# Free Bosonic Vertex Operator Algebras on Genus Two Riemann Surfaces II

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**Abstract** We study $n$-point correlation functions for a vertex operator algebra $V$ on a Riemann surface of genus 2 obtained by attaching a handle to a torus. We obtain closed formulas for the genus two partition function for free bosonic theories and lattice vertex operator algebras $V_L$ and describe their holomorphic and modular properties. We also compute the genus two Heisenberg vector $n$-point function and the Virasoro vector one point function. Comparing with the companion paper, when a pair of tori are sewn together, we show that the partition functions are not compatible in the neighborhood of a two-tori degeneration point. The *normalized* partition functions of a lattice theory $V_L$ *are* compatible, each being identified with the genus two Siegel theta function of $L$.

## 1 Introduction

In previous work [17–20, 34] we developed the general theory of $n$-point functions for a Vertex Operator Algebra (VOA) on a compact Riemann surface $\mathcal{S}$ obtained by sewing together two surfaces of lower genus, and applied this theory to obtain detailed results in the case that $\mathcal{S}$ is obtained by sewing a pair of complex tori—the so-called $\epsilon$-formalism discussed in the companion paper[1] [20]. In the

[1]Reference [20] together with the present paper constitute a much expanded version of [21].

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present paper we consider in detail the situation when $\mathcal{S}$ results from self-sewing a complex torus, i.e., attaching a handle, which we refer to as the $\rho$-formalism. We describe the nature of the resulting $n$-point functions, paying particular attention to the 0-point function, i.e., the genus 2 *partition function*, in the $\rho$-formalism. We find the explicit form of the partition function for the Heisenberg free bosonic string and for lattice vertex operator algebras, and show that these functions are holomorphic on the parameter domain defined by the sewing. We study the generating function for genus two Heisenberg $n$-point functions and show that the Virasoro vector 1-point function satisfies a genus two Ward identity. Many of these results are analogous to those found in the $\epsilon$-formalism discussed in [20] but with significant technical differences. Finally, we compare the results in the two formalisms, and show that the partition functions (and hence all $n$-point functions) are *incompatible*. We introduce *normalized* partition functions, and in the case of $V_L$ show that they *are* compatible; in both formalisms the normalized partition function is the genus two Siegel theta function $\theta_L^{(2)}$.

We now discuss the contents of the paper in more detail. Our approach to genus two correlation functions in both formalisms is to define them in terms of genus one data coming from a VOA $V$. In Sect. 2 we review the $\rho$-formalism introduced in [18]. There, we constructed a genus two surface by self-sewing a torus, and obtained explicit expressions for the genus two normalized 2-form of the second kind $\omega^{(2)}$, a basis of normalized holomorphic 1-forms $\nu_1, \nu_2$, and the period matrix $\Omega$, in terms of genus one data. In particular, we constructed a holomorphic map

$$\begin{aligned} F^{\rho} : \mathcal{D}^{\rho} &\longrightarrow \mathbb{H}_2 \\ (\tau, w, \rho) &\longmapsto \Omega(\tau, w, \rho) \end{aligned} \tag{1}$$

Here, and below, $\mathbb{H}_g$ ($g \geq 1$) is the genus $g$ *Siegel upper half-space*, and $\mathcal{D}^{\rho} \subseteq \mathbb{H}_1 \times \mathbb{C}^2$ is the domain defined in terms of data $(\tau, w, \rho)$ needed to self-sew a torus of modulus $\tau$. Sewing produces a surface $\mathcal{S} = \mathcal{S}(\tau, w, \rho)$ of genus 2, and the map $F^{\rho}$ assigns to $\mathcal{S}$ its period matrix. We also introduce some diagrammatic techniques which provide a convenient way of describing $\omega^{(2)}$, $\nu_1, \nu_2$ and $\Omega$ in the $\rho$-formalism.

Section 3 consists of a brief review of relevant background material on VOA theory, with particular attention paid to the Li-Zamolodchikov or LiZ metric. In Sect. 4, motivated by ideas in conformal field theory [6, 29, 31, 32], we introduce $n$-point functions (at genus one and two) in the $\rho$-formalism for a general VOA with nondegenerate LiZ metric. In particular, the genus two *partition function* $Z_V^{(2)} : \mathcal{D}^{\rho} \to \mathbb{C}$ is formally defined as

$$Z_V^{(2)}(\tau, w, \rho) = \sum_{n \geq 0} \rho^n \sum_{u \in V_{[n]}} Z_V^{(1)}(\bar{u}, u, w, \tau), \tag{2}$$

where the inner sum is taken over any basis for a homogeneous space $V_{[n]}$ of weight $wt[n]$, $Z_V^{(1)}(\bar{u}, u, w, \tau)$ is a genus one 2-point function and $\bar{u}$ is the LiZ metric dual of $u$. In Sect. 4.1 we consider an example of self-sewing a sphere (Theorem 6), while in

Sect. 4.2 we show (Theorem 7) that a particular degeneration of the genus 2 partition function of a VOA $V$ can be described in terms of genus 1 data. Of particular interest here is the interesting relationship between the quasiprimary decomposition of $V$ and the Catalan series.

In Sects. 5 and 6 we consider in detail the case of the Heisenberg free bosonic theory $M^l$ corresponding to $l$ free bosons, and lattice VOAs $V_L$ associated with a positive-definite even lattice $L$. Although (2) is a priori a formal power series in $\rho, w$ and $q = e^{2\pi i\tau}$, we will see that for these two theories it is a holomorphic function on $\mathcal{D}^\rho$. We expect that this result holds in much wider generality. Although our calculations in these two sections generally parallel those for the $\epsilon$-formalism [20], the $\rho$-formalism is far from being a simple translation. Several issues require additional attention, so that the $\rho$-formalism is rather more complicated than its $\epsilon$-counterpart. This arises in part from the fact that $F^\rho$ involves a logarithmic term that is absent in the $\epsilon$-formalism. The moment matrices employed are also more unwieldy.

We establish (Theorem 8) a fundamental formula describing $Z^{(2)}_M(\tau, w, \rho)$ as a quotient of the genus one partition function for $M$ by a certain infinite determinant. This determinant was already introduced in [18], and its holomorphy and nonvanishing in $D^\rho$ (loc. cit.) implies the holomorphy of $Z^{(2)}_M$. We also obtain a product formula for the infinite determinant (Theorem 9), and establish the automorphic properties of $Z^{(2)}_{M^2}$ with respect to the action of a group $\Gamma_1 \cong \mathrm{SL}(2, \mathbb{Z})$ (Theorem 11) that naturally acts on $D^\rho$. In particular, we find that $Z^{(2)}_{M^{24}}$ is a form of weight $-12$ with respect to the action of $\Gamma_1$. These are the analogs in the $\rho$-formalism of results obtained in Sect. 6 of [20] for the genus two partition function of $M$ in the $\epsilon$-formalism.

We also calculate some genus two $n$-point functions for the rank one Heisenberg VOA $M$, specifically the $n$-point function for the weight 1 Heisenberg vector and the 1-point function for the Virasoro vector $\tilde{\omega}$. We show that, up to an overall factor of the genus two partition function, the formal differential forms associated with these $n$-point functions are described in terms of the global symmetric 2-form $\omega^{(2)}$ [33] and the genus two projective connection [11] respectively. Once again, these results are analogous to results obtained in [20] in the $\epsilon$-formalism.

