# Recent Progress on Dimensions of Projections

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**Abstract** This is a survey on recent progress on the question: how do projections effect dimensions generically? I shall also discuss briefly dimensions of plane sections.

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## 1 Introduction

I give a survey on the question how projection-type transformations change dimensions of sets. I shall mainly discuss Hausdorff dimension but packing and Minkowski dimensions will also be briefly looked at. First I review classical Marstrand's projection theorem and give Kaufman's proof for it. Then I present recent partial analogues of Marstrand's projection theorem in Heisenberg groups due to Balogh, Durand-Cartagena, Fässler, Tyson and myself. After that I discuss generalized projections of Peres and Schlag. In Heisenberg groups and other situations one encounters small, restricted, families of transformations. I review recent results of E. Järvenpää, M. Järvenpää, Ledrappier, Leikas and Keleti and of Fässler and Orponen on them. Then I mention briefly older results of Falconer and Howroyd and recent results of Fässler and Orponen on packing and Minkowski dimensions. For them one has generally only inequalities, but Falconer and Howroyd proved also a constancy theorem. I present this and some recent constancy theorem of Fässler and Orponen on Hausdorff dimension for a particular restricted family of projections. Finally we shall have

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a look at Marstrand's classical result on Hausdorff dimension of plane sections and recent analogues of it in Heisenberg groups.

Background on this topic can be found in the books [Fa85, Map2]. Recent related surveys are [Map3, Map4].

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## 2 Marstrand's Projection Theorem

Marstrand proved in 1954 the following theorem in [Ma54]:

**Theorem 2.1** *Suppose* $A \subset \mathbb{R}^2$ *is a Borel set and denote by* $P_\theta, \theta \in [0, \pi)$*, the orthogonal projection onto the line* $L_\theta = \{t(\cos\theta, \sin\theta) : t \in \mathbb{R}\}$*:* $P_\theta(x, y) = (\cos\theta)x + (\sin\theta)y$*.*

(1) *If* $\dim A \leq 1$*, then* $\dim P_\theta(A) = \dim A$ *for almost all* $\theta \in [0, \pi)$.

(2) *If* $\dim A > 1$*, then* $\mathcal{L}^1(P_\theta(A)) > 0$ *for almost all* $\theta \in [0, \pi)$.

Here dim means Hausdorff dimension and $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

Marstrand's original proof was based on the definition and basic properties of Hausdorff measures. Kaufman used in [Ka68] potential theoretic and Fourier analytic methods to give a different proof. To present Kaufman's proof let us first look at the required preliminaries.

The Hausdorff dimension of a Borel set $A \subset \mathbb{R}^n$ can be determined by looking at the behaviour of Borel measures $\mu$ with compact support $\operatorname{spt}\mu \subset A$. Denote the family of such measures $\mu$ with $0 < \mu(A) < \infty$ by $\mathcal{M}(A)$. By the well-known Frostman's lemma $\dim A$ is the supremum of the numbers $s$ such that there exists $\mu \in \mathcal{M}(A)$ for which

$$\mu(B(x, r)) \leq r^s \quad \text{for } x \in \mathbb{R}^n. \tag{2.1}$$

This is easily transformed into an integral condition. Let

$$I_s(\mu) = \iint |x - y|^{-s} d\mu x d\mu y$$

be the $s$-energy of $\mu$. Then $\dim A$ is the supremum of the numbers $s$ such that there exists $\mu \in \mathcal{M}(A)$ for which

$$I_s(\mu) < \infty. \tag{2.2}$$

For a fixed $\mu$ (2.1) and (2.2) may not be equivalent, but they are closely related: (2.2) implies that the restriction of $\mu$ to a suitable set with positive $\mu$ measure satisfies (2.1), and (2.1) implies that $\mu$ satisfies (2.2) for any $s' < s$. Defining the Riesz kernel $k_s$, $k_s(x) = |x|^{-s}$, the $s$-energy of $\mu$ can written as

$$I_s(\mu) = \int k_s * \mu d\mu.$$

For $0 < s < n$ the Fourier transform of $k_s$ is (in the sense of distributions) $\widehat{k_s} = c(s,n)k_{n-s}$. Thus we have by Plancherel's theorem

$$I_s(\mu) = \int \widehat{k_s}|\widehat{\mu}|^2 = c(s,n)\int |x|^{s-n}|\widehat{\mu}(x)|^2 dx.$$

Consequently, $\dim A$ is the supremum of the numbers $s \leq n$ such that there exists $\mu \in \mathcal{M}(A)$ for which

$$\int |x|^{s-n}|\widehat{\mu}(x)|^2 dx < \infty. \tag{2.3}$$

To prove (1) of Theorem 2.1 let $0 < s < \dim A$ and choose by (2.2) a measure $\mu \in \mathcal{M}(A)$ such that $I_s(\mu) < \infty$. Let $\mu_\theta \in \mathcal{M}(P_\theta(A))$ be the push-forward of $\mu$ under $P_\theta$: $\mu_\theta(B) = \mu(P_\theta^{-1}(B))$. Then

$$\begin{aligned}\int_0^\pi I_s(\mu_\theta)d\theta &= \int_0^\pi \iint |P_\theta(x-y)|^{-s} d\mu x d\mu y d\theta \\ &= \iint \int_0^\pi |P_\theta(\tfrac{x-y}{|x-y|})|^{-s} d\theta |x-y|^{-s} d\mu x d\mu y \\ &= c(s)I_s(\mu) < \infty,\end{aligned}$$

where for $v \in S^1$, $c(s) = \int_0^\pi |P_\theta(v)|^{-s} d\theta < \infty$ as $s < 1$. Referring again to (2.2) we see that $\dim P_\theta(A) \geq s$ for almost all $\theta \in [0,\pi)$. By the arbitrariness of $s$, $0 < s < \dim A$, we obtain $\dim P_\theta(A) \geq \dim A$ for almost all $\theta \in [0,\pi)$. The opposite inequality follows from the fact that the projections $P_\theta$ are Lipschitz.

