Spatial Mixing of Coloring Random Graphs*

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Abstract. We study the strong spatial mixing (decay of correlation) property of proper q-colorings of random graph $G(n, d/n)$ with a fixed d. The strong spatial mixing of coloring and related models have been extensively studied on graphs with bounded maximum degree. However, for typical classes of graphs with bounded average degree, such as $G(n, d/n)$, an easy counterexample shows that colorings do not exhibit strong spatial mixing with high probability. Nevertheless, we show that for $q \geq \alpha d + \beta$ with $\alpha > 2$ and sufficiently large $\beta = O(1)$, with high probability proper q-colorings of random graph $G(n, d/n)$ exhibit strong spatial mixing *with respect to an arbitrarily fixed vertex*. This is the first strong spatial mixing result for colorings of graphs with unbounded maximum degree. Our analysis of strong spatial mixing establishes a block-wise correlation decay instead of the standard point-wise decay, which may be of interest by itself, especially for graphs with unbounded degree.

1 Introduction

A proper q-coloring of a graph G is an assignment of q colors $\{1, 2, \ldots, q\}$ to the vertices so that adjacent vertices receive different colors. Each coloring corresponds to a configuration in the q-state zero-temperature antiferromagnetic Potts model. The uniform probability space, known as the Gibbs measure, of proper q-colorings of the graph, receives extensive studies from both Theoretical Computer Science and Statistical Physics.

An important question concerned with the Gibbs measure is about the mixing rate of Glauber dynamics, usually formulated as: on graphs with maximum degree d, assuming $q \geq \alpha d + \beta$, the lower bounds for α and β to guarantee rapidly mixing of the Glauber dynamics over proper q-colorings. (See [9] for a survey.)

Recently, much attention has been focused on the spatial mixing (correlation decay) aspect of the Gibbs measure, which is concerned with the case where the site-to-boundary correlations in the Gibbs measure decay exponentially to zero with distance. In Statistical Physics, spatial mixing implies the uniqueness of infinite-volume Gibbs measure. The notion of *strong spatial mixing* was introduced in Theoretical Computer Science by Weitz [18]. Here, the exponential decay of site-to-boundary correlations is required to hold even conditioning on an arbitrarily fixed boundary. Strong spatial mixing is interesting to Computer

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Science because it may imply efficient approximation algorithms for counting and sampling. This implication was fully understood for two-state spin systems. For multi-state spin systems such as coloring, this algorithmic implication of strong spatial mixing is only known for special classes of graphs, such as neighborhood-amenable (slow-growing) graphs [13]. Strong spatial mixing of proper q-coloring has been proved for classes of degree-bounded graphs, including regular trees [12], lattices graphs [13], and finally the general degree-bounded triangle-free graphs [11], all with the same $\alpha > \alpha^*$ bound where $\alpha^* = 1.763...$ is the unique solution to $x^x = \mathrm{e}$.

All these temporal and spatial mixing results are established for graphs with bounded *maximum degree*. It is then natural to ask what happens for classes of graphs with bounded *average degree*. A natural model for the "typical" graphs with bounded average degree d is the Erdös-Rényi random graph $G(n, d/n)$. In this model, the Gibbs measure of proper q-colorings becomes more complicated because the maximum degree is unbounded and the decision of colorability is nontrivial. Nevertheless, it was discovered in [5] that for $G(n, d/n)$ the rapid mixing of (block) Glauber dynamics over the proper q-colorings can be guaranteed by a $q = O(\log\log n/\log\log\log n)$, much smaller than the maximum degree of $G(n, d/n)$. This upper bound on the number of colors was later reduced to a constant $q = \mathrm{poly}(d)$ in [8] and independently in [15,16], and very recently to a linear $q \geq \alpha d + \beta$ with $\alpha = 5.5$ in [7].

On the spatial mixing side, the strong spatial mixing of the models which are simpler than coloring has been studied on random graph $G(n, d/n)$, or other classes of graphs with bounded average degree. Recently in [17], such average-degree based strong spatial mixing is established for the independent sets of graphs with bounded *connective constant*. Since $G(n, d/n)$ has connective constant $\approx d$ with high probability, this result is naturally translated to $G(n, d/n)$.

It is then an important open question to ask about the conditions for the spatial mixing of colorings of graphs with bounded average degree. The following simple example shows that this can be very hard to achieve: Consider a long path of ℓ vertices, each adjacent to $q-2$ isolated vertices, where q is the number of colors. When the path is sufficiently long, the connective constant of this graph can be arbitrarily close to 1. However, colors of those isolated vertices can be properly fixed to make the remaining path effectively a 2-coloring instance, which certainly has long-range correlation, refuting the existence of strong spatial mixing.

