Morphing Planar Graph Drawings Optimally*

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Abstract. We provide an algorithm for computing a planar morph between any two planar straight-line drawings of any n-vertex plane graph in $O(n)$ morphing steps, thus improving upon the previously best known $O(n^2)$ upper bound. Furthermore, we prove that our algorithm is optimal, that is, we show that there exist two planar straight-line drawings Γ_s and Γ_t of an n-vertex plane graph G such that any planar morph between Γ_s and Γ_t requires $\Omega(n)$ morphing steps.

1 Introduction

A *morph* is a continuous transformation between two topologically equivalent geometric objects. The study of morphs is relevant for several areas of computer science, including computer graphics, animation, and modeling. Many of the geometric shapes that are of interest in these contexts can be effectively described by two-dimensional planar graph drawings. Hence, designing algorithms and establishing bounds for morphing planar graph drawings is an important research challenge. We refer the reader to [7–9, 12, 13] for extensive descriptions of the applications of graph drawing morphs.

It has long been known that there always exists a *planar morph* (that is, a morph that preserves planarity at any time instant) transforming any planar straight-line drawing Γ_s of a plane graph G into any other planar straight-line drawing Γ_t of G. The first proof of such a result, published by Cairns in 1944 [5], was "existential", i.e., no guarantee was provided on the complexity of the trajectories of the vertices during the morph. Almost 40 years later, Thomassen proved in [14] that a morph between Γ_s and Γ_t always exists in which vertices follow trajectories of exponential complexity (in the number of vertices of G). In other words, adopting a setting defined by Grünbaum and Shepard [10] which is also the one we consider in this paper, Thomassen proved that a sequence $\Gamma_s = \Gamma_1, \Gamma_2, \dots, \Gamma_k = \Gamma_t$ of planar straight-line drawings of G exists such that, for $1 \leq i \leq k-1$, the *linear morph* transforming Γ_i into Γ_{i+1} is planar, where a linear morph moves each vertex at constant speed along a straight-line trajectory.

A breakthrough was recently obtained by Alamdari *et al.* by proving that a planar morph between any two planar straight-line drawings of the same n-vertex connected

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plane graph exists in which each vertex follows a trajectory of polynomial complexity [1]. That is, Alamdari *et al.* showed an algorithm to perform the morph in $O(n^4)$ *morphing steps*, where a morphing step is a linear morph. The $O(n^4)$ bound was shortly afterwards improved to $O(n^2)$ by Angelini *et al.* [3].

In this paper, we provide an algorithm to compute a planar morph with $O(n)$ morphing steps between any two planar straight-line drawings Γ_s and Γ_t of any n-vertex connected plane graph G. Also, we prove that our algorithm is optimal. That is, for every n, there exist two drawings Γ_s and Γ_t of the same n-vertex plane graph (in fact a path) such that any planar morph between Γ_s and Γ_t consists of $\Omega(n)$ morphing steps. To the best of our knowledge, no super-constant lower bound was previously known.

The schema of our algorithm is the same as in [1, 3]. Namely, we morph Γ_s and Γ_t into two drawings Γ_s^x and Γ_t^x in which a certain vertex v can be contracted onto a neighbor x. Such contractions generate two straight-line planar drawings Γ'_s and Γ'_t of a smaller plane graph G'. A morph between Γ'_s and Γ'_t is recursively computed and suitably modified to produce a morph between Γ_s and Γ_t. The main ingredient for our new bound is a drastically improved algorithm to morph Γ_s and Γ_t into Γ_s^x and Γ_t^x. In fact, while the task of making v contractible onto x is accomplished with $O(n)$ morphing steps in [1, 3], we devise and use properties of monotone drawings, level planar drawings, and hierarchical graphs to perform it with $O(1)$ morphing steps.

The idea behind the lower bound is that linear morphs can poorly simulate rotations, that is, a morphing step rotates an edge of an angle whose size is $O(1)$. We then consider two drawings Γ_s and Γ_t of an n-vertex path P, where Γ_s lies on a straight-line, whereas Γ_t has a spiral-like shape, and we prove that in any planar morph between Γ_s and Γ_t there is one edge of P whose total rotation describes an angle whose size is $\Omega(n)$.

Because of space limitations, some proofs are omitted and can be found in [2].

2 Preliminaries

Drawings and Embeddings. A *planar straight-line drawing* of a graph maps each vertex to a distinct point in the plane and each edge to a straight-line segment between its endpoints so that no two edges cross. A planar drawing partitions the plane into topologically connected regions, called *faces*. The bounded faces are *internal*, while the unbounded face is the *outer face*. A planar straight-line drawing is *convex* if each face is delimited by a convex polygon. A planar drawing of a graph determines a circular ordering of the edges incident to each vertex, called *rotation system*. Two drawings of a graph are *equivalent* if they have the same rotation system and the same outer face. A *plane embedding* is an equivalence class of planar drawings. A graph with a plane embedding is called a *plane graph*. A plane graph is *maximal* if no edge can be added to it while maintaining its planarity.

Subgraphs and Connectivity. A *subgraph* $G'(V', E')$ of a graph $G(V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$; G' is *induced* if, for every $u, v \in V'$, $(u, v) \in E'$ if and only if $(u, v) \in E$. If G is a plane graph, then a subgraph G' of G is regarded as a plane graph whose plane embedding is the one obtained from G by removing all the vertices and edges not in G'.

