Mandelbrot Cascades and Related Topics

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Abstract This article is an extended version of the talk given by the author at the conference "Advances in fractals and related topics", in December 2012 at the Chinese Hong-Kong University. It gathers recent advances in Mandelbrot cascades theory and related topics, namely branching random walks, directed polymers on disordered trees, multifractal analysis, and dynamical systems.

1 Introduction

In the late sixties, motivated by Kolmogorov's work [Kol62] on turbulence in which the "lognormal hypothesis" appeared, Mandelbrot introduced lognormal multiplicative processes (see [Man72]) to build random measures obtained as limit of martingales, and describing the distribution of the energy dissipation in intermittent turbulence. As his model turned out to be too difficult to found and study in complete rigor, Mandelbrot defined (not necessarily log-normal or conservative) multiplicative cascades on homogeneous trees [Man74a, Man74b], now called Mandelbrot cascades, to provide a mathematically easier to define and, a priori, tractable model of turbulence. Some of his conjectures on the model behavior were then proved by Kahane and Peyrière in [Kah74, Pey74, KP76], and some questions remained open for a long time. In the eighties, Kahane developed multiplicative chaos theory [Kah85, Kah87a] to give a completely rigorous framework to Mandelbrot original lognormal multiplicative processes, and go beyond. This theory (which covers the case of Mandelbrot cascades) and its applications have been particularly enriched by the regular introduction of new models of multiplicative cascades [BF05, BJM10, BJM10a, BM02, BM03, BM04, BM04a, BM09, CRV13, Fan89, Fan97, LRV00, Pey77, Pey79, RV10, WW69], and during the last five years by the rigorous

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connexion between lognormal multiplicative chaos and KPZ relations in quantum gravity [BJRV13, BS09, DS11, KPZ98, RV08] and the connexion between this chaos and SLE curves [AJKS11, She00], as well as by fine renormalization results for degenerate multiplicative chaos [AS14, BRV12, DRSV00a, DRSV00, JW11, Mad00, Web11] (the interested reader should consult the recent survey [RV13] on lognormal multiplicative chaos for more details).

This paper focuses on Mandelbrot cascades, and aims at gathering recent advances in its theory and related topics, namely branching random walks, directed polymers on disordered trees, multifractal analysis, and dynamical systems (other surveys dealing with Mandelbrot cascades, multiplicative chaos and applications are [BFP10, Fan04, Kah91, Pey00]).

2 Mandelbrot Cascades

Consider the set $\mathscr{A} = \{0, \dots, b-1\}$, where $b \geq 2$. Set $\mathscr{A}^* = \bigcup_{n\geq 0} \mathscr{A}^n$, where, by convention, $\mathscr{A}^0$ is the singleton $\{\epsilon\}$ whose the only element is the empty word ϵ. If $w \in \mathscr{A}^*$, we denote by $|w|$ the integer such that $w \in \mathscr{A}^{|w|}$. If $n \geq 1$ and $w = w_1 \cdots w_n \in \mathscr{A}^n$ then for $1 \leq k \leq n$ the word $w_1 \cdots w_k$ is denoted by $w|k$. By convention, $w|0 = \epsilon$.

Given v and w in $\mathscr{A}^n$, $v \wedge w$ is defined to be the longest prefix common to both v and w, i.e., $v|n_0$, where $n_0 = \sup\{0 \leq k \leq n \ : \ v|k = w|k\}$.

Let $\mathscr{A}^\omega$ stand for the set of infinite sequences $w = w_1 w_2 \cdots$ of elements of $\mathscr{A}$. Also, for $x \in \mathscr{A}^\omega$ and $n \geq 0$, let $x|n$ stand for the projection of x on $\mathscr{A}^n$.

If $w \in \mathscr{A}^*$, we consider the cylinder $[w]$ consisting of infinite words in $\mathscr{A}^\omega$ whose w is a prefix.

We index the closed b-adic subintervals of $[0, 1]$ by $\mathscr{A}^*$: for $w \in \mathscr{A}^*$, we set

$$I_w = \left[\sum_{1\leq k\leq |w|} w_k b^{-k}, \sum_{1\leq k\leq |w|} w_k b^{-k} + b^{-|w|} \right].$$

Let W be a non negative random variable such that $\mathbb{E}(W) = 1$ and $\big(W(w)\big)_{w\in\mathscr{A}^*}$ a family of independent random variables, identically distributed with W. Denote by $(\Omega, \mathcal{A}, \mathbb{P})$ the probability space over which these random variables are defined.

Then the non negative martingale

$$Y_n = b^{-n} \sum_{w\in\mathscr{A}^n} W(w|1)W(w|2)\cdots W(w|n) \tag{2.1}$$

converges to a non negative random variable Y, and the sequence of Borel positive measures $(\mu_n)_{n\geq 1}$ defined on $[0, 1]$ by

$$\mu_n(\mathrm{d}x) = W(w|1)W(w|2)\cdots W(w|n)\,\mathrm{d}x \quad \text{if } x \in I_w,\ w \in \mathscr{A}^n, \tag{2.2}$$

weakly converges to a non negative measure $\mu = \mu_W$ with total mass $\|\mu\| = Y$ almost surely. In the literature, the terminology *Mandelbrot cascade*, or *Mandelbrot martingale* denotes the martingale $(Y_n)_{n\geq 1}$ or the measure-valued sequence $(\mu_n)_{n\geq 1}$, while μ is called a *Mandelbrot measure*.

For each $0 \leq k \leq b-1$, substituting $\big(W(kw)\big)_{w\in\mathscr{A}^*}$ to $\big(W(w)\big)_{w\in\mathscr{A}^*}$ yields a copy $Y(k)$ of Y, so that the $Y(k)$ are independent, and independent of $(W(0),\ldots,W(b-1))$. The statistical self-similarity of the construction is summarized in the fundamental almost sure equation

$$Y = b^{-1}\sum_{k=0}^{b-1} W(k)Y(k). \tag{2.3}$$

Moreover, defining more generally

$$Y_n(w) = b^{-n}\sum_{v\in\mathscr{A}^n} W(w\cdot v|1)W(w\cdot v|2)\cdots W(w\cdot v|n), \quad (w \in \mathscr{A}^*),$$

and $Y(w) = \lim_{n\to\infty} Y_n(w)$, $Y(w)$ is a copy of $\|\mu\|$ and one gets the following multiplicative structure for the the mass of b-adic intervals:

$$\mu_{|w|+n}(I_w) = b^{-|w|}Y_n(w)\prod_{1\leq j\leq |w|} W(w|j),$$

which leads

$$\mu(I_w) = b^{-|w|}Y(w)\prod_{1\leq j\leq |w|} W(w|j), \tag{2.4}$$

since b-adic numbers cannot be atoms of μ (indeed, fixing $x_0 \in [0,1]$, and, for $\epsilon > 0$, f_ϵ a non negative function bounded by 1, with compact support in $[x_0-\epsilon, x_0+\epsilon]$ and taking the value 1 over $[x_0-\epsilon/2, x_0+\epsilon/2]$, we have $\mathbb{E}(\mu(\{x_0\})) \leq \mathbb{E}(\mu(f_\epsilon)) \leq \liminf_{n\to\infty}\mathbb{E}(\mu_n(f_\epsilon)) \leq \mathrm{Leb}(f_\epsilon) \leq 2\epsilon$, where we used the martingale property of $(\mu_n)_{n\geq 1}$).

Finally, μ possesses the statistical self-similar structure:

$$\mu_{|I_w} = \Big(b^{-|w|}\prod_{1\leq j\leq |w|} W(w|j)\Big)\mu^{(w)}\circ f_w^{-1} \quad (w \in \mathscr{A}^*), \tag{2.5}$$

where f_w is the direct similitude mapping $[0,1]$ onto I_w and $\mu^{(w)}$ is the copy of μ built from $(W(w\cdot v))_{v\in\mathscr{A}*}$.

It turns out that μ may vanish almost surely. This had been observed by Mandelbrot, who considered the function

$$\varphi(q) = \varphi_W(q) = q - 1 - \log_b \mathbb{E}(W^q) \quad (q \in \mathbb{R}_+), \tag{2.6}$$

and conjectured that $\mathbb{P}(\mu \neq 0) > 0$ if and only if $\varphi'(1^-) > 0$. It was Kahane who proved this result in [Kah74, KP76] (this was also proved independently and a bit later by Biggins [Big77] in the more general context of the branching random walk, with a different approach; another approach is presented in [WW69]).

Theorem 1 *One has* $\mathbb{P}(\mu \neq 0) > 0$ *if and only if* $\mathbb{E}(W \log W) < \log(b)$*, i.e.* $\varphi'_W(1^-) > 0$*. Moreover, in this case,* $(Y_n)_{n\geq 1}$ *is uniformly integrable*: $\mathbb{E}(Y) = 1$.

The necessity of $\varphi'(1^-) \geq 0$ is easy to see, for otherwise since $\varphi(1) = 0$ one has $\varphi(h) > 0$ for h close to 1^-, so that

$$\mathbb{E}(Y_n^h) \leq b^{-nh}\mathbb{E} \sum_{w\in\mathscr{A}^n} W(w|1)^h W(w|2)^h \cdots W(w|n)^h = b^{-n\varphi(h)}$$

tends to 0, and $Y = 0$ almost surely by applying Fatou's lemma. The sufficiency of $\varphi'(1^-) > 0$ is also quite direct if one assumes $\varphi(h) > -\infty$ for some $h > 1$. Indeed, in this case choosing h close enough to 1^+ yields $\varphi(h) < 0$. Then, following Kahane in [KP76], if we write $Y_n = b^{-1}\sum_{k=0}^{b-1} W(k)Y_{n-1}(k)$ and take $h \in (1, 2]$, then $Y_n^h = ((b^{-1}\sum_{k=0}^{b-1} W(k)Y_{n-1}(k))^{h/2})^2 \leq b^{-h}(\sum_{k=0}^{b-1} W(k)^{h/2}Y_{n-1}(k)^{h/2})^2$, so $\mathbb{E}(Y_n^h) \leq b^{-h}b\mathbb{E}(W^h)\mathbb{E}(Y_{n-1}^h)+b^{-h}b(b-1)\mathbb{E}(W^{h/2})^2\mathbb{E}(Y_{n-1}^{h/2})^2 \leq b^{\varphi(h)}\mathbb{E}(Y_{n-1}^h)+ b^{-h}b(b-1)$ since $\mathbb{E}(W) = \mathbb{E}(Y_{n-1}) = 1$, and finally $\mathbb{E}(Y_n^h) \leq (1 - b^{\varphi(h)})^{-1}b^{-h}b(b-1)$ after noting that $\mathbb{E}(Y_{n-1}^h) \leq \mathbb{E}(Y_n^h)$ by Jensen's inequality. Thus $(Y_n)_{n\geq 1}$ is bounded in L^h, hence uniformly integrable. The sharp result is much more delicate.

When $\mathbb{E}(W \log W) \geq \log(b)$, i.e. $\varphi'(1^-) \leq 0$, Mandelbrot naturally asked in [Man74a] for the existence of a normalizing positive sequence $(A_n)_{n\geq 1}$ so that $(Y_n/A_n)_{n\geq 1}$ converges, at least in law, to some non degenerate random variable. He observed that the convergence in law of Y_n/A_n to some random variable Z imposes that A_{n+1}/A_n converges to a positive constant c, so that Z must satisfy the equation

$$Z \overset{\text{dist}}{=} \sum_{k=0}^{b-1} \frac{W(k)}{bc} Z(k), \tag{2.7}$$

where $\overset{\text{dist}}{=}$ means equality in distribution, the $Z(k)$ are independent, identically distributed with Z, and independent of $(W(0), \ldots, W(b-1))$.

The non trivial solutions of this equation and its generalization to the branching random walk context, also called fixed points of the smoothing transformation, have been studied intensively, starting with the fundamental paper [DL83] by Durrett and Liggett (their motivation came from interacting particle systems) and followed by regular notable advances. We will see (Sect. 3.1) that its general solutions are natural combinations of stable laws and the laws of random variables of the same nature as $\|\mu\|$, or combinations of stable laws and the laws of the total mass of *critical* Mandelbrot measures. These solutions have counterparts in terms of statistically self-similar measures, namely derivatives of stable Lévy subordinators in Mandelbrot time, or critical Mandelbrot time. These statistically self-similar measures, as well as Mandelbrot and critical Mandelbrot measures provide fundamental illustrations of the multifractal formalism, a notion that was pointed out by Frisch and Parisi to provide a fine geometric description of the energy dissipation in intermittent turbulence, with Mandelbrot measures as main illustration [FP88]. This will be explained in Sect. 5, where we present recent progress in the multifractal analysis of these objects, as well as in the control of the modulus of continuity of Mandelbrot and critical Mandelbrot measures. These moduli of continuity controls are partly based on the remarkable advances achieved in the solution to the original normalization question (see Sect. 3.2). It turns out that combining these renormalization results with fixed points of the smoothing transformation theory yields, in the case of a second order phase transition, a precise description of the asymptotic behavior of the partition functions and the limiting laws of Gibbs measures associated with polymers on disordered trees expressed in terms of the statistically self-similar measures mentioned above (Sect. 4).

In fact, in his notes on multiplicative cascades [Man74a], Mandelbrot starts by raising the following general problem: assume only that W is a random variable taking values in $\mathbb{R}$, and consider the sequence of functions

$$F_n(x) = F_{W,n}(x) = \int_0^x Q_n(u)\,\mathrm{d}u, \tag{2.8}$$

where $Q_n(x) = W(w|1)W(w|2)\cdots W(w|n)\,\mathrm{d}x$ if $x \in I_w,\ w \in \mathscr{A}^n$; under which condition does there exist a normalizing sequence $(A_n)_{n\geq 1}$ such that $F_n(x)/A_n$ converges in law, or in a stronger sense?

We will give some results in this direction in Sect. 6. In Sect. 7 we explain how Mandelbrot cascades define a natural dynamical system on fixed points of the smoothing transformation with finite expectation, to which is associated a functional CTL whose limit process is obtained as the limit of an *additive cascade* on $\mathscr{A}^*$.

Finally, let us mention that a lot of results presented in this paper are not specific to multiplicative cascades on regular trees, and have extensions in the context of branching random walks on Galton-Watson trees. Also, while we consider measures on the interval [0, 1], the results of Sect. 5 can be extended to Mandelbrot measures on $[0, 1]^d$ ($d \geq 2$). The only difference in that one must use a different way to define the Lévy-Mandelbrot measures defined in Sect. 3.1.2; the procedure is explained in [BJRV13, BRV12].

In Sects. 3–5, to simplify the discussion we assume that $W > 0$ and the probability distribution of $\log(W)$ is non lattice.

3 Fixed Points of the Smoothing Transformation and Associated Statistically Self-similar Measures; Renormalization of Mandelbrot Cascades

This section gathers old and new information related to the solutions of the functional equation pointed out by Mandelbrot in connexion with the renormalization of the martingale $(Y_n)_{n\geq 1}$ when $\varphi'(1) \leq 0$.

3.1 Fixed Points of the Smoothing Transformation

If Z is a non negative random variable, denote by ϕ_Z the Laplace transform of its probability distribution.

Consider the positive random variable W and do not assume a priori that $\mathbb{E}(W) = 1$. Then consider the equation

$$Z \overset{\text{dist}}{=} b^{-1} \sum_{k=0}^{b-1} W(k) Z(k), \tag{3.1}$$

where the random variables $W(k)$, $0 \leq k \leq b-1$, are independent copies of W, the random variables $Z(k)$ are independent copies of the non negative random variable Z, and $(W(0), \dots, W(b-1))$ and $(Z(0), \dots, Z(b-1))$ are independent. If Z is not identically equal to 0, then it is easily deduced from (3.1) that in fact Z is positive almost surely; moreover any positive multiple of Z satisfies (3.1).

In terms of Laplace transform, (3.1) means

$$\phi_Z = \left(\mathbb{E}(\phi_Z(b^{-1} t W))\right)^b, \tag{3.2}$$

so that ϕ_Z is a fixed point of the *smoothing transformation* T_W defined on the space of Laplace transforms of probability distributions on $\mathbb{R}_+$ as

$$T_W(\phi) = (\mathbb{E}(\phi(b^{-1} t W)))^b.$$

As indicated in the previous section, non trivial solutions of Eq. (3.1) have been studied intensively [DL83, Gui90, BK97, Liu98, Liu00, Liu01, BK04, BK05, AM12, ABM12] (see also [AM13] for the study of non necessarily positive solutions, still with $W > 0$).

Let φ be defined as in Sect. 5.1.3.

