Efficiency Guarantees in Auctions with Budgets

Shahar Dobzinski[1] and Renato Paes Leme[2]

[1] Weizmann Institute
dobzin@gmail.com
[2] Google Research NYC
renatoppl@google.com

Abstract. In settings where players have limited access to liquidity, represented in the form of budget constraints, efficiency maximization has proven to be a challenging goal. In particular, the social welfare cannot be approximated by a better factor than the number of players. Therefore, the literature has mainly resorted to Pareto-efficiency as a way to achieve efficiency in such settings. While successful in some important scenarios, in many settings it is known that either exactly one truthful auction that always outputs a Pareto-efficient solution, or that no truthful mechanism always outputs a Pareto-efficient outcome. Moreover, since Pareto-efficiency is a binary property (is either satisfied or not), it cannot be circumvented as usual by considering approximations. To overcome impossibilities in important setting such as multi-unit auctions with decreasing marginal values and private budgets, we propose a new notion of efficiency, which we call *liquid welfare*. This is the maximum amount of revenue an omniscient seller would be able to extract from a certain instance. For the aforementioned setting, we give a deterministic $O(\log n)$-approximation for the liquid welfare in this setting.

We also study the liquid welfare in the traditional setting of additive values and public budgets. We present two different auctions that achieve a 2-approximation to the new objective. Moreover, we show that no truthful algorithm can guarantee an approximation factor better than 4/3 with respect to the liquid welfare.

1 Introduction

Auctions started being regularly held in Europe around the middle of the 18th century - originally being used to sell antiques and artwork in English auction houses and agricultural produce such as flowers in the Netherlands [21]. In the last decades of the 20th century, however, auctions started being deployed in an incredibly larger scale: privatization auctions in Eastern Europe, sale of spectrum in the US, auctions for rights to explore natural resources, among others. The new scale brought various new challenges – among them, how to deal with the disconnect between players *willingness to pay* (value) and *ability to pay* (budget). Studying the FCC auctions, Bulow, Levin and Milgrom [7] observe the following:

> *"According to our theory, it is bidders budgets, as opposed to their license values, that determine average prices in a spectrum auction."*

J. Esparza et al. (Eds.): ICALP 2014, Part I, LNCS 8572, pp. 392–404, 2014.

In fact, in any setting where the magnitude of the financial transactions is very large, budgets play a major role in the auction. One of the prime examples in internet advertisement – the choice of budget to spend is the first question asked to advertisers in the interface of Google Adwords, even before they are asked bids or keywords. Much work has been devoted to understanding the impact of budget constraints in sponsored search auctions [1,15,10]. For a more extensive discussion on the source of financial constraints, we refer to Che and Gale [9].

Despite being widely relevant in practice, it is not clear how to design *efficient* auctions in the presence of budget constraints. When efficiency means social welfare maximization, a folklore result states that, when n is the number of players, no incentive-compatible auction can do better than an n-approximation. Even weaker notions such as Pareto efficiency, are impossible to be achieved through incentive compatible mechanisms in important settings such as *private budgets* and *decreasing marginal values*. Our main goal in this paper is to search for alternative notions of efficiency that are more suitable for budgeted settings.

Due to its practical relevance, many theoretical investigations have been devoted to analyzing auctions for budget constrained agents. The impact of budgets on the revenue of standard auctions was analyzed in Che and Gale [9] and Benoit and Krishna [4], and mechanisms that optimize (exactly or approximately) revenue were designed by Laffont and Robert [17], Malakhov and Vohra [19], Pai and Vohra [23], Borgs et al [6] and Chawla et al [8].

When the objective is welfare efficiency rather then revenue, the literature has early stumbled upon impossibility results. The traditional social welfare measure, the sum of player's values for their outcomes, is known to be very poorly approximable under budget constraints, even when budgets are known to the auctioneer. This motivates the search for truthful auctions satisfying weaker notions of efficiency. Dobzinski, Lavi and Nisan [12] suggest studying Pareto-efficient auctions: the outcome of an auction is *Pareto-efficient* if there is no other outcome (allocation and payments) where no agent (bidders or auctioneer) is worse-off and at least one agent is better off. When budgets are public, they give a truthful and Pareto-efficient multi-unit auction based on Ausubel's clinching framework [2]. Furthermore, they show that this auction is the unique truthful auction that always produces Pareto-efficient solution. A sequence of follow-ups designed Pareto-efficient auctions for budget to different settings: Bhattacharya et al [5], Fiat et al [14], Colini-Baldeschi et al [10] and Goel et al [15,16].