A graph G is *k-connected* if removing any $k-1$ vertices leaves G connected; a *separating k-set* is a set of k vertices whose removal disconnects G. A 3-cycle in a plane graph G is *separating* if it contains vertices both in its interior and in its exterior. Every separating 3-set in a maximal plane graph G induces a separating 3-cycle.

Monotonicity. An *arc* $\boldsymbol{xy}$ is a line segment directed from a point x to a point y; $\boldsymbol{xy}$ is *monotone* with respect to an oriented straight line $\boldsymbol{d}$ if it has a *positive projection* on $\boldsymbol{d}$, i.e., for any two distinct points p and q in this order along $\boldsymbol{xy}$ from x to y, the projection of p on $\boldsymbol{d}$ precedes the projection of q on $\boldsymbol{d}$ according to the orientation of $\boldsymbol{d}$. A path $P = (u_1, \dots, u_n)$ is *$\boldsymbol{d}$-monotone* if the straight-line arc $\boldsymbol{u_i u_{i+1}}$ is monotone with respect to $\boldsymbol{d}$, for $i = 1, \dots, n-1$; a polygon Q is *$\boldsymbol{d}$-monotone* if it contains two vertices s and t such that the two paths between s and t that delimit Q are both $\boldsymbol{d}$-monotone. A path P (a polygon Q) is *monotone* if there exists an oriented straight line $\boldsymbol{d}$ such that P (resp. Q) is $\boldsymbol{d}$-monotone. We show two lemmata about monotone paths and polygons.

Lemma 1. *Let Q be any convex polygon and let $\mathbf{d}$ be any oriented straight line not perpendicular to any straight line through two vertices of Q. Then Q is $\mathbf{d}$-monotone.*

Lemma 2. *Any simple polygon Q with at most* 5 *vertices is monotone.*

Morphing. A *linear morph* between two straight-line planar drawings Γ_1 and Γ_2 of a plane graph G is a continuous transformation from Γ_1 to Γ_2 such that each vertex moves at constant speed along a straight line from its position in Γ_1 to the one in Γ_2. A linear morph between Γ_1 and Γ_2 is denoted by $\langle \Gamma_1, \Gamma_2 \rangle$. A linear morph is *planar* if no crossing or overlap occurs between any two edges or vertices during the transformation. A planar linear morph is also called a *morphing step*. In the remainder of the paper, we will construct *unidirectional* linear morphs, that were defined in [4] as linear morphs in which the straight-line trajectories of the vertices are parallel.

A *morph* $\langle \Gamma_s, \dots, \Gamma_t \rangle$ between two straight-line planar drawings Γ_s and Γ_t of a plane graph G is a finite sequence of morphing steps that transforms Γ_s into Γ_t. A *unidirectional morph* is such that each of its morphing steps is unidirectional.

Let Γ be a planar straight-line drawing of a plane graph G. The *kernel* of a vertex v of G is the open convex region R such that placing v at any point of R while maintaining the position of every other vertex unchanged yields a planar straight-line drawing of G. If a neighbor x of v lies on the boundary of the kernel of v in Γ, we say that v is *x-contractible*. The *contraction of v onto x* in Γ is the operation resulting in: (i) a simple graph $G' = G/(v, x)$ obtained from G by removing v and by replacing each edge (v, w), where $w \neq x$, with an edge (x, w) (if it does not already belong to G); and (ii) a planar straight-line drawing Γ' of G' such that each vertex different from v is mapped to the same point as in Γ. Also, the *uncontraction of v from x into Γ* is the reverse operation of the contraction of v onto x in Γ, i.e., the operation that produces a planar straight-line drawing Γ of G from a planar straight-line drawing Γ' of G'. A vertex v in a plane graph G is *quasi-contractible* if (i) $\deg(v) \leq 5$ and (ii) for any two neighbors u and w of v connected by an edge, cycle (u, v, w) delimits a face of G. We have the following.

Lemma 3. *(Angelini et al. [3]) Every plane graph contains a quasi-contractible vertex.*

In the remainder of the paper, even when not explicitly specified, we will only consider and perform contractions of quasi-contractible vertices.

Let Γ_1 and Γ_2 be two straight-line planar drawings of the same plane graph G. We define a *pseudo-morph* of Γ_1 into Γ_2 inductively, as follows:
(A) a unidirectional morph with m morphing steps of Γ_1 into Γ_2 is a pseudo-morph with m steps of Γ_1 into Γ_2;
(B) a unidirectional morph with m_1 morphing steps of Γ_1 into a straight-line planar drawing Γ_1^x of G, followed by a pseudo-morph with m_2 steps of Γ_1^x into a straight-line planar drawing Γ_2^x of G, followed by a unidirectional morph with m_3 morphing steps of Γ_2^x into Γ_2 is a pseudo-morph of Γ_1 into Γ_2 with $m_1 + m_2 + m_3$ steps; and
(C) let Γ_1' (Γ_2') be the straight-line planar drawing of the plane graph G' obtained by contracting a quasi-contractible vertex v of G onto x in Γ_1 (in Γ_2); the contraction of v onto x, followed by a pseudo-morph with m steps of Γ_1' into Γ_2' and by the uncontraction of v from x into Γ_2 is a pseudo-morph with $m + 2$ steps of Γ_1 into Γ_2.

Pseudo-morphs have two useful and powerful features.