In Sect. 6.1 we establish (Theorem 14) a basic formula for the genus two partition function for lattice theories in the $\rho$-formalism. The result is

$$Z^{(2)}_{V_L}(\tau, w, \rho) = Z^{(2)}_{M^l}(\tau, w, \rho)\theta^{(2)}_L(\Omega), \tag{3}$$

where $\theta^{(2)}_L(\Omega)$ is the genus two Siegel theta function attached to $L$ [7] and $\Omega = F^\rho(\tau, w, \rho)$; indeed, (3) is an identity of formal power series. The holomorphy and automorphic properties of $Z^{(2)}_{V_L,\rho}$ follow from (3) and those of $Z^{(2)}_{M^l}$ and $\Theta^{(2)}_L$. Heisenberg $n$-point functions and a genus two Ward identity involving the Virasoro 1-point function are also discussed.

Section 7 is devoted to a *comparison* of genus two $n$-point functions, and especially partition functions, in the $\epsilon$- and $\rho$-formalisms. There are strong formal similarities between $Z^{(2)}_{M^l,\epsilon}(\tau_1,\tau_2,\epsilon)$ and $Z^{(2)}_{M^l,\rho}(\tau,w,\rho)$ so it is natural to ask if they are equal in some sense.[2] In the very special case that $V$ is holomorphic (i.e., it has a *unique* irreducible module), one knows (e.g., [33]) that the genus 2 conformal block is one-dimensional, in which case an identification of the two partition functions might seem inevitable. On the other hand, the partition functions are defined on quite different domains, so there is no question of them being literally equal. Indeed, we argue in Sect. 7 that $Z^{(2)}_{M^l,\epsilon}(\tau_1,\tau_2,\epsilon)$ and $Z^{(2)}_{M^l,\rho}(\tau,w,\rho)$ are *incompatible*, i.e., there is *no* sensible way in which they can be identified.

We therefore introduce *normalized* partition functions, defined as

$$\hat{Z}^{(2)}_{V,\rho}(\tau,w,\rho) := \frac{Z^{(2)}_{V,\rho}(\tau,w,\rho)}{Z^{(2)}_{M^l,\rho}(\tau,w,\rho)},\quad \hat{Z}^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon) := \frac{Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)}{Z^{(2)}_{M^l,\epsilon}(\tau_1,\tau_2,\epsilon)},$$

associated to a VOA $V$ of central charge $l$. For $M^l$, the normalized partition functions are equal to 1. The relation between the normalized partition functions for lattice theories $V_L$ ($\mathrm{rk}\, L = l$) in the two formalisms can be displayed in the diagram

$$\begin{array}{ccccc} D^{\epsilon} & \xrightarrow{F^{\epsilon}} & \mathbb{H}_2 & \xleftarrow{F^{\rho}} & D^{\rho} \\ & \hat{Z}^{(2)}_{V,\epsilon} \searrow & \downarrow \theta^{(2)}_L & \swarrow \hat{Z}^{(2)}_{V,\rho} & \\ & & \mathbb{C} & & \end{array} \tag{4}$$

That this is a *commuting* diagram combines formula (3) in the $\rho$-formalism, and Theorem 14 of [20] for the analogous result in the $\epsilon$-formalism. Thus, the *normalized* partition functions for $V_L$ are *independent of the sewing scheme*. They can be identified, via the sewing maps $F^{\bullet}$, with *a genus two Siegel modular form of weight* $l/2$, the Siegel theta function. It is therefore the normalized partition function(s) which can be identified with an element of the conformal block, and with each other. It would obviously be useful to have available a result that provides an a priori guarantee of this fact. A partial confirmation of this fact is described in [12] where it is shown that the normalized partition functions for any VOA $V$ agree in the degeneration limit where one torus is pinched down to a Riemann sphere. Section 8 contains a brief further discussion of these issues in the light of related ideas in string theory and algebraic geometry.

[2] Here we include an additional subscript of either $\epsilon$ or $\rho$ to distinguish between the two formalisms.

## 2 Genus Two Riemann Surface from Self-sewing a Torus

In this section we review some relevant results of [18] based on a general sewing formalism due to Yamada [36]. In particular, we review the construction of a genus two Riemann surface formed by self-sewing a twice-punctured torus. We refer to this sewing scheme as the $\rho$-formalism. We discuss the explicit form of various genus two structures such as the period matrix $\Omega$. We also review the convergence and holomorphy of an infinite determinant that naturally arises later on. An alternative genus two surface formed by sewing together two tori, which we refer to as the $\epsilon$-formalism, is utilised in the companion paper [20].

### *2.1 Some Elliptic Function Theory*

We begin with the definition of various modular and elliptic functions [17, 18]. We define

$$\begin{aligned} P_2(\tau, z) &= \wp(\tau, z) + E_2(\tau) \\ &= \frac{1}{z^2} + \sum_{k=2}^{\infty}(k-1)E_k(\tau)z^{k-2}, \end{aligned} \tag{5}$$

where $\tau \in \mathbb{H}_1$, the complex upper half-plane and where $\wp(\tau, z)$ is the Weierstrass function (with periods $2\pi\mathrm{i}$ and $2\pi\mathrm{i}\tau$) and $E_k(\tau) = 0$ for $k$ odd, and for $k$ even is the Eisenstein series. Here and below, we take $q = \exp(2\pi\mathrm{i}\tau)$. We define $P_0(\tau, z)$, up to a choice of the logarithmic branch, and $P_1(\tau, z)$ by

$$P_0(\tau, z) = -\log(z) + \sum_{k\geq 2}\frac{1}{k}E_k(\tau)z^k, \tag{6}$$

$$P_1(\tau, z) = \frac{1}{z} - \sum_{k\geq 2}E_k(\tau)z^{k-1}. \tag{7}$$

$P_0$ is related to the elliptic prime form $K(\tau, z)$, by [27]

$$K(\tau, z) = \exp\left(-P_0(\tau, z)\right). \tag{8}$$

Define elliptic functions $P_k(\tau, z)$ for $k \geq 3$

$$P_k(\tau, z) = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}P_1(\tau, z). \tag{9}$$

Define for $k, l \geq 1$

$$C(k,l) = C(k,l,\tau) = (-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}E_{k+l}(\tau), \tag{10}$$

$$D(k,l,z) = D(k,l,\tau,z) = (-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}P_{k+l}(\tau,z). \tag{11}$$