Theorem 2 *Equation* (3.1) (*or equivalently* (3.2)) *has a non trivial solution if and only if there exists* $\alpha \in (0, 1]$ *such that* $\varphi_W(\alpha) = 0$ *and* $\varphi_W'(\alpha^-) \geq 0$. *Moreover, if* Z_1 *and* Z_1 *are two such solutions, there exists* $c > 0$ *such that* $Z_1 \overset{\text{dist}}{=} c Z_2$.

This was proved by Durrett and Liggett under the assumption $\varphi(1+\epsilon) > -\infty$ for some $\epsilon > 0$. This assumption has been removed thanks to recent advances due to Alsmeyer, Biggins, Kyprianou and Meiners works in [ABM12, AM12, BK04, BK05]. We do not try to outline the extremely involved proof of this result; we just mention that renewal theory and fluctuation theory of random walks on $\mathbb{R}$ play a central role here.

It turns out that assuming that $\varphi''(\alpha^-) > -\infty$ whenever $\varphi'(\alpha^-) = 0$ simplifies the description of the solutions. This is what we will do in the next subsections.

3.1.1 Special Solutions from Mandelbrot and Critical Mandelbrot Cascades When $\alpha = 1$

For $q \in \mathbb{R}_+$ and $n \geq 1$ set

$$Y_{q,n} = b^{n(\varphi(q)-q)} \sum_{w\in\mathscr{A}^n} W(w|1)^q W(w|2) \cdots W(w|n)^q,$$

and

$$\widetilde{Y}_n = -\frac{\mathrm{d}Y_{q,n}}{\mathrm{d}q}\Big|_{q=1}.$$

Notice that if $\varphi(1) = 0$, i.e. $\mathbb{E}(W) = 1$, then $Y_{1,n} = Y_n$ (see (2.1)). If, moreover, $\varphi'(1) = 0$, we are in the critical case of degeneracy of Mandelbrot cascades, and $\widetilde{Y}_n$ takes the form

$$\widetilde{Y}_n = -\sum_{w\in\mathscr{A}^n} (b^{-n} W(w|1)W(w|2)\cdots W(w|n)) \log(b^{-n} W(w|1)W(w|2)\cdots W(w|n));$$

in this case it is a martingale with respect to $(\sigma(W(w) : w \in \mathscr{A}^k,\ 1 \leq k \leq n))_{n\geq 1}$ and it is called *derivative martingale*.

The following result gathers information about the non trivial solutions of (3.1) when $\varphi(1) = 0$ and $\varphi'(1) \leq 0$.

Theorem 3 *Assume that $\varphi(1) = 0$.*

(1) *Suppose that $\varphi'(1^-) > 0$. Let Y be the almost sure limit of $(Y_n)_{n\geq 1}$ given by Theorem 1.*

  (a) *For $q > 1$ one has $\mathbb{E}(Y^q) < \infty$ if and only if $\varphi(q) > 0$;*
  (b) *If $\varphi(q_0) = 0$ and $\varphi'(q_0) > -\infty$ for some $q_0 > 1$ (such a q_0 is necessarily unique due to the concavity of φ), one has $\mathbb{P}(Y > x) \sim_\infty Ax^{-q_0}$ for some $A > 0$. (c) For $q < 0$, $\varphi(q/b) > -\infty$ implies $E(Y^{q'}) < \infty$ for all $q' \in (q, 0)$, and $E(Y^q) < \infty$ implies $\varphi(q/b) > -\infty$.*

Moreover, any non trivial solution Z of (3.1) satisfies $Z \overset{dist}{=} cY$ for some $c > 0$.

(2) *Suppose that* $\varphi'(1^-) = 0$ *and* $\varphi''(1^-) > -\infty$. *Then* $(\widetilde{Y}_n)_{n\geq 1}$ *is a martingale which converges almost surely to a positive random variable* $\widetilde{Y}$ *satisfying the following properties*:

  (a) *for* $q > 0$ *one has* $\mathbb{E}(Y)^q < \infty$ *if and only if* $q < 1$;
  (b) $\phi_{\widetilde{Y}}(t) \sim_{0^+} 1 - ct\log(1/t)$ *for some* $c > 0$;
  (c) *if* $\varphi(1+\epsilon) > -\infty$ *for some* $\epsilon > 0$, $\mathbb{P}(\widetilde{Y} > x) \sim_\infty Ax^{-1}$ *for some* $A > 0$.
  (d) *For* $q < 0$, $\varphi(q/b) > -\infty$ *implies* $E(Y^{q'}) < \infty$ *for all* $q' \in (q, 0)$, *and* $E(Y^q) < \infty$ *implies* $\varphi(q/b) > -\infty$.

Moreover, any non trivial solution Z *of* (3.1) *satisfies* $Z \overset{\text{dist}}{=} c\widetilde{Y}$ *for some* $c > 0$.

Under the assumption that $\varphi(1+\epsilon) > -\infty$ for some $\epsilon > 0$, Durrett and Liggett showed that when $\varphi(1) = 0$ and $\varphi'(1^-) = 0$, Eq. (3.2) possesses non trivial solutions satisfying $\phi(t) \sim_{0^+} 1 - ct\log(1/t)$ for some $c > 0$, and unique up to a scaling factor in their argument. Then Liu [Liu98] used this behavior of ϕ at 0^+ to identify the solutions as the Laplace transform of $\widetilde{Y}$ by proving that $\widetilde{Y}_n$ converges to a non trivial limit (the point is that for any fixed $t > 0$, $\prod_{w\in\mathscr{A}^n} \phi(tW(w|1)W(w|2)\cdots W(w|n))$ is a bounded non negative martingale with expectation in (0, 1), which is asymptotically equivalent to $e^{-tc\widetilde{Y}_n}$). The weaker condition used in Theorem 3 is established in [BK04, BK05].

The necessary and sufficient condition for the finitness of moments of order greater than 1 when $\varphi'(1) > 0$ (conjectured in [Man74a]) was established by Kahane in [Kah74, KP76] by generalizing the argument presented just after Theorem 1. The right tail behaviors of solutions when $\varphi'(1) > 0$ and $\varphi'(1) = 0$ are due Guivarc'h [Gui90] (conjectured in [Man74a]) and Buraczewski [Bur09] respectively. The proofs strongly rely on random difference equations and renewal theories.

The result on moments of negative orders is first established in [Kah91] (see also [Liu01, Mol96]). Its proof consists in estimating from (3.2) the asymptotic behavior of ϕ_Y or $\phi_{\widetilde{Y}}$ at ∞.

Associated statistically self-similar measures Under the assumptions of Theorem 3(1), we already now the Mandelbrot measure μ.

Under the assumptions of Theorem 3(2), defining

$$\widetilde{Y}_n(w) = -\sum_{v\in\mathscr{A}^n} \frac{W(w\cdot v|1)\cdots W(w\cdot v|n)}{b^n}\log\frac{W(w\cdot v|1)\cdots W(w\cdot v|n)}{b^n} \quad (w\in\mathscr{A}^*)$$

and $\widetilde{Y}(w) = \lim_{n\to\infty}\widetilde{Y}_n(w)$, $\widetilde{Y}(w)$ is a copy of $\|\widetilde{\mu}\|$ and

$$\widetilde{\nu}([w]) = b^{-|w|}\widetilde{Y}(w)\prod_{1\leq j\leq |w|} W(w|j), \tag{3.3}$$

defines almost surely a measure on $\mathscr{A}^\omega$. One can show (see [Bar00]) $\widetilde{\nu}$-amost every $t \in \mathscr{A}^\omega$ is normal, so that $\widetilde{\nu}$ has no atom in the set of infinite branches of $\mathscr{A}^\omega$

encoding b-adic numbers of [0, 1]. Thus $\widetilde{\nu}$ naturally projects on [0, 1] to a measure $\widetilde{\mu} = \widetilde{\mu}_W$, called *critical Mandelbrot measure*, such that for any $w \in \mathscr{A}^*$,

$$\widetilde{\mu}_{|I_w} = \Big(b^{-|w|} \prod_{1\le j\le |w|} W(w|j)\Big)\widetilde{\mu}^{(w)} \circ f_w^{-1} \quad (w \in \mathscr{A}^*), \tag{3.4}$$

where $\widetilde{\mu}^{(w)}$ is the copy of $\widetilde{\mu}$ built from $(W(w \cdot v))_{v\in\mathscr{A}^*}$.

3.1.2 General Form of the Solutions

Theorem 4 *Assume that $\varphi(\alpha) = 0$ and $\varphi'(\alpha^-) \ge 0$ for some $\alpha \in (0, 1]$ (this α is then unique by concavity of φ). Let L_α be a stable Lévy subordinator of index α if $\alpha \in (0, 1)$ and the identity map of $\mathbb{R}_+$ if $\alpha = 1$. Assume that L_α is independent of $(W(w))_{w\in\mathscr{A}^*}$. Let $W_\alpha = W^\alpha/\mathbb{E}(W^\alpha)$.*

(1) *Suppose that $\varphi'(\alpha^-) > 0$. Then W_α satisfies the assumptions of Theorem 3(1). Let Y_α be the limit of the associated Mandelbrot cascade built from $(W_\alpha(w))_{w\in\mathscr{A}^*}$. Any non trivial solution Z of (3.1) satisfies $Z \overset{dist}{=} cL_\alpha(Y_\alpha)$ for some $c > 0$. Moreover, $\phi_Z(t) \sim_{0^+} 1 - c't^\alpha$ for some $c' > 0$.*
(2) *Suppose that $\varphi'(\alpha^-) = 0$ and $\varphi''(\alpha^-) > -\infty$. Then W_α satisfies the assumptions of Theorem 3(2). Let $\widetilde{Y}_\alpha$ be the limit of the associated derivative martingale built from $(W_\alpha(w))_{w\in\mathscr{A}^*}$. Any non trivial solution Z of (3.1) satisfies $Z \overset{dist}{=} cL_\alpha(\widetilde{Y}_\alpha)$ for some $c > 0$. Moreover, $\phi_Z(t) \sim_{0^+} 1-c't^\alpha \log(1/t)$ for some $c' > 0$.*

Durrett and Liggett obtained the same sets of solutions under the stronger assumption mentioned above, and the sharp result, without the assumption $\varphi''(\alpha^-) > -\infty$ when $\varphi'(\alpha^-) = 0$, is obtained in [ABM12]. Notice that checking that $cL_\alpha(Y_\alpha)$ is solution of (3.1) in case (1) and $cL_\alpha(\widetilde{Y}_\alpha)$ is solution of (3.1) in case (2) is a simple exercise using that $\phi_{L_\alpha}(t) = e^{-\gamma t^\alpha}$ for some $\gamma > 0$.

Associated statistically self-similar measures Denote by μ and $\widetilde{\mu}$ the Mandelbrot measure and critical Mandelbrot measure considered at the end of Sect. 3.1.1. Let $\alpha \in (0, 1)$ and L_α an α-stable Lévy subordinator independent of $(W(w))_{w\in\mathscr{A}^*}$. Then let μ^α (resp. $\widetilde{\mu}^\alpha$) be the positive measure obtained as the derivative of $L_\alpha(\mu([0, \cdot]))$ (resp. $L_\alpha(\widetilde{\mu}([0, \cdot]))$). Let us also call μ^α a *Lévy-Mandelbrot measure* and $\widetilde{\mu}^\alpha$ a *critical Lévy-Mandelbrot measure*. These measures are statistically self-similar in the sense that we have for any $w \in \mathscr{A}^*$

$$(\mu^\alpha)_{|I_w} \overset{\text{dist}}{=} \Big(b^{-|w|} \prod_{1\le j\le |w|} W(w|j)\Big)^{1/\alpha} \mu^{\alpha,(w)} \circ f_w^{-1} \quad (w \in \mathscr{A}^*), \tag{3.5}$$

where $\mu^{\alpha,(w)}$ is a copy of μ^α independent of $\prod_{1\le j\le |w|} W(w|j)$, and the same holds for $\widetilde{\mu}_\alpha$.

In the context of Theorem 4, (3.5) reads

$$(\mu^\alpha_{W_\alpha})_{|I_w} \stackrel{\text{dist}}{=} \Big(b^{-|w|} \prod_{1\le j\le |w|} W(w|j)\Big)\mu^{\alpha,(w)}_{W_\alpha} \circ f_w^{-1} \quad (w \in \mathscr{A}^*),$$

which is similar to (2.5) and (3.4).

3.2 Renormalization of Mandelbrot Cascades

We now suppose that $\varphi(1) = 0$ and $\varphi'(1^-) \le 0$ and come to the existence of a sequence $(A_n)_{n\ge 1}$ such that Y_n/A_n converges in law to a non trivial limit. When $\log W$ is Gaussian, the solution to this question has been conjectured by Derrida and Spohn [DS88] in the context of directed polymers on disordered trees and rigorously established recently by Webb [Web11], while in the general result presented below, the first part has been obtained by Aidekon and Shi [AS14], and the second one combines a convergence in law result obtained by Madaule [Mad00] with an identification of the limiting law using Theorem 4.

Theorem 5

(1) *Suppose that* $\varphi(1) = 0$, $\varphi'(1^-) = 0$ *and* $\varphi''(1^-) > -\infty$. *Then* $(\sqrt{n}\, Y_n)_{n\ge 1}$ *converges in probability to* $\sqrt{\dfrac{-2}{\pi\varphi''(0)}}\,\widetilde{Y}$.

(2) *Suppose that* $\varphi(1) = 0$ *and* $\varphi'(1^-) < 0$. *Let* α *be the unique solution of* $\varphi'(\alpha) = \varphi(\alpha)/\alpha$ *in* $(0,1)$. *The random variable* $W_\alpha = W^\alpha/\mathbb{E}(W^\alpha)$ *satisfies the assumptions of Theorem* 4(2). *Let* $\widetilde{Y}_\alpha$ *be the associated limit of the derivative martingale built from* $(W_\alpha(w))_{w\in\mathscr{A}^*}$.

The sequence $((n^{3/2}b^{n(\varphi(\alpha)-\alpha)})^{1/\alpha}\, Y_n)_{n\ge 1}$ *converges in law to* $L_\alpha(\widetilde{Y}_\alpha)$, *where* L_α *is a stable Lévy subordinator of index* α *independent of* $\widetilde{Y}_\alpha$.

Theorem 5(2) is a special case of a more general renormalization results for Y_n when one assumes only that $W > 0$ and there exists $\alpha \in (0,1)$ such that $\varphi'(\alpha) = \varphi(\alpha)/\alpha$ (see Sect. 4.2).

The limiting law of μ_n/A_n is described in the next section.

4 Directed Polymers on $\mathscr{A}^*$: Partition Functions, Free Energies and Gibbs Measures

Here we consider the random variables $(W(w))_{w\in\mathscr{A}^*}$ of the previous sections and we only assume that $W > 0$. If we define the potential $V = \log(b) - \log W$ and set $V(w) = \log(b) - \log W(w)$ $(w \in \mathscr{A}^*)$, in the setting of [DS88] the branching random walk $H(w) = V(w|1) + \cdots + V(w|n)$, $w \in \mathscr{A}^n$, defines for each $n \geq 1$ a polymer on the tree $\mathscr{A}^n$ in the random medium $(V(w))_{w \in \bigcup_{1 \leq k \leq n} \mathscr{A}^n}$ by associating the energy $H(w)$ to each path of length n; moreover, this model possesses *logarithmic correlations*.

Then, of first importance is the asymptotic behavior, as n tends to ∞, of thermodynamical objects such as the partition function

$$Z_n(\beta) = \sum_{w \in \mathscr{A}^n} e^{-\beta H(w)} \quad (\beta \geq 0),$$

the free energy

$$P_n(\beta) = \frac{\log Z_n(\beta)}{n \log(b)},$$

and the Gibbs measures defined on [0, 1] by

$$\nu_{\beta,n}(\mathrm{d}x) = b^{-n} \frac{e^{-\beta H(w)}}{Z_n(\beta)} \mathrm{d}x \quad \text{if } x \in I_w \text{ and } w \in \mathscr{A}^n,$$

where β stands for the inverse of the temperature.

Writing $e^{-\beta H(w)} = (b^{-n} W(w|1) \cdots W(w|n))^\beta$ shows the direct connexion with Mandelbrot cascades.

The understanding of these asymptotic behaviors has made enormous progress in the recent years. We still define $\varphi(\beta)$ as in Sect. 5.1.3.

Four different situations can occur; they are described in the Sects. 4.1–4.4 and depend on the behavior of φ at

$$\beta_c = \sup\{\beta > 0 : \varphi'(\beta^-)\beta - \varphi(\beta) > 0\},$$

with the convention $\sup(\emptyset) = 0$.

