The work tape of M will be split into k sections each consisting of $\log(n)$ sequential bits of memory. The ith DFA, D_i, will keep track of the ith section and verify that it is managed correctly. In addition, all of the DFA's will keep track of the input and work tape head positions. We will achieve a better simulation in Theorem 7 where we split up the management of the tape head positions to separate DFA's. The following two concepts are essential to our construction.

A *section i configuration* of M is a tuple of the form

(state, input position, work position, ith section of work tape).

A *forgetful configuration* of M is a tuple of the form

(state, input position, work position).

We say that a section i configuration r extends a forgetful configuration a if r agrees with a on state, input position, and work position. We say that a section i configuration r_1 transitions to a section i configuration r_2 on input s if either the work position for r_1 is in the ith section and r_2 correctly represents how the tape positions and the ith section could change in one step of the computation on s, or r_1 is not in the ith section and r_1 and r_2 agree on the ith section of the work tape.

The states of D_i are identified with section i configurations. The alphabet characters are identified with forgetful configurations. For D_i, each alphabet character a transitions from a state r_1 to a state r_2 if and only if r_2 extends a and r_1 transitions to r_2 on input s.

We assert without proof that for every string x, x represents a valid accepting computation of M on s if and only if $x \in \bigcap_{i\in[k]} L(D_i)$. Therefore, M accepts s if and only if $\bigcap_{i\in[k]} L(D_i)$ is non-empty.

We show that the D_i's have size at most $p(n)$ for some fixed polynomial p. Each D_i consists of a start state, a list of final states, and a list of transitions where each transition consists of two states and an alphabet character. Each state is represented by a section i configuration and each alphabet character is represented by a forgetful configuration. Let m denote the number of states in M. Therefore, in total there are $m\cdot n\cdot k\log(n)\cdot 2^{\log(n)}$ section i configurations and $m \cdot n \cdot k\log(n)$ forgetful configurations. Hence, there exists a fixed two variable polynomial q such that each D_i has at most $q(n,k)$ states. Since k is fixed, one can blow up the degree of q to get a polynomial p such that p doesn't depend on k and each D_i has size at most $p(n)$.

It should be clear from the preceding that there is a fixed polynomial $t(n)$ such that for every k, $N^S_{k\log}$ is $t(n)$-time reducible to k-IE$_{\mathcal{D}}$. However, we want to show that there is a constant c such that for every k, $N^S_{k\log}$ is $c\log(n)$-space reducible to k-IE$_{\mathcal{D}}$. We accomplish this by describing how to print the string encoding of the D_i's to an auxiliary write only output tape using at most $c\log(n)$ space for some constant c.

We will describe how to print the transitions for each D_i and leave the remaining encoding details to the reader. We use a bit string i to represent the current DFA and two bit strings j_1 and j_2 to represent section i configurations. We iterate through every combination of i, j_1, and j_2. If D_i has a transition from j_1 to j_2, then we print (i, j_1, a, j_2) where a is the forgetful configuration such that j_2 extends a. We assert that checking whether to print (i, j_1, a, j_2) requires no more than $d\log(k) + d\log(n)$ bits for some constant d. Therefore, in printing the encoding of the D_i's, we use no more than $c\log(k) + c\log(n)$ bits for some constant c. For each k, when n is sufficiently large, the $\log(k)$ term goes away. It follows that for every k, $N^S_{k\log}$ is $c\log(n)$-space reducible to k-IE$_{\mathcal{D}}$. □

Corollary 6. $\mathrm{S_{IE}}(k) = \Theta(\mathrm{S_{NL}}(k)) = \Theta(k)$.

Proof. By Corollary 3, we have $\mathrm{S_{NL}}(k) = \Theta(k)$. Applying Proposition 4 and Theorem 5, we get that $\mathrm{S_{IE}}(k) = \Theta(\mathrm{S_{NL}}(k)) = \Theta(k)$. □

Theorem 7. $\mathrm{IE}_{\mathcal{D}} \notin \mathrm{NSPACE}(o(\frac{n}{\log(n)\log(\log(n))}))$.

Proof. By the non-deterministic space hierarchy theorem, we may choose a problem Q such that $Q \in \mathrm{NSPACE}(n)$, but $Q \notin \mathrm{NSPACE}(o(n))$. Choose $c \in \mathbb{N}$ and a non-deterministic Turing machine M that solves Q using at most cn bit cells. We optimize the construction from the proof of Theorem 5 to show that if $\mathrm{IE}_{\mathcal{D}} \in \mathrm{NSPACE}(o(\frac{n}{\log(n)\log(\log(n))}))$, then $Q \in \mathrm{NSPACE}(o(n))$. Since we know that $Q \notin \mathrm{NSPACE}(o(n))$, it follows that $\mathrm{IE}_{\mathcal{D}} \notin \mathrm{NSPACE}(o(\frac{n}{\log(n)\log(\log(n))}))$.