Beyond Pareto Efficiency? Pareto efficiency has therefore emerged as the de-facto standard for measuring efficiency when bidders are budget constrained. Indeed, most of the aforementioned papers provide positive results by offering new auctions. Yet, this is far from being a complete solution from both theoretical and practical point of views. We now elaborate on this issue.

In a sense, the uniqueness result of Dobzinski et al [12] for public budgets – that shows that the clinching auction is the only Pareto efficient, truthful auction – may be viewed negatively. Rare are the cases in practice in which the designer sole goal is to obtain a Pareto efficient allocation. A more realistic view is that theory provides the designer a toolbox of complementary methods and techniques designed to obtain various different goals (efficiency, revenue maximization, fairness, computational efficiency, etc.) and balance between them. The composition of these tools as well as their adaptation to the specifics of the setting and fine tuning is the designer's task. A uniqueness result – although extremely appealing from a pure theoretical perspective – implies that the designer's toolbox contains only one tool, obviously an undesirable scenario.

Furthermore, although from a technical point of view the analysis of the existing algorithms is very challenging and the proof techniques are quite unique to each setting, the auctions themselves are all variants of the same basic clinching idea of Ausubel [2]. Again, it is obviously preferable to have more than just one bunny in the hat that will help us design auctions for these important settings.

The situation is obviously even more severe in more complicated settings, where even this lonely bunny is not available. For example, for private budgets and additive multi-unit auctions, an impossibility was given by Dobzinski et al [12], for heterogenous items and public budgets by Fiat, Leonardi, Saia and Sankowski [14] and Dütting, Henzinger and Starnberger [13] and for as multi-unit auctions with subadditive valuations and public budgets by Goel, Mirrokni and Paes Leme [15] and Lavi and May [18].

Alternatives to Pareto Efficiency. Our main goal is to research alternatives to Pareto efficiency for budget constrained agents. We start by observing that a Pareto efficiency is a binary notion: an allocation is either Pareto efficient or not, and there is no sense of one allocation being "more Pareto efficient" than the other. This is in contrast with efficiency in quasi linear environments where the traditional welfare objective induces a total order on the allocations.

The main goal of this paper is to suggest a new measure of efficiency for budgeted settings. The desiderata for this measure are: (i) it is quantifiable, i.e., attaches a value to each outcome; (ii) is achievable, i.e., can be approximated by truthful mechanisms and (iii) allows different designs that approximate welfare.

The measure we propose is called the *liquid welfare*. Before defining it, we give a revenue-motivated definition of the traditional social welfare in unbudgeted settings and show how it naturally generalizes to budgeted settings. One can view the traditional welfare of a certain outcome as the maximum revenue an omniscient seller can obtain from that outcome. If each agent i has value $v_i(x_i)$ for a certain outcome x_i, the omniscient seller can extract revenue arbitrarily close to $\sum_i v_i(x_i)$ by offering this outcome to each player i for price $v_i(x_i) - \epsilon$. This definition generalizes naturally to budgeted settings. Given an outcome, x_i, the *willingness-to-pay* of agent i is $v_i(x_i)$, which is the maximum he would give for this outcome in case he had unlimited resources. His *ability-to-pay*, however, is B_i, which is the maximum amount of money available to him. We define his *admissibility-to-pay* as the maximum value he would admit to pay for this outcome, which is the minimum between his willingness-to-pay and his ability-to-pay. The liquid welfare of a certain outcome is defined as the total admissibility-to-pay. Formally $\bar{\mathbf{W}}(x) = \sum_i \min(v_i(x_i), B_i)$.

An alternative view is as follows: efficiency should be measured only with respect to the funds available to the bidder at the time of the auction, and not the additional liquidity he might gain after receiving the goods he won. The liquid welfare objective frees the auctioneer from considering the hypothetical use the bidders will make of the items they win in the auction, and thus can focus only on the resources available to them at the time of the auction.

This objective satisfies our first requirement: it associates each outcome with an objective measure. Also, it is achievable. In fact, the clinching auction [12], which is the base for all auction achieving Pareto-efficient outcomes for budgeted settings, provide a 2-approximation for the liquid welfare objective. To show that this allows flexibility in the design, we show a different auction that also provides a 2-approximation and reveals a connection between our liquid welfare objective and the notion of market equilibrium.