For $\beta \geq 0$ and $n \geq 1$, we denote by $\mu_{\beta,n}$ the measure defined by (2.2) when the weight W is taken equal to

$$W_\beta = \frac{W^\beta}{\mathbb{E}(W^\beta)}. \tag{4.1}$$

Notice that by construction we have

$$\|\mu_{\beta,n}\| = \frac{Z_n(\beta)}{\mathbb{E}(Z_n(\beta))} \quad \text{and} \quad \varphi'_{W_\beta}(1^-) = \varphi'(\beta^-)\beta - \varphi(\beta).$$

Section 4.5 presents a unified result for the free energy behavior.

4.1 No Phase Transition: $\beta_c = +\infty$

Theorem 6 *With probability 1, for all $\beta \in [0, \beta_c)$, $(\mu_{\beta,n})_{n\geq 1}$ weakly converges to a non degenerate Mandelbrot measure μ_β.*

Consequently, $\left(\frac{Z_n(\beta)}{\mathbb{E}(Z_n(\beta))}\right)_{n\geq 1}$ converges to $\|\mu_\beta\|$, $P_n(\cdot)$ converges to the analytic function $-\varphi(\cdot)$ uniformly on compact subintervals of $[0, \beta_c)$, and $(\nu_{\beta,n})_{n\geq 1}$ weakly converges to $\frac{\mu_\beta}{\|\mu_\beta\|}$, as $n \to \infty$.

That the result holds for each fixed β almost surely essentially follows from Theorem 1 applied when W is taken equal to W_β. The uniform version of this result essentially follows from [Big92] and [Bar00]. Biggins considers in [Big92] the analytic extensions of the mappings $\mathscr{Y}_n : \beta \mapsto \frac{Z_n(\beta)}{\mathbb{E}(Z_n(\beta))}$ in a complex neighborhood of $\{\beta : \varphi'(\beta^-)\beta - \varphi(\beta) > 0\}$. He proves the almost sure uniform convergence of these extensions on a common domain U of $\mathbb{C}$ by combining Cauchy's formula with the fact that, for each compact subset of U, $\mathscr{Y}_n(z) - \mathscr{Y}_{n-1}(z)$ converges uniformly exponentially to 0 in L^p for some $p > 1$. It remains to prove that almost surely, for all β such that $\varphi'(\beta^-)\beta - \varphi(\beta) > 0$, we have $\lim_{n\to\infty} \mathscr{Y}_n(\beta) > 0$; this is done in [Bar00] (Fig. 1).

4.2 Second Order Phase Transition: $\beta_c \in (0, \infty)$ and $\varphi'_{W_{\beta_c}}(1^-) = \varphi'(\beta_c^-)\beta_c - \varphi(\beta_c) = 0$

This situation is illustrated by the case where the potential V is Gaussian (Fig. 2).

Theorem 7

(1) *The same conclusions as in Theorem 6 hold over $[0, \beta_c)$.*

(2) *Suppose that $\mathbb{E}(W^{\beta_c}|\log(W)|^2) < \infty$, i.e. $\varphi''_{W_{\beta_c}}(1^-) > -\infty$. Let $\widetilde{\mu}_{\beta_c}$ be the critical Mandelbrot measure built from $(W_{\beta_c}(w))_{w\in\mathscr{A}^*}$ in Sect. 3.1.1. Then $\left(\sqrt{n}\frac{Z_n(\beta_c)}{\mathbb{E}(Z_n(\beta_c))}\right)_{n\geq 1}$ converges in probability to $c\|\widetilde{\mu}_{\beta_c}\|$ for some explicit $c > 0$, and $(\nu_{\beta_c,n})_{n\geq 1}$ weakly converges in probability to $\frac{\widetilde{\mu}_{\beta_c}}{\|\widetilde{\mu}_{\beta_c}\|}$ as $n \to \infty$.*

(3) *Suppose that $\mathbb{E}(W^{\beta_c}|\log(W)|^3) < \infty$. For $\beta > \beta_c$, let $\widetilde{\mu}_{\beta_c}^{\frac{\beta_c}{\beta}}$ be the critical Lévy-Mandelbrot measure built from $\widetilde{\mu}_{\beta_c}$ and a stable Lévy subordinator of index β_c/β in Sect. 3.1.2.*

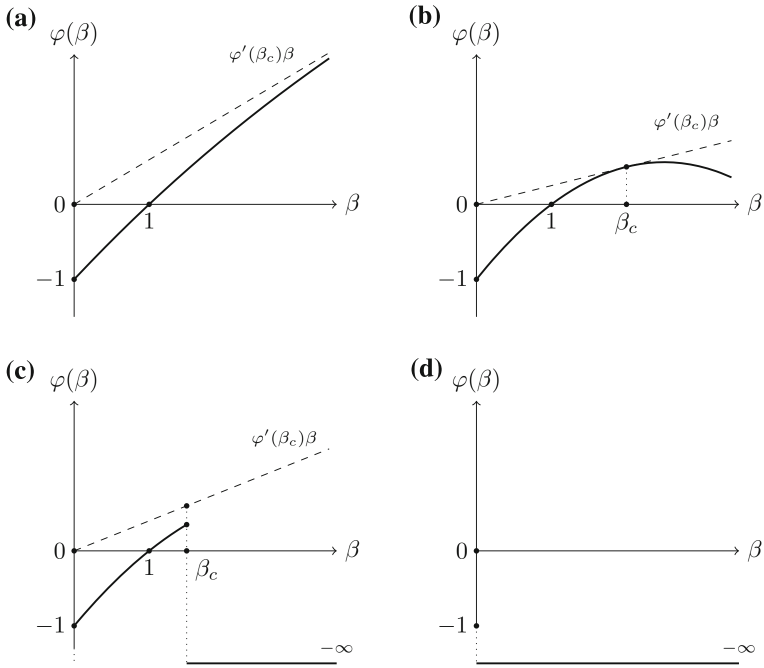

Fig. 1 The four possible behaviors of φ (by design, when $\beta_c > 0$, we make the pictures with $\varphi(1) = 0$ and $\varphi'(1) > 0$ to link these behaviors with multifractal analysis of Mandelbrot measures in Sect. 5). **a** No phase transition. **b** Second order phase transition. **c** First order phase transition. **d** The degenerate case

Then, for all $\beta > \beta_c$, $\left(n^{\frac{3}{2}\frac{\beta}{\beta_c}} \frac{Z_n(\beta)}{(\mathbb{E}(Z_n(\beta_c)))^{\frac{\beta}{\beta_c}}}\right)_{n\geq 1}$ *weakly converges in distribution to* $c\|\widetilde{\mu}_{\beta_c}^{\frac{\beta_c}{\beta}}\|$ *for some* $c > 0$, *and* $(\nu_{\beta,n})_{n\geq 1}$ *weakly converges in distribution to* $\widetilde{\mu}_{\beta_c}^{\frac{\beta_c}{\beta}}/\|\widetilde{\mu}_{\beta_c}^{\frac{\beta_c}{\beta}}\|$ *as* $n \to \infty$.

(4) $P_n(\cdot)$ *converges almost surely, uniformly over the compact subsets of* $\mathbb{R}_+$ *to*

$$P(\beta) = \begin{cases} -\varphi(\beta) & \text{if } 0 \leq \beta \leq \beta_c, \\ \dfrac{-\varphi(\beta_c)}{\beta_c}\beta = -\varphi'(\beta_c^-)\beta & \text{if } \beta > \beta_c \end{cases}.$$

Thus P *is analytic except at* β_c *where it is differentiable with a discontinuity of its second derivative, hence it presents a second order phase transition at* β_c.

Part (1) is obtained in a similar way as in the previous case.

For parts (2) and (3), the convergences of the renormalized partition functions are established in [Web11] for the log-normal case (with convergence in law for the

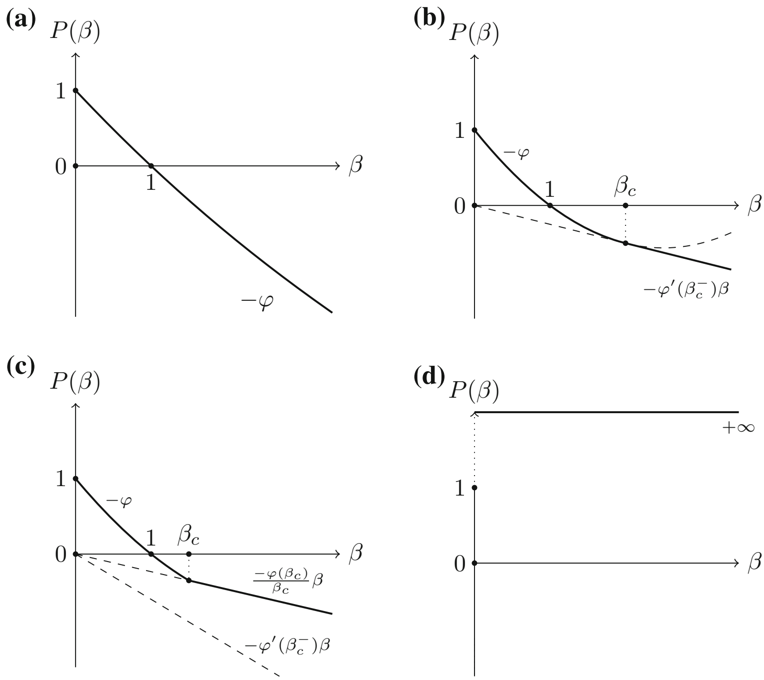

Fig. 2 The corresponding pressure function. **a** No phase transition. **b** Second order phase transition. **c** First order phase transition. **d** The degenerate case

part (2)), and [AS14] and [Mad00] for part (2) and (3) respectively. The identification of the limit in case (3) follows easily from (2.7) and Theorem 4.

In part (3), the exponent $\frac{3}{2}$ is reminiscent from the asymptotic behavior of $M_n = \min\{\beta_c H(w) - n\varphi(\beta_c) : w \in \mathscr{A}^n\}$, for which one has, due to [Web11] for the log-normal case, and [Aid13] for the general case.

Theorem 8 *Suppose that* $\mathbb{E}(W^{\beta_c}|\log(W)|^2) < \infty$. *Then, there exists* $c > 0$ *such that for all* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}(M_n \geq \frac{3}{2}n + x) = \mathbb{E}(e^{-ce^x \|\widetilde{\mu}_{\beta_c}\|}).$$

The asymptotic behavior of the Gibbs measures reflects the fact that when the temperature $1/\beta$ becomes lower than $1/\beta_c$, the systems is in its glassy phase: asymptotically with n, the energy concentrates around a few local minima of $H(x|n)$, a phenomenon amplified as β tends to ∞, and also called freezing transition (see [CD01]).

It is out of our scope to describe the techniques developed in [Aid13, AS14, Mad00, Web11]. The convergence of the Gibbs measures in case (2) and (3) requires additional arguments provided in [JW11] for the case $\beta = \beta_c$, and in [BRV12] for $\beta \geq \beta_c$.

Regarding point (4), this result or its analogue for Mandelbrot measures, has been obtained by several authors [CK92, Fra95, HW92, Mol96, OW00, WW13]. Due to (1), only the case $\beta \geq \beta_c$ must be considered. Due to the convexity of the functions P_n, it is quite direct that $\liminf_{n\to\infty} P_n(\beta) \geq -\varphi'(\beta_c^-)\beta$ for $\beta \geq \beta_c$. Moreover, for $\beta \geq \beta_c$, if $0 < \beta' < \beta_c$, we have $Z_n(\beta) = \sum_{w\in\mathscr{A}^n} e^{-\beta H(w)} = \sum_{w\in\mathscr{A}^n} (e^{-\beta' H(w)})^{\beta/\beta'} \leq Z_n(\beta')^{\beta/\beta'}$. Consequently, since $P_n(\beta')$ converges to $-\varphi(\beta')$, we get $\limsup_{n\to\infty} P_n(\beta) \leq \frac{\beta}{\beta'}(-\varphi(\beta'))$. Letting β' tend to β_c yields the result.

4.3 First Order Phase Transition: $\beta_c \in (0, \infty)$ and $\varphi'_{W_{\beta_c}}(1^-) = \varphi'(\beta_c^-)\beta_c - \varphi(\beta_c) > 0$

Notice that in this case one necessarily has $\varphi(\beta) = -\infty$ for $\beta > \beta_c$. Also, the measures $\mu_{\beta_c,n}$ weakly converge almost surely to the non degenerate Mandelbrot measure μ_{β_c}, for which Theorem 1 is optimal in the sense that $\varphi_{W_{\beta_c}}(\beta) = -\infty$ for $\beta > 1$, hence in this case $(\|\mu_{\beta_c,n}\|)_{n\geq 1}$ is not bounded in any L^q, $q > 1$.

We have the following result.

Theorem 9

(1) *The same conclusions as in Theorem* 6 *hold over* $[0, \beta_c]$.
(2) *With probability* 1*,* $P_n(\cdot)$ *converges almost surely, uniformly over the compact subsets of* $\mathbb{R}_+$ *to*

$$P(\beta) = \begin{cases} -\varphi(\beta) & \text{if } 0 \leq \beta \leq \beta_c, \\ \dfrac{-\varphi(\beta_c)}{\beta_c}\beta > -\varphi'(\beta_c^-)\beta & \text{if } \beta > \beta_c \end{cases}.$$

Thus P *is analytic except at* β_c *where it is continuous and not differentiable, hence it presents a first order phase transition at* β_c*.*

Remark 1 The expression of P shows that the concave Legendre-Fenchel transform of $-P$, i.e. the mapping $\alpha \in \mathbb{R} \mapsto \inf\{\alpha\beta + P(\beta) : \beta \geq 0\}$ is non negative at $\alpha = \varphi(\beta_c)/\beta_c$. This proves that $|\beta_c H(w) - n\varphi(\beta_c)|$ behaves sub-linearly at some $w \in \mathscr{A}^n$, like in the case of the previous section.

Consequently, this raises the following questions: does $\min\{\beta_c H(w) - n\varphi(\beta_c) - s_n : w \in \mathscr{A}^n\}$ converge in law for some sub-linear sequence $(s_n)_{n\geq 1}$? If so, is there a freezing phenomenon like in the previous case?

Then, one can wondering if for $\beta > \beta_c$, $\left(e^{\frac{\beta}{\beta_c}s_n}\dfrac{Z_n(\beta)}{(\mathbb{E}(Z_n(\beta_c)))^{\frac{\beta}{\beta_c}}}\right)_{n\geq 1}$ weakly converges in distribution to $c\|\mu_{\beta_c}^{\frac{\beta_c}{\beta}}\|$ for some $c > 0$, and $(\nu_{\beta,n})_{n\geq 1}$ weakly converges in distribution to $\mu_{\beta_c}^{\frac{\beta_c}{\beta}}/\|\mu_{\beta_c}^{\frac{\beta_c}{\beta}}\|$ as $n \to \infty$, where $\mu_{\beta_c}^{\frac{\beta_c}{\beta}}$ is the Lévy-Mandelbrot measure built from μ_{β_c} and a stable Lévy subordinator of index β_c/β in Sect. 3.1.2.

The upper bound for $\limsup_{n\to\infty} P_n(\beta)$ over (β_c, ∞) is obtained like in the previous case. The lower bound for $\liminf_{n\to\infty} P_n(\beta)$ is more involved. It was obtained by Molchan in [Mol96] only along a deterministic subsequence. However, we believe that Molchan's approach can give the convergence along the whole sequence. Anyway, in [AB14], an alternative approach gives the result in connection with the multifractal analysis of $H(w)$ (see Sect. 5.1.3).

4.4 The Degenerate Case: $\beta_c = 0$, i.e. $\varphi = -\infty$ Over $\mathbb{R}_+^*$

In this case, we have:

Theorem 10 *With probability* 1*, P_n converges pointwise to $-\varphi$ as $n \to \infty$.*

This convergence result is shown in [Mol96] along a deterministic subsequence, and in this form in [AB14].

Remark 2 One can wonder if there is a precise superexponential speed of divergence of $Z_n(\beta)$ to ∞ when $\beta > 0$.

4.5 A Uniform Point of View for the Free Energy

We can deduce from the previous results the following synthetic presentation:

Theorem 11 *With probability* 1, $\lim_{n\to\infty} P_n(\beta) = \beta \inf\{-\varphi(\beta')/\beta' : \beta' \in (0, \beta]\} = \inf\{-\varphi(\theta\beta)/\theta : \theta \in (0, 1]\}$ *for all* $\beta > 0$.

5 Fine Geometric Properties of Statistically Self-similar Measures

This section presents recent results about the modulus of continuity of Mandelbrot and critical Mandelbrot measures, as well as recent progress in their multifractal analysis, with consequences for the multifractal analysis of their discrete companions, i.e. Lévy-Mandelbrot and critical Lévy-Mandelbrot measures. We also give results about the dimension of these measures. Finally, we say a word about KPZ formula.