First, it is easy to design an inductive algorithm for constructing a pseudo-morph between any two planar straight-line drawings Γ_1 and Γ_2 of the same n-vertex plane graph G. Namely, consider any quasi-contractible vertex v of G and let x be any neighbor of v. Morph unidirectionally Γ_1 and Γ_2 into two planar straight-line drawings Γ_1^x and Γ_2^x, respectively, in which v is x-contractible. Now contract v onto x in Γ_1^x and in Γ_2^x obtaining two planar straight-line drawings Γ_1' and Γ_2', respectively, of the same $(n-1)$-vertex plane graph G'. Then, the algorithm is completed by inductively computing a pseudo-morph of Γ_1' into Γ_2'.

Second, computing a pseudo-morph between Γ_1 and Γ_2 leads to computing a planar unidirectional morph between Γ_1 and Γ_2, as formalized in Lemma 4. We remark that, although Lemma 4 has never been stated as below, its proof can be directly derived from the results of [1, 3] and, mainly, of Barrera-Cruz *et al.* [4].

Lemma 4. *Let Γ_s and Γ_t be two straight-line planar drawings of a plane graph G. Let $\mathcal{P}$ be a pseudo-morph with m steps transforming Γ_s into Γ_t. It is possible to construct a planar unidirectional morph M with m morphing steps transforming Γ_s into Γ_t.*

Hierarchical Graphs and Level Planarity. A *hierarchical graph* is defined as a tuple $(G, \boldsymbol{d}, L, \gamma)$ where: (i) G is a graph; (ii) $\boldsymbol{d}$ is an oriented straight line in the plane; (iii) L is a set of parallel lines (sometimes called *layers*) that are orthogonal to $\boldsymbol{d}$; the lines in L are assumed to be ordered in the same order as they are intersected by $\boldsymbol{d}$ when traversing such a line according to its orientation; and (iv) γ is a function that maps each vertex of G to a line in L in such a way that, if an edge (u, v) belongs to G, then $\gamma(u) \neq \gamma(v)$. A *level drawing* of $(G, \boldsymbol{d}, L, \gamma)$ (sometimes also called *hierarchical drawing*) maps each vertex v of G to a point on the line $\gamma(v)$ and each edge (u, v) of G such that line $\gamma(u)$ precedes line $\gamma(v)$ in L to an arc $\boldsymbol{uv}$ monotone with respect to $\boldsymbol{d}$. A *hierarchical plane graph* is a hierarchical graph $(G, \boldsymbol{d}, L, \gamma)$ such that G is a plane graph and such that a level planar drawing Γ of $(G, \boldsymbol{d}, L, \gamma)$ exists that "respects" the embedding of G (that is, the rotation system and the outer face of G in Γ are the same as in the plane embedding of G). Given a hierarchical plane graph $(G, \boldsymbol{d}, L, \gamma)$, an *st-face* of G is a face delimited by two paths $(s = u_1, u_2, \ldots, u_k = t)$ and $(s = v_1, v_2, \ldots, v_l = t)$ such that $\gamma(u_i)$

precedes $\gamma(u_{i+1})$ in L, for every $1 \leq i \leq k-1$, and such that $\gamma(v_i)$ precedes $\gamma(v_{i+1})$ in L, for every $1 \leq i \leq l-1$. We say that $(G, \boldsymbol{d}, L, \gamma)$ is a *hierarchical plane st-graph* if every face of G is an st-face. Let Γ be any straight-line level planar drawing of a hierarchical plane graph $(G, \boldsymbol{d}, L, \gamma)$ and let f be a face of G; then, it is easy to argue that f is an st-face if and only if the polygon delimiting f in Γ is $\boldsymbol{d}$-monotone.

In this paper we will use an algorithm by Hong and Nagamochi that constructs convex straight-line level planar drawings of hierarchical plane st-graphs [11]. Here we explicitly formulate a weaker version of their main theorem.[1]

Theorem 1. *(Hong and Nagamochi [11]) Every* 3*-connected hierarchical plane st-graph* $(G, \boldsymbol{d}, L, \gamma)$ *admits a convex straight-line level planar drawing.*

Consider any straight-line level planar drawing Γ of a hierarchical plane graph $(G, \boldsymbol{d}, L, \gamma)$. Since each edge (u, v) of G is represented in Γ by a $\boldsymbol{d}$-monotone arc, the fact that (u, v) intersects a line $l_i \in L$ does not depend on the actual drawing Γ, but only on the fact that l_i lies between lines $\gamma(u)$ and $\gamma(v)$ in L. Assume that each line $l_i \in L$ is oriented so that $\boldsymbol{d}$ cuts l_i from the right to the left of l_i. We say that an edge e *precedes* (*follows*) a vertex v on a line l_i in Γ if $\gamma(v) = l_i$, e intersects l_i in a point $p_i(e)$, and $p_i(e)$ precedes (resp. follows) v on l_i when traversing such a line according to its orientation. Also, we say that an edge e *precedes* (*follows*) an edge e' on a line l_i in Γ if e and e' both intersect l_i at points $p_i(e)$ and $p_i(e')$, and $p_i(e)$ precedes (resp. follows) $p_i(e')$ on l_i when traversing such a line according to its orientation.

Now consider two straight-line level planar drawings Γ_1 and Γ_2 of a hierarchical plane graph $(G, \boldsymbol{d}, L, \gamma)$. We say that Γ_1 and Γ_2 are *left-to-right equivalent* if, for any line $l_i \in L$, for any vertex or edge x of G, and for any vertex or edge y of G, we have that x precedes (follows) y on l_i in Γ_1 if and only if x precedes (resp. follows) y on l_i in Γ_2. We are going to make use of the following lemma.