More devastatingly, it is easy to see that for any constant q, with high probability the random graph $G(n, d/n)$ contains a path of length $\ell = \Theta(\log n)$ in which every vertex has degree $q-2$. As in the above example, even in a weaker sense of site-to-site correlation which was considered in [13], this forbids the strong spatial mixing up to a distance $\Theta(\log n)$. Meanwhile, it is well known that the diameter of $G(n, d/n)$ is $O(\log n)$ with high probability. So the strong spatial mixing of colorings of random graph $G(n, d/n)$ cannot hold except for a narrow range of distances in $\Theta(\log n)$.

In this case, inspired by the studies of spatial mixing in rooted trees, where only the decay of correlation to the root is considered, we propose to study the strong spatial mixing *with respect to a fixed vertex*, instead of all vertices.

Assumption 1. *We make following assumptions:*

- $d > 1$ *is fixed, and* $q \geq \alpha d + \beta$ *for* $\alpha > 2$ *and sufficiently large* $\beta = O(1)$ *(*$\beta \geq 23$ *is fine);*
- $v \in V$ *is arbitrarily fixed and* $G = (V, E)$ *is a random graph drawn from* $G(n, d/n)$*, where* n *is sufficiently large.*

Note that vertex v is fixed independently of the sampling of random graph. With these assumptions we prove the following theorem.

Theorem 2. *Let* q, v *and* G *satisfy Assumption 1, and* $t(n) = \omega(1)$ *an arbitrary super-constant function. With high probability,* G *is* q*-colorable and the following holds: for any region* $R \subset V$ *containing* v*, whose vertex boundary is* ∂R*, for any feasible colorings* $\sigma, \tau \in [q]^{\partial R}$ *partially specified on* ∂R *which differ only at vertices that are at least* $t(n)$ *distance away from* v *in* G*, for some constants* $C_1, C_2 > 0$ *depending only on* d *and* q*, it holds that*

$$|\Pr[c(v) = x \mid \sigma] - \Pr[c(v) = x \mid \tau]| \leq C_1 \exp(-C_2 \cdot \mathrm{dist}(v, \Delta)),$$

for a uniform random proper q*-coloring* c *of* G *and any* $x \in [q]$*, where* $\Delta \subset \partial R$ *is the vertex set on which* σ *and* τ *differ, and* $\mathrm{dist}(v, \Delta)$ *denotes the shortest distance in* G *between* v *and any vertex in* Δ*.*

This is the first strong spatial mixing result for colorings of graphs with unbounded maximum degree. Our technique is developed upon the error function method introduced in [11], which uses a cleverly designed error function to measure the discrepancy of marginal distributions, and the strong spatial mixing is implied by an exponential decay of errors measured by this function.

In all existing techniques for strong spatial mixing of colorings, when the degree of a vertex is unbounded, a multiplicative factor of ∞ is contributed to the decay of correlation, which unavoidably ruins the decay. However, in the real case for colorings of graphs with unbounded degree, a large-degree vertex may at most locally "freeze" the coloring, rather than nullify the existing decay of correlation. This limitation on the effect of large-degree vertex has not been addressed by any existing techniques for spatial mixing.

We address this issue by considering a block-wise correlation decay, so that within a block the coloring might be "frozen", but between blocks, the decay of correlation is as in that between vertices in the degree-bounded case. This analysis of block-wise correlation decay can be seen as a spatial analog to the block dynamics over colorings of random graphs, and is the first time that such an idea is used in the analysis of spatial mixing.

Related Work. As one of the most important random CSP, the decision problem of coloring sparse random graphs has been extensively studied, e.g. in [1, 3].

Monte Carlo algorithms for sampling random coloring in sparse random graphs were studied in [4,7,8,15,16], and in [6], a non-Monte-Carlo algorithm was given for the same problem which uses less colors but has worse error dependency than the Monte-Carlo algorithms. In [10,14] the correlation decay on computation tree for coloring was studied which implies FPTAS for counting coloring.

2 Preliminaries

Graph coloring. Let $G = (V, E)$ be an undirected graph. For each vertex $v \in V$, let $d_G(v)$ denote the degree of v. For any $u, v \in V$, let $\text{dist}_G(u, v)$ denote the distance between u and v in G; and for any vertex sets $S, T \subseteq V$, let $\text{dist}_G(u, S) = \min_{v \in S} \text{dist}_G(u, v)$ and $\text{dist}_G(S, T) = \min_{u \in S, v \in T} \text{dist}_G(u, v)$. The subscripts can be omitted if graph G is assumed in context. For any vertex set $S \subset V$, we use $\partial S = \{v \notin S \mid uv \in E, u \in S\}$ to denote the *vertex boundary* of S, and use $\delta S = \{uv \in E \mid u \in S, v \notin S\}$ to denote the *edge boundary* of S.