5.1 Dimension, Modulus of Continuity, and Multifractal Analysis of Mandelbrot and Critical Mandelbrot Measures

Here we suppose that $\mathbb{E}(W) = 1$.

5.1.1 Dimension

Recall that a positive and finite Borel measure ν on $[0, 1]$ is said to be exact dimensional with dimension D if $\lim_{r\to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)} = D$, ν-almost everywhere, or equivalently $\lim_{n\to\infty} \frac{\log(\mu(I_n(x)))}{-n\log(b)} = D$, ν-almost everywhere, where $I_n(x)$ stands for the closure of the semi-open to the right b-adic interval of generation$x$ which contains x when $x \in [0, 1)$ and $[1 - b^{-n}, 1]$ if $x = 1$.

Theorem 12

(1) *Suppose that* $\varphi'(1^-) > 0$. *Then the Mandelbrot measure* μ *is exact dimentional with dimension* $\varphi'(1^-)$*; in particular it is continuous.*
If, moreover, $\mathbb{E}(W|\log W|^3) < \infty$ *and* $-\varphi''(1^-) > 0$ *(i.e.* $W \overset{dist}{\not\equiv} 1$*), then for* μ*-almost every* x *we have*

$$\liminf_{n\to\infty}(\textit{resp.}\ \limsup_{n\to\infty})\frac{\log(\mu(I_n(x))) + n\log(b)\varphi'(1^-)}{\sqrt{2\log(b)(-\varphi''(1^-))n\log\log(n)}} = -1(\textit{resp.}\ -1).$$

(2) *Suppose that* $\varphi'(1^-) = 0$ *and* $\varphi''(1^-) > -\infty$. *Then the critical Mandelbrot measure* $\widetilde{\mu}$ *is exact dimensional with dimension 0. If, moreover,* $\varphi(1+\epsilon) > -\infty$ *for some* $\epsilon > 0$*, then* $\widetilde{\mu}$ *is continuous.*
Also, if $\log(W)$ *is Gaussian, for* $\widetilde{\mu}$*-almost every* x*, for all* $\alpha > 1/3$ *and* $k \in \mathbb{N}$*, for* n *large enough, we have*

$$\exp\left(-\sqrt{6\log(b)}\sqrt{n\,(\log n + \alpha\log\log n)}\right) \le \widetilde{\mu}(I_n(x)) \le n^{-k}. \qquad (5.1)$$

For part (1), the fact that μ is exact dimensional, conjectured in [Man74b], was proved by Peyrière in [KP76, Pey74] under the assumption $\mathbb{E}(Y\log^+(Y)) < \infty$, which is shown to be equivalent to $\varphi''(1) > -\infty$ in [Big79]. In [Kah87], Kahane used a powerful percolation argument combined with Theorem 1 to eliminate the assumption $\mathbb{E}(Y\log^+(Y)) < \infty$.

The law of the iterated logarithm is stated in [Liu00] assuming $\mathbb{E}(W|\log W|^2) < \infty$. To get such a law, consider the now called "Peyrière measure" $\mathbb{Q}$ introduced in [Pey74] and defined on $(\Omega \times [0, 1], \boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{B}}([0, 1]))$ by

$$\mathbb{Q}(A) = \mathbb{E}\left(\int_{[0,1]} \mathbf{1}_A(\omega, x)\,\mu(\mathrm{d}x)\right),$$

so that "$\mathbb{Q}$-almost everywhere" means "almost surely, μ-almost everywhere". Its proof first uses the standard law of the iterated logarithm applied to the sequence of centered i.i.d. random variables $\log(W(x|k)) - \log(b) + \log(b)\varphi'(1^-)$ with variance $-\log(b)\varphi'(1^-)$ with respect to $\mathbb{Q}$, to control $\frac{\sum_{k=1}^n \log(W(x|k)) - \log(b) + \log(b)\varphi''(1^-)}{\sqrt{2\log(b)(-\varphi''(1^-))n\log\log(n)}}$. However the control of $Y(x|n)$ as $o(\sqrt{n\log\log(n)})$ has a gap in [Liu00]. Assuming $\mathbb{E}(W|\log W|^3) < \infty$, we know from [Big79] that $\mathbb{E}(Y\log(Y)^2) < \infty$. Then, for $\epsilon > 0$, a calculation yields

$$\begin{aligned}
\mathbb{Q}\left(\{|\log Y(x|n)| \ge \sqrt{n}\epsilon\}\right) &= \mathbb{Q}\left(\{Y(x|n) \ge e^{\sqrt{n}\epsilon}\}\right) + \mathbb{Q}\left(\{Y(x|n) < e^{-\sqrt{n}\epsilon}\}\right)\\
&= \mathbb{E}\left(Y \cdot \mathbf{1}_{\{Y \ge e^{\sqrt{n}\epsilon}\}}\right) + \mathbb{E}\left(Y \cdot \mathbf{1}_{\{Y < e^{-\sqrt{n}\epsilon}\}}\right)\\
&\le \mathbb{E}\left(Y \cdot \mathbf{1}_{\{Y \ge e^{\sqrt{n}\epsilon}\}}\right) + e^{-\sqrt{n}\epsilon}.
\end{aligned}$$

Applying the elementary inequality $\sum_{n\ge 1} \mathbf{1}_{\{X \ge \sqrt{n}\}} \le X^2$ we get

$$\begin{aligned}
\sum_{n\ge 1} \mathbb{Q}\left(\{|\log Y(x|n)| \ge \sqrt{n}\epsilon\}\right) &\le \sum_{n\ge 1} \mathbb{E}\left(Z \cdot \mathbf{1}_{\{Z \ge e^{\sqrt{n}\epsilon}\}}\right) + \sum_{n\ge 1} e^{-\sqrt{n}\epsilon}\\
&= \mathbb{E}\left(Z \cdot \sum_{n\ge 1} \mathbf{1}_{\left\{\frac{\log Z}{\epsilon} \ge \sqrt{n}\right\}}\right) + \sum_{n\ge 1} e^{-\sqrt{n}\epsilon}\\
&\le \epsilon^{-2}\mathbb{E}(Z(\log Z)^2) + \sum_{n\ge 1} e^{-\sqrt{n}\epsilon} < \infty.
\end{aligned}$$

Then the Borel-Cantelli lemma yields $|\log Y(x|n)| = o(\sqrt{n})$, μ-almost everywhere.

For part (2), the fact that $\widetilde{\mu}$ is exact dimensional with dimension 0 is established in [Bar00] using large deviations estimates to prove that $\sum_{n\ge 1} \widetilde{\mu}(\{x : \widetilde{\mu}(I_n(x)) \le b^{-n\epsilon}\}) < \infty$ for all $\epsilon > 0$. This is refined in [BKNSW14a], using Theorem 3(2, 3), to get the lower bound in (5.1) (this bound is easy to extend to general distributions for W). The fact that $\widetilde{\mu}$ is atomless under the assumption $\varphi(1+\epsilon) > -\infty$ for some $\epsilon > 0$ is also established in [BKNSW14a]; this exploits Theorems 3(2, 3) and 5(1) to prove that $n^\gamma \max_{w\in\mathscr{A}^n} \mu(I_w)$ converge to 0 in probability as $n \to \infty$ for all $\gamma \in [0, 1/2)$. The upper bound in (5.1) is more involved than the lower one; we refer to [BKNSW14a] for the details of (5.1).

Remark 3 The bounds (5.1) are not completely satisfactory since we do not know whether they are sharp or not; in fact, we believe that at least the second one is not sharp, and also that the order of magnitude of the sharp upper bound should differ from that of the lower bound and reflect the fluctuations of random walks conditioned to stay positive.

5.1.2 Modulus of Continuity

At first we notice that if μ is a non degenerate Mandelbrot measure then $\beta_c \geq 1$ and $\beta_c > 1$ if $\varphi'_{W_{\beta_c}}(1^-) = 0$, and if $\widetilde{\mu}$ is a critical Mandelbrot measure, then $\beta_c = 1$, where β_c is defined in Sect. 4.

Also, it is important to have in mind that for either of these measures, the smallest pointwise Hölder exponent is $\varphi(\beta_c)/\beta_c$, which equals $\varphi'(\beta_c^-)$ if $\varphi'_{W_{\beta_c}}(1^-) = 0$ (see Sect. 5.1.3). It is natural to complete this information by estimating the modulus of continuity of these measures.

Theorem 13

(1) *Assume that* $\varphi'(1) > 0$, $\varphi'_{W_{\beta_c}}(1^-) = 0$, *and there exists* $\beta > \beta_c$ *such that* $\varphi(\beta) > -\infty$.
With probability 1, *for all* $\gamma \in (0, 1/2)$, *there exists* $C > 0$ *such that for all subintervals* $I \subset [0, 1]$, *the Mandelbrot measure* μ *satisfies*

$$\mu(I) \leq C|I|^{\varphi(\beta_c)/\beta_c} \left(\log\left(1 + \frac{1}{|I|}\right)\right)^{-\gamma/\beta_c}. \tag{5.2}$$

(2) *Assume that* $\varphi(-\beta) > -\infty$ *and* $\varphi(1 + \beta) > -\infty$ *for some* $\beta > 0$, *as well as* $\varphi'(1) = 0$.
With probability 1, *for all* $\gamma \in (0, 1/2)$, *there exists* $C > 0$ *such that for all subintervals* $I \subset [0, 1]$, *the critical Mandelbrot measure* $\widetilde{\mu}$ *satisfies*

$$\widetilde{\mu}(I) \leq C \left(\log\left(1 + \frac{1}{|I|}\right)\right)^{-\gamma}. \tag{5.3}$$

Moreover, one cannot take $\gamma > 1/2$ *in the above statement.*

This result is proved in [BKNSW14a].

For (1), one needs that $\mathbb{E}(Y^\beta) < \infty$ for some $\beta > \beta_c$ to use the fact that $h(x) = \mathbb{P}(Y \geq x) \leq cx^{-\beta}$ for such a β; this is the case by Theorem 3(1) since $\varphi(\beta_c) > 0$ and our assumption imply $\varphi(\beta) > 0$ near β_c^+. This is combined in a non obvious way with the upper bound for $\mathbb{E}\Big(\Big(n^{\frac{3}{2}\frac{\beta}{\beta_c}} \frac{Z_n(\beta)}{(\mathbb{E}(Z_n(\beta_c)))^{\frac{\beta}{\beta_c}}}\Big)^\theta\Big)$ provided for $\theta \in (0, 1)$ in [Mad00] (and used here with $\theta = \beta_c/\beta$) to prove that

$$\begin{aligned}\mathbb{P}\Big(\max_{w\in\mathscr{A}^n}\{\mu(I_w)\} \geq 2^{-n\frac{\varphi(\beta_c)}{\beta_c}} n^{-\frac{\gamma}{\beta_c}}\Big) &= \mathbb{E}\Big(\prod_{w\in\mathscr{A}^n}(1 - h(e^{\beta_c H(w) - n\varphi(\beta_c)} n^{-\gamma}))\Big)\\ &\leq C_\epsilon n^{(1-\epsilon)(\gamma-3/2)}\end{aligned}$$

for all $\epsilon > 0$ and $\gamma \in (0, 3/2)$. This yields (5.2).

Similarly, the upper bound for $\mathbb{E}\big((n^{\frac{1}{2}} Z_n(1))^\theta\big)$ provided for $\theta \in (0, 1)$ in [HS09] is combined with the information $\mathbb{P}(\widetilde{Y} \geq x) \leq cx^{-1}$ provided by Theorem 3(2, 3) to prove that $\mathbb{P}(\max_{w\in\mathscr{A}^n} \mu(I_w) \geq n^{-\gamma}) \leq C_\epsilon n^{(1-\epsilon)(\gamma-1/2)}$ for all $\epsilon > 0$ and $\gamma \in (0, 1/2)$. This yields (5.3). The fact that one cannot take $\gamma > 1/2$ in (5.3) is proved by using Theorems 5(1) and 8, as well as the information $\mathbb{P}(\widetilde{Y} \geq x) \geq c'x^{-1}$ $(x \geq 1)$ provided by Theorem 3(2, 3) to obtain that $\mathbb{P}(\max_{w\in\mathscr{A}^n} \mu(I_w) < n^{-\gamma})$ tends to 0 for all $\gamma > 1/2$.

Remark 4

(1) It is not known if the choice $\gamma < 1/2$ in (5.2) can be improved.
(2) Property (5.3) is extended to critical lognormal multiplicative chaos measures in [BKNSW14+].
(3) Theorem 13 only deals with the second order phase transition case (in the frame of Sect. 4). To get similar results in the first order phase transition case is desirable. This is related to Remark 1.

5.1.3 Multifractal Analysis

Recall that given a positive Borel measure ν supported on [0, 1], its multifractal analysis consists in computing the Hausdorff dimension, denoted dim, of the level sets of the pointwise Hölder exponent of ν, namely the sets

$$E_\nu(\gamma) = \Big\{x \in [0, 1] : \gamma(\nu, x) = \gamma\Big\} \quad (\gamma \geq 0),$$

where

$$\gamma(\nu, x) = \liminf_{r\to 0^+} \frac{\log \nu(B(x, r))}{\log(r)}.$$

The L^q-spectrum of ν is defined as

$$\tau_\nu : q \in \mathbb{R} \mapsto \liminf_{r\to 0^+} \frac{\log \sup \Big\{ \sum_i \mu(B(x_i, r))^q \Big\}}{\log(r)}, \tag{5.4}$$

where the supremum is taken over all the centered packing of [0, 1] by closed balls of radius r. Define also

$$\widetilde{\tau}_\nu : q \in \mathbb{R} \mapsto \liminf_{n\to\infty} \frac{-1}{n\log(b)} \log \sum_{w\in\mathscr{A}^n} \mu(I_w))^q, \tag{5.5}$$

Throughout, we adopt the convention that a set has a negative dimension if and only if it is empty.

One always has

$$\dim E_\nu(\gamma) \le \tau_\nu^*(\gamma),$$

and one says that ν obeys the multifractal formalism at γ if $\dim E_\nu(\gamma) = \tau_\nu^*(\gamma)$ (see [Ols95]).

Let us naturally extend φ to the real line as

$$\varphi(q) = \varphi_W(q) = q - 1 - \log_b \mathbb{E}(W^q) \quad (q \in \mathbb{R}),$$

Given a function $\psi : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, define its concave Legendre-Fenchel conjugate as

$$\psi^* : \gamma \in \mathbb{R} \mapsto \inf\{\gamma q - \psi(q) : q \in \mathbb{R}\}.$$

Let

$$f : \gamma \mapsto \begin{cases} \varphi^*(\gamma) & \text{if } \varphi^*(\gamma) \ge 0 \\ -\infty & \text{otherwise} \end{cases}.$$

For non degenerate Mandelbrot measures and critical Mandelbrot measures, one has the following result (for point (3) we refer the reader to appendix for the definition of Hausdorff measures).

Theorem 14 *Let ν be the non degenerate Mandelbrot measure μ if $\varphi'(1^-) > 0$, or the critical Mandelbrot measure $\widetilde{\mu}$ if $\varphi'(1^-) = 0$ and $\varphi''(1^-) > -\infty$.*

Suppose that $\varphi(-\epsilon) > -\infty$ for some $\epsilon > 0$. Then, with probability 1,

(1) $\tau_\nu(0) = -1$ *and for all* $q \in \mathbb{R}^*$, $\tau_\nu(q) = \sup\{\varphi(\theta q)/\theta : \theta \in (0, 1]\}$. *Moreover,* $\tau_\nu = f^*$, $f = \tau_\nu^*$, *and in* (5.4), lim inf *can be replaced by* lim.
(2) *For all* $\gamma \ge 0$, $\dim E_\nu(\gamma) = \tau_\nu^*(\gamma) = f(\gamma)$.
(3) $E_\nu(\tau_\nu'(0))$ *is of full Lebesgue measure;*
(0–∞ *law) for all* $\gamma \ge 0$ *such that* $\dim 0 < \tau_\nu^*(\gamma) < 1$, *for all gauge functions* g, *we have* $\mathcal{H}^g(E_\nu(\gamma)) = \infty$ *if* $\limsup_{t\to 0^+} \log(g(t))/\log(t) \le \tau_\nu^*(\gamma)$ *and* $\mathcal{H}^g(E_\nu(\gamma)) = 0$ *otherwise.*
(4) *For* $n \ge 1$, $\gamma \in \mathbb{R}_+$ *and* $\epsilon > 0$ *let*

$$f(n, \gamma, \epsilon) = n^{-1} \log_b \#\Big\{w \in \mathscr{A}^n : \frac{\log \mu(I_w)}{-n \log(b)} \in [\gamma - \epsilon, \gamma + \epsilon]\Big\}.$$

We have

$$\text{for all } \gamma \ge 0, \quad \lim_{\epsilon\to 0^+} \liminf_{n\to\infty} f(n, \gamma, \epsilon) = \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} f(n, \gamma, \epsilon) = f(\gamma).$$

Let us sketch the ideas of the proof of this result, and start with a remark.