To prove (2) choose by (2.3) a measure $\mu \in \mathcal{M}(A)$ such that $\int |x|^{-1}|\hat{\mu}(x)|^2 dx < \infty$. Directly from the definition of the Fourier transform we see that $\widehat{\mu_\theta}(t) = \hat{\mu}(t(\cos\theta, \sin\theta))$ for $t \in \mathbb{R}$, $\theta \in [0,\pi)$. Integrating in polar coordinates we obtain

$$\int_0^{\pi}\int_{-\infty}^{\infty} |\widehat{\mu_\theta}(t)|^2 dt d\theta = 2\int_0^{\pi}\int_0^{\infty} |\widehat{\mu}(t(\cos\theta, \sin\theta))|^2 dt d\theta$$
$$= \int |x|^{-1}|\widehat{\mu}(x)|^2 dx < \infty.$$

Thus for almost all $\theta \in [0, \pi)$, $\widehat{\mu_\theta} \in L^2(\mathbb{R})$ which means that $\mu_\theta$ is absolutely continuous with $L^2$-density and hence $\mathcal{L}^1(P_\theta(A)) > 0$.

It is not difficult to prove (2) without Fourier transform: application of Fubini's theorem and some simple estimates yield

$$\int_0^{\pi}\int \liminf_{\delta\to 0} \delta^{-1}\mu_\theta(x-\delta, x+\delta) d\mu_\theta x d\theta \leq C I_1(\mu), \tag{2.4}$$

from which (2) follows by standard results on differentiation of measures.

Theorem 2.1 has the following generalization:

**Theorem 2.2** *Suppose $A \subset \mathbb{R}^2$ is a Borel set.*

(1) *If* $0 \leq t \leq \dim A \leq 1$*, then*

$$\dim\{\theta \in [0, \pi) : \dim P_\theta(A) < t\} \leq t.$$

(2) *If* $\dim A > 1$*, then*

$$\dim\{\theta \in [0, \pi) : \mathcal{L}^1(P_\theta(A)) = 0\} \leq 2 - \dim A.$$

Part (1) was proved by Kaufman with a similar method as above; one uses Frostman's lemma also for the exceptional set of directions. Part (2) was proved by Falconer with a Fourier-analytic method.

To formulate the higher dimensional version of Theorem 2.2, denote by $G(n, m)$ the Grassmannian manifold of linear $m$-dimensional subspaces of $\mathbb{R}^n$. For $V \in G(n, m)$, let

$$P_V : \mathbb{R}^n \to V$$

be the orthogonal projection. As above, we shall often write also $P_V : \mathbb{R}^n \to \mathbb{R}^m$ in a natural way. Identifying $V$ with $P_V$, $G(n, m)$ becomes a smooth submanifold of dimension $m(n-m)$ of $\mathbb{R}^{n^2}$.

**Theorem 2.3** *Suppose $A \subset \mathbb{R}^n$ is a Borel set.*

(1) *If* $\dim A \leq m$*, then*

$$\dim\{V \in G(n, m) : \dim P_V(A) < t\} \leq m(n-m) - m + t.$$

(2) *If* $\dim A > m$*, then*

$$\dim\{V \in G(n,m) : \mathcal{L}^1(P_V(A)) = 0\} \leq m(n-m) + m - \dim A.$$

Part (1) was proved in [Map5] and (2) in [Fa82]. The bound in (1) is sharp when $t = \dim A$. This was shown by Kaufman and myself in [KM75] with examples based on Jarnik's results on dimension and diophantine approximation, see also [Fa85], Sect. 8.5. Similar examples work also for (2). As far as I know the sharp bound in (1) for $t < \dim A$ is unkown. Anyway, the one given in Theorem 2.2 is not always sharp due to the following result of Bourgain in [Bo10] and Oberlin in [Ob12]:

**Theorem 2.4** *Suppose* $A \subset \mathbb{R}^2$ *is a Borel set. Then*

$$\dim\{\theta \in [0,\pi) : \dim P_\theta(A) < \dim A/2\} = 0.$$

The construction in [KM75] can be used to get for any $0 < t \leq s < 2$ a compact set $A \subset \mathbb{R}^2$ with $\dim A = s$ such that

$$\dim\{\theta \in [0,\pi) : \dim P_\theta(A) \geq t\} \geq 2t - s.$$

Could $2t - s$ be the sharp upper bound in the range $s/2 \leq t \leq \min\{1, s\}$? In any case this shows that to get dimension 0 for the exceptional set, the bound $\dim A/2$ is the best possible.

Bourgain's estimate is somewhat stronger than the above. He obtained his result as part of deep investigations in additive combinatorics, whereas Oberlin's proof is much simpler and more direct. Oberlin also had another exceptional set estimate in [Ob13].

Some improvements on part (2) of Theorem 2.3 will be given soon in Sect. 4 in a more general setting.