We consider the *list-coloring problem*, which is a generalization of q-coloring problem. Let $q > 0$ be a finite integer, a pair $(G, \mathcal{L})$ is called a *list-coloring instance* if $G = (V, E)$ is an undirected graph, and $\mathcal{L} = (L(v) : v \in V)$ is a sequence of lists where for each vertex $v \in V$, $L(v) \subseteq [q]$ is a list of colors from $[q] = \{1, 2, \ldots, q\}$ associated with vertex v. A $\sigma \in [q]^V$ is a *proper coloring* of $(G, \mathcal{L})$ if $\sigma(v) \in L(v)$ for every vertex $v \in V$ and no two adjacent vertices in G are assigned with the same color by σ. A list-coloring instance $(G, \mathcal{L})$ is said to be *feasible* or *colorable* if there exists a proper coloring of $(G, \mathcal{L})$. A coloring can also be partially specified on a subset of vertices in G. For $S \subseteq V$, let $L(S) = \{\sigma \in [q]^S \mid \forall v \in V, \sigma(v) \in L(v)\}$ denote the set of all possible colorings (not necessarily proper) of the vertices in S. A coloring $\sigma \in L(S)$ partially specified on a subset $S \subseteq V$ of vertices is said to be *feasible* if there is a proper coloring τ of $(G, \mathcal{L})$ such that σ and τ are consistent over set S. A coloring $\sigma \in L(S)$ partially specified on a subset $S \subseteq V$ of vertices is said to be *proper* or *locally feasible* if σ is a proper coloring of $(G[S], \mathcal{L}_S)$ where $G[S]$ is the subgraph of G induced by S and $\mathcal{L}_S = (L(v) : v \in S)$ denotes the sequence $\mathcal{L}$ of lists restricted on set S of vertices. For any $S \subseteq V$, we use $L^*(S)$ to denote the set of proper colorings of S.

When $L(v) = [q]$ for all vertices $v \in V$, a list-coloring instance $(G, \mathcal{L})$ becomes an instance for q-coloring, which we denote as $(G, [q])$.

Self-avoiding Walk (SAW) Tree. Given a graph $G(V, E)$ and a vertex $v \in V$, a tree T rooted by v can be naturally constructed from all self-avoiding walks starting from v so that each walk corresponds to a vertex in T, and each walk p is the parent of walks (p, u) where $u \in V$ is a vertex. We use $T_{\text{SAW}}(G, v) = T$ to denote this tree constructed as above, and call it a *self-avoiding walk tree (SAW)* of graph G.

Gibbs Measure and Strong Spatial Mixing. A feasible list-coloring instance $(G, \mathcal{L})$ gives rise to a natural probability distribution $\mu = \mu_{G,\mathcal{L}}$, which is the uniform distribution over all proper list-colorings. This distribution μ is also called the *Gibbs* *measure* of list-colorings. We also a notation of $\mathrm{P}_{G,\mathcal{L}}(\text{event}(c)) = \Pr[\text{event}(c)]$ to evaluate probability of an event defined on a uniform random proper coloring c of $(G, \mathcal{L})$. Let $B \subset V$ and $\Lambda \subset V$. For any feasible coloring $\sigma \in L(\Lambda)$ partially specified on vertex set Λ, we use $\mu_B^\sigma = \mu_{G,\mathcal{L},B}^\sigma$ to denote the marginal distribution over colorings of vertices in B conditioning on that the coloring of vertices in Λ is as specified by σ. And when $B = \{v\}$, we write $\mu_v^\sigma = \mu_{G,\mathcal{L},v}^\sigma = \mu_{G,\mathcal{L},\{v\}}^\sigma$. The list-coloring instance $(G, \mathcal{L})$ in the subscripts can be omitted if it is assumed in context. Formally, for a uniformly random proper coloring c of $(G, \mathcal{L})$, we have

$$\forall x \in L(v), \quad \mu_v^\sigma(x) = \mathrm{P}_{G,\mathcal{L}}(c(v) = x \mid \sigma),$$
$$\forall \pi \in L(B), \quad \mu_B^\sigma(\pi) = \mathrm{P}_{G,\mathcal{L}}(c(B) = \pi \mid \sigma).$$

The notion strong spatial mixing is introduced in [18,19] for independent sets and extended to colorings in [11,13].

Definition 3 (Strong Spatial Mixing). *The Gibbs measure on proper q-colorings of a family $\mathcal{G}$ of finite graphs exhibits* strong spatial mixing (SSM) *if there exist constants $C_1, C_2 > 0$ such that for any graph $G(V, E) \in \mathcal{G}$, any $v \in V, \Lambda \subseteq V$, and any two feasible q-colorings $\sigma, \tau \in [q]^\Lambda$, we have*

$$\|\mu_v^\sigma - \mu_v^\tau\|_{\mathrm{TV}} \le C_1 \exp(-C_2 \mathrm{dist}(v, \Delta)),$$

where $\Delta \subseteq \Lambda$ is the subset on which σ and τ differ, and $\|\cdot\|_{\mathrm{TV}}$ is the total variation distance.

When the exponential bound relies on $\mathrm{dist}(v, \Lambda)$ instead of $\mathrm{dist}(v, \Delta)$, the definition becomes *weak spatial mixing (WSM).* The difference is SSM requires the exponential correlation decay continues to hold even conditioning on the coloring of a subset $\Lambda \setminus \Delta$ of vertices being arbitrarily (but feasibly) specified.