It is appropriate to discuss the applicability and limitations of the liquid welfare objective. We start by illustrating a setting for which it is *not* applicable. If one were to auction hospital beds or access to doctors, it would be morally repugnant to privilege players based on their ability-to-pay. Therefore, we are not interested in claiming that the liquid welfare objective is the only alternative to Pareto efficiency, but rather argue that in *some* settings it produces reasonable results. Developing other notions of efficiency is an important future direction.

Yet, in many settings capping the welfare of the agents by their budgets makes perfect sense. Consider designing a market like internet advertising which aims at a good mix of good efficiency and revenue. In practice, players that bring more money to the market provide health to the market and improve efficiency. In real markets, there are practices to encourage wealthier players to enter the market. Therefore, privileging such players in the objective is somewhat natural.

An interesting question is whether one can have a truthful mechanism for additive valuations with public budgets that provides an approximation ratio better than 2. We show a lower bound of $\frac{4}{3}$. Closing the gap remains an open question, but we do show that for the special case of 2 players with identical public budgets there is a truthful auction that provides a matching upper bound.

We then move on to consider a setting in which truthful auction that always output Pareto-efficient solution do not exist: multi-unit auctions with decreasing marginal valuations and private budgets. For this setting we borrow ideas from Bartal, Gonen and Nisan [3] and provide a deterministic $O(\log n)$ approximation to the liquid welfare. This can be adapted to the case of subadditive valuations with an approximation of $O(\log^2 n)$ and to indivisible goods with $O(\log m)$-approximation where m is the number of goods.

Related Work. We have already surveyed results designing mechanisms for budget-constrained agents. We now focus on surveying results directly related to our efficiency measure and to our philosophical approach to efficiency maximization. As far as we know, the liquid welfare was first appeared in Chawla et al. [8] as an implicit upper bound on the revenue that a mechanism can extract.

Independently and simultaneously, two other approaches were proposed to provide quantitative guarantees for budgeted settings. Devanur, Ha and Hartline [11] show that the welfare of the clinching auction is a 2-approximation to the welfare of the best envy-free equilibrium. Their approach, however, is restricted to settings with *common budgets*, i.e., all agents have the same budget.

Syrgkanis and Tardos [24] leave the realm of truthful mechanisms and study the set of Nash and Bayes-Nash equilibria of simple mechanisms. For a wide class of mechanisms they show that the *traditional* welfare in equilibrium of such mechanism is a constant fraction of the optimal liquid welfare objective (which they call *effective welfare*). Their approach differs from ours in two ways: first they study auctions in equilibrium while we focus on incentive compatible auctions. Second, the guarantee in their mechanism is that the welfare of the allocation obtained is always greater than some fraction of the liquid welfare. The guarantee of our mechanisms is stronger: we construct mechanisms in which the *liquid welfare* is always greater than some fraction of the liquid welfare, which implies in particular that the welfare is greater than some fraction of the liquid welfare (since the welfare of an allocation is at least its liquid welfare).

Summary of Our Results. In this paper we have proposed to study the liquid welfare. We provided two truthful algorithms that guarante a 2 approximation to this objective for the setting of multi-unit auctions with public budgets. For the harder setting of multi-unit auctions with subadditive valuations and private budgets we provided a truthful $O(\log^2 n)$-approximation algorithm. For submodular bidders, the same mechanism provides an $O(\log n)$ approximation.

The main problem that we leave open is to determine whether there is a constant-approximation mechanism for multi unit auctions with private budgets. This is even open if all valuations are additive. More generally, are there truthful algorithms that provides a good approximation for *combinatorial* auctions? On top of that, notice that computational issues might come into play: while all of the constructions that we present in this paper happen to be computationally efficient, there might be a gap between the power of truthful algorithms in general and the power of computationally efficient truthful algorithms.

2 Preliminaries

2.1 Environments of Interest and Auction Basics

We consider n players and a set X of outcomes (also called environment). For each player, let $v_i : X \to \mathbb{R}_+$ be the valuation function for player i. We consider that agents are budgeted quasi-linear, i.e., each agent i has a budget B_i and for an outcome x and for payments $\pi_1, \ldots, \pi_n$, the utility of agent i is: $u_i = v_i(x_i) - \pi_i$ if $\pi_i \leq B_i$ and $-\infty$ o.w. Below, we list a set of environments we are interested:

1. *Divisible-multi-unit auctions and additive bidders:* $X = \{(x_1, \ldots, x_n); \sum_i x_i = s\}$ for some constant s and $v_i(x_i) = v_i \cdot x_i$, so we can represent the valuation function of each agent by a single real number $v_i \geq 0$.