Lemma 5. *Let* Γ_1 *and* Γ_2 *be two left-to-right equivalent straight-line level planar drawings of the same hierarchical plane graph* $(G, \boldsymbol{d}, L, \gamma)$*. Then the linear morph* $\langle \Gamma_1, \Gamma_2 \rangle$ *transforming* Γ_1 *into* Γ_2 *is planar and unidirectional.*

3 A Morphing Algorithm

In this section we describe an algorithm to construct a planar unidirectional morph with $O(n)$ steps between any two straight-line planar drawings Γ_s and Γ_t of the same n-vertex plane graph G. The algorithm relies on two subroutines, called FAST CONVEXIFIER and CONTRACTIBILITY CREATOR, which are described in Sections 3.1 and 3.2, respectively. The algorithm is described in Section 3.3.

[1] We make some remarks. First, the main result in [11] proves that a convex straight-line level planar drawing of $(G, \boldsymbol{d}, L, \gamma)$ exists even if a convex polygon representing the cycle delimiting the outer face of G is arbitrarily prescribed. Second, the result holds for a super-class of the 3-connected planar graphs, namely for all the graphs that admit a convex straight-line drawing [6, 15]. Third, the result assumes that the lines in L are horizontal; however, a suitable rotation of the coordinate axes shows how that assumption is not necessary. Fourth, looking at the figures in [11] one might get the impression that the lines in L need to be equidistant; however, this is nowhere used in their proof, hence the result holds for any set of parallel lines.

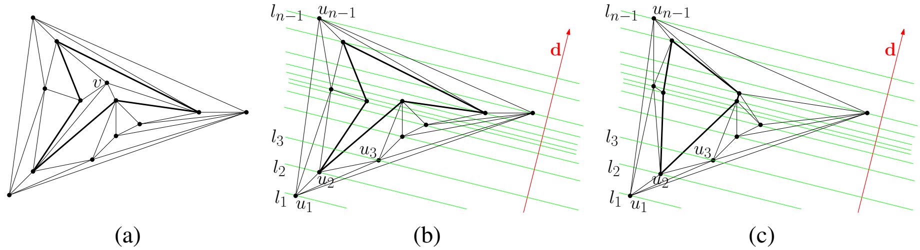

Fig. 1. (a) Straight-line planar drawing Γ of G. (b) Straight-line level planar drawing Γ' of $(G', \boldsymbol{d}, L', \gamma')$. (c) Convex straight-line level planar drawing Γ'_M of $(G', \boldsymbol{d}, L', \gamma')$.

3.1 Fast Convexifier

Consider a straight-line planar drawing Γ of an n-vertex maximal plane graph G, for some $n \geq 4$. Let v be a quasi-contractible internal vertex of G and let C_v be the cycle of G induced by the neighbors of v. See Fig. 1(a). In this section we show an algorithm, that we call FAST CONVEXIFIER, morphing Γ into a straight-line planar drawing Γ_M of G in which C_v is convex with a single unidirectional morphing step.

Let G' be the $(n-1)$-vertex plane graph obtained by removing v and its incident edges from G. Also, let Γ' be the straight-line planar drawing of G' obtained by removing v and its incident edges from Γ. As v is quasi-contractible, we have the following.

Lemma 6. *Graph G' is 3-connected.*

Consider the polygon Q_v representing C_v in Γ and in Γ'. By Lemma 2, Q_v is $\boldsymbol{d}$-monotone, for some oriented straight line $\boldsymbol{d}$. Slightly perturb the slope of $\boldsymbol{d}$ so that no line through two vertices of G in Γ is perpendicular to $\boldsymbol{d}$. If the perturbation is small enough, then Q_v is still $\boldsymbol{d}$-monotone. Denote by $u_1, \dots, u_{n-1}$ the vertices of G' ordered according to their projection on $\boldsymbol{d}$. For $1 \leq i \leq n-1$, denote by l_i the line through u_i orthogonal to $\boldsymbol{d}$. Let $L' = \{l_1, \dots, l_{n-1}\}$; note that the lines in L' are parallel and distinct. Let γ' be the function that maps u_i to l_i, for $1 \leq i \leq n-1$. See Fig. 1(b).

Lemma 7. *$(G', \boldsymbol{d}, L', \gamma')$ is a hierarchical plane st-graph.*

Proof: By construction, Γ' is a straight-line level planar drawing of $(G', \boldsymbol{d}, L', \gamma')$, hence $(G', \boldsymbol{d}, L', \gamma')$ is a hierarchical plane graph. Further, every polygon delimiting a face of G' in Γ' is $\boldsymbol{d}$-monotone. This is true for Q_v by construction and for every other polygon Q_i delimiting a face of G' in Γ' by Lemma 1, given that Q_i is a triangle and hence it is convex. Since every polygon delimiting a face of G' in Γ' is $\boldsymbol{d}$-monotone, every face of G' is an st-face, hence $(G', \boldsymbol{d}, L', \gamma')$ is a hierarchical plane st-graph. □

By Lemmata 6 and 7, $(G', \boldsymbol{d}, L', \gamma')$ is a 3-connected hierarchical plane st-graph. By Theorem 1, a convex straight-line level planar drawing Γ'_M of $(G', \boldsymbol{d}, L', \gamma')$ exists. Denote by Q_v^M the convex polygon representing C_v in Γ'_M. See Fig. 1(c).