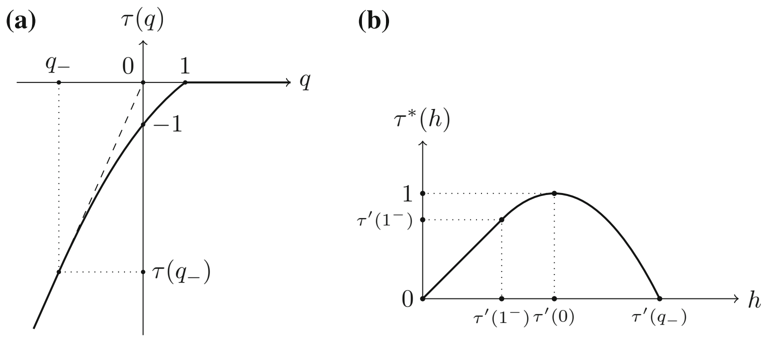

Fig. 3 Multifractal nature of a Mandelbrot measure with a second order phase transition at q_- and a first order phase transition at $q_+ = 1$, i.e. in the case where Theorem 1 is sharp. **a** The L^q spectrum of μ. **b** Its Legendre transform

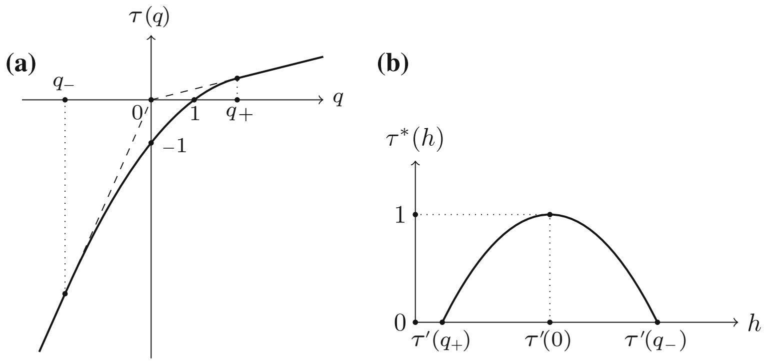

Fig. 4 Multifractal nature of a Mandelbrot measure with a second phase transition at both q_- and q_+. This situation occurs when W is lognormal. **a** The L^q spectrum of μ. **b** Its Legendre transform

Remark 5

(1) It turns out that $\widetilde{\tau}_\nu = \tau_\nu$, so according to Part(1), $\tau_\nu(q)$ is on $\mathbb{R}_+$ the opposite of the free energy $P(q)$ of the directed polymer associated with $H(w)$ defined in Sect. 4, and on $\mathbb{R}_-$, $\tau_\nu(q)$ is the opposite of $P(-q)$, where P is the free energy of the directed polymer associated with $-H(w)$. Thus, there may be two phase transitions for τ_ν: one on $\mathbb{R}_+$ according to $q_+ = \sup\{q > 0 : \varphi'(q)q - \varphi(q) > 0\}$ is finite or not, and one on $\mathbb{R}_-$ according to $q_- = \inf\{q < 0 : \varphi'(q)q - \varphi(q) > 0\}$ is finite or not. This yields nine possible situations under our assumptions in the case of Mandelbrot measures, and three in the case of critical Mandelbrot measures, since on $\mathbb{R}_+$ there is automatically a second order phase transition at 1. Some of these possibilities are illustrated in Figs. 3, 4, 5 and 6.

(2) Due to (2.4) and (3.3), and Parts (2) and (4) of Theorem 14 are geometric and statistical counterparts in $\mathscr{A}^\omega$ and $\mathscr{A}^*$ of Cramer's theorem and its extension by Bahadur and Zabell (see [DZ98]), which ensures that for all $\gamma \in \mathbb{R}$, over any fixed

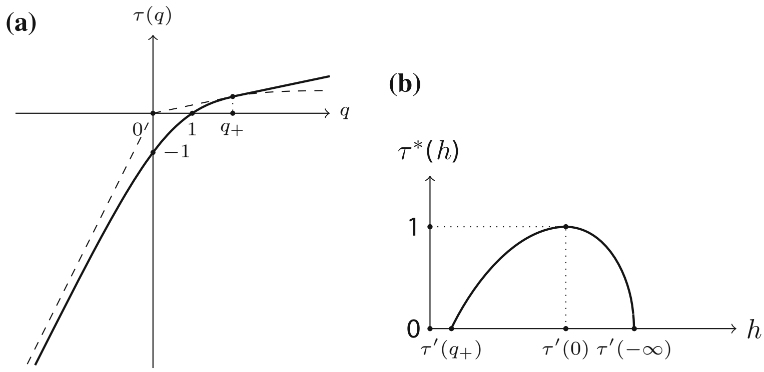

Fig. 5 Multifractal nature of a Mandelbrot measure with $q_- = -\infty$ and a second order phase transition at q_+. **a** The L^q spectrum of μ. **b** Its Legendre transform

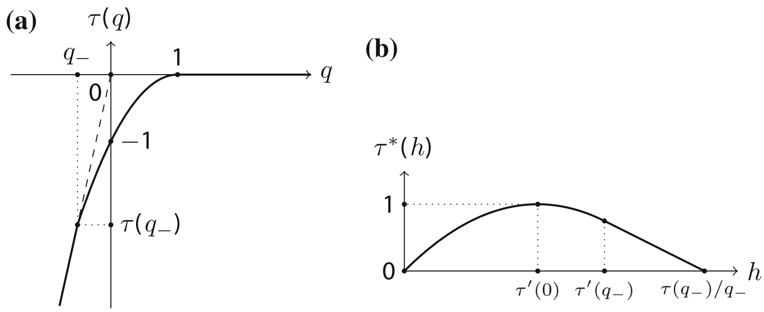

Fig. 6 Multifractal nature of a critical Mandelbrot measure (one always has $q_+ = 1$, where a second order phase transition occurs), with a first order phase transition at q_-. **a** The L^q spectrum of $\widetilde{\mu}$. **b** Its Legendre transform

infinite branch $w_1 \cdots w_n \cdots$ of $\mathscr{A}^\omega$, after setting $L(q) = \log_b \mathbb{E}((bW^{-1})^q)$, one has

$$\begin{aligned} r(\gamma) &:= \inf_{q\in\mathbb{R}} L(q) - \gamma q \\ &= \lim_{\epsilon\to 0} \liminf_{n\to\infty} n^{-1} \log_b \mathbb{P}(|\gamma - (n\log b)^{-1} \sum_{k=1}^n \log(b) - \log(W(w_1\cdots w_n)| \leq \epsilon) \\ &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} n^{-1} \log_b \mathbb{P}(|\gamma - (n\log b)^{-1} \sum_{k=1}^n \log(b) - \log(W(w_1\cdots w_n)| \leq \epsilon). \end{aligned}$$

Indeed, heuristically, due to the rate of growth of the trees $\{\mathscr{A}\}^n$, $n \geq 1$, almost surely, for all $\gamma \in \mathbb{R}$ such that $1 + r(\gamma) \geq 0$, one should have the "logarithmic frequency"

$$\begin{aligned}\lim_{\epsilon\to 0^+}\liminf_{n\to\infty} f(n,\gamma,\epsilon) &= \lim_{\epsilon\to 0^+}\limsup_{n\to\infty} f(n,\gamma,\epsilon)\\ &= \lim_{\epsilon\to 0}\liminf_{n\to\infty} n^{-1}\log_b\Big(\#(\mathscr{A}^n)\cdot\mathbb{P}(|\gamma-(n\log b)^{-1}\sum_{k=1}^n \log(b)\\ &\qquad\qquad -\log(W(w_1\cdots w_n)|\le\epsilon)\Big)\\ &= 1+r(\gamma)=\varphi^*(\gamma).\end{aligned}$$

This observation about point (4) was made by Mandelbrot in [Man74a].

We notice that the formula given by Theorem 14 for τ_ν simplifies to $\tau_\nu=\varphi$ over (q_-,q_+), and also $\tau_\nu^*(\gamma)=\varphi^*(\gamma)$ for $\gamma=\varphi'(q)$, $q\in(q_-,q_+)$.

The lower bound $\tau_\nu(q)\ge T(q)=\sup\{\varphi(\theta q)/\theta:\theta\in(0,1]\}$ is quite easy. At first the problem reduces to packings by b-adic intervals, since $\tau_\nu=\widetilde{\tau}_\nu$. Then, notice that for $q\in(q_-,q_+)$, after Theorem 3 we have $\mathbb{E}(\|\nu\|^q)<\infty$, so (using (2.5) or (3.4))

$$\sum_{n\ge 1}\mathbb{E}\Big(\sum_{w\in\mathscr{A}^n}\nu(I_w)^q b^{n(\varphi(q)-\epsilon)}\Big)=\mathbb{E}(\|\nu\|^q)\sum_{n\ge 1} b^{-n\epsilon}<\infty$$

for all $\epsilon>0$. This yields $\widetilde{\tau}_\nu(q)\ge\varphi(q)$, but $\varphi(q)=\sup\{\varphi(\theta q)/\theta:\theta\in(0,1]\}$ if $q\in(q_-,q_+)$. For the other values of q, the argument giving the lower bound for $\widetilde{\tau}_\nu$ is similar to that used to obtain the upper bound for $\limsup_{n\to\infty}P_n$ in Sect. 4.2 when $\beta\ge\beta_c$. Moreover, one can show that $T^*=f$ and $f=T^*$ (see [AB14] for the details).

Thus, for all $\gamma\ge 0$, $\dim E_\nu(\gamma)\le\tau_\nu^*(\gamma)\le f(\gamma)\le\varphi^*(\gamma)$.

Let $\gamma\ge 0$ such that $\varphi^*(\gamma)\ge 0$. When $\nu=\mu$, the sharp lower bound $\varphi^*(\gamma)$ for $\dim E_\nu(\gamma)$ is determined in several papers (under different kind of assumptions, and sometimes for the level sets associated with $\widetilde{\gamma}(\nu,x)$ rather than $\gamma(\nu,x)$) in the case that $\gamma=\varphi'(q)$ with $q\in(q_-,q_+)$ [Bar00, BBP03, BHJ11, BJ10, Fal94, HW92, Kah91, Mol96, Ols94] or $q\in\{q_-,q_+\}$ [Bar00, BJ10]. This fully describes the range of γ for which $E_\nu(\gamma)\neq\emptyset$ in the cases for which q_- and q_+ correspond to no phase transition or a second order phase transition: one can prove that, with probability 1, for all $q\in(q_-,q_+)$, the Mandelbrot measure μ_q associated with the weights $W_q(w)$ (see (4.1)) is exact dimensional with a Hausdorff dimension equal to $\varphi^*(\varphi'(q))=\varphi'(q)q-\varphi(q)$ and is carried by $E_\nu(\alpha)$ [Bar00] (in [BHJ11, Fal94, HW92, Kah91, Mol96, Ols94], one finds the weaker version: for each fixed $q\in(q_-,q_+)$, $\dim E_\nu(\varphi'(q))\ge\varphi^*(\varphi'(q))$ almost surely). For $\gamma=\varphi'(q)$ with $q\in\{q_-,q_+\}$, if q is finite then one shows that the critical Mandelbrot measure $\widetilde{\mu}_q$ is carried by $E_\nu(\varphi'(q))$ [Bar00], otherwise one builds an inhomogeneous Mandelbrot measure carried by $E_\nu(\varphi'(q))$ and whose Hausdorff dimension is $\varphi^*(\varphi'(q))$ [BJ10].

The same approach is used in [Bar00] to achieve the multifractal analysis of $\widetilde{\mu}$ in absence of first order phase transition on $\mathbb{R}_-$.

However, in case of first order phase transition at $q\in\{q_-,q_+\}$, the previous method does not make it possible to study $E_\nu(\gamma)$ for the exponents γ in

$[\varphi(q)/q, \varphi'(q^-))$ if $q = q_+$ or in $(\varphi'(q^+), \varphi(q)/q]$ if $q = q_-$. Moreover, in any situation, they cannot provide the 0–∞ law obtained in point (3).

In [AB14], a new approach is developed to treat all the possible types of phase transition in the multifractal analysis of branching random walks. Moreover, following the approach developed in [BBP03] or [BJ10], one can reduce the study of the level sets of $\gamma(\nu, x)$ to that of the level sets of $\liminf_{n\to\infty} \frac{\log \nu(I_n(x))}{-n\log(b)}$. Also, the assumption $\varphi(-\epsilon) > -\infty$ for some $\epsilon > 0$ assures that $\mathbb{E}(\|\nu\|^{-\epsilon}) < \infty$, and $\mathbb{E}(\|\nu\|^{\epsilon}) < \infty$ for $\epsilon \in (0,1)$, so that the study of the level sets of $\liminf_{n\to\infty} \frac{\log \nu(I_n(x))}{-n\log(b)}$ reduces to the level sets of $\liminf_{n\to\infty} \frac{H(x|n)}{n\log(b)}$, by using (2.4).

One can also assume that $W \overset{\text{dist}}{\neq} 1$ without loss of generality. Then (we still use the notations of Sect. 4), there exists $A_0 > 0$ such that for $A \geq A_0$, $\mathbb{P}(|V| \leq A) > 0$. One considers the random weights

$$W_{q,A}(w) = \frac{\mathbf{1}_{\{|V(w)|\leq A\}} W^q(w)}{\mathbb{E}(\mathbf{1}_{\{|V(w)|\leq A\}} W^q(w))}$$

and the functions

$$\varphi_A(q) = -1 + q - \log_b \mathbb{E}(\mathbf{1}_{\{|V(w)|\leq A\}} W^q).$$

One has $\varphi_A \searrow \varphi$ pointwise as $A \to \infty$, and it can be shown that this implies that $\varphi_A^* \nearrow \varphi^*$ pointwise on the interior of $\{\gamma : \varphi^*(\gamma) > -\infty\}$ as $A \to \infty$. It follows that one can find an increasing sequence $(A_k)_{k\geq 1}$ converging to ∞ and for each $k \geq 1$ a finite set $D_k \subset \{q : \varphi_{A_k}^*(\varphi'_{A_k}(q)) > 0\}$ such that for each γ such that $\varphi^*(\gamma) \geq 0$, there exists $(q_k)_{k\geq 1} \in \prod_{k\geq 1} D_k$ such that $\lim_{k\to\infty} \varphi'_{A_k}(q_k) = \gamma$ and $\lim_{k\to\infty} \varphi_{A_k}^*(\varphi'_{A_k}(q_k)) = \varphi^*(\gamma)$.

Instead of considering Mandelbrot measures like the measures μ_q, $q \in (q_-, q_+)$, one considers the family of inhomogeneous Mandelbrot measures obtained as follows.