### 2.2 The ρ-Formalism for Self-sewing a Torus

Consider a compact Riemann surface $\mathcal{S}$ of genus 2 with standard homology basis $a_1, a_2, b_1, b_2$. Let

$$\omega(x,y) = \left(\frac{1}{(x-y)^2} + \text{regular terms}\right) dxdy \tag{12}$$

be the normalized differential of the second kind [4, 36] for local coordinates $x, y$ with normalization $\oint_{a_i} \omega(x,\cdot) = 0$ for $i = 1, 2$. Then

$$\nu_i(x) = \oint_{b_i} \omega(x,\cdot), \tag{13}$$

for $i = 1, 2$ is a basis of holomorphic 1-forms with normalization $\oint_{a_i} \nu_j = 2\pi \mathrm{i}\delta_{ij}$. The genus 2 period matrix $\Omega \in \mathbb{H}_2$ is defined by

$$\Omega_{ij} = \frac{1}{2\pi\mathrm{i}}\oint_{b_i} \nu_j. \tag{14}$$

We now review a general method due to Yamada [36], and discussed at length in [18], for calculating $\omega(x,y)$, $\nu_i(x)$ and $\Omega_{ij}$ on the Riemann surface formed by sewing a handle to an oriented torus $\mathcal{S} = \mathbb{C}/\Lambda$ with lattice $\Lambda = 2\pi\mathrm{i}(\mathbb{Z}\tau \oplus \mathbb{Z})$ and $\tau \in \mathbb{H}_1$. Consider discs centered at $z = 0$ and $z = w$ with local coordinates $z_1 = z$ and $z_2 = z - w$, and positive radius $r_a < \frac{1}{2}D(q)$ with $1 \leq a \leq 2$. Here, we have introduced the minimal lattice distance

$$D(q) = \min_{(m,n)\neq(0,0)} 2\pi|m + n\tau| > 0. \tag{15}$$

Note that $r_1, r_2$ must be sufficiently small to ensure that the discs do not intersect on $\mathcal{S}$. Introduce a complex parameter $\rho$ where $|\rho| \leq r_1 r_2$ and excise the discs $\{z_a, |z_a| \leq |\rho|/r_{\bar{a}}\}$ to obtain a twice-punctured torus (illustrated in Fig. 1)

$$\hat{\mathcal{S}} = \mathcal{S}\backslash\{z_a, |z_a| \leq |\rho|/r_{\bar{a}}\} \;\; (1 \leq a \leq 2).$$

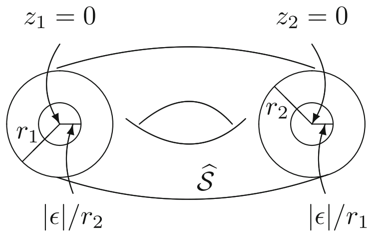


**Fig. 1** Self-sewing a torus

Here, and below, we use the convention

$$\overline{1} = 2, \qquad \overline{2} = 1. \tag{16}$$

Define annular regions $\mathcal{A}_a = \{z_a, |\rho| r_{\overline{a}}^{-1} \leq |z_a| \leq r_a\} \in \hat{\mathcal{S}}$ $(1 \leq a \leq 2)$, and identify $\mathcal{A}_1$ with $\mathcal{A}_2$ as a single region via the sewing relation

$$z_1 z_2 = \rho. \tag{17}$$

The resulting genus two Riemann surface (excluding the degeneration point $\rho = 0$) is parameterized by the domain

$$\mathcal{D}^{\rho} = \left\{(\tau, w, \rho) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} : \ |w - \lambda| > 2|\rho|^{1/2} > 0 \text{ for all } \lambda \in \Lambda\right\}, \tag{18}$$

where the first inequality follows from the requirement that the annuli do not intersect. The Riemann surface inherits the genus one homology basis $a_1, b_1$. The cycle $a_2$ is defined to be the anti-clockwise contour surrounding the puncture at $w$, and $b_2$ is a path between identified points $z_1 = z_0$ to $z_2 = \rho/z_0$ for some $z_0 \in \mathcal{A}_1$.

$\omega, \nu_i$ and $\Omega$ are expressed as a functions of $(\tau, w, \rho) \in \mathcal{D}^{\rho}$ in terms of an infinite matrix of $2 \times 2$ blocks $R(\tau, w, \rho) = (R(k, l, \tau, w, \rho))$ $(k, l \geq 1)$ where [18]

$$R(k, l, \tau, w, \rho) = -\frac{\rho^{(k+l)/2}}{\sqrt{kl}} \begin{pmatrix} D(k, l, \tau, w) & C(k, l, \tau) \\ C(k, l, \tau) & D(l, k, \tau, w) \end{pmatrix}, \tag{19}$$

for $C, D$ of (10) and (11). $I - R$ and $\det(I - R)$ play a central rôle in our discussion, where $I$ denotes the doubly-indexed identity matrix and $\det(I - R)$ is defined by

$$\log \det(I - R) = \operatorname{Tr} \log(I - R) = -\sum_{n \geq 1} \frac{1}{n} \operatorname{Tr} R^n. \tag{20}$$

In particular (op. cit., Proposition 6 and Theorem 7)

**Theorem 1.** *We have*

*(a)*

$$(I - R)^{-1} = \sum_{n\geq 0} R^n \tag{21}$$

*is convergent in $\mathcal{D}^{\rho}$.*

*(b)* $\det(I - R)$ *is nonvanishing and holomorphic in $\mathcal{D}^{\rho}$.* □

We define a set of 1-forms on $\hat{\mathcal{S}}$ given by

$$\begin{aligned} a_1(k,x) &= a_1(k,x,\tau,\rho) = \sqrt{k}\rho^{k/2}P_{k+1}(\tau,x)dx, \\ a_2(k,x) &= a_2(k,x,\tau,\rho) = a_1(k,x-w), \end{aligned} \tag{22}$$

indexed by integers $k \geq 1$. We also define the infinite row vector $a(x) = (a_a(k,x))$ and infinite column vector $\overline{a}(x)^T = (a_{\overline{a}}(k,x))^T$ for $k \geq 1$ and block index $1 \leq a \leq 2$. We find (op. cit., Lemma 11, Proposition 6 and Theorem 9):

**Theorem 2.**

$$\omega(x,y) = P_2(\tau,x-y)dxdy - a(x)(I-R)^{-1}\overline{a}(y)^T. \quad \square \tag{23}$$

Applying (13) results in (op. cit., Lemma 12 and Theorem 9)

**Theorem 3.**

$$\begin{aligned} \nu_1(x) &= dx - \rho^{1/2}\sigma\left((a(x)(I-R)^{-1})(1)\right) \\ \nu_2(x) &= (P_1(\tau,x-w) - P_1(\tau,x))\,dx - a(x)(I-R)^{-1}\overline{d}^T. \end{aligned} \tag{24}$$

*$d = (d_a(k))$ is a doubly-indexed infinite row vector*[3]

$$\begin{aligned} d_1(k) &= -\frac{\rho^{k/2}}{\sqrt{k}}\left(P_k(\tau,w) - E_k(\tau)\right), \\ d_2(k) &= (-1)^k\frac{\rho^{k/2}}{\sqrt{k}}\left(P_k(\tau,w) - E_k(\tau)\right), \end{aligned} \tag{25}$$

*with $\overline{d}_a = d_{\overline{a}}$. (1) refers to the $(k) = (1)$ entry of a row vector and $\sigma(M)$ denotes the sum over the finite block indices for a given $1 \times 2$ block matrix $M$.* □

$\Omega$ is determined (op. cit., Proposition 11) by (14) as follows:

[3]Note that $d$ is denoted by $\beta$ in [18].