Denote by r and s the minimum and the maximum index such that u_r and u_s belong to C_v, respectively. Denote by $l(v)$ the line through v orthogonal to $\boldsymbol{d}$ in Γ. If $l(v)$

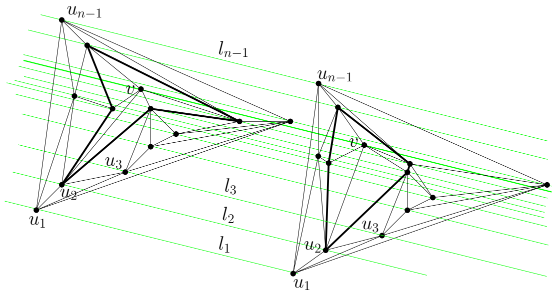

Fig. 2. Morphing Γ into a straight-line planar drawing Γ_M of G in which the polygon Q_v^M representing C_v is convex. The thick line parallel to l_i is $l(v)$.

were contained in the half-plane delimited by l_r and not containing l_s, then v would not lie inside Q_v in Γ, as the projection of every vertex of Q_v on $\boldsymbol{d}$ would follow the projection of v on $\boldsymbol{d}$. Analogously, $l(v)$ is not contained in the half-plane delimited by l_s and not containing l_r. It follows that $l(v)$ is "in-between" l_r and l_s, that is, $l(v)$ lies in the strip defined by l_r and l_s. Construct a straight-line planar drawing Γ_M of G from Γ'_M by placing v on any point at the intersection of $l(v)$ and the interior of Q_v^M. Such an intersection is always non-empty, given that l_r and l_s have non-empty intersection with Q_v^M, given that $l(v)$ is in-between l_r and l_s, and given that Q_v^M is a convex polygon.

Algorithm FAST CONVEXIFIER consists of a single linear morph $\langle \Gamma, \Gamma_M \rangle$ transforming Γ into Γ_M. See Fig. 2. Note that the polygon Q_v^M representing C_v in Γ_M is convex. Also, let γ be the function that maps v to $l(v)$ and u_i to l_i, for $1 \leq i \leq n-1$. We have that Γ and Γ_M are left-to-right equivalent straight-line level planar drawings of $(G, \boldsymbol{d}, L' \cup \{l(v)\}, \gamma)$, hence, by Lemma 5, $\langle \Gamma, \Gamma_M \rangle$ is unidirectional and planar.

3.2 Contractibility Creator

We now describe algorithm CONTRACTIBILITY CREATOR, which receives a straight-line planar drawing Γ of a plane graph G, a quasi-contractible vertex v of G, and a neighbor x of v, and returns a planar unidirectional morph with $O(1)$ morphing steps transforming Γ into a straight-line planar drawing Γ' of G in which v is x-contractible.

Denote by $u_1, \dots, u_k$ the clockwise order of the neighbors of v. If $k = 1$, then v is x-contractible in Γ, hence algorithm CONTRACTIBILITY CREATOR returns $\Gamma' = \Gamma$.

If $k \geq 2$, consider any pair of consecutive neighbors of v, say u_i and u_{i+1} (where $u_{k+1} = u_1$). See Fig. 3(a). If edge (u_i, u_{i+1}) belongs to G, then cycle (u_i, v, u_{i+1}) delimits a face of G, given that v is quasi-contractible. Otherwise, we aim at morphing Γ into a straight-line planar drawing of G where a dummy edge (u_i, u_{i+1}) can be introduced while maintaining planarity and while ensuring that cycle (u_i, v, u_{i+1}) delimits a face of the augmented graph $G \cup \{(u_i, u_{i+1})\}$. (This insertion might not be performed directly in Γ, see Fig. 3(b).) The required morphing is constructed as follows:

(Step 1) We add two dummy vertices r and r', and six dummy edges (r, v), (r, u_i), (r, u_{i+1}), (r', u_i), (r', u_{i+1}), and (r, r') to Γ and G, obtaining a straight-line planar

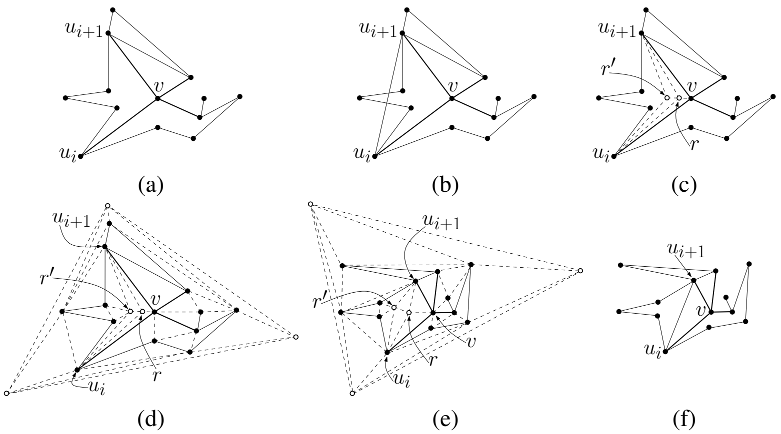

Fig. 3. (a) Drawing Γ of G. (b) Drawing (u_i, u_{i+1}) in Γ might cause a crossing. (c) Drawing Γ^+ of G^+. (d) Drawing Γ^* of G^*. (e) Drawing Γ^*_M of G^*. (f) Drawing Γ_M of $G \cup \{(u_i, u_{i+1})\}$.

drawing Γ^+ of a plane graph G^+, in such a way that Γ^+ is planar and cycles (v, r, u_i), (v, r, u_{i+1}), (r', r, u_i), and (r', r, u_{i+1}) delimit faces of G^+. See Fig. 3(c). **(Step 2)** We add dummy vertices and edges to Γ^+ and G^+, obtaining a straight-line planar drawing Γ^* of a graph G^*, in such a way that Γ^* is planar, that G^* is a maximal planar graph, and that edges (u_i, u_{i+1}) and (r', v) do not belong to G^*. Observe that r is a quasi-contractible vertex of G^*. See Fig. 3(d). **(Step 3)** We apply algorithm FAST CONVEXIFIER to morph Γ^* with one unidirectional morphing step into a straight-line planar drawing Γ^*_M of G^* such that the polygon of the neighbors of r is convex. See Fig. 3(e). **(Step 4)** We remove from Γ^*_M all the dummy vertices and edges that belong to G^* and do not belong to G, and we add edge (u_i, u_{i+1}) to Γ^*_M and G, obtaining a straight-line planar drawing Γ_M of graph $G \cup \{(u_i, u_{i+1})\}$. See Fig. 3(f).