## 3 Projection Theorems in Heisenberg Groups

Heisenberg group $\mathbb{H}^n$ is $\mathbb{R}^{2n+1}$ equipped with a non-abelian group structure, with a left invariant metric and with natural dilations. The first Heisenberg group $\mathbb{H}^1$ is the simplest of these. We can write $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$, where the points are written as $p = (w, s), q = (z, t) \in \mathbb{H}^1$. Then the product of $p$ and $q$ is

$$p \cdot q = (w + z, s + t + 2Im(w\bar{z})).$$

To define the distance between $p$ and $q$, set first

$$\|p\| = (|z|^4 + t^2)^{1/4},$$

and then

$$d(p,q) = \|p^{-1} \cdot q\| = (|z-w|^4 + |t-s-2Im(w\bar{z})|^2)^{1/4}.$$

It is easy to check that $d$ really is a metric and that it is left invariant. We have the dilations

$$\delta_r(p) = (rz, r^2 t)$$

for which

$$d(\delta_r(p), \delta_r(q)) = rd(p,q).$$

When the distance is restricted to the $t$-axis $\{0\} \times \mathbb{R}$ it is just the square root distance. Essentially because of this the Heisenberg Hausdorff dimension of $\mathbb{H}^1$ is

$$\dim_H \mathbb{H}^1 = 4.$$

Here $\dim_H$ refers to the Hausdorff dimension with respect to the Heisenberg metric. Always dim will refer to the Hausdorff dimension with respect to the Euclidean metric. It is easy to check that

$$\dim(A) \leq \dim_H(A) \leq 2\dim(A),\ A \subset \mathbb{H}^1.$$

These inequalities are sharp. For example, if $A$ is a subset of the $x$-axis, $\dim_H(A) = \dim(A)$, and if $A$ is a subset of the $t$-axis, $\dim_H(A) = 2\dim(A)$. However, one can improve them for sets $A$ with $\dim(A) > 1$. Very precise inequalities were obtained by Balogh et al. [BT09].

We define the projections in $\mathbb{H}^1$ in the group sense. Good subgroups of $\mathbb{H}^1$ for this purpose are those which are invariant under the dilations and have a complementary subgroup in the sense described below. They are precisely the horizontal lines

$$V_\theta = \{te_\theta : t \in \mathbb{R}\},\ e_\theta = (\cos\theta, \sin\theta, 0),\ 0 \leq \theta < \pi,$$

and the vertical planes

$$W_\theta = V_\theta^\perp.$$

The horizontal lines $V_\theta$ are Euclidean, the distance restricted to them is the Euclidean distance, whereas the vertical planes $W_\theta$ are non-Euclidean; for them $\dim_H W_\theta = \dim W_\theta + 1$. We have the splitting

$$\mathbb{H}^1 = W_\theta \cdot V_\theta,$$

that is, for $p \in \mathbb{H}^1$ we have the unique factorization

$$p = Q_\theta(p) \cdot P_\theta(p), \;\; P_\theta(p) \in V_\theta, \, Q_\theta(p) \in W_\theta.$$

Thus we get the group projections

$$P_\theta : \mathbb{H}^1 \to V_\theta, \, Q_\theta : \mathbb{H}^1 \to W_\theta, 0 \le \theta < \pi.$$

Writing $p = (z, t) = (x + iy, t) \in \mathbb{H}^1$ we have the explicit formulas

$$P_\theta(p) = ((x \cos\theta + y \sin\theta)e_\theta, 0),$$

$$Q_\theta(p) = ((y \cos\theta - x \sin\theta)e_\theta^\perp, t - 2\cos(2\theta)xy + \sin(2\theta)(x^2 - y^2)),$$

where $e_\theta^\perp = (-\sin\theta, \cos\theta)$. So $P_\theta$ is the standard linear projection, essentially the one we considered above in $\mathbb{R}^2$, but $Q_\theta$ is a non-linear projection. $P_\theta$ is nice, it is Lipschitz and group homomorphism, but $Q_\theta$ is neither of those, it is only Hölder continuous with exponent $1/2$.

Now we have the following analogue for horizontal projections of Marstrand's projection theorem from [BD13]:

**Theorem 3.1** *Let $A \subset \mathbb{H}^1$ be a Borel set. Then for almost all $\theta \in [0, \pi)$,*

$$\dim_H P_\theta(A) \ge \dim_H A - 2 \textit{ if } \dim_H A \le 3,$$

$$\mathcal{H}^1(P_\theta(A)) > 0 \textit{ if } \dim_H A > 3.$$

This is sharp: consider $A = \{(x, 0, t) : x \in C, t \in [0, 1]\}, C \subset \mathbb{R}$. Then $\dim_H A = \dim C + 2$ and

$$\dim_H P_\theta(A) = \dim P_\theta(A) = \dim P_\theta(C) = \dim C$$

for all but one $\theta$.

Theorem 3.1 follows easily applying Marstrand's projection theorem to the projection of $A$ on $\mathbb{C} \times \{0\}$.

For the vertical projections we have:

**Theorem 3.2** *Let $A \subset \mathbb{H}^1$ be a Borel set. If $\dim_H A \le 1$, then for almost all $\theta \in [0, \pi)$,*

$$\dim_H A \le \dim_H Q_\theta(A) \le 2 \dim_H A.$$

For $A$ with $\dim_H A \le 1$ this is sharp:
if $A \subset t$-axis, $\dim_H Q_\theta(A) = \dim_H A$ for all $\theta$,
if $A \subset x$-axis, $\dim_H Q_\theta(A) = 2 \dim_H A$ for all but one $\theta$.