Random graph model. The Erdös-Rényi random graph $G(n, p)$ is the graph with n vertices V and random edges E where for each pair $\{u, v\}$, the edge uv is chosen independently with probability p. We consider $G(n, d/n)$ with fixed $d > 1$.

We say an event occurs *with high probability (w.h.p.)* if the probability of the event is $1 - o(1)$.

3 Correlation Decay along Self-avoiding Walks

In this section, we analyze the propagation of errors between marginal distributions measured by a special norm introduced in [11] in general degree-unbounded graphs. Throughout this section, we assume $(G, \mathcal{L})$ to be a list-coloring instance with $G = (V, E)$ and $\mathcal{L} = (L(v) : v \in V)$ where each $L(v) \subseteq [q]$.

The following error function is introduced in [11].

Definition 4 (Error Function). *Let $\mu_1 : \Omega \to [0, 1]$ and $\mu_2 : \Omega \to [0, 1]$ be two probability measures over the same sample space Ω. We define*

$$\mathcal{E}(\mu_1, \mu_2) = \max_{x,y \in \Omega} \left(\log\left(\frac{\mu_1(x)}{\mu_2(x)}\right) - \log\left(\frac{\mu_1(y)}{\mu_2(y)}\right) \right),$$

with the convention that $0/0 = 1$ and $\infty - \infty = 0$.

We assume $(G, \mathcal{L})$ to be feasible so that for vertex set $B \subset V$ and feasible colorings $\sigma, \tau \in L(\Lambda)$ of vertex set $\Lambda \subset V$, the marginal probabilities μ_B^σ and μ_B^τ are well-defined. The strong spatial mixing is proved by establishing a propagation of errors $\mathcal{E}(\mu_B^\sigma, \mu_B^\tau)$. Note that unlike in bounded-degree graphs, in general the value of $\mathcal{E}(\mu_B^\sigma, \mu_B^\tau)$ can be infinite, which occurs when the possibility of a particular coloring of B is changed by conditioning on σ and τ. This is avoided when a vertex cut with certain "permissive" property separating B from the boundary. The following proposition is proved in the full version.

Proposition 5. *If there is a $S \subset V \setminus (B \cup \Lambda)$ such that $|L(v)| > d(v) + 1$ for every $v \in S$ and removing S disconnects B and Λ, then $\mathcal{E}(\mu_B^\sigma, \mu_B^\tau)$ is finite for any feasible colorings $\sigma, \tau \in L(\Lambda)$.*

This motivates the following definition of permissive vertex and vertex set.

Definition 6. *Given a list-coloring instance $(G, \mathcal{L})$, a vertex v is said to be* permissive *in $(G, \mathcal{L})$ if for all neighbors u of v and $u = v$, it holds that $|L(u)| > d(u) + 1$. A set S of vertices is said to be permissive if all vertices in S are permissive.*

Let $T = T_{\text{SAW}}(G, v)$ be the self-avoiding walk tree of graph G expanded from vertex v. Recall that every vertex u in T can be naturally identified (many-to-one) with the vertex in G at which the corresponding self-avoiding walk ends (which we also denote by the same letter u).

Definition 7. *Given a list-coloring instance $(G, \mathcal{L})$, let $v \in V$, $T = T_{\text{SAW}}(G, v)$, and S a set of vertices in T. Suppose that the root v has m children $v_1, v_2, \ldots, v_m$ in T and for $i = 1, 2 \ldots, m$, let T_i denote the subtree rooted by v_i. The quantity $\mathcal{E}_{T,\mathcal{L},S}$ is recursively defined as follows*

$$\mathcal{E}_{T,\mathcal{L},S} = \begin{cases} \sum_{i=1}^{m} \delta(v_i) \cdot \mathcal{E}_{T_i,\mathcal{L},S} & \text{if } v \notin S, \\ 3q & \text{if } v \in S, \end{cases}$$

where $\delta(u)$ is a piecewise function defined as that for any vertex u in T,

$$\delta(u) = \begin{cases} \frac{1}{|L(u)|) - d_G(u) - 1} & \text{if } |L(u)| > d_G(u) + 1, \\ 1 & \text{otherwise}, \end{cases}$$

where $d_G(v)$ is the degree in the original graph G instead of the degree in SAW-tree T.

In particular, when $(G, \mathcal{L})$ is a q-coloring instance $(G, [q])$, we denote this quantity as $\mathcal{E}_{T,[q],S}$.

To state the main theorem of this section, we need one more definition.

Definition 8. *Let $G = (V, E)$, $v \in V$, $\Delta \subset V$, and $T = T_{\text{SAW}}(G, v)$. A set S of vertices in T is a* cutset *in T for v and Δ if: (1) no vertex in S is identified to*

v or any vertex u with $\operatorname{dist}(u, \Delta) < 2$ *by* $T_{\text{SAW}}(G, v)$*; and (2) any self-avoiding walk from v to a vertex in Δ must intersect S in T. A cutset S in T for v and Δ is said to be permissive in $(G, \mathcal{L})$ if every vertex in S is identified with a permissive vertex in $(G, \mathcal{L})$ by $T_{\text{SAW}}(G, v)$.*

The following theorem is the main theorem of this section, which bounds the error function $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau)$ by the $\mathcal{E}_{T,\mathcal{L},S}$ defined in Definition 7 when there is a good cutset in the SAW tree.