Let an input string s for M of length n be given. Our task is to construct $(c+1)\cdot n$ DFA's each with at most $d\log(n)$ states for some constant d such that M accepts s if and only if the DFA's have a non-empty intersection. The DFA's will read in a bit string that represents a computation of M on s and verify that the computation is valid and accepting. In this construction, we split up the management of the tape head positions to separate DFA's. There are n DFA's, denoted by $< I_i >_{i\in[n]}$, that manage the input tape and there are cn DFA's, denoted by $< W_i >_{i\in[cn]}$, that manage the work tape. The following concept is essential to our construction.

An *informative configuration* of M is a tuple of the form

(state, input position, current input bit, work position, current work bit).

The DFA's will read in a sequence of informative configurations that are encoded as bit strings. In contrast to the previous construction, the DFA's will have a binary input alphabet.

Each DFA is assigned to manage a bit position of either the input tape or work tape. Each I_i stores the ith input tape bit and operates as follows. It reads each informative configuration and checks if it represents the input position i. If it does not, then it ignores the informative configuration and moves on to the next one. However, if it does represent the input position i, then it checks that the stored bit matches the current input bit and uses the current work bit to check that the input position and state validly transition to the next informative configuration. Each W_i stores the ith work tape bit and operates as follows. It reads each informative configuration and checks if it represents the work position i. If it does not, then it ignores the informative configuration and moves on to the next one. However, if it does represent position i, then it checks that the stored bit matches the current work bit and uses the current input bit to modify the stored bit and check that the work position and state validly transition to the next informative configuration. It's important to remark that DFA's for boundary positions such as I_1, I_n, W_1, and W_{cn} cannot allow the input position or work position to go outside $[n]$ or $[cn]$, respectively.

We assert without proof that for every bit string x, x represents a valid accepting computation of M on s if and only if $x \in \bigcap_{i\in[n]} \mathrm{L}(I_i)$ and $x \in \bigcap_{i\in[cn]} \mathrm{L}(W_i)$. Therefore, M accepts s if and only if there exists a string x such that $x \in \bigcap_{i\in[n]} \mathrm{L}(I_i)$ and $x \in \bigcap_{i\in[cn]} \mathrm{L}(W_i)$.

A DFA with $\log(cn)$ states can be constructed to recognize a fixed binary number $i \in [cn]$. Since a tape position i could only transition to $i-1$, i, or $i+1$ in one step, it follows that a DFA with $d\log(n)$ states for some constant d can be constructed to check the validity of transitioning to the next informative configuration. Therefore, we can construct each DFA with at most $d\log(n)$ states for some constant d.

We described how to construct $(c+1)\cdot n$ DFA's each with at most $d\log(n)$ states for some constant d whose intersection is non-empty if and only if M accepts s. Since the total length of the string encoding of $< I_i >_{i\in[n]}$ combined with $< W_i >_{i\in[cn]}$ is at most $n\log(n)\log(\log(n))$, it follows that $\mathrm{IE}_{\mathcal{D}} \in \mathrm{NSPACE}(o(\frac{n}{\log(n)\log(\log(n))}))$ implies $Q \in \mathrm{NSPACE}(o(n))$. We obtain the desired result because $Q \notin \mathrm{NSPACE}(o(n))$. □

5 Space vs Time

We introduce functions $\mathrm{R}_{\mathrm{NL}}(k)$ and $\mathrm{R}_{\mathrm{IE}}(k)$ that measure the actual time complexities of $N^S_{k\log}$ and k-$\mathrm{IE}_{\mathcal{D}}$, respectively. In particular, $\mathrm{R}_{\mathrm{NL}}(k)$ and $\mathrm{R}_{\mathrm{IE}}(k)$ are defined as follows:

$$\mathrm{R}_{\mathrm{NL}}(k) := \min\{\, d \in \mathbb{N} \mid N^S_{k\log} \in \mathrm{DTIME}(n^d) \,\} \tag{4}$$

$$\mathrm{R}_{\mathrm{IE}}(k) := \min\{\, d \in \mathbb{N} \mid k\text{-}\mathrm{IE}_{\mathcal{D}} \in \mathrm{DTIME}(n^d) \,\}. \tag{5}$$

In this section, we show that if there exists a function $f(k) = o(k)$ such that for every k, $N^S_{k\log} \in \mathrm{DTIME}(n^{f(k)})$, then P $\neq$ NL. Using the function $\mathrm{R}_{\mathrm{NL}}(k)$ we can express this result as if $\mathrm{R}_{\mathrm{NL}}(k) = o(k)$, then P $\neq$ NL. Notice that by using

the reduction from Theorem 5, we also have $\mathrm{R}_{\mathrm{IE}}(k) = \Theta(\mathrm{R}_{\mathrm{NL}}(k))$. It follows that if $\mathrm{R}_{\mathrm{IE}}(k) = o(k)$, then $\mathrm{P} \neq \mathrm{NL}$.