2. *Divisible-multi-unit auctions with decreasing marginal bidders:* $X = \{(x_1, \dots, x_n); \sum_i x_i = s\}$ for some constant s and $v_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone non-decreasing concave function. A generalization of decreasing marginal valuations is subadditive valuations, i.e., $v_i(x_1 + x_2) \leq v_i(x_1) + v_i(x_2)$ for every x_1, x_2.
3. 0/1 *environments:* $X \subseteq \{0, 1\}^n$ and $v_i(x_i) = v_i$ if $x_i = 1$ and $v_i(x_i) = 0$ otherwise. Again the valuation is represented by a single $v_i \geq 0$.

An auction for a particular setting elicits the valuations of the players and budgets $B_1, \dots, B_n$ and outputs an outcome $x \in X$ and payments $\pi_1, \dots, \pi_n$ for each agents respecting budgets, i.e., such that $\pi_i \leq B_i$ for each agent. We will distinguish between *public budgets* and *private budgets* mechanisms. In the former, the auctioneer has access to the true budget of each agent[1]. In the later case, agents need to be incentivized to report their true budget. In either case, the valuations of each agent are private. We will focus on designing mechanisms that are incentive compatible (a.k.a. truthful), i.e., are such that agents utilities are maximized once they report their true value in the public budget case and their true value and budget in the private budget case. We will also require mechanisms to be individually rational, i.e., agents always derive non-negative utility upon bidding their true value.

In the case of divisible multi-unit auctions and additive bidders, the valuations can be represented by real numbers, So we can see the auctions as a pair of functions $x : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ and $\pi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ that map (v, B) to a vector of allocations $x(v, B) \in \mathbb{R}_+^n$ and a vector of payments $\pi(v, B) \in \mathbb{R}_+^n$. The set of functions that induce incentive compatible and individually rational auctions are characterized by Myerson's Lemma:

Lemma 1 (Myerson [22]). *A pair of functions (x, π) define an incentive-compatible and individually rational auction iff (i) for each v_{-i}, $x_i(v_i, v_{-i})$ is monotone non-decreasing in v_i and (ii) the payments are such that: $\pi_i(v_i, v_{-i}) = v_i \cdot x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(u, v_{-i}) du$.*

2.2 Efficiency Measures

The traditional efficiency measure in mechanism design is the *social welfare* which associates for each outcome x, the objective: $\mathbf{W}(x) = \sum_i v_i(x)$. It is known that one cannot even approximate the optimal welfare in budgeted settings in an incentive-compatible way, even if the budgets are known and equal. The result is folklore. We sketch the proof in the full version for completeness.

Lemma 2 (Folklore). *Consider the divisible-multi-unit auctions and additive bidders. There is no α-approximate, incentive compatible and individually rational mechanism $x(v), \pi(v)$ with $\alpha < n$. For $\alpha = n$ there is the mechanism that allocates the item at random to one player and charges nothing.*

[1] Most of the literature on auctions for budgeted settings [12,14,15,10,16] falls in this category, including classical references (Laffont and Robert [17] and Maskin [20]).

Due to impossibility results of this flavor, efficiency was mainly achieved in the literature through *Pareto efficiency.* We say that an outcome (x, π) with $x \in X$ and $\pi_i \leq B_i$ is Pareto-efficient if there is no alternative outcome where the utility of all the agents involved (including the auctioneer, being his utility the revenue $\sum_i \pi_i$) does not decrease and at least one agent improves. Formally, (x, π) is Pareto optimal iff there is no (x', π'), $x' \in X$, $\pi'_i \leq B_i$ such that:

$$u'_i = v_i \cdot x'_i - \pi'_i \geq u_i = v_i \cdot x_i - \pi_i, \forall i \text{ and } \sum_i \pi'_i \geq \sum_i \pi_i \text{ and } \sum_i v_i x'_i > \sum_i v_i x_i$$

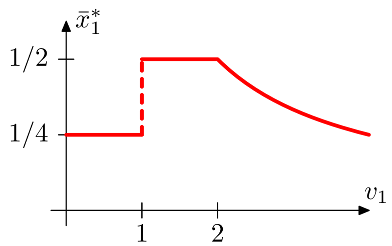
In particular, if the budgets are infinity (or simply very large), the only Pareto-optimal outcomes are those maximizing social welfare. For divisible-multi-unit auctions with additive bidders, this is achieved by the Adaptive Clinching Auction of Dobzinski, Lavi and Nisan [12]. Moreover, the authors show that this is the only incentive-compatible, individually-rational auction that achieves Pareto-optimal outcomes. The auction is further analyzed in Bhattacharya et al [5] and Goel et al [16]. In this paper we propose the *liquid welfare* objective:

Definition 1 (Liquid Welfare). *In a budgeted setting, we define the liquid welfare associated with outcome $x \in X$ by $\bar{\mathbf{W}}(x) = \sum_i \min\{v_i(x), B_i\}$.*

We will refer to the optimal liquid welfare as $\bar{\mathbf{W}}^* = \max_{x \in X} \bar{\mathbf{W}}(x)$. It is instructive yet straightforward to see that:

Lemma 3. *For divisible-multi-unit auctions and additive bidders, the optimal liquid welfare $\bar{\mathbf{W}}^*$ occurs for $\bar{x}_i^* = \min\left(\frac{B_i}{v_i}, [1 - \sum_{j<i} \bar{x}_j^*]^+\right)$ where players are sorted in non-increasing order of value, i.e., $v_1 \geq v_2 \geq \ldots \geq v_n$.*

An easy observation is that the optimal allocation for $\bar{\mathbf{W}}^*$ is not monotone in v_i, and hence cannot be implemented truthfully. For example, consider 3 agents with values $v_1 = v, v_2 = 1, v_3 = 2$ and budgets $B_1 = 1, B_2 = \frac{1}{4}, B_3 = 1$. Now, notice that $\bar{x}_1^*(v_1)$ is not monotone in v_1 as depicted in Figure 1.



$$\bar{x}_1^*(v_1, v_{-1}) = \begin{cases} 1/4, & 0 \leq v_1 \leq 1 \\ 1/2, & 1 \leq v_1 \leq 2 \\ 1/v_1, & 2 \leq v_1 \end{cases}$$

Fig. 1. Depiction of the first component of $\bar{x}^* = \text{argmax}_x \bar{\mathbf{W}}(x)$ for a 3 agent instance with $v = (v_1, 1, 2)$ and $B = (1, 1/4, 1)$. The figure highlights the non-monotonicity of the optimal solution $\bar{x}^*(v)$.

2.3 VCG and the Liquid Welfare Objective

The reader might suspect, however, that a modification of VCG might take care of optimizing the liquid welfare benchmark. This is indeed true for a couple of very simple settings. For example, for selling one indivisible item, a simple Vickrey auction on modified values: $\bar{v}_i = \min\{v_i, B_i\}$ provides a truthful mechanism that exactly optimizes the liquid welfare objective. More generally:

Theorem 2 (0/1 Environments). *Given a* $0/1$*-environment* $X \subseteq \{0,1\}^n$ *with valuations* $v_i(x) = v_i$ *if* $x_i = 1$ *and zero otherwise. Then running VCG on modified values* $\bar{v}_i = \min\{v_i, B_i\}$ *is incentive compatible and exactly optimized the liquid welfare objective* $\bar{\mathbf{W}}^*$*.*

The proof is trivial. This slightly generalizes to other simple environments of interest, for example, matching markets, where there are n agents and n indivible items and each agent i has a value v_{ij} for item j and possible outcomes are perfect matchings. Running VCG on $\bar{v}_{ij} = \min\{v_{ij}, B_i\}$ provides an incentive compatible mechanism that also exactly approximated $\bar{\mathbf{W}}^*$.

This technique, however, does not generalize past those few special cases as we show in the full version.

3 A First 2-Approximation: The Clinching Auction

In the previous section we defined our proposal for an efficiency measure in budgeted settings: the liquid welfare objective $\bar{\mathbf{W}}$. The second item in the desiderata for a new efficiency measure is that it is achievable, i.e., it could be optimized or well-approximated by an incentive compatible mechanism. In this section we show, for the setting of divisible multi-unit auctions with additive bidders, a mechanism that provides a 2-approximation for the liquid welfare, while still producing Pareto-efficient outcomes. The mechanism we use is the Adaptive Clinching Auction [12,5,16]. In the next section we provide a different truthful auction that is also a 2-approximation, and show that with respect to the liquid welfare the new auction is better on an instance-by-instance basis.

The clinching auction can be described by means of an ascending price clock procedure. The auction starts with the good unallocated and for every price p (considered in increasing order and in ϵ increments), the demand of each agent is computed, i.e., the maximum amount of the good an agent would like to be allocated at any given price. The amount an agent can *clinch* at price p is the amount leftover of the good minus the amount demanded by all other agents. At any given price, an agent is allocated his *clinched* amount at the current price.