If $k = 2$, then after the above described algorithm is performed, we have that v is x-contractible in $\Gamma' = \Gamma_M$, both if $x = u_1$ or if $x = u_2$, given that (v, u_1, u_2) delimits a face of $G \cup \{(u_1, u_2)\}$. If $3 \leq k \leq 5$, then the above described algorithm is repeated at most k times (namely once for each pair of consecutive neighbors of v that are not adjacent in G), at each time inserting an edge between a distinct pair of consecutive neighbors of v. Eventually, we obtain a straight-line planar drawing Φ of plane graph $G \cup \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_1)\}$ in which v is quasi-contractible. Then we add dummy vertices and edges to Φ, obtaining a straight-line planar drawing Σ of a graph H, in such a way that H is a maximal planar graph and that v is quasi-contractible in Σ. We apply algorithm FAST CONVEXIFIER to morph Σ with one unidirectional morphing step into a straight-line planar drawing Ψ of H such that the polygon of the neighbors of v is convex. Hence, v is contractible onto any of its neighbors in Ψ. Then, we remove the edges of H not in G, obtaining a straight-line planar drawing Γ' of G in which v is contractible onto any of its neighbors; hence, v is x-contractible in Γ'. Finally, observe that Γ' is obtained from Γ in at most $k + 1 \leq 6$ unidirectional morphing steps.

3.3 The Algorithm

We now describe an algorithm to construct a pseudo-morph $\mathcal{P}$ with $O(n)$ steps between any two straight-line planar drawings Γ_s and Γ_t of the same n-vertex plane graph G.

The algorithm works by induction on n. If $n = 1$, then $\mathcal{P}$ consists of a single unidirectional morphing step transforming Γ_s into Γ_t. If $n \geq 2$, then let v be a quasi-contractible vertex of G, which exists by Lemma 3, and let x be any neighbor of v. Let M_s and M_t be the planar unidirectional morphs with $O(1)$ morphing steps produced by algorithm CONTRACTIBILITY CREATOR transforming Γ_s and Γ_t into straight-line planar drawings Γ_s^x and Γ_t^x of G, respectively, such that v is x-contractible both in Γ_s^x and in Γ_t^x. Let G' be the $(n-1)$-vertex plane graph obtained by contracting v onto x in G, and let Γ_s' and Γ_t' be the straight-line planar drawings of G' obtained from Γ_s^x and Γ_t^x, respectively, by contracting v onto x. Further, let $\mathcal{P}'$ be the inductively constructed pseudo-morph between Γ_s' and Γ_t'. Then, pseudo-morph $\mathcal{P}$ is defined as unidirectional morph M_s transforming Γ_s into Γ_s^x, followed by the contraction of v onto x in Γ_s^x, followed by the pseudo-morph $\mathcal{P}'$ between Γ_s' and Γ_t', followed by the uncontraction of v from x into Γ_t^x, followed by the unidirectional morph M_t^{-1} transforming Γ_t^x into Γ_t. Observe that $\mathcal{P}$ has a number of steps which is a constant plus the number of steps of $\mathcal{P}'$. Hence, $\mathcal{P}$ consists of $O(n)$ steps. A unidirectional planar morph M between Γ_s and Γ_t can be constructed with a number of morphing steps equal to the number of steps of $\mathcal{P}$, by Lemma 4. This proves the following:

Theorem 2. *Let Γ_s and Γ_t be any two straight-line planar drawings of the same n-vertex plane graph G. There exists an algorithm to construct a planar unidirectional morph with $O(n)$ morphing steps transforming Γ_s into Γ_t.*

4 A Lower Bound

In this section we show two straight-line planar drawings Γ_s and Γ_t of an n-vertex path $P = (v_1, \ldots, v_n)$, and we prove that any planar morph M between Γ_s and Γ_t requires $\Omega(n)$ morphing steps. In order to simplify the description, we consider each edge $e_i = (v_i, v_{i+1})$ as oriented from v_i to v_{i+1}, for $i = 1, \ldots, n-1$.

Drawing Γ_s (see Fig. 4(a)) is such that all the vertices of P lie on a horizontal straight line with v_i to the left of v_{i+1}, for each $i = 1, \ldots, n-1$. Drawing Γ_t (see Fig. 4(b)) is such that: (a) for each $i = 1, \ldots, n-1$ with $i \bmod 3 \equiv 1$, e_i is horizontal with v_i to the left of v_{i+1}; (b) for each $i = 1, \ldots, n-1$ with $i \bmod 3 \equiv 2$, e_i is parallel to line $y = \tan(\frac{2\pi}{3})x$ with v_i to the right of v_{i+1}; and (c) for each $i = 1, \ldots, n-1$ with $i \bmod 3 \equiv 0$, e_i is parallel to line $y = \tan(-\frac{2\pi}{3})x$ with v_i to the right of v_{i+1}.