Fix an increasing sequence of integers $(N_k)_{k\geq 0}$ with $N_0 = 0$. Set $M_k = \sum_{i=1}^{k} N_i$ and for $n \geq 1$ define k_n so that

$$M_{k_n} + 1 = 1 + \sum_{k=1}^{k_n} N_k \leq n \leq \sum_{k=1}^{k_n+1} N_k.$$

For every sequence $B = (q_k)_{k\geq 1} \in \prod_{k=1}^{\infty} D_k$, consider the inhomogeneous branching random walk

$$H_B(w) = n\log(b) - \sum_{k=1}^{k_n} \sum_{i=1}^{N_k} \log W_{q_k,A_k}(w|M_{k-1}+i) - \sum_{i=1}^{n-M_{k_n}} \log W_{q_k,A_k}(w|M_{k_n}+i)$$

defined on $E_{B,n} = \{w \in \mathscr{A}^n : |V(w|M_k + i)| \le A_k,\ \forall k, i \text{ such that } M_k + i \le n\}$. Then define the random measures

$$\mu_{B,n}(\mathrm{d}x) = \mathbf{1}_{E_{B,n}}(w) e^{-H_B(w)}\, b^n \mathrm{d}x, \text{ if } x \in I_w.$$

They form a martingale which converges almost surely to a random measure μ_B supported on $E_B = \bigcap_{n\ge 1} \bigcup_{w\in E_{B,n}} I_w$. Moreover, it is possible to choose $(N_k)_{k\ge 1}$ suitably so that all the measures μ_B are simultaneously defined and non degenerate conditionally on $E_B \neq \emptyset$. The sequence $(N_k)_{k\ge 1}$ can also be fixed so that for all γ such that $\varphi^*(\gamma) \ge 0$, each time $B = (q_k)_{k\ge 1}$ is such that $\lim_{k\to\infty} \varphi'_{A_k}(q_k) = \gamma$ and $\lim_{k\to\infty} \varphi^*_{A_k}(\varphi'_{A_k}(q_k)) = \varphi^*(\gamma)$, then, conditionally on $\{E_B \neq \emptyset\}$, μ_B is exact dimensional with dimension $\varphi^*(\gamma)$ and carried both by the set $E_H(\gamma) = \{x : \lim_{n\to\infty} \frac{H(x|n)}{n\log(b)} = \gamma\}$ and the set $\widetilde{E}_\nu(\gamma) = \Big\{x \in [0,1] : \lim_{n\to\infty} \frac{\log \nu(I_n(x))}{-n\log(b)} = \gamma\Big\} \cap \Big\{x \in [0,1] : \lim_{r\to 0^+} \frac{\log\nu(B(x,r))}{\log(r)} = \gamma\Big\}$. Moreover, there are uncountably many such measures for a given γ, and two such measures μ_B and $\mu_{B'}$ are mututally singular if B and B' are not ultimately equal. Since, moreover, $\mathbb{P}(\{E_B \neq \emptyset\})$ tends to 1 as the first terms A_1 of $(A_k)_{k\ge 1}$ tends to ∞, the measures μ_B are the main tool to get parts (2) of the theorem. Part (3) requires additional work, and uses in an essential way the reach family exhibited above of inhomogeneous measures of dimension $\varphi^*(\gamma)$ supported by $E_\nu(\gamma)$.

To get what remains of part (1), it is first not hard to deduce point (4) from the previous estimations for the Hausdorff dimensions: we have

$$\text{a.s., for all } \gamma \ge 0, \quad \lim_{\epsilon\to 0^+} \liminf_{n\to\infty} f(n,\gamma,\epsilon) = \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} f(n,\gamma,\epsilon) = f(\gamma).$$

Then the conclusion follows from Varadhan's integral Lemma [DZ98, Theorem 4.3.1].

Remark 6

(1) The measures of type μ_B can be used to control the Hausdorff and packing dimensions of the wider family of sets

$$\mathbb{E}_\nu(\gamma;\gamma') = \Big\{x \in [0,1] : \liminf_{n\to\infty} \frac{\log\nu(I_n(x))}{-n\log(b)} = \gamma,\ \limsup_{n\to\infty} \frac{\log\nu(I_n(x))}{-n\log(b)} = \gamma'\Big\},$$

$0 \le \gamma \le \gamma'$, (see [AB14] for the details).

(2) Consider the branching random walk $H(w)$, $w \in \mathscr{A}^*$, for itself, and do not assume any integrability properties for H (or W). Set

$$\widetilde{f}(n,\gamma,\epsilon) = n^{-1}\log_b \#\Big\{w \in \mathscr{A}^n : \frac{H_n(w)}{n\log(b)} \in [\gamma-\epsilon, \gamma+\epsilon]\Big\}.$$

The same approach as above yields that with probability 1,

$$\text{for all } \gamma \in \mathbb{R}, \quad \dim E_H(\gamma) = \lim_{\epsilon\to 0^+} \liminf_{n\to\infty} \widetilde{f}(n,\gamma,\epsilon)$$
$$= \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \widetilde{f}(n,\gamma,\epsilon) = f(\gamma).$$

(see [AB14]).
This includes in particular the both sided degenerate case $\varphi(q) = -\infty$ for all $q \neq 0$ (see Sect. 4.4 for the definiton of the degenerate case), and in this case one has $\dim E_H(\gamma) = 1$ for all $\gamma \in \mathbb{R}$.

5.2 *Multifractal Analysis of Lévy-Mandelbrot and Critical Lévy-Mandelbrot Measures*

We now describe the multifractal nature of the discrete statistically self-similar measures defined in Sect. 3.1.2 (notice that by construction these measures are exact dimensional with dimension 0) (Fig. 7).

Theorem 15 *Let $\alpha \in (0,1)$ and ν the non degenerate Mandelbrot measure μ if $\varphi'(1^-) > 0$, or the critical Mandelbrot measure $\widetilde{\mu}$ if $\varphi'(1^-) = 0$ and $\varphi''(1^-) > -\infty$. Let ν^α be the associated Lévy-Mandelbrot or critical Lévy-Mandelbrot measure in Sect. 3.1.2.*

Suppose that $\varphi(-\epsilon) > -\infty$ for some $\epsilon > 0$. With probability 1, we have

(1) *for all $q \in \mathbb{R}$, $\tau_{\nu^\alpha}(q) = \min(\tau_\nu(q/\alpha), 0)$;*
(2) *for all $\gamma \in \mathbb{R}_+$, $\dim E_{\nu^\alpha}(\gamma) = \tau^*_{\nu^\alpha}(\gamma)$.*

Remark 7 There is a phase transition at $q_+ = \alpha$; it is of first order when $\nu = \mu$ and of second order when $\nu = \widetilde{\mu}$. On $\mathbb{R}_-$, we have the same three possibilities as for ν.

We are going to see that when $\nu = \mu$, the first order phase transition at $q = \alpha$ also corresponds to a transition in the geometric properties responsible for the Hausdorff

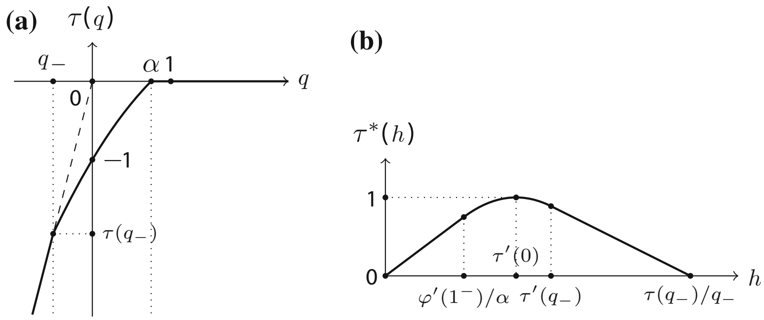

Fig. 7 Multifractal nature of a Lévy-Mandelbrot measure, with the necessary first order phase transition at $\alpha = q_+$, and another first order phase transition at q_-. **a** The L^q spectrum of μ^α. **b** Its Legendre transform

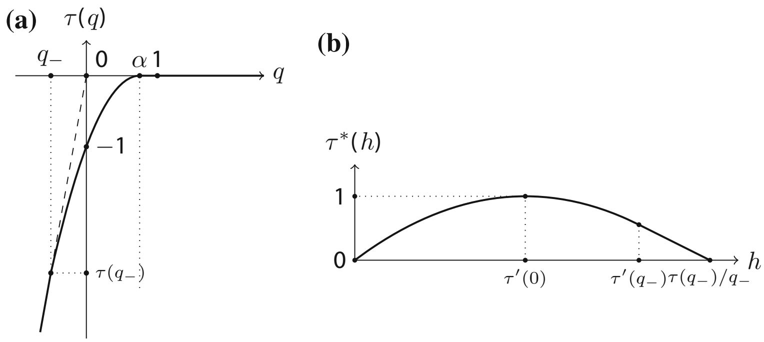

Fig. 8 Multifractal nature of a critical Lévy-Mandelbrot measure, with the necessary second order phase transition at $\alpha = q_+$, and a first order phase transition at q_-. **a** The L^q spectrum of $\widetilde{\mu}^\alpha$. **b** Its Legendre transform

dimensions of the sets $E_{\nu^\alpha}(\gamma)$ for $\gamma \in [0, \varphi'(1^-)/\alpha]$: for these exponents the rate of approximations of the elements of $E_{\nu^\alpha}(\gamma)$ by the atoms of ν^α come into play, while for the other exponents, and the same holds when $\nu = \widetilde{\mu}$, the Hausdorff dimension of $E_{\nu^\alpha}(\gamma)$ can be captured essentially in the same way as for the sets $E_\nu(\alpha\gamma)$ (Fig. 8).

We can reduce the study of the L^q-spectrum to packings by b-adic intervals. For $\alpha q_- < q < \alpha$ we have, using (3.5)

$$\sum_{n\geq 1} \mathbb{E}\Big(\sum_{w\in\mathscr{A}^n} \nu^\alpha(I_w)^q b^{n(\varphi(q/\alpha)-\epsilon)} \Big) = \mathbb{E}(\|\nu^\alpha\|^q) \sum_{n\geq 1} b^{-n\epsilon} < \infty$$

for all $\epsilon > 0$. This yields $\tau_\nu(q) \geq \varphi(q/\alpha)$, and with manipulations similar to those use in the sketch of proof of Theorem 14, we get the lower bound $\tau_{\nu^\alpha}(q) \geq \min(\tau_\nu(q/\alpha), 0)$ almost surely. In particular, $\tau_{\nu^\alpha}(\alpha) \geq 0$, so that since $\tau_{\nu^\alpha}(1) = 0$ and τ_{ν^α} is non decreasing and concave, we must have $\tau_{\nu^\alpha} = 0$ over $[\alpha, \infty)$. The upper bound for τ_{ν^α} is a consequence of the lower bound for the Hausdorff dimension in part (2) of the theorem, as well as the inverse Legendre transform for φ^*.

Part (2) is proven in [Jaff99] when ν is the Lebesgue measure restricted to [0, 1] ($W = 1$ almost surely), i.e. ν^α is just the derivative of a Lévy subordinator, in the following form: $\dim E_{\nu^\alpha}(\gamma) = \alpha\gamma$ if $\gamma \in [0, 1/\alpha]$, and $\dim E_{\nu^\alpha}(\gamma) = -\infty$ otherwise.

Fix $T > 0$ and let $\mathcal{P} = \{(y_n, r_n) : n \geq 1\}$ be a Poisson point process with intensity $\mathrm{d}y \otimes \frac{\mathrm{d}r}{r^2}$ in $[0, T] \times \mathbb{R}_+^*$, so that the sequence $(r_n)_{n\geq 1}$ tends to 0 as $n \to \infty$. Then take for L_α the α-stable Lévy subordinator $L_\alpha(y) = \sum_{n: y_n \leq y} r_n^{1/\alpha}$, so that $\nu^\alpha = \sum_{n\geq 1} r_n^{1/\alpha} \delta_{y_n} := \rho$ when ν is the Lebesgue measure. The multifractal analysis in this case uses the following facts. At first, it results from quite direct estimates that for any $y \in \mathbb{R}_+ \setminus \{y_n : n \geq 1\}$, the pointwise exponent $\gamma(\rho, y)$ equals $1/(\alpha s(y))$, where $s(y)$ is the rate of approximation of y by the family $\{(y_n, r_n) : n \geq 1\}$, defined as

$$s(y) = \limsup_{n\to\infty} \frac{\log(|y - y_n|)}{\log(r_n)}.$$

By a theorem of Shepp [She72], one has $s(y) \geq 1$ for all $y \in [0, T] \setminus \{y_n : n \geq 1\}$, hence $\gamma(\rho, y) \leq 1/\alpha$, so $E_\rho(\gamma) = \emptyset$ if $\gamma > 1/\alpha$. Moreover, as a consequence of a theorem on the "ubiquitous systems" established independently in [Jaf00a] and [DMPV95] and applied to the family $\{(y_n, r_n) : n \geq 1\}$, one has $\dim\{y : s(y) : s\} = 1/s$ for all $s \geq 1$, hence the Hausdorff dimension of $E_\rho(\gamma) = \{y \in [0, T] : s(y) : 1/(\alpha\gamma)\} = \alpha\gamma$.

The general case is treated in [BS07] under stronger assumptions on φ that only allow no phase transition or a second order phase transition on $\mathbb{R}_-$. Conditionally on ν, the Lévy subirdinator L_α is considered over $[0, T]$, with $T = \|\nu\|$. The composition by the indefinite integral of ν when $W \overset{\text{dist}}{\neq} 1$ induces distorsions with respect to the situation explained above. For $\gamma \geq \varphi'(1^-)/\alpha$ such that $\varphi^*(\alpha\gamma) \geq 0$, one takes a measure μ_B of Hausdorff dimension $\varphi^*(\alpha\gamma) \geq 0$ carried by $E_\nu(\alpha\gamma)$ as in the proof of Theorem 14 and proves that for μ_B-almost very x, noting $F(x) = \nu([0, x])$, we have $s(F(x)) = 1$, which combined with $\lim_{r\to 0^+} \frac{\log(\nu(B(x,r)))}{\log(r)} = \alpha\gamma$ and $\gamma(\varrho, F(x)) = 1/(\alpha s(F(x)))$ yields $\gamma(\nu^\alpha, x) = \gamma$. Hence $\dim E_{\nu^\alpha}(\gamma) \geq \varphi^*(\alpha\gamma) = (\min(\tau_\nu(q/\alpha), 0))^*(\gamma)$.

When $\nu = \widetilde{\mu}$, since $\varphi'(1^-) = 0$, this yields the whole spectrum for $\gamma > 0$. Since $E_{\nu^\alpha}(0)$ contains the atoms of ν^α, i.e. $\{F^{-1}(\{y_n : n \geq 1\})$, we also have $\dim E_{\nu^\alpha}(0) \geq 0 = (\min(\tau_\nu(q/\alpha), 0))^*(0)$, and this is also valid when $\nu = \mu$.

The most delicate sets are the $E_{\nu^\alpha}(\gamma)$ for $\gamma \in (0, \varphi'(1^-)/\alpha)$ when $\nu = \mu$, for which we must prove $\dim E_{\nu^\alpha}(\gamma) \geq \alpha\gamma$. This requires a non trivial extension of classical "ubiquitous systems" (which deserve to be called homogeneous) to "heterogeneous ubiquitous systems". This is achieved in [BS07], and applied to the present situation in [BS07]. The main tool provided by these papers is, for each $s > 1$, a Borel probability measure ρ_s on $[0, 1]$ such that for ρ_s-almost every x, $\liminf_{r\to 0^+} \frac{\log(\rho_s(B(x,r)))}{\log(r)} \geq \frac{\varphi'(1^-)}{s}$ (which implies that for any Borel set E, one has $\rho_s(E) = 0$ if $\dim E < \frac{\varphi'(1^-)}{s}$) and there exists a decreasing sequence $(\epsilon_j)_{j\geq 1}$ and a subsequence $(n_j)_{j\geq 1}$ such that for each $j \geq 1$ one has $F^{-1}(y_{n_j}) \in B(x, r_{n_j}^{s/\varphi'(1^-)-\epsilon_j})$. This second property implies that $\nu^\alpha(B(x, r_{n_j}^{s/\varphi'(1^-)-\epsilon_j})) \geq r_{n_j}^{1/\alpha}$, hence $\gamma(\nu^\alpha, x) \leq \varphi'(1^-)/\alpha s$. In particular for $\gamma = \varphi'(1^-)/(\alpha s)$, $\rho_s(F_{\nu^\alpha}(\gamma)) = 1$, where $F_{\nu^\alpha}(\gamma) = \{x \in [0, 1] : \gamma(\nu^\alpha, x) \leq \gamma\}$.

Now, we notice that for $\gamma \in (0, \varphi'(1^-)/\alpha)$, we have $(\min(\tau_\nu(q/\alpha), 0))^*(\gamma) = \alpha\gamma$. Moreover, it also comes from the multifractal formalism [Ols95] that $\dim F_{\nu^\alpha}(\gamma') \leq \tau^*_{\nu^\alpha}(\gamma')$ for $\gamma' \in [0, \tau'_{\nu^\alpha}(0^-)]$. Since $\tau_{\nu^\alpha}(q) \geq \min(\tau_\nu(q/\alpha), 0)$ almost surely for all q, we get $\dim F_{\nu^\alpha}(\gamma') \leq \tau^*_{\nu^\alpha}(\gamma') \leq \alpha\gamma'$ for all $0 \leq \gamma' < \gamma < \varphi'(1^-)/\alpha)$. Now, setting $\gamma = \varphi'(1^-)/\alpha s$, since $\liminf_{r\to 0^+} \frac{\log(\rho_s(B(x,r)))}{\log(r)} \geq \frac{\varphi'(1^-)}{s} = \alpha\gamma$, we get $\rho_s(F_{\nu^\alpha}(\gamma')) = 0$ for all $0 \leq \gamma' < \gamma$, hence $\rho_s\Big(\bigcup_{0\leq\gamma'<\gamma} F_{\nu^\alpha}(\gamma')\Big) = 0$

noting that the sets $F_{\nu^\alpha}(\gamma')$ are non decreasing. Finally, writing $E_{\nu^\alpha}(\gamma) = F_{\nu^\alpha}(\gamma) \setminus \bigcup_{0\le\gamma'<\gamma} F_{\nu^\alpha}(\gamma')$, we obtain $\rho_s(E_{\nu^\alpha}(\gamma)) > 0$ hence $\dim E_{\nu^\alpha}(\gamma)) \ge \alpha\gamma$.