The upper bound $2 \dim_H A$ follows from the Hölder continuity of $Q_\theta$. For the lower bound we use again the energy integrals. Let

$$p = (z, t), q = (\zeta, \tau) \in \mathbb{H}^1$$

and denote

$$\varphi_1 = \arg(z-\zeta), \varphi_2 = \arg(z+\zeta).$$

Then one can check that

$$d(p,q)^4 = |z-\zeta|^4 + (t-\tau+|z^2-\zeta^2|\sin(\varphi_1-\varphi_2))^2$$

and

$$d(Q_\theta(p), Q_\theta(q))^4 = |z-\zeta|^4 \sin^4(\varphi_1-\theta) \\ + (t-\tau-|z^2-\zeta^2|\sin(\varphi_2+\varphi_1-2\theta))^2$$

To get $\int_0^\pi d(Q_\theta(p), Q_\theta(q))^{-s} d\theta \lesssim d(p,q)^{-s}$, one needs for $a \in \mathbb{R}$,

$$\int_0^\pi \frac{d\theta}{|a+\sin\theta|^{s/2}} \lesssim 1, \tag{3.1}$$

which is easy to check when $s < 1$.

If $\dim_H A > 1$, we have some estimates which quite likely are not sharp. For example, we do not know if $\dim_H A > 3$ implies $\mathcal{H}^2(Q_\theta(A)) > 0$ for almost all $\theta \in [0,\pi)$. Here $\mathcal{H}^2$ is the Euclidean two-dimensional Hausdorff measure. When restricted to a vertical plane it agrees with the three-dimensional Heisenberg Hausdorff measure, both give the Haar measure for this subgroup.

A related Euclidean question is: does $\dim A > 2$ imply $\mathcal{H}^2(Q_\theta(A)) > 0$ for almost all $\theta \in [0,\pi)$?

Let us now consider higher dimensions, these were treated in [BF12]. Then the basic notions and facts are

- $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $p = (w,s)$, $q = (z,t) \in \mathbb{H}^n$,
- $\omega(w,z) = 2Im(w \cdot z) = 2\sum_{j=1}^n (v_j x_j - u_j y_j)$, $w = (u_j + iv_j)$, $z = (x_j + iy_j)$,
- $p \cdot q = (w+z, s+t+\omega(w,z))$,
- $||p|| = (|z|^4 + t^2)^{1/4}$,
- $d(p,q) = ||p^{-1} \cdot q|| = (|z-w|^4 + |t-s-\omega(w,z)|^2)^{1/4}$,
- $\delta_r(p) = (rz, r^2 t)$,
- $d(\delta_r(p), \delta_r(q)) = r d(p,q)$,
- $d(p \cdot q_1, p \cdot q_2) = d(q_1, q_2)$,
- $\dim_H \mathbb{H}^n = 2n+2$.

The subgroups invariant under dilations split again to horizontal and vertical subgroups. The horizontal ones are those $m$-dimensional linear subspaces of $\mathbb{R}^{2n}$, $0 < m \le n$, on which the bilinear form $\omega$ vanishes. That is, the elements of

$$G_h(n,m) = \{V \in G(2n,m) : \omega(w,z) = 0 \ \forall w, z \in V\}.$$

They are called isotropic subspaces. The unitary group $U(n) \subset O(2n)$ acts transitively on $G_h(n,m)$; by definition $g \in U(n)$ if $\omega(g(w), g(z)) = \omega(w,z)$ for all $w, z \in \mathbb{C}^n$. The vertical subgroups are all linear subspaces of $\mathbb{R}^{2n+1}$ which contain the $t$-axis. The horizontal subgroups are again Euclidean and for the vertical subgroups $W$ we have $\dim_H W = \dim W + 1$. Then

$$\mathbb{H}^n = V^\perp \cdot V, V^\perp \subset \mathbb{R}^{2n+1}, V \in G_h(n,m),$$

$$p = Q_V(p) \cdot P_V(p), \ P_V(p) \in V, Q_V(p) \in V^\perp, \text{ for } p \in \mathbb{H}^n.$$

Again $P_V : \mathbb{H}^n \to V$ is the standard linear projection, but $Q_V : \mathbb{H}^n \to V^\perp$,

$$Q_V(z,t) = (P_{V^\perp}(z), t - \omega((P_{V^\perp}(z), P_V(z))),$$

is a non-linear projection.

Notice that in the above splitting the linear dimension of $V$ is always at most $n$. The vertical subgroups $W$ of linear dimension $1 \le \dim W \le n$ have no complementary subgroups in the above sense.

We have the following horizontal projection theorem in $\mathbb{H}^n$:

**Theorem 3.3** *Let $A \subset \mathbb{H}^n$ be a Borel set. If* $\dim_H A \le m+2$*, then*

$$\dim P_V(A) \ge \dim_H A - 2$$

*for $\mu_{n,m}$ almost all $V \in G_h(n,m)$. Furthermore, if* $\dim_H A > m+2$*, then*

$$\mathcal{H}^m(P_V(A)) > 0 \textit{ for } \mu_{n,m} \textit{ almost } V \in G_h(n,m).$$

This is again sharp. Above $\mu_{n,m}$ is the unique $U(n)$-invariant Borel probability measure on $G_h(n,m)$.