Let $M = \langle \Gamma_s = \Gamma_1, \ldots, \Gamma_x = \Gamma_t \rangle$ be any planar morph transforming Γ_s into Γ_t.

For $i = 1, \ldots, n$ and $j = 1, \ldots, x$, we denote by v_i^j the point where vertex v_i is placed in Γ_j and by e_i^j the directed straight-line segment representing edge e_i in Γ_j.

For $1 \leq j \leq x-1$, we define the *rotation* ρ_i^j of e_i around v_i during the morphing step $\langle \Gamma_j, \Gamma_{j+1} \rangle$ as follows (see Figs. 5(a)–(b)). Translate e_i at any time instant of $\langle \Gamma_j, \Gamma_{j+1} \rangle$ so that v_i stays fixed at a point a during the entire morphing step. After this translation, the morph between e_i^j and e_i^{j+1} is a rotation of e_i around a (where e_i might vary its

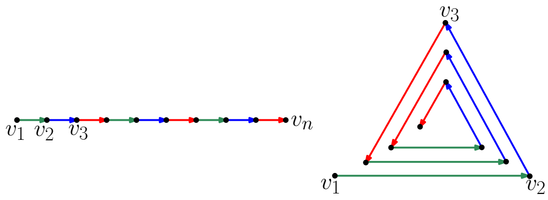

Fig. 4. Drawings Γ_s (a) and Γ_t (b)

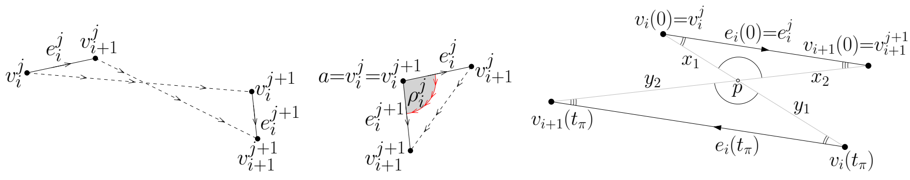

Fig. 5. (a) Morph between e_i^j and e_i^{j+1}. (b) Translation of the positions of e_i during $\langle \Gamma_j, \Gamma_{j+1} \rangle$, resulting in e_i spanning an angle ρ_i^j around v_i. (c) Illustration for the proof of Lemma 8.

length during $\langle \Gamma_j, \Gamma_{j+1} \rangle$) spanning an angle ρ_i^j, where we assume $\rho_i^j > 0$ if the rotation is counter-clockwise, and $\rho_i^j < 0$ otherwise. We have the following.

Lemma 8. *For each $j = 1, \ldots, x-1$ and $i = 1, \ldots, n-1$, we have $|\rho_i^j| < \pi$.*

Proof: Assume, for a contradiction, that $|\rho_i^j| \geq \pi$, for some $1 \leq j \leq x-1$ and $1 \leq i \leq n-1$. Also assume, w.l.o.g., that the morphing step $\langle \Gamma_j, \Gamma_{j+1} \rangle$ happens between time instants $t = 0$ and $t = 1$. For any $0 \leq t \leq 1$, denote by $v_i(t)$, $v_{i+1}(t)$, $e_i(t)$, and $\rho_i^j(t)$ the position of v_i, the position of v_{i+1}, the drawing of e_i, and the rotation of e_i around v_i at time instant t, respectively. Note that $v_i(0) = v_i^j$, $v_{i+1}(0) = v_{i+1}^j$, $e_i(0) = e_i^j$, $\rho_i^j(0) = 0$, and $\rho_i^j(1) = \rho_i^j$. Since a morph is a continuous transformation and since $|\rho_i^j| \geq \pi$, there exists a time instant t_π with $0 < t_\pi \leq 1$ such that $|\rho_i^j(t_\pi)| = \pi$.

We prove that there exists a time instant t_r with $0 < t_r \leq t_\pi$ in which $v_i(t)$ and $v_{i+1}(t)$ coincide, thus contradicting the assumption that morph $\langle \Gamma_j, \Gamma_{j+1} \rangle$ is planar.

Since $|\rho_i^j(t_\pi)| = \pi$, it follows that $e_i(t_\pi)$ is parallel to $e_i(0)$ and oriented in the opposite way. This easily leads to conclude that t_r exists if $e_i(t_\pi)$ and $e_i(0)$ are aligned. Otherwise, the straight-line segments $\overline{v_i(0)v_i(t_\pi)}$ and $\overline{v_{i+1}(0)v_{i+1}(t_\pi)}$ meet in a point p. Refer to Fig. 5(c). Let $x_1 = |\overline{pv_i(0)}|$, $x_2 = |\overline{pv_{i+1}(0)}|$, $y_1 = |\overline{pv_i(t_\pi)}|$, and $y_2 = |\overline{pv_{i+1}(t_\pi)}|$. By the similarity of triangles $(v_i(0), p, v_{i+1}(0))$ and $(v_i(t_\pi), p, v_{i+1}(t_\pi))$, we have $\frac{x_1}{y_1} = \frac{x_2}{y_2}$ and hence $\frac{x_1}{x_1+y_1} = \frac{x_2}{x_2+y_2}$. Thus, $v_i(\frac{x_1}{x_1+y_1} t_\pi)$ and $v_{i+1}(\frac{x_1}{x_1+y_1} t_\pi)$ are coincident with p. This contradiction proves the lemma. □

For $j = 1, \ldots, x-1$, we denote by M_j the subsequence $\langle \Gamma_1, \ldots, \Gamma_{j+1} \rangle$ of M; also, for $i = 1, \ldots, n-1$, we define the *total rotation* $\rho_i(M_j)$ of edge e_i around v_i during morph M_j as $\rho_i(M_j) = \sum_{m=1}^{j} \rho_i^m$.