Theorem 9. *Let $(G, \mathcal{L})$ be a feasible list-coloring instance where $G = (V, E)$ and $\mathcal{L} = (L(v) \subseteq [q] : v \in V)$. Let $v \in V$, $\Lambda \subset V$ and $\Delta \subseteq \Lambda$ be arbitrary, and $T = T_{\text{SAW}}(G, v)$. If there is a permissive cutset S in T for v and Δ, then for any feasible colorings $\sigma, \tau \in L(\Lambda)$ which differ only on Δ, it holds that*

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \mathcal{E}_{T,\mathcal{L},S}.$$

This theorem is implied by the following weak spatial mixing version of the theorem.

Lemma 10. *Let $(G, \mathcal{L})$ be a feasible list-coloring instance where $G = (V, E)$ and $\mathcal{L} = (L(v) \subseteq [q] : v \in V)$. Let $v \in V$ and $\Delta \subseteq \Lambda$ be arbitrary, and $T = T_{\text{SAW}}(G, v)$. If there is a permissive cutset S in T for v and Δ, then for any feasible colorings $\sigma, \tau \in L(\Delta)$, it holds that*

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \mathcal{E}_{T,\mathcal{L},S}.$$

The implication from Lemma 10 to Theorem 9 is quite standard, whose proof is in the full version. It now remains to prove Lemma 10.

3.1 The Block-Wise Correlation Decay

Now our task is to prove Lemma 10. This is done by establishing the decay of $\mathcal{E}(\mu_B^\sigma, \mu_B^\tau)$ along walks among blocks B with the following good property.

Definition 11. *Given a list-coloring instance $(G, \mathcal{L})$, a vertex set $B \subseteq V$ is a* permissive block around *v in $(G, \mathcal{L})$ if $v \in B$ and $|L(u)| > d_G(u) + 1$ for every vertex u in the vertex boundary ∂B.*

For permissive blocks B, a coloring of B is globally feasible if and only if it is locally feasible (i.e. proper on B).

Lemma 12. *Let $\Delta \subset V$ and $B \subset V$ a permissive block such that* $\operatorname{dist}(B, \Delta) \geq 2$. *Then for any feasible coloring $\sigma \in L(\Delta)$, for any coloring $\pi \in L(B)$, it holds that $\mu_B^\sigma(\pi) > 0$ if and only if π is proper on B.*

Proof. Let $S = \partial B$. Note that with $\operatorname{dist}(B, \Delta) \geq 2$ and S must be a vertex cut separating B and Δ. Then the lemma can be proved by the same argument as in the proof of Proposition 5.

Notations. We now define some notations which are used throughout this section. Let $B \subset V$ be a permissive block in a feasible list-coloring instance $(G, \mathcal{L})$. Let $\delta B = \{uw \in E \mid u \in B \text{ and } w \notin B\}$ be the edge boundary of B. We enumerate these boundary edges as $\delta B = \{e_1, e_2, \ldots, e_m\}$. For $i = 1, 2, \ldots, m$, we assume $e_i = u_i v_i$ where $u_i \in B$ and $v_i \notin B$. Note that in this notation more than one u_i or v_i may refer to the same vertex in G. Let $G_B = G[V \setminus B]$ be the subgraph of G induced by vertex set $V \setminus B$. For a coloring $\pi \in L(B)$ and $1 \leq i \leq m$, we denote $\pi_i = \pi(u_i)$. For $1 \leq i \leq m$ and $\pi, \rho \in L(B)$, let $\mathcal{L}_{i,j,\pi,\rho} = (L'(v) : v \in V \setminus B)$ be obtained from $\mathcal{L}$ by removing the color π_k from the list $L(v_k)$ for all $k < i$ and removing the color ρ_k from the list $L(v_k)$ for all $k > i$ (if any of these lists do not contain the respective color then no change is made to them).

With this notation, the following lemma (proved in the full version) generalizes a recursion introduced in [11] for bounded-degree graphs to general graphs.