For the vertical projections we have

**Theorem 3.4** *Let $A \subset \mathbb{H}^n$ be a Borel subset with* $\dim_H A \le 1$*. Then for $\mu_{n,m}$ almost $V \in G_h(n,m)$,*

$$\dim_H A \le \dim_H Q_V A \le 2 \dim_H A.$$

This is sharp when $\dim_H A \le 1$. Some, probably rather imprecise, partial results are known when $\dim_H A > 1$. One might expect that the methods would yield this theorem for $\dim_H A \le m$, but there are some serious obstacles. Let us see what they are. We can now write

$$d_H(p,q) = \sqrt[4]{|z-w|^4 + (t-s-2\omega(\zeta,z))^2},$$

and

$$d_H(Q_V(p), Q_V(q))^4 = |P_{V^\perp}(z-w)|^4 + (t - s - \omega(P_{V^\perp}(z), P_V(z)) \\ + \omega(P_{V^\perp}(w), P_V(w)) - \omega(P_{V^\perp}(w), P_{V^\perp}(z)))^2.$$

The key estimate in the proof is

$$\int_{G_h(n,m)} |a - \omega(v, P_V(w))|^{-s/2} d\mu_{n,m} V \lesssim 1$$

for all $0 < s < 1$, $a \in \mathbb{R}$ and $v, w \in S^{2n-1}$. In local coordinates for $V$ the expression $a - 2\omega(v, P_V(w))$ is a second degree polynomial which can vanish to second order. Because of this the above estimate is false for $s \geq 1$ and it seems to be difficult to find anything to replace it.

There are various other results in the papers [BD13, BF12]. In particular, quite precise information is obtained on inequalities that hold for all projections.

## 4 Generalized Projections

Studying Kaufman's proof of Marstrand's projection theorem one notices quickly that it applies to much more general families of mappings than orthogonal projections onto lines and planes. Peres and Schlag developed this idea in [PS00] much farther. The following is still a special case of their general setting:

Let $(\Omega, d)$ be a compact metric space, $Q \subset \mathbb{R}^k$ an open connected set. We have mappings

$$\pi_\lambda\colon \Omega \to \mathbb{R}^m, \quad \lambda \in Q,$$

such that the mapping $\lambda \mapsto \pi_\lambda(x)$ is in $C^\infty(Q)$ for every fixed $x \in \Omega$, and to every compact $K \subset Q$ and any multi-index $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$ there corresponds a finite constant $C_{\eta,K} > 0$ such that

$$|\partial_\lambda^\eta \pi_\lambda(x)| \leq C_{\eta,K}, \qquad \lambda \in K. \tag{4.1}$$

**Definition 4.1** Define

$$\Phi_\lambda(x,y) = \frac{\pi_\lambda(x) - \pi_\lambda(y)}{d(x,y)}.$$

The family $\{\pi_\lambda, \lambda \in Q\}$ is said to be *transversal*, if there exists a finite constant $C_0 > 0$ such that

$$|\Phi_\lambda(x,y)| \leq C_0 \implies \det(D_\lambda \Phi_\lambda(x,y)(D_\lambda \Phi_\lambda(x,y)^t)) \geq C_0 \tag{4.2}$$

for $\lambda \in Q$ and $x, y \in \Omega, x \neq y$. The family $\{\pi_\lambda, \lambda \in Q\}$, is said to be *regular*, if to every multi-index $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$ there correspond a finite constant $C_\eta > 0$ such that

$$|\Phi_\lambda(x, y)| \leq C_0 \implies \left|\partial_\lambda^\eta \Phi_\lambda(x, y)\right| \leq C_\eta \tag{4.3}$$

for $\lambda \in Q$ and $x, y \in \Omega, x \neq y$.

Orthogonal projections when restricted to some compact set are easily seen to form transversal and regular families of mappings. When considering projections from $\mathbb{R}^n$ onto $m$-planes, we can take $k = \dim G(n, m) = m(n - m)$.

Since we are looking for lower bounds for dimensions of projections, the bad pairs of points are such $x$ and $y$ which are mapped close to each other. The point in transversality is that if $(x, y)$ is a pair of bad points for some $\lambda$, then it becomes quickly better when $\lambda$ moves a bit. For real-valued maps ($m = 1$), such as projections onto lines, the transversality means

$$|\Phi_\lambda(x, y)| \leq C_0 \implies |\nabla_\lambda \Phi_\lambda(x, y)| \geq C_0.$$

Here is a special case of a theorem of Peres and Schlag:

**Theorem 4.2** *Suppose the above transversality and regularity conditions hold. Let $A \subset \Omega$ be a Borel set and $s = \dim A$.*

(a) *If $s \leq m$ and $t \in (0, s)$, then*

$$\dim\{\lambda \in Q : \dim \pi_\lambda(A) < t\} \leq k - m + t.$$

(b) *If $s > m$, then*

$$\dim\{\lambda \in Q : \dim \pi_\lambda(A) < t\} \leq k - s + t$$

*and*

$$\dim\{\lambda \in Q : \mathcal{L}^m(\pi_\lambda(A)) = 0\} \leq k - s + m.$$

(c) *If $s > 2m$, then*

$$\dim\{\lambda \in Q : \textit{the interior of } \pi_\lambda(A) \textit{ is empty}\} \leq n - s + 2.$$

In addition to being applicable to many families of mappings, this theorem also improves Theorem 2.3 in the case of orthogonal projections. As Peres and Schlag showed it can be applied in many interesting situations, for example to Bernoulli convolutions, sum sets and pinned distance sets.