**Theorem 4.** *There is a holomorphic map*

$$F^{\rho} : \mathcal{D}^{\rho} \to \mathbb{H}_2,$$
$$(\tau, w, \rho) \mapsto \Omega(\tau, w, \rho), \tag{26}$$

*where* $\Omega = \Omega(\tau, w, \rho)$ *is given by*

$$2\pi \mathrm{i} \Omega_{11} = 2\pi \mathrm{i}\tau - \rho\sigma \left((I - R)^{-1}(1, 1)\right), \tag{27}$$
$$2\pi \mathrm{i} \Omega_{12} = w - \rho^{1/2}\sigma \left(d(I - R)^{-1}(1)\right), \tag{28}$$
$$2\pi \mathrm{i} \Omega_{22} = \log\left(-\frac{\rho}{K(\tau, w)^2}\right) - d(I - R)^{-1}\overline{d}^T. \tag{29}$$

*$K$ is the elliptic prime form (8), $(1, 1)$ and $(1)$ refer to the $(k, l) = (1, 1)$, respectively, $(k) = (1)$ entries of an infinite matrix and row vector respectively. $\sigma(M)$ denotes the sum over the finite block indices for a given $2 \times 2$ or $1 \times 2$ block matrix $M$.* □

$\mathcal{D}^{\rho}$ admits an action of the Jacobi group $J = \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ as follows:

$$(a, b).(\tau, w, \rho) = (\tau, w + 2\pi \mathrm{i} a\tau + 2\pi \mathrm{i} b, \rho) \quad ((a, b) \in \mathbb{Z}^2), \tag{30}$$
$$\gamma_1.(\tau, w, \rho) = (\frac{a_1\tau + b_1}{c_1\tau + d_1}, \frac{w}{c_1\tau + d_1}, \frac{\rho}{(c_1\tau + d_1)^2}) \quad (\gamma_1 \in \Gamma_1), \tag{31}$$

with $\Gamma_1 = \left\{ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right\} = \mathrm{SL}(2, \mathbb{Z})$. Due to the branch structure of the logarithmic term in (29), $F^{\rho}$ is not equivariant with respect to $J$. (See Sect. 6.3 of [18] for details.)

There is a natural injection $\Gamma_1 \to \mathrm{Sp}(4, \mathbb{Z})$ defined by

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{32}$$

through which $\Gamma_1$ acts on $\mathbb{H}_2$ by the standard action

$$\gamma.\Omega = (A\Omega + B)(C\Omega + D)^{-1}, \quad \left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z})\right). \tag{33}$$

We then have (op. cit., Theorem 11, Corollary 2)

**Theorem 5.** *$F^{\rho}$ is equivariant with respect to the action of $\Gamma_1$, i.e. there is a commutative diagram for $\gamma_1 \in \Gamma_1$,*

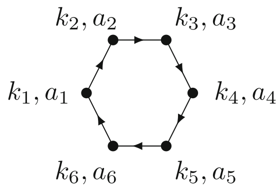


**Fig. 2** Doubly-indexed cycle

$$\begin{array}{ccc} \mathcal{D}^\rho & \stackrel{F^\rho}{\to} & \mathbb{H}_2 \\ \gamma_1 \downarrow & & \downarrow \gamma_1 \\ \mathcal{D}^\rho & \stackrel{F^\rho}{\to} & \mathbb{H}_2 \end{array} \qquad \square$$

## 2.3 Graphical Expansions

We present a graphical approach to describing the expressions for $\omega, \nu_i, \Omega_{ij}$ reviewed above. These also play an important rôle in the analysis of genus two partition functions for the Heisenberg vertex operator algebra. A similar approach is described in [20] suitable for the $\epsilon$-sewing scheme. Here we introduce *doubly-indexed* cycles construed as (clockwise) oriented, labelled polygons $L$ with $n$ nodes for some integer $n \geq 1$, nodes being labelled by a pair of integers $k, a$ where $k \geq 1$ and $a \in \{1, 2\}$. Thus, a typical doubly-indexed cycle looks as in Fig. 2.

We define a weight function[4] $\zeta$ with values in the ring of elliptic functions and quasi-modular forms $\mathbb{C}[P_2(\tau, w), P_3(\tau, w), E_2(\tau), E_4(\tau), E_6(\tau)]$ as follows: if $L$ is a doubly-indexed cycle then $L$ has edges $E$ labelled as $\stackrel{k,a}{\bullet} \to \stackrel{l,b}{\bullet}$, and we set

$$\zeta(E) = R_{ab}(k, l, \tau, w, \rho), \tag{34}$$

with $R_{ab}(k, l)$ as in (19) and

$$\zeta(L) = \prod \zeta(E),$$

where the product is taken over all edges of $L$.

We also introduce *doubly-indexed necklaces* $\mathcal{N} = \{N\}$. These are connected graphs with $n \geq 2$ nodes, $(n-2)$ of which have valency 2 and two of which have valency 1 together with an orientation, say from left to right, on the edges. In this case, each vertex carries two integer labels $k, a$ with $k \geq 1$ and $a \in \{1, 2\}$. We define the degenerate necklace $N_0$ to be a single node with no edges, and set $\zeta(N_0) = 1$.

[4]Denoted by $\omega$ in Sect. 6.2 of [18].

We define necklaces with distinguished end nodes labelled $k, a; l, b$ as follows:

$$\underset{k,a}{\bullet} \longrightarrow \underset{k_1,a_1}{\bullet} \ldots \underset{k_2,a_2}{\bullet} \longrightarrow \underset{l,b}{\bullet} \qquad \text{(type } k, a; l, b)$$

and set[5]

$$\mathcal{N}(k, a; l, b) = \{\text{isomorphism classes of necklaces of type } k, a; l, b\}. \tag{35}$$

We define

$$\begin{aligned}
\zeta(1; 1) &= \sum_{a_1,a_2=1,2} \sum_{N\in\mathcal{N}(1,a_1;1,a_2)} \zeta(N), \\
\zeta(d; 1) &= \sum_{a_1,a_2=1,2} \sum_{k\geq 1} d_{a_1}(k) \sum_{N\in\mathcal{N}(k,a_1;1,a_2)} \zeta(N), \\
\zeta(d; \overline{d}) &= \sum_{a_1,a_2=1,2} \sum_{k,l\geq 1} d_{a_1}(k)\overline{d}_{a_2}(l) \sum_{N\in\mathcal{N}(k,a_1;l,a_2)} \zeta(N).
\end{aligned} \tag{36}$$

Then we find

**Proposition 1 ([18], Proposition 12).** *The period matrix is given by*

$$\begin{aligned}
2\pi i\Omega_{11} &= 2\pi i\tau - \rho\zeta(1; 1), \\
2\pi i\Omega_{12} &= w - \rho^{1/2}\zeta(d; 1), \\
2\pi i\Omega_{22} &= \log\left(-\frac{\rho}{K(\tau, w)^2}\right) - \zeta(d; \overline{d}). \quad \square
\end{aligned}$$

We can similarly obtain necklace graphical expansions for the bilinear form $\omega(x, y)$ and the holomorphic one forms $\nu_i(x)$. We introduce further distinguished valence one nodes labelled by $x \in \hat{\mathcal{S}}$, the punctured torus. The set of edges $\{E\}$ is augmented by edges with weights defined by:

$$\begin{aligned}
\zeta(\overset{x}{\bullet} \longrightarrow \overset{y}{\bullet}) &= P_2(\tau, x - y)dxdy, \\
\zeta(\overset{x}{\bullet} \longrightarrow \overset{k,a}{\bullet}) &= a_a(k, x), \\
\zeta(\overset{k,a}{\bullet} \longrightarrow \overset{y}{\bullet}) &= -a_{\overline{a}}(k, y),
\end{aligned} \tag{37}$$

for 1-forms (22).