In the full version we formally describe the clinching auction and introduce the concept of the *clinching interval* – which correspond to the price interval in which agents actively acquire goods. Finally. we prove our main result:

Theorem 3. *The clinching auction is a* 2*-approximation to the liquid welfare objective. That is, given* n *agents with values per unit* v_i *and budgets* B_i*, let* x, π *be the outcome of the clinching auction for such input. Then,* $\bar{\mathbf{W}}(x) \geq \frac{1}{2}\bar{\mathbf{W}}^*$*.*

In the full version we show that the bound proved in Theorem 3 is tight. Also, the following corollary about the revenue of the clinching auction follows from our proof:

Corollary 1 (Revenue). *If the clinching auction allocates items to more than one player, then its revenue is at least* $\frac{1}{2} \cdot \bar{\mathbf{W}}^*$.

4 A 2-Approximation via Market Equilibrium

We defined a quantifiable measure of efficiency (Section 2) and showed it can be approximated by an incentive-compatible mechanism (Section 3). The remaining item in the list of desiderata was to show that our efficiency measure allows for different designs. Here we show that we have "an extra bunny in the hat", an auction that also achieves a 2-approximation to the liquid welfare objective and is *not* based on Ausubel's clinching technique. Instead, it is based on the concept of Market Equilibrium.

Borrowing inspiration from general equilibrium theory, consider a market with n buyers each endowed with B_i dollars and willing to pay v_i per unit for a certain divisible good. This is the special case where there is only one product in the market. In this case, a price p is called a *market clearing price* if each buyer can be assigned an optimal basket of goods (in the particular of a single product, an optimal amount of the good) such that there is no surplus or deficiency of any good. Observe that there is one such price and that allocations can be computed once the price is found. Our Uniform Price Auction simply computes the market clearing price and allocates according to it. This defines the allocation. The payments are computed using the Myerson's formula for this allocation and happen to be different than the clearing price.

Definition 4 (Uniform Price Auction). *Consider n agents with values $v_1 \geq \ldots \geq v_n$ (i.e., ordered without loss of generality) and budgets B_i. Consider the auction that allocates one unit of a divisible good in the following way: let k be the maximum integer such that $\sum_{j=1}^{k} B_j \leq v_k$, then:*

- *Case I: if $\sum_{j=1}^{k} B_j > v_{k+1}$ allocate $x_i = \frac{B_i}{\sum_{j=1}^{k} B_j}$ for $i = 1, \ldots, k$ and nothing for the remaining players.*
- *Case II: if $\sum_{j=1}^{k} B_j \leq v_{k+1}$ allocate $x_i = \frac{B_i}{v_{k+1}}$ for $i = 1, \ldots, k$, $x_{k+1} = 1 - \sum_{j=1}^{k} x_j$ and nothing for the remaining players.*

Payments are defined through Myerson's integral (Lemma 1).

Case I corresponds to the case where the market clearing price of the Fisher Market instance is $p = \sum_{j=1}^{k} B_j$. Case II coresponds to the case where the Market clearing price is $p = v_{k+1}$. First we show that this auction induces an incentive-compatible auction that does not exceed the budgets of the agents – and thus is a valid auction for this setting. Then we show that it is a 2-approximation to the liquid welfare benchmark. Proofs are in the full version.

Lemma 4 (Monotonicity). *The allocation function of the Uniform Price Auction is monotone, i.e.,* $v_i \mapsto x_i(v_i, v_{-i})$ *is non-decreasing.*

Lemma 5 (Budget Feasibility). *The payments that make this auction incentive-compatible do not exceed the budgets.*

Theorem 5. *The Uniform Price Auction is an incentive compatible* 2*-approximation to the liquid welfare objective.*

The same example used for showing that the analysis for the Clinching Auction was tight can be used for showing that the analysis for the Uniform Price Auction is tight. We discuss it in detail in the full version.

One of the advantages in having a quantifiable measure of efficiency is that we can compare two different outcomes and decide which one is "better". In this section we show that although the worst-case guarantees of the clinching auction and of the uniform-price auction are identical, the liquid welfare of the uniform-price auction is *always* (weakly) dominates that of the clinching auction. We refer to the full version for a proof.