Proposition 8. $\mathrm{R}_{\mathrm{IE}}(k) = \Theta(\mathrm{R}_{\mathrm{NL}}(k))$.

Theorem 9. *If* $\mathrm{R}_{\mathrm{NL}}(k) = o(k)$, *then* $\mathrm{NL} \neq \mathrm{P}$.

Proof. Suppose that $\mathrm{NL} = \mathrm{P}$. Since $D_n^T \in \mathrm{P}$, we have $D_n^T \in \mathrm{NL}$. Choose $d \in \mathbb{N}$ such that $D_n^T \in \mathrm{NSPACE}^2(d\log(n))$. Further, by using padding to reduce $D_{n^k}^T$ to D_n^T for every k, one can show that there exists d' such that for all k, $D_{n^k}^T \in \mathrm{NSPACE}^2(d'k\log(n))$. Choose such a constant d' satisfying for all k, $D_{n^k}^T \in \mathrm{NSPACE}^2(d'k\log(n))$.

Suppose for sake of contradiction that $\mathrm{R}_{\mathrm{NL}}(k) = o(k)$. By Proposition 2, we may choose c such that for all k sufficiently large

$$N_{k\log}^S \notin \mathrm{NSPACE}^2(\left\lfloor \frac{k}{c} \right\rfloor \log(n)). \tag{6}$$

Since $\mathrm{R}_{\mathrm{NL}}(k) = o(k)$, for all k sufficiently large

$$\mathrm{R}_{\mathrm{NL}}(k) < \left\lfloor \frac{k}{cd'} \right\rfloor. \tag{7}$$

Choose m satisfying $N_{m\log}^S \notin \mathrm{NSPACE}^2(\lfloor \frac{m}{c} \rfloor \log(n))$ and $\mathrm{R}_{\mathrm{NL}}(m) < \lfloor \frac{m}{cd'} \rfloor$. Therefore,

$$N_{m\log}^S \in \mathrm{DTIME}(o(n^{\lfloor \frac{m}{cd'} \rfloor})). \tag{8}$$

Since $D_{n^k}^T \in \mathrm{NSPACE}^2(d'k\log(n))$ for all k,

$$D^T_{n^{\lfloor \frac{m}{cd'} \rfloor}} \in \mathrm{NSPACE}^2(d' \left\lfloor \frac{m}{cd'} \right\rfloor \log(n)) \subseteq \mathrm{NSPACE}^2(\left\lfloor \frac{m}{c} \right\rfloor \log(n)). \tag{9}$$

Since we can trivially reduce every problem in $\mathrm{DTIME}(o(n^{\lfloor \frac{m}{cd'} \rfloor}))$ to $D^T_{n^{\lfloor \frac{m}{cd'} \rfloor}}$,

$$N_{m\log}^S \in \mathrm{DTIME}(o(n^{\lfloor \frac{m}{cd'} \rfloor})) \subseteq \mathrm{NSPACE}^2(\left\lfloor \frac{m}{c} \right\rfloor \log(n)) \tag{10}$$

which is a contradiction because $N_{m\log}^S \notin \mathrm{NSPACE}^2(\lfloor \frac{m}{c} \rfloor \log(n))$. □

Corollary 10. *If* $\mathrm{R}_{\mathrm{IE}}(k) = o(k)$, *then* $\mathrm{NL} \neq \mathrm{P}$.

Next, we show that if $\mathrm{R}_{\mathrm{NL}}(k)$ is unbounded, then P does not contain any space complexity class larger than NL. Since $\mathrm{R}_{\mathrm{IE}}(k) = \Theta(\mathrm{R}_{\mathrm{NL}}(k))$, it follows that if $\mathrm{R}_{\mathrm{IE}}(k)$ is unbounded, then P does not contain any space complexity class larger than NL.

For every function f, let N_f^S denote the acceptance problem for $f(n)$-space bounded non-deterministic Turing machines. N_f^S is of particular interest to us if it is non-deterministically solvable in $f(n)$ space.