Lemma 13. *Let $(G, \mathcal{L})$ be a feasible list-coloring instance, $B \subset V$ a permissive block with edge boundary $\delta B = \{e_1, e_2, \ldots, e_m\}$ where $e_i = u_i v_i$ for each $i = 1, 2, \ldots, m$, and $\pi, \rho \in L^*(B)$ any two proper colorings of B. For every $1 \leq i \leq m$,*

- *if a vertex $u \notin B$ is permissive in $(G, \mathcal{L})$, then it is permissive in the new instance $(G_B, \mathcal{L}_{i,\pi,\rho})$;*
- *the new instance $(G_B, \mathcal{L}_{i,\pi,\rho})$ is feasible.*

For any feasible coloring $\sigma \in L(\Delta)$ of a vertex set $\Delta \subset V$ with $\mathrm{dist}(B, \Delta) \geq 2$, *we have*

$$\frac{\mathrm{P}_{G,\mathcal{L}}(c(B) = \pi \mid \sigma)}{\mathrm{P}_{G,\mathcal{L}}(c(B) = \rho \mid \sigma)} = \prod_{i=1}^{m} \frac{1 - \mathrm{P}_{G_B,\mathcal{L}_{i,\pi,\rho}}(c(v_i) = \pi_i \mid \sigma)}{1 - \mathrm{P}_{G_B,\mathcal{L}_{i,\pi,\rho}}(c(v_i) = \rho_i \mid \sigma)}.$$

The following marginal bounds are standard and are proved in full version.

Lemma 14. *Given a feasible list-coloring instance $(G, \mathcal{L})$, if vertex v has $|L(v)| > d(v) + 1$ and $v \notin \Delta$, then for any feasible coloring $\sigma \in L(\Delta)$ and any $x \in L(v)$, we have*

$$\mathrm{P}_{G,\mathcal{L}}(c(v) = x \mid \sigma) \leq \frac{1}{|L(v)| - d(v)}.$$

If vertex v is permissive in $(G, \mathcal{L})$ and $\mathrm{dist}(v, \Delta) \geq 2$, *then for any feasible coloring $\sigma \in L(\Delta)$ and any $x \in L(v)$, we have*

$$\mathrm{P}_{G,\mathcal{L}}(c(v) = x \mid \sigma) \geq \frac{1}{|L(v)| 2^{d(v)}}.$$

The recursion in Lemma 13 can imply the following bound for the block-wise decay of error function $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau)$. The proof generalizes the analysis of the point-wise decay in degree-bounded graphs in [11], and is put to the full version.

Lemma 15. *Let $(G, \mathcal{L})$ be a feasible list-coloring instance, $v \in V$ and $B \subset V$ a permissive block around v with edge boundary $\delta B = \{e_1, e_2, \ldots, e_m\}$ where*

$e_i = u_i v_i$ for each $i = 1, 2, \ldots, m$. Let $\Delta \subset V$ be a vertex set with $\mathrm{dist}(B, \Delta) \geq 2$*, and $\sigma, \tau \in L(\Delta)$ any two feasible colorings of Δ. Assume $\pi, \rho \in L^*(B)$ to be two proper colorings of B achieving the maximum in the error function:*

$$\mathcal{E}(\mu_B^\sigma, \mu_B^\tau) = \max_{\pi,\rho \in L^*(B)} \left(\log\left(\frac{\mu_B^\sigma(\pi)}{\mu_B^\tau(\pi)}\right) - \log\left(\frac{\mu_B^\sigma(\rho)}{\mu_B^\tau(\rho)}\right)\right).$$

It holds that

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \sum_{i=1}^{m} \frac{1}{|L(v_i)| - d(v_i) - 1} \cdot \mathcal{E}(\mu_i^\sigma, \mu_i^\tau),$$

where $\mu_i^\sigma = \mu^\sigma_{G_B,\mathcal{L}_{i,\pi,\rho},v_i}$ and $\mu_i^\tau = \mu^\tau_{G_B,\mathcal{L}_{i,\pi,\rho},v_i}$ are the respective marginal distributions of coloring of vertex v_i conditioning on σ and τ in the new list-coloring instance $(G_B, \mathcal{L}_{i,\pi,\rho})$.

With the above block-wise decay, we are now ready to prove Lemma 10, which implies Theorem 9.

Proof (Proof of Lemma 10). Given a feasible list-coloring instance $(G, \mathcal{L})$ and a vertex v, let $T = T_{\mathrm{SAW}}(G, v)$ and S a permissive cutset in T separating v and Δ. We consider the following procedure:

1. Let B be the *minimal* permissive block around v with edge boundary $\delta B = \{e_1, e_2, \ldots, e_m\}$, where $e_i = u_i v_i$ for $i = 1, 2, \ldots, m$ (note that more than one u_i or v_i may refer to the same vertex). By Lemma 15, we have