The construction of the measures ρ_s is rather involved, and uses a combination of Shepp's theorem and the statistical self-similarity and exact dimensionality properties of μ.

5.3 KPZ Formula

Here we give, in the Mandelbrot cascade context, results related to KPZ formula [KPZ98] of two dimensional quantum gravity. The KPZ formula was reformulated and proved by Duplantier and Sheffield [DS09, DS11] as a relation between the box counting dimensions of sets in the Euclidean geometry and the expecting box counting dimensions of sets computed using a random measure given by exponential of the Gaussian Free Field (a fundamental example of log-Gaussian multiplicative chaos), and reformulated by Benjamini and Schramm [BS09] and Rhodes and Vargas [RV08] as a relation between Hausdorff dimensions of sets computed using the Lebesgue measure to measure the size of balls, and the Hausdorff dimensions of sets when the Lebesgue measure is replaced by a non degenerate Mandelbrot measure or the limit of a non degenerate log-infinitely divisible cascade. In dimension 1, this can be directly interpreted in terms of a change of metric. Then, a rigorous proof of the "dual" KPZ formula was given in [BJRV13], using discrete random measures which are the analogue of the measure μ^α of Sect. 3.1.2 in the log-normal multiplicative chaos framework (see [RV13] for a review of this). This has been extended to the cascade case in [BKNSW14a], with similar formulas for critical Mandelbrot measures $\widetilde{\mu}$ and the associated discrete measures $\widetilde{\mu}^\alpha$.

Since we work in dimension 1, it is convenient to present KPZ formulas as a relation between the Hausdorff dimension of a set and the Hausdorff dimension of its image by the indefinite integral of a given statistically self-similar measure:

Theorem 16 *Let ν be the non degenerate Mandelbrot measure μ if $\varphi'(1^-) > 0$, or the critical Mandelbrot measure $\widetilde{\mu}$ if $\varphi'(1^-) = 0$ and $\varphi''(1^-) > -\infty$. Denote also ν by ν^1. Suppose that $\varphi(-q) < -\infty$ for all $q \in (0, 1/b)$.*

Let $\alpha \in (0, 1]$ and define $F_\alpha(t) = \nu^\alpha([0, t])$ for $t \in [0, 1]$ (i.e. F_α is the α-stable Lévy process L_α in independent multifractal time $F : t \mapsto \nu([0, t])$). If E is a Borel subset of $[0, 1]$ of Hausdorff dimension ξ_0, then, with probability 1, the Hausdorff dimension of $F_\alpha(E)$ is the unique solution ξ of the equation

$$1 + \varphi(\xi/\alpha) = \xi_0.$$

Notice that when $\alpha = 1$, $F_1 = F$. In this case, when $\nu = \mu$, the result is proved in [BS09], since the authors prove that the dimension ξ of E under the metric $d(x, y) = \mu([x, y])$ is given by the above formula. The formula is quite easy to guess using a natural covering argument and the fact that for all $x, x' \in [0, 1]$,

$\mathbb{E}(d(x,y)^s) \leq 8|x-y|^{1+\varphi(s)}$ for all $s \in [0,1]$ to find an upper bound of $\dim E$ under the metric d. For the lower bound, one first reduces the situation to E being compact. If $\xi_0 = 0$ there is nothing to prove. Otherwise, for each $t \in [0, \xi_0)$ one fixes a Frostman Borel measure ρ such that the energy $\int\int \frac{\rho(\mathrm{d}x)\rho(\mathrm{d}y)}{|y-x|^t}$ is finite, sets s the solution of $1+\varphi(s) = t$, and then shows that the measures $\rho_n(\mathrm{d}x) = W_s(w|1)W_s(w|2)\cdots W_s(w|n)\,\rho(\mathrm{d}x)$ if $x \in I_w,\ w \in \mathscr{A}^n$ weakly converge to a non degenerate measure ρ_s supported on E almost surely and such that $\int\int \frac{\rho_s(\mathrm{d}x)\rho_s(\mathrm{d}y)}{d(x,y)^s} < \infty$ almost surely, which implies that the Hausdorff dimension of E under d is at least s. Since s tends to ξ as t tends to ξ_0, this is enough to conclude.

This approach can be adapted to get the result when $\alpha = 1$ and $\nu = \widetilde{\mu}$ (see [BKNSW14a]). The general case then follows from the fact that a.s. $L_\alpha(A) = \alpha \dim A$ for all subsets A of [0, 1] [Ber96, III.5].

6 On Signed and Complex Multiplicative Cascades

We give some convergence and renormalization results proved in [BM09, BJM10, BJM10a] for the continuous function-valued sequence $(F_n = F_{W,n})_{n\geq 1}$ defined in (2.8). The asymptotic behavior of $(F_n)_{n\geq 1}$ is far from being completely understood in general, and deserves to be further explored. An interesting related and earlier work is [DES93] about the mean field theory of directed polymers with random complex weights whose modulus is independent of the argument; under this kind of assumptions, renormalization results in the space of distributions have been obtained very recently in [LRV00] for complex Gaussian multiplicative chaos on $\mathbb{R}^d$ when the modulus and argument which are independent.

When $(F_n)_{n\geq 1}$ has a non degenerate limit F, the multifractal analysis of F is similar to that of Mandelbrot measures, but new multifractal phenomena can emerge when working with conservative complex cascades (see [BJ10]), a situation excluded in the present setting. This classical multifractal analysis can be completed by the natural notion of multifractal analysis of the graph roughness, a notion explored for non degenerate limits of $(F_n)_{n\geq 1}$ in [Jin11].

For simplicity we suppose that $|W| > 0$ almost surely. Also, we assume that $\mathbb{P}(W \in \mathbb{C} \setminus \mathbb{R}_+) > 0$ and the normalization $E(W) = 1$ holds, so that $(F_n)_{n\geq 1}$ is a martingale. Then we set

$$\varphi(q) = \varphi_W(q) = q - 1 - \log_b \mathbb{E}(|W|^q).$$

6.1 Some Convergence Theorems

6.1.1 Strong Convergence of Complex Cascades

There are conditions sufficient to imply the convergence of $F_n = F_{W,n}$, which in view of the positive case (Theorems 1 and 3(b)) seem optimal. The following result is proved in [BJM10a] (see [BJM10] for an extended version to general complex multiplicative chaos).

Theorem 17 *Assume that there exists* $q > 1$ *such that* $\varphi_W(q) > 0$. *Also suppose* $q \in (1, 2]$ *or* $\varphi_W(2) > 0$.

(1) $(F_n)_{n\geq 1}$ *uniformly converges, with probability 1, and in* L^q *norm, when* n *goes to* ∞, *towards a Hölder function* $F = F_W$.

(2) $F = \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]}\Big(F(i/b)+W(i)\, F_i\circ f_i^{-1}\Big)$, *where* $(W(0), \dots, W(b-1))$, $F_0, \dots, F_{b-1}$ *are independent,* F_i *is equidistributed with* F, *and the equality holds with probability* 1.

With respect to the convergence of $(\mu_n)_{n\geq 1}$ when $W > 0$, the proof necessitates an additional compactness argument.

Remark 8

(1) The statistical self-affinity expressed by Theorem 17(2) implies, setting $Z = F(1) - F(0)$ and $Z(i) = F_i(1) - F_i(0)$:

$$Z = b^{-1} \sum_{k=0}^{b-1} W(k)\, Z(k),$$

which could be considered as an extension of (2.3) to the case of complex weights $W(k)$.

(2) When the weight W is real and such that $|W| = b^{1-H}$, Theorem 17 yields a limit function F which is a monofractal object obtained by a multiplicative cascade, which shares lots of properties with the fractional brownian motion of index H, but its construction is more straightforward. This remarkable fact was one of Mandelbrot's first motivations to consider the signed cascades. Nevertheless there is a constraint on the exponent H: it should lie within the interval $(1/2, 1]$ (see [BM09] for a specific study of the monofractal case).
For a pair $B_H = (F_1, F_2)$ of two independent copies of such a monofractal process, Jin has proved in [Jin14] an analogue of Kaufman's theorem about the Hausdorff dimension of the image of Borel subsets of $\mathbb{R}_+$, namely $\mathbb{P}(\forall\, E \in \mathcal{B}([0, 1]), \dim B_H(E) = H^{-1} \dim E) = 1$.

(3) In [BJM10a], when $|W|$ is not constant, the natural question of deciding whether the limit F can be decomposed as a monofractal process B composed with

the indefinite integral of a Mandelbrot measure μ is discussed. Under stronger assumptions than that of Theorem 17, denoting by β the unique solution of $\varphi(q) = 0$ in $[1, 2]$, and setting $W_\beta = |W|^\beta/\mathbb{E}(|W|^\beta)$, such a decomposition exists, B being of index $H = 1/\beta$, and $\mu = \mu_{W_\beta}$.

6.1.2 Weak Convergence Towards a Brownian Motion in Multifractal Time When W Is Real Valued

When one cannot use Theorem 17, it is natural to seek for a suitable normalisation of the process $F_{W,n}$ in order it be convergent in distribution. We present one result in this direction.

Assume

(1) $W \in \mathbb{R}$ almost surely.
(2) $\varphi_W > -\infty$ on $\mathbb{R}_+$.
(3) $\varphi_W(2) \leq 0$ and φ_W is non-decreasing.

It follows from these hypotheses that $|W| \leq b$ a.s., and the hypotheses of Theorem 17 are not fulfilled. Also, a direct computation shows that the martingale $F_n(1)$ is not bounded in L^2. More precisely

$$\sigma_n^2 = \mathbb{E}(|F_n(1)|^2) \sim \begin{cases} \sigma^2 b^{-n\varphi_W(2)} \text{ with } \sigma^2 = \dfrac{\mathbb{E}\big(\big|\sum_{i=0}^{b-1} W(i)\big|^2\big) - b^2}{\mathbb{E}\big(\sum_{i=0}^{b-1} |W(i)|^2\big) - b^2} & \text{if } \varphi_W(2) < 0 \\ \sigma^2 n \text{ with } \sigma^2 = b^{-2} \sum_{i \neq j} \mathbb{E}(W(i)W(j)) & \text{if } \varphi_W(2) = 0 \end{cases}.$$

Moreover, (3) implies $\varphi_{W_2}(q) > 0$ for all $q > 1$ (where $W_2 = W^2/\mathbb{E}(W^2)$. So, the non decreasing function F_{W_2}, which is nothing but the indefinite integral of the Mandelbrot measure μ_{W_2}, is non degenerate, and $F_{W_2}(1)$ is bounded in L^q for all $q > 1$ after Theorem 3(1).

Theorem 18 *Under the above assumptions, the random continuous function F_n/σ_n converges in distribution towards $B \circ F_{W(2)}$, where B is a standard brownian motion independent of $F_{W(2)}$.*

This is established in [BJM10a]. The assumption (3) is used to identify the law of the limit process thanks to the moments method and the equation $F_n(1) = \sum_{k=0}^{b-1} b^{-1} W(k) F_{n-1,k}$, after proving the tightness of the normalized sequence. However, it is desirable to relax this assumption.

Remark 9 A few additional information about the asymptotic behavior of F_n when the assumptions of Theorems 17 and 18 fail can be found in [BJM10a]. When $W/|W|$ and $|W|$ are independent, precise information about the convergence in probability of the free energy $n^{-1} \log |F_n(1)|$ can be found in [DES93], with three main possible phases. The recent related work achieved in [LRV00] leads to believe that in general, possible limit of F_n/A_n when the conditions of Theorem 17 fail are Brownian motion in critical Mandelbrot or critical Lévy-Mandelbrot time when φ presents a second order phase transition at β_c.

6.2 Multifractal Analysis of Roughness in the Graph of F

The multifractal analysis of the limit function F in Theorem 17 is a refinement of that of the Mandelbrot measure μ: one considers the Hölder exponent associated with the oscillations of order 1 of F,

$$h_F(t) = \liminf_{r \to 0^+} \frac{\log \mathrm{Osc}_F([t-r, r+r])}{\log r}$$

and the corresponding level sets

$$E_F(h) = \{t \in [0,1] : h_F(t) = h\} \quad (h \geq 0).$$

The exponent $h_F(\cdot)$ should be considered as a measure of the aspect more or less rough of the graph of F at each point: the smaller $h_F(t)$, the larger roughness at t is.

Let

$$\tau_F(q) = \liminf_{r \to 0} \frac{\log \sup \left\{ \sum_i \mathrm{Osc}_F(B_i)^q \right\}}{\log r},$$

where the supremum is taken over all families of pairwise disjoint closed intervals B_i, of diameter $2r$, and centered in $\mathrm{supp}(F')$, and F' is the derivative of F in the distribution sense. The following statement is similar to Theorem 14. It is originally proved in [BJ10] under much stronger assumptions (also, in [BJ10] a finer multifractal analysis is achieved; also see [BFP10, BS05, Jaf98, Jaf00, Jaf04] to have a substantial overview of the multifractal formalism for functions), but the techniques introduced in [AB14] makes it possible to relax them, and now cover cases presenting first order phase transition.

Theorem 19 *Under the assumptions of Theorem* 17*, if* $\varphi(-\epsilon) > -\infty$ *for some* $\epsilon > 0$*, then with probability* 1*, all the conclusions of Theorem* 14 *hold with* ν *replaced by* F*.*

The sets $E_F(h)$ are only indirectly linked to the apparent roughness of the graph because they lie in the support of F. This leads to consider the *roughness spectrum* which gives the Hausdorff dimension of the sets obtained by lifting on the graph of F the sets $E_F(h)$. In other terms, this spectrum is the Hausdorff dimensions of the sets

$$G_F(h) = \{(t, F(t)) : t \in E_F(h)\} \quad (h \geq 0).$$

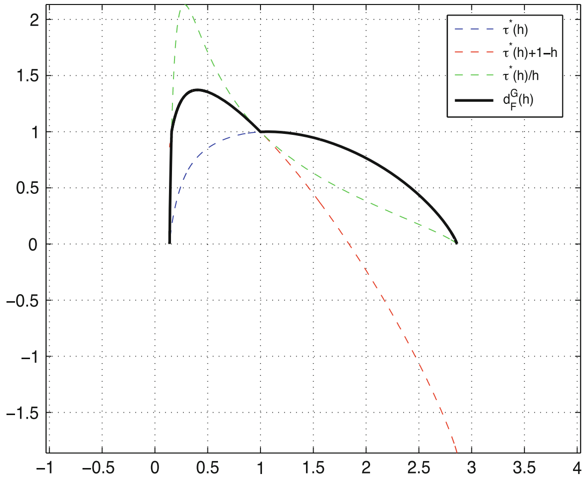

Fig. 9 Example of roughness spectrum $d_F^G : h \mapsto \dim G_F(h)$

The following result [Jin11] is the first of this kind for multifractal stochastic processes. We present here a stronger version using the assumptions of Theorem 19.

Theorem 20 *Under the assumptions of Theorem* 19*, if W is real valued, with probability* 1*, one has*

$$\dim \mathrm{Graph}(F) = 1 - \tau(1),$$

and, for all $h \geq 0$ such that $E_F(h) \neq \emptyset$, one has

$$\dim E_F(h) = \tau_F^*(h)$$

and

$$\dim G_F(h) = \Big(\frac{\dim E_F(h)}{h} \wedge \big(\dim E_F(h) + 1 - h\big)\Big) \vee \dim E_F(h).$$

The upper bounds for Hausdorff dimensions follows from quite standard arguments and is true in general. For the lower bound, the proof uses the auxiliary measures introduced to find a lower bound for $\dim E_F(h)$. These measures are lifted to the graph of F, and the lifted measures do have the desired lower Hausdorff dimension. This dimension is estimated by showing that they have a finite energy with respect to suitable Riesz kernels. However, while for the calculation of the dimension of the graph this method works quite straightforward, for the sets $G_F(h)$ in general the study is very delicate (see [Jin11] for details) (Fig. 9).

7 Mandelbrot Cascades as a Dynamical System

We present the results obtained in [BPW09], showing that Mandelbrot cascades define a dynamical system on a subset of the fixed points of the smoothing transformation. The asymptotic behavior of this dynamical systems exhibits a functional central limit theorem whose Gaussian limit process is, unexpectedly, the limit of an additive cascade on a tree. Fine properties of this process are also detailed.