Theorem 11. *If* $\mathrm{R_{NL}}(k)$ *is unbounded, then* $N_f^S \notin \mathrm{P}$ *for all functions* $f(n) = \omega(\log(n))$.

Proof. We will prove the contrapositive. Suppose that $N_f^S \in \mathrm{P}$ for some function $f(n) = \omega(\log(n))$. By assumption, we may choose $c \in \mathbb{N}$ and a deterministic Turing machine T such that T solves N_f^S in at most $O(n^c)$ time. Let $k \in \mathbb{N}$ be given. Choose a non-deterministic Turing machine M that solves $N_{k\log}^S$ using at most $O(\log(n))$ bit cells. We can deterministically solve $N_{k\log}^S$ in at most $O(n^c)$ time by feeding T an encoding of M and the input string. Since k is arbitrary, $N_{k\log}^S$ is solvable in $O(n^c)$ time for every k. It follows that $\mathrm{R_{NL}}(k)$ is bounded. □

Corollary 12. *If* $\mathrm{R_{NL}}(k)$ *is unbounded, then* $\mathrm{NSPACE}(f(n)) \not\subseteq \mathrm{P}$ *for all* $f(n) = \omega(\log(n))$ *such that* f *is space-constructible.*

Proof. Suppose $\mathrm{R_{NL}}(k)$ is unbounded. Let a function $f(n) = \omega(\log(n))$ such that f is space-constructible be given. Apply the preceding theorem to get that $N_f^S \notin \mathrm{P}$. Since f is space-constructible, one can use the simulation found in any common proof of the space hierarchy theorem to show that $N_f^S \in \mathrm{NSPACE}(f(n))$. Since $N_f^S \notin \mathrm{P}$ and $N_f^S \in \mathrm{NSPACE}(f(n))$, it follows that $\mathrm{NSPACE}(f(n)) \not\subseteq \mathrm{P}$. □

Corollary 13. *If* $\mathrm{R_{IE}}(k)$ *is unbounded, then* $\mathrm{NSPACE}(f(n)) \not\subseteq \mathrm{P}$ *for all* $f(n) = \omega(\log(n))$ *such that* f *is space-constructible.*

6 Conclusion

In Section 4, we showed that $\mathrm{S_{NL}}(k) = \mathrm{S_{IE}}(k) = \Theta(k)$. Therefore, we think of intersection non-emptiness for DFA's as characterizing the complexity class NL. Further, we showed that $\mathrm{IE}_{\mathcal{D}} \notin \mathrm{NSPACE}(o(\frac{n}{\log(n)\log(\log(n))}))$. In Section 5, we showed that if $\mathrm{R_{IE}}(k) = o(k)$, then NL $\neq$ P and if $\mathrm{R_{IE}}(k)$ is unbounded, then $\mathrm{NSPACE}(f(n)) \not\subseteq \mathrm{P}$ for all $f(n) = \omega(\log(n))$ such that f is space-constructible. Therefore, the asymptotic complexity of $\mathrm{R_{IE}}(k)$ determines the relationship between space and time complexity classes.

There are several related problems that appear to be harder than k-$\mathrm{IE}_{\mathcal{D}}$, but easier than $N_{k\log}^S$. For example, consider intersection non-emptiness for k NFA's, non-emptiness for k-turn 2DFA's, and intersection non-emptiness for k DFA's and a one-counter automaton. We can use $\mathrm{S_{NL}}(k) = \mathrm{S_{IE}}(k)$ and $\mathrm{R_{NL}}(k) = \mathrm{R_{IE}}(k)$ as squeeze theorems to show that all of these problems are of "equivalent" difficulty. Also, one could define a function that maps the k-$\mathrm{IE}_{\mathcal{D}}$ problems to their actual circuit complexities. The asymptotic complexity of such a function could determine the relationship between NL vs NP and P/poly vs space complexity classes [4].

Several related intersection non-emptiness problems have been studied. There are two such problems that we would like to mention. In [10], intersection non-emptiness for acyclic DFA's, which are DFA's without directed cycles, was shown

to be NP-complete. We assert that one could modify the construction from the proof of Theorem 5 to reduce the acceptance problem for n-time and $k\log(n)$-space bounded non-deterministic Turing machines to intersection non-emptiness for k acyclic DFA's. Also, in [11], intersection non-emptiness for tree automata was shown to be EXPTIME-complete. In an upcoming paper, the author and Joseph Swernofsky introduce time complexity lower bounds for intersection non-emptiness for tree automata.

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