We also consider doubly-indexed necklaces where one or both end points are $x, y$-labeled nodes. We thus define for $x, y \in \hat{\mathcal{S}}$ two isomorphism classes of oriented

[5] Two graphs are isomorphic if they have the same labelled vertices and directed edges.

doubly-indexed necklaces denoted by $\mathcal{N}(x; y)$, and $\mathcal{N}(x; k, a)$ with the following respective typical configurations

$$\{\overset{x}{\bullet} \longrightarrow \overset{k_1,a_1}{\bullet} \ldots \overset{k_2,a_2}{\bullet} \longrightarrow \overset{y}{\bullet}\}, \tag{38}$$

$$\{\overset{x}{\bullet} \longrightarrow \overset{k_1,a_1}{\bullet} \ldots \overset{k_2,a_2}{\bullet} \longrightarrow \overset{k,a}{\bullet}\}. \tag{39}$$

Furthermore, we define the weights

$$\begin{aligned}
\zeta(x; y) &= \sum_{N \in \mathcal{N}(x;y)} \zeta(N), \\
\zeta(x; 1) &= \sum_{a=1,2} \sum_{N \in \mathcal{N}(x;1,a)} \zeta(N), \\
\zeta(x; \overline{d}) &= \sum_{a=1,2} \sum_{k \geq 1} \sum_{N \in \mathcal{N}(x;k,a)} \zeta(N) \overline{d}_a(k).
\end{aligned} \tag{40}$$

Comparing to (23) and (24) we find the following graphical expansions for the bilinear form $\omega(x, y)$ and the holomorphic one forms $\nu_i(x)$

**Proposition 2.** *For* $x, y \in \hat{\mathcal{S}}$

$$\omega(x, y) = \zeta(x; y), \tag{41}$$

$$\nu_1(x) = dx - \rho^{1/2} \zeta(x; 1), \tag{42}$$

$$\nu_2(x) = (P_1(\tau, x - w) - P_1(\tau, x))dx - \zeta(x; \overline{d}). \tag{43}$$

# 3 Vertex Operator Algebras and the Li-Zamolodchikov Metric

## *3.1 Vertex Operator Algebras*

We review some relevant aspects of vertex operator algebras [8, 9, 13, 15, 22, 23]. A vertex operator algebra (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of a $\mathbb{Z}$-graded complex vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, a linear map $Y : V \to (\mathrm{End}\, V)[[z, z^{-1}]]$, for formal parameter $z$, and a pair of distinguished vectors (states), the vacuum $\mathbf{1} \in V_0$, and the conformal vector $\omega \in V_2$. For each state $v \in V$ the image under the $Y$ map is the vertex operator

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}, \tag{44}$$

with modes $v(n) \in \mathrm{End}\,V$ where $\mathrm{Res}_{z=0}z^{-1}Y(v,z)\mathbf{1} = v(-1)\mathbf{1} = v$. Vertex operators satisfy the Jacobi identity or equivalently, operator locality or Borcherds's identity for the modes (loc. cit.).

The vertex operator for the conformal vector $\omega$ is defined as

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}.$$

The modes $L(n)$ satisfy the Virasoro algebra of central charge $c$:

$$[L(m), L(n)] = (m-n)L(m+n) + (m^3 - m)\frac{c}{12}\delta_{m,-n}.$$

We define the homogeneous space of weight $k$ to be $V_k = \{v \in V | L(0)v = kv\}$ where we write $wt(v) = k$ for $v$ in $V_k$. Then as an operator on $V$ we have

$$v(n) : V_m \to V_{m+k-n-1}.$$

In particular, the *zero mode* $o(v) = v(wt(v) - 1)$ is a linear operator on $V_m$. A non-zero vector $v$ is said to be *quasi-primary* if $L(1)v = 0$ and *primary* if additionally $L(2)v = 0$.

The subalgebra $\{L(-1), L(0), L(1)\}$ generates a natural action on vertex operators associated with $\mathrm{SL}(2,\mathbb{C})$ Möbius transformations [2, 3, 9, 13]. In particular, we note the inversion $z \mapsto 1/z$, for which

$$Y(v,z) \mapsto Y^{\dagger}(v,z) = Y\left(e^{zL(1)}\left(-\frac{1}{z^2}\right)^{L(0)} v, \frac{1}{z}\right). \tag{45}$$

$Y^{\dagger}(v,z)$ is the *adjoint* vertex operator [9].

We consider in particular the Heisenberg free boson VOA and lattice VOAs. Consider an $l$-dimensional complex vector space (i.e., abelian Lie algebra) $\mathfrak{H}$ equipped with a non-degenerate, symmetric, bilinear form $(\ ,\ )$ and a distinguished orthonormal basis $a_1, a_2, \ldots a_l$. The corresponding affine Lie algebra is the Heisenberg Lie algebra $\hat{\mathfrak{H}} = \mathfrak{H} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$ with brackets $[k, \hat{\mathfrak{H}}] = 0$ and

$$[a_i \otimes t^m, a_j \otimes t^n] = m\delta_{i,j}\delta_{m,-n}k. \tag{46}$$

Corresponding to an element $\lambda$ in the dual space $\mathfrak{H}^*$ we consider the Fock space defined by the induced (Verma) module

$$M^{(\lambda)} = U(\hat{\mathfrak{H}}) \otimes_{U(\mathfrak{H}\otimes\mathbb{C}[t]\oplus\mathbb{C}k)} \mathbb{C},$$

where $\mathbb{C}$ is the one-dimensional space annihilated by $\mathfrak{H} \otimes t\mathbb{C}[t]$ and on which $k$ acts as the identity and $\mathfrak{H} \otimes t^0$ via the character $\lambda$; $U$ denotes the universal enveloping algebra. There is a canonical identification of linear spaces

$$M^{(\lambda)} = S(\mathfrak{H} \otimes t^{-1}\mathbb{C}[t^{-1}]),$$

where $S$ denotes the (graded) symmetric algebra. The Heisenberg free boson VOA $M^l$ corresponds to the case $\lambda = 0$. The Fock states

$$v = a_1(-1)^{e_1}.a_1(-2)^{e_2}\ldots.a_1(-n)^{e_n}\ldots.a_l(-1)^{f_1}.a_l(-2)^{f_2}\ldots a_l(-p)^{f_p}.\mathbf{1}, \quad (47)$$

for non-negative integers $e_i,\ldots,f_j$ form a basis of $M^l$. The vacuum $\mathbf{1}$ is canonically identified with the identity of $M^l_0 = \mathbb{C}$, while the weight 1 subspace $M^l_1$ may be naturally identified with $\mathfrak{H}$. $M^l$ is a simple VOA of central charge $l$.

Next we consider the case of a lattice vertex operator algebra $V_L$ associated to a positive-definite even lattice $L$ (cf. [2, 8]). Thus $L$ is a free abelian group of rank $l$ equipped with a positive definite, integral bilinear form $(\,,\,) : L \otimes L \to \mathbb{Z}$ such that $(\alpha,\alpha)$ is even for $\alpha \in L$. Let $\mathfrak{H}$ be the space $\mathbb{C} \otimes_{\mathbb{Z}} L$ equipped with the $\mathbb{C}$-linear extension of $(\,,\,)$ to $\mathfrak{H} \otimes \mathfrak{H}$ and let $M^l$ be the corresponding Heisenberg VOA. The Fock space of the lattice theory may be described by the linear space

$$V_L = M^l \otimes \mathbb{C}[L] = \sum_{\alpha\in L} M^l \otimes \mathrm{e}^{\alpha}, \quad (48)$$

where $\mathbb{C}[L]$ denotes the group algebra of $L$ with canonical basis $\mathrm{e}^{\alpha}$, $\alpha \in L$. $M^l$ may be identified with the subspace $M^l \otimes \mathrm{e}^0$ of $V_L$, in which case $M^l$ is a subVOA of $V_L$ and the rightmost equation of (48) then displays the decomposition of $V_L$ into irreducible $M^l$-modules. $V_L$ is a simple VOA of central charge $l$. Each $\mathbf{1}\otimes \mathrm{e}^{\alpha} \in V_L$ is a primary state of weight $\frac{1}{2}(\alpha,\alpha)$ with vertex operator (loc. cit.)