7.1 A Dynamical System

Let $\mathcal{P}$ the set of Borel probability measures on $\mathbb{R}_+$. If $\mu \in \mathcal{P}$ and $p > 0$, we denote by $\mathbf{m}_p(\mu)$ the moment of order p of μ, i.e.,

$$\mathbf{m}_p(\mu) = \int_{\mathbb{R}_+} x^p \, \mu(\mathrm{d}x).$$

Then let $\mathcal{P}_1$ be the set of elements of $\mathcal{P}$ whose first moment equals 1:

$$\mathcal{P}_1 = \{\rho \in \mathcal{P} \; : \; \mathbf{m}_1(\rho) = 1\}.$$

Using the characterizations of the elements of $\mathcal{P}$ by their Laplace transform, the smoothing transformation S_ρ associated with $\rho \in \mathcal{P}$ considered in Sect. 3.1 is nothing but the mapping from $\mathcal{P}$ to itself so defined: If $\nu \in \mathcal{P}$, one considers $2b$ independent random variables, $Z(0),\ Z(1), \dots,\ Z(b-1)$, whose common probability distribution is ν, and $W(0),\ W(1), \dots,\ W(b-1)$ whose common probability distribution is ρ; then $\mathsf{S}_\rho \nu$ is the probability distribution of the random variable $b^{-1} \sum_{0 \le j < b} W(j)\, Z(j)$.

Since the measure ρ is in $\mathcal{P}_1$, S_ρ maps $\mathcal{P}_1$ into itself. We have seen (Theorem 1) that the condition $\int x \log(x)\, \rho(\mathrm{d}x) < \log b$ is necessary and sufficient for the weak convergence of the sequence $\mathsf{S}_\rho^n \delta_1$ (where δ_1 stands for the Dirac mass at point 1) towards a probability measure ν, which therefore is a fixed point of S_ρ. Indeed, under this assumption $\mathsf{S}_\rho^n \delta_1$ is the probability distribution of the uniformly integrable non-negative martingale $(Y_n)_{n \ge 1}$ whose limit Y has probability distribution ν belonging to $\mathcal{P}_1$ and satisfies $\mathsf{S}_\rho \nu = \nu$ due to (2.3). In this case, we denote the measure ν by $\mathsf{T}\rho$. It is natural to try and iterate T. But, in general this is not possible because $\nu = \mathsf{T}\rho$ may not inherit the property $\int x \log(x)\, \nu(\mathrm{d}x) < \log b$. So, we have to find a domain stable under the action of T. This is done by imposing conditions on moments.

Indeed, it is easily seen that the sequence $(Y_n)_{n \ge 1}$ defined by (2.1) remains bounded in L^2 norm if and only if $\mathbb{E}(W^2) = \mathbf{m}_2(\rho) < b$, and that in this case Formula (2.3) yields

$$\mathbb{E}Y^2 = \frac{b-1}{b-\mathbb{E}W^2} \tag{7.1}$$

(since the random variables $W(j)$ and $Y(j)$ are independent and of expectation 1, squaring both sides of Formula (2.3) yields $b^2\mathbb{E}Y^2 = b\mathbb{E}W^2\mathbb{E}Y^2 + b(b-1)$). It follows that if $b \geq 3$ and $1 \leq \mathbb{E}W^2 < b-1$, we have $\mathbb{E}Y^2 \leq \mathbb{E}W^2$ (the equality holding only if $W = 1$). Therefore, since the condition $\mathbb{E}W^2 < b$ is stronger than $\mathbb{E}(W \log W) < \log b$ when $\mathbb{E}W = 1$ (since the function $t \mapsto \log \mathbb{E}W^t$ is convex), T is a transformation on the subset of $\mathcal{P}_1$ defined by

$$\mathscr{P}_b = \left\{\rho \in \mathcal{P}_1 \ : \ 1 < \mathbf{m}_2(\rho) < b-1\right\}.$$

If $\rho \in \mathscr{P}_b$, due to (2.3), we can associate with each $n \geq 0$ a random variable W_{n+1} as well as $2b$ independent random variables $W_n(0), \ldots, W_n(b-1)$ and $W_{n+1}(0), \ldots, W_{n+1}(b-1)$ such that

$$W_{n+1} = \frac{1}{b}\sum_{k=0}^{b-1} W_n(k) W_{n+1}(k), \tag{7.2}$$

$\mathsf{T}^n\rho$ is the probability distribution of $W_n(k)$ for every k such that $0 \leq k \leq b-1$, and $\mathsf{T}^{n+1}\rho$ is the probability distribution of W_{n+1} and $W_{n+1}(k)$ for every $0 \leq k \leq b-1$. We advise the reader that if one writes Formula (7.2) with $n-1$ instead of n, the variables $W_n(k)$ which then appear are different from the previous ones.

We have seen in Sect. 2 that the random variable Y represents the increment between 0 and 1 of the non-decreasing continuous function F on $[0, 1]$ obtained as the almost sure uniform limit of the sequence of non-decreasing continuous functions F_n defined in (2.8), and F is nothing but the indefinite integral of the Mandelbrot measure μ. Thus, (2.4) rewrites, for $w \in \mathscr{A}^*$,

$$\Delta(F, I_w) = b^{-|w|} Y(w) \prod_{1 \leq j \leq |w|} W(w|j), \tag{7.3}$$

where for any bounded $f : [0, 1] \mapsto \mathbb{R}$, for every sub-interval $I = [\alpha, \beta]$ of $[0, 1]$, we denote by $\Delta(f, I)$ the increment $f(\beta) - f(\alpha)$ of f over the interval I.

Let us denote by $\Phi(\rho)$ the probability distribution of the limit F, considered as a random continuous function.

We can describe the asymptotic behavior of the dynamical system $(\mathscr{P}_b, \mathsf{T})$, as well as the asymptotic behavior of $\left(\mathsf{T}^n\rho, \Phi(\mathsf{T}^{n-1}\rho)\right)_{n\geq 1}$ as n goes to ∞.

We need some more definitions. For $b \geq 3$, set

$$w_2(b) = \min\left(b-1,\ b\frac{b^4 - 4b^2 + 12b - 8}{b^4 + 8b^2 - 12b + 4}\right)$$

and, for t such that $1 < t < w_2(b)$,

$$w_3(b,t) = \frac{b^2}{2} + \frac{1}{2}\sqrt{\frac{b(b^4-4b^2+12b-8) - t\big(b^4+8b^2-12b+4\big)}{b-t}}.$$

One always has $w_3(b,t) < b^2 - 1$.

Also set

$$\mathscr{D}_b = \Big\{\rho \in \mathcal{P} \ :\ \mathbf{m}_1(\rho) = 1,\ 1 < \mathbf{m}_2(\rho) < w_2(b),\ \ \text{and}\ \mathbf{m}_3(\rho) < w_3\big(b, \mathbf{m}_2(\rho)\big)\Big\}.$$

Theorem 21 *Suppose $b \geq 3$. Let $\rho \in \mathscr{P}_b$, and, for $n \geq 0$, define $\sigma_n = \left(\int (x-1)^2\, \mathsf{T}^n\mu(dx)\right)^{1/2}$. Then*

(1) *The limit of $(b-1)^{n/2}\sigma_n$ exists and is positive; so $\lim_{n\to\infty} \mathsf{T}^n\mu = \delta_1$.*
More generally, the probability distributions $\Phi(\mathsf{T}^n\rho)$ weakly converges towards δ_{Id}.

(2) *Suppose that ρ lies in the domain $\mathscr{D}_b \subset \mathscr{P}_b$. Then, if W_n is a variable whose distribution is $\mathsf{T}^n\mu$, $\dfrac{W_n - 1}{\sigma_n}$ converges in distribution towards $\mathcal{N}(0,1)$.*
More generally, if h_n is a random function distributed according to $\Phi(\mathsf{T}^{n-1}\rho)$, the distribution of $\dfrac{h_n - Id}{\sigma_n}$ weakly converges towards the distribution of the unique continuous Gaussian process $(X_t)_{t\in[0,1]}$, such that $X(0) = 0$ and, for all $j \geq 1$, the covariance matrix M_j of the vector $\big(\Delta(X, I_w)\big)_{w\in\mathscr{A}^j}$ is given by

$$\mathsf{M}_j(w,w') = \begin{cases} b^{-2j}\big(1 + (b-1)|w|\big) & \textit{if } w = w', \\ b^{-2j}(b-1)|w \wedge w'| & \textit{otherwise}. \end{cases}$$

Part (1) of the theorem follows from an easy calculation. For part (2), setting $Z_n = \frac{W_n - 1}{\sigma_n}$ one first exploits (7.2) to prove that $(|W_n|)_{n\geq 1}$, and then $(|Z_n|)_{n\geq 1}$, is bounded in L^3. This uses a recursion in which the domain of attraction $\mathscr{D}_b$ is introduced. Then, using (7.2) again one gets, after setting,

$$R_n = \frac{1}{b}\sum_{j=0}^{b-1} Z_n(j)Z_{n-1}(j)\sigma_{n-1} + \frac{1}{b}\left(\frac{\sigma_{n-1}}{\sigma_n} - \sqrt{b-1}\right)\sum_{j=0}^{b-1} Z_{n-1}(j),$$

$$Z_n = R_n + \frac{\sqrt{b-1}}{b}\sum_{k=0}^{b-1} Z_{n-1}(k) + \frac{1}{b}\sum_{k=0}^{b-1} Z_n(k). \tag{7.4}$$

Due to the equivalence $\sigma_n \sim c(b-1)^{-n/2}$, the situation is essentially reducible to the relation $Z_n = \frac{\sqrt{b-1}}{b}\sum_{k=0}^{b-1} Z_{n-1}(k) + \frac{1}{b}\sum_{k=0}^{b-1} Z_n(k)$. Using the relation (7.4) recursively n times yields a relation of the form $Z_n \overset{\text{dist}}{=} o(1) + \sum_{k=1}^{N_n} a_{n,k} Z_{n,k}$, where the $Z_{n,k}$ are independent centered variables, each of which is distributed like one of the Z_k, $1 \leq k \leq n$, $\lim_{n\to\infty} \sup\{a_{n,k} = 1 \leq k \leq N_n\} = 0$ and $\sum_{k=1}^{N_n} a_{n,k}^2 = 1$. An application of Lindeberg theorem (using the boundedness of $(|Z_k|)_{k\geq 1}$ in L^3) yields the convergence in law of $(Z_n)_{n\geq 1}$ to $\mathcal{N}(0, 1)$.

For the functional central limit theorem, set $\mathcal{Z}_n = \dfrac{h_n - \mathrm{Id}}{\sigma_n}$. From (7.3) one deduces

$$\begin{aligned}\Delta(\mathcal{Z}_n, I_w) &= b^{-j}\sigma_n^{-1}\left[W_n(w)\prod_{k=1}^{j} W_{n-1}(w|k) - 1\right]\\ &= b^{-j} Z_n(w)\prod_{k=1}^{j} W_{n-1}(w|k)\\ &\quad + b^{-j}\sum_{l=1}^{j}\frac{\sigma_{n-1}}{\sigma_n} Z_{n-1}(w|l)\prod_{k=1}^{l-1} W_{n-1}(w|k).\end{aligned} \tag{7.5}$$

From this relation one can both show the tightness of the distributions of the processes $\mathcal{Z}_n$, $n \geq 1$, and the convergence in distribution of the increments by simply passing to the limit and using the independences between the $W_{n-1}(w|k)$ and $W_n(w)$: there exist $\big(\mathcal{N}(v)\big)_{v\in\bigcup_{k=1}^{j}\mathscr{A}^k}$ and $\big(\widetilde{\mathcal{N}}(w)\big)_{w\in\mathscr{A}^j}$ two families of $\mathcal{N}(0, 1)$ random variables so that all the random variables involved in these families are independent, and

$$\lim_{n\to\infty}\big(\Delta(\mathcal{Z}_n, I_w)\big)_{w\in\mathscr{A}^j} \overset{\text{dist}}{=} b^{-j}\left(\widetilde{\mathcal{N}}(w) + \sqrt{b-1}\sum_{k=1}^{j}\mathcal{N}(w|k)\right)_{w\in\mathscr{A}^j}. \tag{7.6}$$

This is enough to derive the convergence in law of the $\mathcal{Z}_n$ to the Gaussian process X of the statement.

Remark 10 It is natural to seek for a larger domain of attraction than $\mathscr{D}_b$. This requires to be able to keep controls similar to the previous ones in $L^{2+\epsilon}$ (if not in L^2) rather than in L^3.

7.2 The Limit Process X as the Limit of an Additive Cascade

Recall that, if $v \in \mathscr{A}^*$, $[v]$ stands for the cylinder in $\mathscr{A}^\omega$ consisting of sequences beginning by v.

Let $(\xi(w))_{w\in\mathscr{A}^*}$ be a sequence of independent $\mathcal{N}(0, 1)$ random variables. For each $w \in \mathscr{A}^*$,

$$\zeta_n(w) = \sqrt{b-1}\sum_{j=1}^{n} b^{-j}\sum_{v\in\mathscr{A}^j}\xi(wv),$$

is a martingale bounded in L^2 norm, whose limit $\zeta(w)$ is a $\mathcal{N}(0,1)$ random variable, independent of the $\xi(v)$, $|v| \le |w|$. Moreover, all the $\zeta(w)$, $w \in \mathscr{A}^n$ are independent.

One can then check that

$$M([w]) = b^{-|w|}\left(\zeta(w) + \sqrt{b-1}\sum_{1\le k\le |w|}\xi(w|k)\right) \tag{7.7}$$

defines a finitely additive Gaussian random measure defined on the cylinders of $\mathscr{A}^\omega$ Then, the limit process X of the previous sections can be seen as the primitive of the projection of M on $[0,1]$, and the structure of the increment $\Delta(X, I_w)$ given by (7.7) is an additive counterpart to the multiplicative structure of the increment of F given by (7.3).

Of course (7.7) makes sense even for $b = 2$.

7.3 Fine Properties of X

Due to the structure of the increments of X, it is natural to consider for all $\alpha \in \mathbb{R}$ the sets

$$\overline{E}_\alpha = \left\{t \in [0,1) : \limsup_{n\to\infty} \frac{\Delta(X, I_n(t))}{nb^{-n}} = \alpha\sqrt{b-1}\right\},$$
$$\underline{E}_\alpha = \left\{t \in [0,1) : \liminf_{n\to\infty} \frac{\Delta(X, I_n(t))}{nb^{-n}} = \alpha\sqrt{b-1}\right\},$$

and

$$E_\alpha = \underline{E}_\alpha \bigcap \overline{E}_\alpha,$$

where $I_n(t)$ stands for the semi-open to the right b-adic interval of generation n containing t. One has

Theorem 22 *With probability 1,*

(1) *the modulus of continuity of X is a $O\big(\delta\log(1/\delta)\big)$,*
(2) *X does not belong to the Zygmund class,*
(3) *the set E_0 contains a set of full Lebesgue measure at each point of which X is not differentiable,*
(4) $\dim E_\alpha = \dim \underline{E}_\alpha = \dim \overline{E}_\alpha = 1 - \dfrac{\alpha^2}{2\log b}$ *if $|\alpha| \le \sqrt{2\log b}$, and $E_\alpha = \emptyset$ if $|\alpha| > \sqrt{2\log b}$.*

The multifractal analysis part (4) is directly deduced from the multifractal analysis of the branching random walk $S(w) = \sum_{1\le j\le |w|}\xi(w|j)$ which follows from the approach explained in Sect. 5.1.3. Part (3) is a consequence of the law of the iterated logarithm with respect to $\mathbb{P} \otimes$ Leb. Points (1) and (2) follow from quite direct calculations.

It would be good to decide whether or not X is nowhere differentiable.

Appendix: Hausdorff Measures and Dimension

Given $g : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous non-decreasing function near 0 and such that $g(0) = 0$, and E a subset of $[0, 1]$, the Hausdorff measure of E with respect to the gauge function g is defined as

$$\mathcal{H}^g(E) = \lim_{\delta \to 0^+} \inf \Big\{ \sum_{i \in \mathbb{N}} g(\mathrm{diam}(U_i)) \Big\},$$

the infimum being taken over all the countable coverings $(U_i)_{i \in \mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ.

If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $\mathcal{H}^g(E)$ is also denoted $\mathcal{H}^s(E)$ and called the s-dimensional Hausdorff measure of E. Then, the Hausdorff dimension of E is defined as

$$\dim E = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\},$$

with the convention $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

For more information the reader is referred to [Fal03, Mat95].

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