Theorem 6. *Consider* n *players with valuations* $v_1 \geq \ldots \geq v_n$ *and budgets* $B_1, \ldots, B_n$*. Let* x^{c} *and* x^{u} *be the outcomes of the Clinching and Uniform Price Auctions respectively. Then:* $\bar{\mathbf{W}}(x^{\mathrm{u}}) \geq \bar{\mathbf{W}}(x^{\mathrm{c}})$*.*

5 A Lower Bound and Some Matching Upper Bounds

In the previous sections, we showed two different auctions that are incentive compatible 2-approximations to the optimal liquid welfare for the setting of multi-unit auctions with additive valuations. Inthe full version we investigate the limits of the approximability of the liquid welfare. By the observation depicted in Figure 1, it is clear that an exact incentive compatible mechanism is not possible for this setting. First, we present a $\frac{4}{3}$ lower bound and show matching upper bounds for some special cases. We refer to the appendix for the full details.

6 Subadditive Bidders with Private Budgets

Finally, we consider the setting where players have subadditive valuations and private budgets. This is a notoriously hard setting for Pareto-optimality. In fact, considering either subadditive valuations or privated budgets alone already produces an impossibility result for achieving Pareto-efficient outcomes.

We will have one divisible good and each player has a subadditive valuation $v_i : [0, 1] \to \mathbb{R}_+$ and a budget B_i. This setting differs from the previously considered in the sense that budgets B_i are private information of the players.

The auction we propose is inspired in a technique by Bartal, Gonen and Nisan [3]. To describe it, we use the following notation: $\bar{v}(x_i) = \min\{v_i(x_i), B_i\}$. Now, consider the following selling procedure:

Definition 7 (Sell-Without-r). *Let r be a player. Consider the following mechanism to sell half the good, to players $i \neq r$ using the information about $\bar{v}_r(\frac{1}{2})$.*

Divide the segment $[0, \frac{1}{2}]$ into $k = 8\log(n)$ parts, each of size $\frac{1}{2k}$. Associate part $i = 1, \ldots, k$ with price per unit $p_i = \frac{2^i}{8}\bar{v}_r(\frac{1}{2})$. Order arbitrarily all players but player r. Each player different than r, in his turn, takes his most profitable (unallocated) subset of $[0, \frac{1}{2}]$ under the specified prices. Players are not allowed to pay more than their budget.

More precisely, let $p : [0, \frac{1}{2}] \to \mathbb{R}_+$ be such that for $x \in [\frac{1}{2k}(i-1), \frac{1}{2k}i]$, $p(x) = p_i = \frac{2^i}{8}\bar{v}_r(\frac{1}{2})$. Now, for $i = 1, \ldots, r-1, r+1, \ldots, n$, let x_i maximize $v_i(x_i) - \int_{z_i}^{z_i+x_i} p(t)dt$ where $z_i = \sum_{j<i} x_j$, conditioned on the payment being below the budget, i.e., $\int_{z_i}^{z_i+x_i} p(t)dt \leq B_i$. Set the payment as: $\pi_i = \int_{z_i}^{z_i+x_i} p(t)dt$.

Sell-Without-r is used in our main construction for this section:

Definition 8 (Estimate-and-Price). *Given one divisible good and n players with valuations $v_i(\cdot)$ and budgets B_i, consider the following auction: let $r_1 = \arg\max_i \bar{v}_i(\frac{1}{2})$ and $r_2 = \arg\max_{i \neq r_1} \bar{v}_i(\frac{1}{2})$. We say that r_1 is the* pivot player. *Let (x, π) be the outcome of Sell-Without-r_1 for players $[n] \setminus r_1$ and let (x', π') be the outcome of Sell-Without-r_2 for players $[n] \setminus r_2$.*

For players $i \neq r_1$, allocate x_i and charge π_i. For r_1 if $v_{r_1}(x'_{r_1}) - \pi'_{r_1} \geq v_{r_1}(\frac{1}{2}) - 2 \cdot \bar{v}_{r_2}(\frac{1}{2})$ allocate him x'_{r_1} and charge π'_{r_1} and if not, allocate $\frac{1}{2}$ and charge $2 \cdot \bar{v}_{r_2}(\frac{1}{2})$.

First, notice that the auction defined above is feasible, since r_1 is allocated at most half of the good and the players in $[n] \setminus r_1$ get allocated at most half of the good. Then we argue that this auction is incentive compatible (in the full version).

Lemma 6. *The Estimate-and-Price auction is incentive compatible for players with private budgets.*

This leads to our main result for submodular and subadditive bidders: we show that the Estimate and Price Auction is a logarithmic approximation for the liquid welfare objective. The details as well as a discussion of extensions of this result can be found in the full version.