$$\begin{aligned} Y(\mathbf{1}\otimes \mathrm{e}^{\alpha}, z) &= Y_-(\alpha,z)Y_+(\alpha,z)\mathrm{e}^{\alpha}z^{\alpha}, \\ Y_{\pm}(\alpha,z) &= \exp\left(\mp\sum_{n>0}\frac{\alpha(\pm n)}{n}z^{\mp n}\right). \end{aligned} \quad (49)$$

The operators $\mathrm{e}^{\alpha} \in \mathbb{C}[L]$ obey

$$\mathrm{e}^{\alpha}\mathrm{e}^{\beta} = \epsilon(\alpha,\beta)\mathrm{e}^{\alpha+\beta} \quad (50)$$

for a bilinear 2-cocycle $\epsilon(\alpha,\beta)$ satisfying $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{(\alpha,\beta)}$.

## 3.2 The Li-Zamolodchikov Metric

A bilinear form $\langle\,,\,\rangle : V \times V \longrightarrow \mathbb{C}$ is called *invariant* in case the following identity holds for all $a,b,c \in V$ [9]:

$$\langle Y(a,z)b, c\rangle = \left\langle b, Y^{\dagger}(a,z)c\right\rangle, \quad (51)$$

with $Y^{\dagger}(a,z)$ the adjoint operator (45). If $V_0 = \mathbb{C}\mathbf{1}$ and $V$ is self-dual (i.e. $V$ is isomorphic to the contragradient module $V'$ as a $V$-module) then $V$ has a unique non-zero invariant bilinear form up to scalar [16]. Note that $\langle\ ,\rangle$ is necessarily symmetric by a theorem of [9]. Furthermore, if $V$ is simple then such a form is necessarily non-degenerate. All of the VOAs that occur in this paper satisfy these conditions, so that normalizing $\langle \mathbf{1},\mathbf{1}\rangle = 1$ implies that $\langle\ ,\rangle$ is unique. We refer to such a bilinear form as the *Li-Zamolodchikov metric* on $V$, or LiZ-metric for short [20]. We also note that the LiZ-metric is multiplicative over tensor products in the sense that LiZ metric of the tensor product $V_1 \otimes V_2$ of a pair of simple VOAs satisfying the above conditions is by uniqueness, the tensor product of the LiZ metrics on $V_1$ and $V_2$.

For a quasi-primary vector $a$ of weight $wt(a)$, the component form of (51) becomes

$$\langle a(n)b,c\rangle = (-1)^{wt(a)}\langle b, a(2wt(a)-n-2)c\rangle\,. \tag{52}$$

In particular, for the conformal vector $\omega$ we obtain

$$\langle L(n)b,c\rangle = \langle b, L(-n)c\rangle. \tag{53}$$

Taking $n = 0$, it follows that the homogeneous spaces $V_n$ and $V_m$ are orthogonal if $n \neq m$.

Consider the rank one Heisenberg VOA $M = M^1$ generated by a weight one state $a$ with $(a,a) = 1$. Then $\langle a,a\rangle = -\langle \mathbf{1}, a(1)a(-1)\mathbf{1}\rangle = -1$. Using (46), it is straightforward to verify that the Fock basis (47) is orthogonal with respect to the LiZ-metric and

$$\langle v,v\rangle = \prod_{1\le i\le n}(-i)^{e_i}e_i!. \tag{54}$$

This result generalizes in an obvious way to the rank $l$ free boson VOA $M^l$ because the LiZ metric is multiplicative over tensor products.

We consider next the lattice vertex operator algebra $V_L$ for a positive-definite even lattice $L$. We take as our Fock basis the states $\{v\otimes \mathrm{e}^{\alpha}\}$ where $v$ is as in (47) and $\alpha$ ranges over the elements of $L$.

**Lemma 1.** *If $u,v \in M^l$ and $\alpha,\beta \in L$, then*

$$\begin{aligned}\left\langle u\otimes \mathrm{e}^{\alpha}, v\otimes \mathrm{e}^{\beta}\right\rangle &= \langle u,v\rangle\langle \mathbf{1}\otimes \mathrm{e}^{\alpha}, \mathbf{1}\otimes \mathrm{e}^{\beta}\rangle\\ &= (-1)^{\frac{1}{2}(\alpha,\alpha)}\epsilon(\alpha,-\alpha)\langle u,v\rangle\delta_{\alpha,-\beta}.\end{aligned}$$

*Proof.* It follows by successive applications of (52) that the first equality in the lemma is true, and that it is therefore enough to prove it in the case that $u = v = \mathbf{1}$. We identify the primary vector $\mathbf{1}\otimes \mathrm{e}^{\alpha}$ with $\mathrm{e}^{\alpha}$ in the following. Then $\left\langle \mathrm{e}^{\alpha}, \mathrm{e}^{\beta}\right\rangle = \left\langle \mathrm{e}^{\alpha}(-1)\mathbf{1}, \mathrm{e}^{\beta}\right\rangle$ is given by

$$
\begin{aligned}
&(-1)^{\frac{1}{2}(\alpha,\alpha)}\left\langle \mathbf{1}, \mathrm{e}^{\alpha}((\alpha,\alpha)-1)\mathrm{e}^{\beta}\right\rangle \\
&= (-1)^{\frac{1}{2}(\alpha,\alpha)}\,\mathrm{Res}_{z=0} z^{(\alpha,\alpha)-1}\left\langle \mathbf{1}, Y(\mathrm{e}^{\alpha},z)\mathrm{e}^{\beta}\right\rangle \\
&= (-1)^{\frac{1}{2}(\alpha,\alpha)}\epsilon(\alpha,\beta)\,\mathrm{Res}_{z=0} z^{(\alpha,\beta)+(\alpha,\alpha)-1}\left\langle \mathbf{1}, Y_{-}(\alpha,z).\mathrm{e}^{\alpha+\beta}\right\rangle .
\end{aligned}
$$

Unless $\alpha+\beta=0$, all states to the left inside the bracket $\langle\ ,\rangle$ on the previous line have positive weight, hence are orthogonal to $\mathbf{1}$. So $\langle \mathrm{e}^{\alpha}, \mathrm{e}^{\beta}\rangle = 0$ if $\alpha+\beta\neq 0$. In the contrary case, the exponential operator acting on the vacuum yields just the vacuum itself among weight zero states, and we get $\langle \mathrm{e}^{\alpha}, \mathrm{e}^{-\alpha}\rangle = (-1)^{\frac{1}{2}(\alpha,\alpha)}\epsilon(\alpha,-\alpha)$ in this case. □

**Corollary 1.** *We may choose the cocycle so that $\epsilon(\alpha,-\alpha)=(-1)^{\frac{1}{2}(\alpha,\alpha)}$ (cf. (132) in Appendix). In this case, we have*

$$
\left\langle u\otimes \mathrm{e}^{\alpha}, v\otimes \mathrm{e}^{\beta}\right\rangle = \langle u,v\rangle \delta_{\alpha,-\beta}. \tag{55}
$$