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \sum_{i=1}^{m} \frac{1}{|L(v_i)| - d(v_i) - 1} \cdot \mathcal{E}(\mu_i^\sigma, \mu_i^\tau), \tag{1}$$

   where $\mu_i^\sigma = \mu^\sigma_{G_B,\mathcal{L}_{i,\pi,\rho},v_i}$ and $\mu_i^\tau = \mu^\tau_{G_B,\mathcal{L}_{i,\pi,\rho},v_i}$ are the respective marginal distributions at v_i in the new list-coloring instance $(G_B, \mathcal{L}_{i,\pi,\rho})$ for the $\pi, \rho \in L^*(B)$ defined in Lemma 15. By Lemma 13, all these new list-coloring instances are feasible.
2. We identify each v_i with a distinct self-avoiding walk in G from v to v_i through only vertices in B and approaching v_i via the edge $e_i = u_i v_i$. Such self-avoiding walk must exist or otherwise B is not minimal. If there are more than one such self-avoiding walk for a v_i, choose an arbitrary one to identify v_i with. We use w_i to denote this walk to v_i.
   Note that along every such self-avoiding walk w_i from v to v_i, all vertices u except v and v_i must have $|L(u)| \leq d(u) + 1$ in $(G, \mathcal{L})$ or otherwise B is not minimal. Thus by Definition 7, in quantity $\mathcal{E}_{T,\mathcal{L},S}$, along every walk w_i from v to v_i, at each intermediate vertex $u \notin \{v, v_i\}$, only a factor of $\delta(u) = 1$ is multiplied in $\mathcal{E}_{T,\mathcal{L},S}$, so we have

$$\mathcal{E}_{T,\mathcal{L},S} \geq \sum_{i=1}^{m} \frac{1}{|L(v_i)| - d(v_i) - 1} \mathcal{E}_{T_{w_i},\mathcal{L},S}, \tag{2}$$

   where T_{w_i} denotes the subtree of the SAW tree T rooted by the self-avoiding walk w_i.

3. For each $1 \leq i \leq m$, if the self-avoiding walk w_i corresponds to a vertex in the permissive cutset S in the SAW tree T, then v_i itself must be permissive in $(G, \mathcal{L})$ and $\text{dist}(v_i, \Delta) \geq 2$, both of which continue to hold in the new instance $(G_B, \mathcal{L}_{i,\pi,\rho})$. By Lemma 14, we have $\mu_i^\sigma(x), \mu_i^\tau(x) \in \left[\frac{1}{q2^{q-2}}, \frac{1}{2}\right]$ for any $x \in L(v_i)$, thus

$$\mathcal{E}\left(\mu_i^\sigma, \mu_i^\tau\right) \leq 2\left(\ln q + q \ln 2\right) \leq 3q; \tag{3}$$

and if otherwise, w_i is not in S in the SAW tree T, we repeat from the first step for vertex v_i in the new instance $(G_B, \mathcal{L}_{i,\pi,\rho})$.

We can then apply an induction to prove that $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \mathcal{E}_{T,\mathcal{L},S}$, with (3) as basis, and (1) and (2) as induction step. We only need to clarify that each application of (1) creates new instances $(G_B, \mathcal{L}_{i,\pi,\rho})$, while $\mathcal{E}_{T,\mathcal{L},S}$ is defined using only the original instance $(G, \mathcal{L})$. This will not cause any issue because by Lemma 13, every new instance $(G_B, \mathcal{L}_{i,\pi,\rho})$ created during this procedure must be feasible. Moreover, the operation the new instance $(G_B, \mathcal{L}_{i,\pi,\rho})$ applying on $(G, \mathcal{L})$ never makes any vertex less permissive, and never increases the multiplicative factor $\frac{1}{|L(v_i)|-d(v_i)-1}$ in the recursion.

4 Strong Spatial Mixing on Random Graphs

In this section, we prove Theorem 2, the strong spatial mixing of q-coloring of random graph $G(n, d/n)$ with respect to a fixed vertex. The theorem is proved by applying Theorem 9 to random graph $G(n, d/n)$. The following lemma states the existing with high probability of a good permissive cutset in the self-avoiding walk tree of a random graph $G(n, d/n)$. The proof is in the full version.

Lemma 16. *Let $d > 1$, $q \geq \alpha d + \beta$ for $\alpha > 2$ and $\beta \geq 23$, and $t(n) = \omega(1)$ an arbitrary super-constant function. Let $v \in V$ be arbitrarily fixed and $G = (V, E)$ a random graph draw from $G(n, d/n)$. The following event holds with high probability: for any $t(n) \leq t \leq \frac{\ln n}{\ln d}$ and any vertex set $\Delta \subset V$ satisfying $\text{dist}_G(v, \Delta) > 2t$, there exists a permissive cutset S in $T = T_{\text{SAW}}(G, v)$ for v and Δ such that $t \leq \text{dist}_T(v, u) < 2t$ for all vertices $u \in S$.*

We then observe that the quantity $\mathcal{E}_{T,\mathcal{L},S}$ decays fast on average. The proof is also in the full version.

Lemma 17. *Let $f_q(x)$ be a piecewise function defined as*

$$f_q(x) = \begin{cases} \frac{1}{q-x-1} & \text{if } x \leq q-2, \\ 1 & \text{otherwise.} \end{cases}$$

Let X be a random variable distributed according to binomial distribution $B(n, \frac{d}{n})$ where $d = o(n)$. For $q \geq 2d + 4$, it holds that $\mathbb{E}\left[f_q(X)\right] < \frac{1}{d}$.

We then prove a strong spatial mixing theorem with the norm of error function $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau)$.