We will show in Lemma 10 that there exists an edge e_i, for some $1 \leq i \leq n-1$, whose total rotation $\rho_i(M_{x-1}) = \rho_i(M)$ is $\Omega(n)$. In order to do that, we first analyze the relationship between the total rotation of two consecutive edges of P.

Lemma 9. *For each $j = 1, \ldots, x-1$ and for each $i = 1, \ldots, n-2$, we have that $|\rho_{i+1}(M_j) - \rho_i(M_j)| < \pi$.*

Proof: Suppose, for a contradiction, that $|\rho_{i+1}(M_j) - \rho_i(M_j)| \geq \pi$ for some $1 \leq j \leq x-1$ and $1 \leq i \leq n-2$. Assume that j is minimal under this hypothesis. Since each vertex moves continuously during M_j, there exists an intermediate drawing Γ^* of P, occurring during morphing step $\langle \Gamma_j, \Gamma_{j+1} \rangle$, such that $|\rho_{i+1}(M^*) - \rho_i(M^*)| = \pi$, where $M^* = \langle \Gamma_1, \ldots, \Gamma_j, \Gamma^* \rangle$ is the morph obtained by concatenating M_{j-1} with the morphing step transforming Γ_j into Γ^*. Recall that in Γ_1 edges e_i and e_{i+1} lie on the same straight line and have the same orientation. Then, since $|\rho_{i+1}(M^*) - \rho_i(M^*)| = \pi$, in Γ^* edges e_i and e_{i+1} are parallel and have opposite orientations. Also, since edges e_i and e_{i+1} share vertex v_{i+1}, they lie on the same line. This implies that such edges overlap, contradicting the hypothesis that M^*, M_j, and M are planar. □

We now prove the key lemma for the lower bound.

Lemma 10. *There exists an index i such that $|\rho_i(M)| \in \Omega(n)$.*

Proof: Refer to Fig. 4. For every $1 \leq i \leq n-2$, edges e_i and e_{i+1} form an angle of π radiants in Γ_s, while they form an angle of $\frac{\pi}{3}$ radiants in Γ_t. Hence, $\rho_{i+1}(M) = \rho_i(M) + \frac{2\pi}{3} + 2z_i\pi$, for some $z_i \in \mathbb{Z}$. In order to prove the lemma, it suffices to prove that $z_i = 0$, for every $i = 1, \ldots, n-2$. Namely, in this case $\rho_{i+1}(M) = \rho_i(M) + \frac{2\pi}{3}$ for every $1 \leq i \leq n-2$, and hence $\rho_{n-1}(M) = \rho_1(M) + \frac{2\pi}{3}(n-2)$. This implies $|\rho_{n-1}(M) - \rho_1(M)| \in \Omega(n)$, and thus $|\rho_1(M)| \in \Omega(n)$ or $|\rho_{n-1}(M)| \in \Omega(n)$. Assume, for a contradiction, that $z_i \neq 0$, for some $1 \leq i \leq n-2$. If $z_i > 0$, then $\rho_{i+1}(M) \geq \rho_i(M) + \frac{8\pi}{3}$; further, if $z_i < 0$, then $\rho_{i+1}(M) \leq \rho_i(M) - \frac{4\pi}{3}$. Since each of these inequalities contradicts Lemma 9, the lemma follows. □

We are now ready to state the main theorem of this section.

Theorem 3. *There exists two straight-line planar drawings Γ_s and Γ_t of an n-vertex path P such that any planar morph between Γ_s and Γ_t requires $\Omega(n)$ morphing steps.*

Proof: The two drawings Γ_s and Γ_t of path $P = (v_1, \ldots, v_n)$ are those illustrated in Fig. 4. By Lemma 10, there exists an edge e_i of P, for some $1 \leq i \leq n-1$, such that $|\sum_{j=1}^{x-1} \rho_i^j| \in \Omega(n)$. Since, by Lemma 8, we have that $|\rho_i^j| < \pi$ for each $j = 1, \ldots, x-1$, it follows that $x \in \Omega(n)$. This concludes the proof of the theorem. □

5 Conclusions

In this paper we presented an algorithm to construct a planar morph between two planar straight-line drawings of the same n-vertex plane graph in $O(n)$ morphing steps.

We also proved that this bound is tight (note that our lower bound holds for any morphing algorithm in which the vertex trajectories are polynomial functions of constant degree).

In our opinion, the main challenge in this research area is the one of designing algorithms to construct planar morphs between straight-line planar drawings with good resolution and within polynomial area (or to prove that no such algorithm exists). In fact, the algorithm we presented, as well as other algorithms known at the state of the art [1, 3, 5, 14], construct intermediate drawings in which the ratio between the lengths of the longest and of the shortest edge is exponential. Guaranteeing good resolution and small area seems to be vital for making a morphing algorithm of practical utility.

Finally, we would like to mention an original problem that generalizes the one we solved in this paper and that we repute very interesting. Let Γ_s and Γ_t be two straight-line drawings of the same (possibly non-planar) topological graph G. Does a morphing algorithm exist that morphs Γ_s into Γ_t and that preserves the topology of the drawing at any time instant? A solution to this problem is not known even if we allow the trajectories followed by the vertices to be of arbitrary complexity.

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