# 4 Partition and *n*-Point Functions for Vertex Operator Algebras on a Genus Two Riemann Surface

In this section we consider the partition and $n$-point functions for a VOA on Riemann surface of genus one or two, formed by attaching a handle to a surface of lower genus. We assume that $V$ has a non-degenerate LiZ metric $\langle\ ,\rangle$. Then for any $V$ basis $\{u^{(a)}\}$, we may define the *dual basis* $\{\overline{u}^{(a)}\}$ with respect to the LiZ metric where

$$
\left\langle u^{(a)}, \overline{u}^{(b)}\right\rangle = \delta_{ab}. \tag{56}
$$

## *4.1 Genus One*

It is instructive to first consider an alternative approach to defining the genus one partition function. In order to define $n$-point correlation functions on a torus, Zhu introduced [37] a second VOA $(V, Y[\ ,\ ], \mathbf{1}, \tilde{\omega})$ isomorphic to $(V, Y(\ ,\ ), \mathbf{1}, \omega)$ with vertex operators

$$
Y[v,z] = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1} = Y\left(q_z^{L(0)}v, q_z-1\right), \tag{57}
$$

and conformal vector $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. Let

$$
Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n]z^{-n-2}, \tag{58}
$$

and write $wt[v] = k$ if $L[0]v = kv$, $V_{[k]} = \{v \in V | wt[v] = k\}$. Similarly, we define the square bracket LiZ metric $\langle\ ,\ \rangle_{\mathrm{sq}}$ which is invariant with respect to the square bracket adjoint.

The (genus one) 1-point function is now defined as

$$Z_V^{(1)}(v, \tau) = \mathrm{Tr}_V \left( o(v) q^{L(0)-c/24} \right). \tag{59}$$

An $n$-point function can be expressed in terms of 1-point functions [17, Lemma 3.1] as follows:

$$\begin{aligned} & Z_V^{(1)}(v_1, z_1; \ldots v_n, z_n; \tau) \\ &= Z_V^{(1)}(Y[v_1, z_1] \ldots Y[v_{n-1}, z_{n-1}] Y[v_n, z_n] \mathbf{1}, \tau) \qquad (60) \\ &= Z_V^{(1)}(Y[v_1, z_{1n}] \ldots Y[v_{n-1}, z_{n-1 n}] v_n, \tau), \qquad (61) \end{aligned}$$

where $z_{in} = z_i - z_n$ $(1 \leq i \leq n-1)$. In particular, $Z_V^{(1)}(v_1, z_1; v_2, z_2; \tau)$ depends only one $z_{12}$, and we denote this 2-point function by

$$\begin{aligned} Z_V^{(1)}(v_1, v_2, z_{12}, \tau) &= Z_V^{(1)}(v_1, z_1; v_2, z_2; \tau) \\ &= \mathrm{Tr}_V \left( o(Y[v_1, z_{12}] v_2) q^{L(0)} \right). \qquad (62) \end{aligned}$$

Now consider a torus obtained by self-sewing a Riemann sphere with punctures located at the origin and an arbitrary point $w$ on the complex plane (cf. [18, Sect. 5.2.2]). Choose local coordinates $z_1$ in the neighborhood of the origin and $z_2 = z - w$ for $z$ in the neighborhood of $w$. For a complex sewing parameter $\rho$, identify the annuli $|\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a$ for $1 \leq a \leq 2$ and $|\rho| \leq r_1 r_2$ via the sewing relation

$$z_1 z_2 = \rho. \tag{63}$$

Define

$$\chi = -\frac{\rho}{w^2}. \tag{64}$$

Then the annuli do not intersect provided $|\chi| < \frac{1}{4}$, and the torus modular parameter is

$$q = f(\chi), \tag{65}$$

where $f(\chi)$ is the Catalan series

$$f(\chi) = \frac{1 - \sqrt{1 - 4\chi}}{2\chi} - 1 = \sum_{n \geq 1} \frac{1}{n} \binom{2n}{n+1} \chi^n. \tag{66}$$

$f = f(\chi)$ satisfies $f = \chi(1+f)^2$ and the following identity, which can be proved by induction on $m$:

**Lemma 2.** *$f(\chi)$ satisfies*

$$f(\chi)^m = \sum_{n\geq m} \frac{m}{n}\binom{2n}{n+m}\chi^n \quad (m \geq 1). \quad \square$$

We now define the genus one partition function in the $\rho$-sewing scheme (63) by

$$Z^{(1)}_{V,\rho}(\rho, w) = \sum_{n\geq 0}\rho^n \sum_{u\in V_n} \mathrm{Res}_{z_2=0} z_2^{-1} \mathrm{Res}_{z_1=0} z_1^{-1} \left\langle \mathbf{1}, Y(u, w+z_2)Y(\overline{u}, z_1)\mathbf{1}\right\rangle, \tag{67}$$

where the inner sum is taken over any basis for $V_n$. This partition function is directly related to the standard one $Z^{(1)}_V(q) = \mathrm{Tr}_V\left(q^{L(0)-c/24}\right)$ as follows:

**Theorem 6.** *In the sewing scheme (63), we have*

$$Z^{(1)}_{V,\rho}(\rho, w) = q^{c/24} Z^{(1)}_V(q), \tag{68}$$

*where $q = f(\chi)$ is given by (65).*

*Proof.* The summand in (67) for $u \in V_n$ is

$$\begin{aligned}\langle \mathbf{1}, Y(u,w)\overline{u}\rangle &= \left\langle Y^\dagger(u,w)\mathbf{1}, \overline{u}\right\rangle \\ &= (-w^{-2})^n \left\langle Y(e^{wL(1)}u, w^{-1})\mathbf{1}, \overline{u}\right\rangle \\ &= (-w^{-2})^n \left\langle e^{w^{-1}L(-1)} e^{wL(1)} u, \overline{u}\right\rangle,\end{aligned}$$

where we have used (45) and $Y(v,z)\mathbf{1} = \exp(zL(-1))v$ (e.g [13, 22, 23]). Hence we find that

$$\begin{aligned}Z^{(1)}_{V,\rho}(\rho, w) &= \sum_{n\geq 0}\left(-\frac{\rho}{w^2}\right)^n \sum_{u\in V_n}\left\langle e^{w^{-1}L(-1)}e^{wL(1)}u, \overline{u}\right\rangle \\ &= \sum_{n\geq 0}\chi^n \,\mathrm{Tr}_{V_n}\left(e^{w^{-1}L(-1)}e^{wL(1)}\right).\end{aligned}$$

Expanding the exponentials yields

$$Z^{(1)}_{V,\rho}(\rho, w) = \mathrm{Tr}_V\left(\sum_{r\geq 0}\frac{L(-1)^r L(1)^r}{(r!)^2}\chi^{L(0)}\right), \tag{69}$$

an expression which depends only on $\chi$.