Theorem 9. *For submodular bidders, the Estimate-and-Price auction is a truthful $O(\log n)$-approximation to the liquid welfare objective. For subadditive bidders, the same auction is an $O(\log^2 n)$-approximation to the liquid welfare.*

Acknowledgments. The first author is the incumbent of the Lilian and George Lyttle Career Development Chair. His work is supported in part by the I-CORE program of the planning and budgeting committee and the Israel Science Foundation 4/11 and by EU CIG grant 618128.

References

1. Aggarwal, G., Muthukrishnan, S., Pál, D., Pál, M.: General auction mechanism for search advertising. In: WWW, pp. 241–250 (2009)
2. Ausubel, L.M.: An efficient ascending-bid auction for multiple objects. American Economic Review 94 (1997)
3. Bartal, Y., Gonen, R., Nisan, N.: Incentive compatible multi unit combinatorial auctions. In: TARK, pp. 72–87 (2003)
4. Benoit, J.-P., Krishna, V.: Multiple-object auctions with budget constrained bidders. Review of Economic Studies 68(1), 155–179 (2001)
5. Bhattacharya, S., Conitzer, V., Munagala, K., Xia, L.: Incentive compatible budget elicitation in multi-unit auctions. In: SODA, pp. 554–572 (2010)
6. Borgs, C., Chayes, J.T., Immorlica, N., Mahdian, M., Saberi, A.: Multi-unit auctions with budget-constrained bidders. In: ACM EC (2005)
7. Bulow, J., Levin, J., Milgrom, P.: Winning play in spectrum auctions. Working Paper 14765, National Bureau of Economic Research (March 2009)
8. Chawla, S., Malec, D.L., Malekian, A.: Bayesian mechanism design for budget-constrained agents. In: ACM EC, pp. 253–262 (2011)
9. Che, Y.-K., Gale, I.: Standard auctions with financially constrained bidders. Review of Economic Studies 65(1), 1–21 (1998)
10. Colini-Baldeschi, R., Henzinger, M., Leonardi, S., Starnberger, M.: On multiple keyword sponsored search auctions with budgets. In: Czumaj, A., Mehlhorn, K., Pitts, A., Wattenhofer, R. (eds.) ICALP 2012, Part II. LNCS, vol. 7392, pp. 1–12. Springer, Heidelberg (2012)
11. Devanur, N.R., Ha, B.Q., Hartline, J.D.: Prior-free auctions for budgeted agents. In: EC (2013)
12. Dobzinski, S., Lavi, R., Nisan, N.: Multi-unit auctions with budget limits. Games and Economic Behavior 74(2), 486–503 (2012)
13. Dütting, P., Henzinger, M., Starnberger, M.: Auctions with heterogeneous items and budget limits. In: Goldberg, P.W. (ed.) WINE 2012. LNCS, vol. 7695, pp. 44–57. Springer, Heidelberg (2012)
14. Fiat, A., Leonardi, S., Saia, J., Sankowski, P.: Single valued combinatorial auctions with budgets. In: ACM EC, pp. 223–232 (2011)
15. Goel, G., Mirrokni, V.S., Paes Leme, R.: Polyhedral clinching auctions and the adwords polytope. In: STOC, pp. 107–122 (2012)
16. Goel, G., Mirrokni, V.S., Paes Leme, R.: Clinching auctions with online supply. In: SODA (2013)
17. Laffont, J.-J., Robert, J.: Optimal auction with financially constrained buyers. Economics Letters 52(2), 181–186 (1996)
18. Lavi, R., May, M.: A note on the incompatibility of strategy-proofness and pareto-optimality in quasi-linear settings with public budgets. In: Chen, N., Elkind, E., Koutsoupias, E. (eds.) Internet and Network Economics. LNCS, vol. 7090, pp. 417–417. Springer, Heidelberg (2011)
19. Malakhov, A., Vohra, R.V.: Optimal auctions for asymmetrically budget constrained bidders. Working paper (December 2005)

20. Maskin, E.S.: Auctions, development, and privatization: Efficient auctions with liquidity-constrained buyers. European Economic Review 44(4-6), 667–681 (2000)
21. McAfee, R.P., McMillan, J.: Auctions and bidding. Journal of Economic Literature 25(2), 699–738 (1987)
22. Myerson, R.: Optimal auction design. Mathematics of Operations Research 6(1), 58–73 (1981)
23. Pai, M., Vohra, R.: Optimal auctions with financially constrained bidders. Working Paper (2008)
24. Syrgkanis, V., Tardos, É.: Composable and efficient mechanisms. In: STOC (2013)