In $\mathbb{R}^{2n}$ the horizontal Grassmannian, the Grassmannian of isotropic subspaces, $G_h(n, m)$, discussed before in the case of Heisenberg groups, is a proper lower dimensional submanifold of the full Grassmannian $G(2n, m)$ when $1 < m \leq n$. Nevertheless Marstrand's projection theorem holds for this submanifold. We proved this in [BF12]. Hovila established it in [Ho18] by verifying that the family $P_V : \mathbb{R}^{2n} \to V, V \in G_h(n, m)$, is transversal. This has two further consequnces: exceptional set estimates and Besicovitch-Federer projection theorem. The first follows from the above results of Peres and Schlag, the second from Hovila's joint work with Järvenpää et al. in [HLJ2]. There they proved Besicovitch-Federer projection theorem for transversal families of generalized projections. The classical Besicovitch-Federer projection theorem says that an $\mathcal{H}^m$ measurable set $A \subset \mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$ p! rojects into zero $\mathcal{H}^m$ measure in almost all $m$-planes $V \in G(n, m)$ if an and only if it meets every $m$-dimensional $C^1$-surface in a set of zero $\mathcal{H}^m$ measure, see [Fe69] or [Map2].

Neither the vertical nor the horizontal projections in Heisenberg groups satisfy transversality; these families are too small for that.

## 5 Restricted Families of Projections

The reason that it is not possible to get precise almost everywhere equalities for dimensions of projections in Heisenberg groups is that we have too few projections. It is of interest to search projection theorems for such restricted families of projections also in Euclidean spaces. That is, one considers a proper lower dimensional submanifold $G$ of the Grassmannian $G(n, m)$ and the projections $P_V, V \in G$. This splits into two cases: $G$ is general allowing flat submanifolds or $G$ is required to possess some curvature properties. What these mean becomes clearer below. In the first case less can be said and it is completely solved by E. Järvenpää, M. Järvenpää and Keleti as we shall see soon. The second case is extremely difficult and some partial results have been obtained by Fässler and Orponen.

One motivation for studying restricted families of projection-type transformations comes from the work of E. Järvenpää, M. Järvenpää, Ledrappier and their co-workers on measures invariant under geodesic flows on manifolds, see [HJL2] and the references given there.

A simple restricted family of projections in $\mathbb{R}^3$ is given by the horizontal projections, or the projections onto the lines $L_\theta = \{t(\cos\theta, \sin\theta, 0) : t \in \mathbb{R}\}$,

$$P_\theta : \mathbb{R}^3 \to \mathbb{R}, P_\theta(x, y, z) = x\cos\theta + y\sin\theta, 0 \le \theta < \pi. \tag{5.1}$$

Since $P_\theta(A) = P_\theta((\pi(A))$ where $\pi(x, y, z) = (x, y)$, and $\dim A \le \dim \pi(A) + 1$, it is easy to conclude using Marstand's projection theorem that for any Borel set $A \subset \mathbb{R}^3$, for almost all $\theta \in [0, \pi)$,

$$\dim P_\theta(A) \ge \dim A - 1 \text{ if } \dim A \le 2,$$
$$\dim P_\theta(A) = 1 \text{ if } \dim A \ge 2.$$

This is sharp by trivial examples; consider product sets $A = B \times C$, $B \subset \mathbb{R}^2$, $C \subset \mathbb{R}$.

A simple example of projections onto planes is given by

$$\Pi_\theta : \mathbb{R}^3 \to \mathbb{R}^2, \Pi_\theta(x, y, z) = (x \sin\theta - y\cos\theta, z), 0 < \theta < \pi. \tag{5.2}$$

These are essentially orthogonal projections onto the orthogonal complements of the lines $L_\theta$.

Also now it is easy to prove that for any Borel set $A \subset \mathbb{R}^3$, for almost all $\theta \in [0, \pi)$,

$$\begin{aligned}
&\dim \Pi_\theta(A) \geq \dim A \text{ if } \dim A \leq 1,\\
&\dim \Pi_\theta(A) \geq 1 \text{ if } 1 \leq \dim A \leq 2,\\
&\dim \Pi_\theta(A) \geq \dim A - 1 \text{ if } \dim A \geq 2.
\end{aligned}$$

Again by easy examples these inequalities are sharp.

Järvenpää et al. proved in [JJ05] that the above sets of inequalities remain in force for any smooth, in a suitable sense non-degenerate, one-dimensional families of orthogonal projections onto lines and planes in $\mathbb{R}^3$. In fact, they proved such inequalities in more general dimensions and in [JJ13] Järvenpää et al. found the complete solution in all dimensions; sharp inequalities for smooth non-degenerate families of orthogonal projections onto $m$-planes in $\mathbb{R}^n$.

Consider now a slightly modified family of one-dimensional projections; let $p_\theta, \theta \in [0, 2\pi)$, be the orthogonal projection onto the line $l_\theta$ spanned by $(\cos\theta, \sin\theta, 1)$. The previous lines $L_\theta$ spanned a plane, but the lines $l_\theta$ span a cone. The trivial counter-examples do not work anymore and in fact one can now improve the above estimates for $p_\theta$'s relatively easily by showing that if $A \subset \mathbb{R}^3$ is a Borel set with $\dim A \leq 1/2$, then

$$\dim p_\theta(A) \geq \dim A \text{ for almost all } \theta \in [0, 2\pi).$$

The restriction $1/2$ comes because using Kaufman's method one is now lead to estimate integrals of the type

$$\int_0^{2\pi} \frac{d\theta}{|a + \sin\theta|^s}$$

for $s < \dim A$, and they are bounded only if $s < 1/2$. So this is the best one get without new ideas. Introducing some geometric arguments Fässler and Orponen were able to prove in [FO50] the following theorem for the packing dimensions, $\dim_p$, of the projections:

**Theorem 5.1** *Let $U \subset \mathbb{R}$ be an open interval and $\gamma : U \to S^2$ be a $C^3$ curve such that for all $\theta \in U$ the vectors $\gamma(\theta)$, $\gamma'(\theta)$ and $\gamma''(\theta)$ span $\mathbb{R}^3$. Let*

$$p_\theta(x) = \gamma(\theta) \cdot x$$

*be the orthogonal projection onto the line $l_\theta$ spanned by $\gamma(\theta)$ and*

$$\pi_\theta(x) = x - (\gamma(\theta)\cdot x)\gamma(\theta)$$

*the orthogonal projection onto the orthogonal complement of $l_\theta$. Suppose $A \subset \mathbb{R}^3$ is a Borel set with* $\dim A = s$.

(1) *If $s > 1/2$, there exists a number $\sigma_1(s) > 1/2$ such that*

$$\dim_p p_\theta(A) \geq \sigma_1(s) \text{ for almost all } \theta \in U.$$

(2) *If $s > 1$, there exists a number $\sigma_1(s) > 1$ such that*

$$\dim_p \pi_\theta(A) \geq \sigma_2(s) \text{ for almost all } \theta \in U.$$

It is not known if here the packing dimension could be replaced by the Hausdorff dimension. In [Orp3] Orponen was able to do this for the special family of orthogonal projections onto the lines $l_\theta$ spanned by $(\cos\theta, \sin\theta, 1)$ and their orthogonal complements.

It would be very interesting to find similar results for some non-linear families of mappings, for example for the vertical projections $Q_\theta$ of the Heisenberg group $\mathbb{H}^1$ considered just as mappings in $\mathbb{R}^3$:

$$Q_\theta(x, y, t) = (y\cos\theta - x\sin\theta, t - 2\cos(2\theta)xy + \sin(2\theta)(x^2 - y^2)), \theta \in [0, \pi).$$

Although, as said before, for the corresponding linear projections $\Pi_\theta$ as in (5.2) nothing more can be said than what we get from Marstrand's theorem, the non-linear mappings might be better.

## 6 Minkowski and Packing Dimensions

The analogue of Marstrand's theorem fails for Minkowski and packing dimensions; the dimension of the set does not prescribe the dimensions of the typical projections. However, Falconer and Howroyd proved in [FH96] the following sharp inequalities:

**Theorem 6.1** *Let $A \subset \mathbb{R}^n$ be a Borel set. Then*

$$\dim_p P_V(A) \geq \frac{\dim_p A}{1 + (1/m - 1/n)\dim_p A} \text{ for almost all } V \in G(n, m).$$

The same inequality holds also for upper and lower Minkowski dimensions. Examples of Järvenpää in [Jm94] show that the lower bound is sharp. In these examples the Hausdorff dimension of $A$ is 0. Later on we proved with Falconer in [Fm96] a version of this result which gives a sharp lower bound for the packing dimension of the typical projections given both Hausdorff and packing dimension of $A$.

Finding good dimension estimates for exceptional sets in packing dimension projection theorems has turned out to be a very delicate question, Rams obtained some results in [Ram2] for self-conformal sets. Orponen proved in [Ort2] several such estimates and constructed many illustrative examples. He also established Baire category results.

## 7 Constancy Results for Projections

Although there is no dimension preservation for the packing and Minkowski dimensions under projections, Falconer and Howroyd proved in [FH97] that given the set $A$, the dimensions equal almost surely a constant called $\mathrm{Dim}_m A$. The number $\mathrm{Dim}_m A$ comes from certain potentials. More precisely, we first define it for measures $\mu \in \mathcal{M}(A)$:

$$\mathrm{Dim}_m \mu = \sup\{t \geq 0 : \liminf_{r \to 0} r^{-t} F_m^{\mu}(x, r) = 0 \text{ for } \mu \text{ almost all } x \in \mathbb{R}^n\},$$

where

$$F_m^{\mu}(x, r) = \int \min\{1, r^m |x - y|^{-m}\} d\mu y.$$

For sets we define

$$\mathrm{Dim}_m A = \sup\{\mathrm{Dim}_m \mu : \mu \in \mathcal{M}(A)\}.$$

The theorem of Falconer and Howroyd now reads

**Theorem 7.1** *Let $A \subset \mathbb{R}^n$ be a Borel set. Then*

$$\dim_p P_V(A) = \mathrm{Dim}_m A \text{ for almost all } V \in G(n, m).$$

The relation of the potentials $F_m^{\mu}(x, r)$ to projections comes from the following observation:

$$\begin{aligned} F_m^{\mu}(x, r) &\approx \int \gamma_{n,m}(\{V \in G(n, m) : |P_V(x - y)| \leq r\}) d\mu y \\ &= \int P_V \mu(B(P_V(x), r)) d\gamma_{n,m} V, \end{aligned}$$

where $P_V \mu$ is the push forward of $\mu$ under $P_V$.

Similar tools were also used in [FH96, Fm96].