Lemma 18. *Let $d > 1$, $q \geq \alpha d + \beta$ for $\alpha > 2$ and $\beta \geq 23$, and $t(n) = \omega(1)$ an arbitrary super-constant function. Let $v \in V$ be arbitrarily fixed and $G = (V, E)$ a random graph draw from $G(n, d/n)$. There exist constants $C_1, C_2 > 0$ depending only on d and q such that with high probability G is q-colorable and*

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq C_1 \exp(-C_2 \mathrm{dist}(v, \Delta))$$

for any feasible q-colorings $\sigma, \tau \in [q]^\Lambda$ partially specified on a subset $\Lambda \subset V$ of vertices, such that σ and τ differ only on a subset $\Delta \subseteq \Lambda$ with $\mathrm{dist}(v, \Delta) \geq t(n)$.

Proof (Sketch of Proof). We only give a sketch of the proof. The detailed proof is given in the full version.

Fix $v \in V$. Let $T = T_{\mathrm{SAW}}(G, v)$ be the self-avoiding walk tree of G. Fix an arbitrary $t(n) \leq t \leq \frac{\ln n}{\ln d}$. Consider $\mathcal{E}_t = \max_S \mathcal{E}_{T,[q],S}$ where the maximum is taken over all vertex set S in T satisfying $t \leq \mathrm{dist}_T(v, u) < 2t$ for all $u \in S$. By enumerating all self-avoiding walks $P = (v, v_1, \ldots, v_k)$ from v to a vertex $v_k \in S$, we have

$$\mathbb{E}\left[\mathcal{E}_t\right] \leq 3q \sum_{k=t}^{2t-1} d^k \cdot \mathbb{E}\left[\prod_{i=1}^{k} f_q(d_G(v_i)) \;\middle|\; P = (v, v_1, \ldots, v_k) \text{ is a path}\right],$$

where the function $f_q(x)$ is as defined in Lemma 17. We then calculate the expectations. Fix a tuple $P = (v, v_1, \ldots, v_k)$. We construct an independent sequence whose product dominates the $\prod_{i=1}^k f_q(d_G(v_i))$.

Conditioning on $P = (v, v_1, \ldots, v_k)$ being a path in G. Let $X_1, X_2, \ldots, X_k$ be such that each X_i is the number of edges between v_i and vertices in $V \setminus \{v_1, \ldots, v_k\}$; and let Y be the number of edges between vertices in $\{v_1, \ldots, v_k\}$ except for the edges in the path $P = (v, v_1, \ldots, v_k)$. Then $X_1, X_2, \ldots, X_k, Y$ are mutually independent binomial random variables, and for each v_i in the path we have $d_G(v_i) = X_i + 2 + Y_i$ for some $Y_1 + Y_2 + \cdots + Y_k = 2Y$.

Due to the property of function $f_q(x)$ we can bound that

$$\prod_{i=1}^{k} f_q(d_G(v_i)) = \prod_{i=1}^{k} f_q(X_i + 2 + Y_i) \leq 4^Y \prod_{i=1}^{k} f_{q-2}(X_i).$$

Since $X_1, X_2, \ldots, X_k, Y$ are mutually independent conditioning on P is a path,

$$\mathbb{E}\left[\prod_{i=1}^{k} f_q(d_G(v_i)) \;\middle|\; P \text{ is a path}\right] \leq \mathbb{E}\left[4^Y \prod_{i=1}^{k} f_{q-2}(X_i)\right] \leq \mathbb{E}\left[4^Y\right] \mathbb{E}\left[f_{q-2}(X)\right]^k,$$

where $\mathbb{E}\left[f_{q-2}(X)\right]$ can be upper bounded by Lemma 17, and $\mathbb{E}\left[4^Y\right]$ by the binomial theorem. Then a calculation gives

$$\mathbb{E}\left[\mathcal{E}_t\right] \leq 3q \sum_{k=t}^{2t-1} \mathbb{E}\left[\prod_{i=1}^{k} f_q(d_G(v_i)) \;\middle|\; P \text{ is a path}\right] \leq \exp\left(-\Omega(t)\right).$$

By Markov's inequality and union bound, with high probability we have $\mathcal{E}_t \leq \exp\left(-\Omega(t)\right)$ for all $t(n) \leq t \leq \frac{\ln n}{\ln d}$.

By [1], w.h.p. G is q-colorable. By Lemma 16, w.h.p. we have good permissive cutset S satisfying the conditions in Lemma 16, which by Theorem 9, implies that $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \mathcal{E}_{T,[q],S} \leq \mathcal{E}_t \leq \exp(-\Omega(t))$. By [2], w.h.p. the diameter of G is in $O(\frac{\ln n}{\ln d})$, thus we can choose $t = \Theta(\mathrm{dist}(v, \Delta))$ with a $t(n) \leq t \leq \frac{\ln n}{\ln d}$, which gives us $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \exp(-\Omega(\mathrm{dist}(v, \Delta)))$.

With Lemma 18, the proof of Theorem 2 is immediate, which is in the full version.

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