Perhaps such constancy results hold also in Heisenberg groups. This is not known but Fässler and Orponen proved in [FO13] constancy results for some restricted families of projections in $\mathbb{R}^3$. They did it in general dimensions but for simplicity I state their result only in $\mathbb{R}^3$:

**Theorem 7.2** *Let $A \subset \mathbb{R}^3$ be a Borel set. Then for the projections $\Pi_\theta$ as in (5.2)*

$$\dim \Pi_\theta(A) = \sup\{\Pi_\theta(A) : \theta \in (0, \pi)\} \textit{ for almost all } \theta \in (0, \pi).$$

Notice that for the projections $P_\theta$ onto lines given in (5.1) the constancy is trivial by Marstrand's projection theorem: for almost all $\theta \in (0, \pi)$,

$$\dim P_\theta(A) = \dim P_\theta(\pi(A)) = \min\{\dim \pi(A), 1\},$$

where $\pi(x, y, z) = (x, y)$.

Fässler and Orponen proved analogous results also for packing and Minkowski dimensions.

## 8 Slicing Theorems

Marstrand's line intersection theorem says that if $A$ is a Borel subset of the plane with $\dim A > 1$, then the typical lines which intersect $A$ intersect it in dimension $\dim A - 1$. Here is a way to state it more precisely and in higher dimensions:

**Theorem 8.1** *Let $A \subset \mathbb{R}^n$ be a Borel set, $s > m$ and $0 < \mathcal{H}^s(A) < \infty$. Then for $\gamma_{n,m}$ almost $V \in G(n, m)$,*

$$\mathcal{H}^m(\{v \in V : \dim(A \cap (V^\perp + v)) = s - m\}) > 0.$$

Here $\gamma_{n,m}$ is the orthogonally invariant Borel probability measure on $G(n, m)$.

With Balogh et al. we proved in [BF12] the analogous result in Heisenberg groups:

**Theorem 8.2** *Let $A \subset \mathbb{H}^n$ be a Borel set, $s > m + 2$ and $0 < \mathcal{H}^s_H(A) < \infty$. Then for $\mu_{n,m}$ almost $V \in G_h(n, m)$,*

$$\mathcal{H}^m(\{v \in V : \dim_H(A \cap (V^\perp \cdot v)) = s - m\}) > 0.$$

Here for $v \in V$, $V^\perp \cdot v$ is the coset $\{p \cdot v : p \in V^\perp\}$. The assumption $\dim_H A > m + 2$ is necessary.

Another way to formulate such a result, actually the one Marstrand used, is

**Theorem 8.3** *Let $A \subset \mathbb{H}^n$ be a Borel set, $s > m + 2$ and $\mathcal{H}^s_H(A) < \infty$. Then for $\mathcal{H}^s_H$ almost all $p \in A$ we have*

$$\dim_H(A \cap (V^{\perp} \cdot p)) = s - m \text{ for } \mu_{n,m} \text{ almost all } V \in G_h(n,m).$$

Here $\mathcal{H}_H^s$ is the $s$-dimensional Hausdorff measure with respect to the Heisenberg metric.

Orponen studied in [Or12] the problem of the dimension of exceptional sets for line sections. A higher dimensional version of his results is

**Theorem 8.4** *Let $A \subset \mathbb{R}^n$ be a Borel set, $s > m$ and $0 < \mathcal{H}^s(A) < \infty$. Then there is a Borel set $E \subset G(n,m)$ such that*

$$\dim E \leq m(n-m) + m - s$$

*and*

$$\mathcal{H}^m(\{a \in V : \dim A \cap (V^{\perp} + a) = s - m\}) > 0 \text{ for all } V \in G(n,m) \setminus E.$$

The upper bound is again sharp. Observe that this strengthens part (2) of Theorem 2.3.

To get his result, Orponen proved the following inequality: if $s > m$, $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $V \in G(n,m)$, then

$$\int_V I_{s-m}(\mu_{V,a})\, \mathrm{d}\mathcal{H}^m a \lesssim \int_{\mathbb{R}^n} |P_{V^{\perp}}(x)|^{s-n} |\widehat{\mu}(x)|^2 dx.$$

Here the measures $\mu_{V,a}$ are natural slices of $\mu$ with planes $V + a$. They have supports in $\operatorname{spt}\mu \cap (V + a)$ and they disintegrate $\mu$ if the projection of $\mu$ on $V$ is absolutely continuous with respect to the $m$-dimensional Hausdorff measure on $V$.

Fraser et al. found in [FOS3] another interesting application for this inequality: they showed that any one-dimensional graph has Fourier dimension 1. More precisely, in general dimensions

**Theorem 8.5** *For any function $f : A \to \mathbb{R}^{n-m}$, $A \subset \mathbb{R}^m$, and for its graph $G_f = \{(x, f(x) : x \in A\}$, if $\mu \in \mathcal{M}(G_f)$, $s > 0$ and*

$$|\widehat{\mu}(x)| \leq |x|^{-s/2} \text{ for } x \in \mathbb{R}^n,$$

*then $s \leq m$.*

Notice that we make no assumptions on $f$, not even measurability. Still, before the work of Fraser, Orponen and Sahlsten this question was open even for Brownian graphs.

# 9 Final Comments

I have restricted here to general sets. Many of the above results are formulated, and are more natural and general, for measures and their dimensions. I have completely ignored the very interesting question on what can be said in various special cases, for example for self-similar and related sets and measures. For these one can often obtain results which hold for all directions or the exceptional directions can be specified. Outstanding work on self-similar and other dynamically generated sets has been recently done by Furstenberg in [Fu08], by Peres and Shmerkin in [PSh9], by Hochman and Shmerkin in [HS12], and by Hochman in [Ho16]. There have also been many results on dimensions of sections in various special cases, for example in [BFS12, LXZ7, MS03, WWXl, WX10].

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