In order to compute (69) we consider the quasi-primary decomposition of $V$. Let $Q_m = \{v \in V_m | L(1)v = 0\}$ denote the space of quasiprimary states of weight $m \geq 1$. Then $\dim Q_m = p_m - p_{m-1}$ with $p_m = \dim V_m$. Consider the decomposition of $V$ into $L(-1)$-descendents of quasi-primaries

$$V_n = \bigoplus_{m=1}^{n} L(-1)^{n-m} Q_m. \tag{70}$$

**Lemma 3.** *Let $v \in Q_m$ for $m \geq 1$. For an integer $n \geq m$,*

$$\sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m} v = \binom{2n-1}{n-m} L(-1)^{n-m} v.$$

*Proof.* First use induction on $t \geq 0$ to show that

$$L(1)L(-1)^t v = t(2m+t-1)L(-1)^{t-1} v.$$

Then by induction in $r$ it follows that

$$\frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m} v = \binom{n-m}{r}\binom{n+m-1}{r} L(-1)^{n-m} v.$$

Hence

$$\begin{aligned}\sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m} v &= \sum_{r \geq 0}^{n-m} \binom{n-m}{r}\binom{n+m-1}{r} L(-1)^{n-m} v,\\ &= \binom{2n-1}{n-m} L(-1)^{n-m} v,\end{aligned}$$

where the last equality follows from a comparison of the coefficient of $x^{n-m}$ in the identity $(1+x)^{n-m}(1+x)^{n+m-1} = (1+x)^{2n-1}$. □

Lemma 3 and (70) imply that for $n \geq 1$,

$$\begin{aligned}\mathrm{Tr}_{V_n}\left(\sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2}\right) &= \sum_{m=1}^{n} \mathrm{Tr}_{Q_m}\left(\sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m}\right)\\ &= \sum_{m=1}^{n} (p_m - p_{m-1})\binom{2n-1}{n-m}.\end{aligned}$$

The coefficient of $p_m$ is

$$\binom{2n-1}{n-m} - \binom{2n-1}{n-m-1} = \frac{m}{n}\binom{2n}{m+n},$$

and hence

$$\mathrm{Tr}_{V_n}\left(\sum_{r\geq 0}\frac{L(-1)^r L(1)^r}{(r!)^2}\right) = \sum_{m=1}^{n}\frac{m}{n}\binom{2n}{m+n}p_m.$$

Using Lemma 2, we find that

$$\begin{aligned}
Z^{(1)}_{V,\rho}(\rho,w) &= 1+\sum_{n\geq 1}\chi^n\sum_{m=1}^{n}\frac{m}{n}\binom{2n}{m+n}p_m,\\
&= 1+\sum_{m\geq 1}p_m\sum_{n\geq m}\frac{m}{n}\binom{2n}{m+n}\chi^n\\
&= 1+\sum_{m\geq 1}p_m(f(\chi))^m\\
&= \mathrm{Tr}_V\left(f(\chi)^{L(0)}\right),
\end{aligned}$$

and Theorem 6 follows. □

## 4.2 Genus Two

We now turn to the case of genus two. Following Sect. 2.2, we employ the $\rho$-sewing scheme to self-sew a torus $\mathcal{S}$ with modular parameter $\tau$ via the sewing relation (17). For $x_1,\ldots,x_n \in \mathcal{S}$ with $|x_i| \geq |\epsilon|/r_2$ and $|x_i - w| \geq |\epsilon|/r_1$, we define the genus two $n$-point function in the $\rho$-formalism by

$$Z^{(2)}_V(v_1,x_1;\ldots v_n,x_n;\tau,w,\rho) = \sum_{r\geq 0}\rho^r\sum_{u\in V_{[r]}}\mathrm{Res}_{z_1=0}z_1^{-1}\mathrm{Res}_{z_2=0}z_2^{-1}Z^{(1)}_V(\overline{u},w+z_2;v_1,x_1;\ldots v_n,x_n;u,z_1;\tau), \quad (71)$$

where the inner sum is taken over any basis for $V_{[r]}$. In particular, with the notation (62), the genus two partition function is

$$Z_V^{(2)}(\tau, w, \rho) = \sum_{n\geq 0} \rho^n \sum_{u\in V_{[n]}} Z_V^{(1)}(\overline{u}, u, w, \tau) \,. \tag{72}$$

Next we consider $Z_V^{(2)}(\tau, w, \rho)$ in the two-tori degeneration limit. Define, much as in (64),

$$\chi = -\frac{\rho}{w^2}, \tag{73}$$

where $w$ denotes a point on the torus and $\rho$ is the genus two sewing parameter. Then one finds that the two-tori degeneration limit is given by $\rho, w \to 0$ for fixed $\chi$, where

$$\Omega \to \begin{pmatrix} \tau & 0 \\ 0 & \frac{1}{2\pi \mathrm{i}} \log(f(\chi)) \end{pmatrix} \tag{74}$$

and $f(\chi)$ is the Catalan series (66) (cf. [18, Sect. 6.4]).

**Theorem 7.** *For fixed* $|\chi| < \frac{1}{4}$, *we have*

$$\lim_{w,\rho\to 0} Z_V^{(2)}(\tau, w, \rho) = f(\chi)^{c/24} Z_V^{(1)}(q) Z_V^{(1)}(f(\chi)).$$

*Proof.* By (62) we have

$$Z_V^{(1)}(\overline{u}, u, w, \tau) = \mathrm{Tr}_V \left( o(Y[\overline{u}, w]u) q^{L(0)} \right),$$

where $u \in V_{[n]}$. Using the non-degeneracy of the LiZ metric $\langle\ ,\ \rangle_{\mathrm{sq}}$ in the square bracket formalism we obtain

$$Y[\overline{u}, w]u = \sum_{m\geq 0} \sum_{v\in V_{[m]}} \langle \overline{v}, Y[\overline{u}, w]u\rangle_{\mathrm{sq}}\, v,$$

summing over any basis for $V_{[m]}$. Arguing much as in the first part of the proof of Theorem 6, we also find

$$\begin{aligned} \langle \overline{v}, Y[\overline{u}, w]u\rangle_{\mathrm{sq}} &= (-w^{-2})^n \left\langle Y[\mathrm{e}^{wL[1]}\overline{u}, w^{-1}]\overline{v}, u\right\rangle_{\mathrm{sq}} \\ &= (-w^{-2})^n \left\langle \mathrm{e}^{w^{-1}L[-1]} Y[\overline{v}, -w^{-1}]\mathrm{e}^{wL[1]}\overline{u}, u\right\rangle_{\mathrm{sq}} \\ &= (-w^{-2})^n \left\langle E[\overline{v}, w]\overline{u}, u\right\rangle_{\mathrm{sq}}, \end{aligned}$$

where

$$E[\overline{v}, w] = \exp\left(w^{-1}L[-1]\right) Y[\overline{v}, -w^{-1}] \exp(wL[1]).$$

Hence

$$Z_V^{(2)}(\tau, w, \rho) = \sum_{m\geq 0} \sum_{v\in V_{[m]}} \sum_{n\geq 0} \chi^n \sum_{u\in V[n]} \langle E[\overline{v}, w]\overline{u}, u\rangle \, Z_V^{(1)}(v, q)$$
$$= \sum_{m\geq 0} \sum_{v\in V_{[m]}} \mathrm{Tr}_V \left(E[\overline{v}, w]\chi^{L[0]}\right) Z_V^{(1)}(v, q).$$

Now consider

$$\mathrm{Tr}_V \left(E[\overline{v}, w]\chi^{L[0]}\right) =$$
$$w^m \sum_{r,s\geq 0} (-1)^{r+m} \frac{1}{r!s!} \mathrm{Tr}_V \left(L[-1]^r \overline{v}[r-s-m-1] L[1]^s \chi^{L[0]}\right).$$

The leading term in $w$ is $w^0$ (arising from $\overline{v} = \mathbf{1}$) and is given by

$$\mathrm{Tr}_V \left(E[\mathbf{1}, w]\chi^{L[0]}\right) = f(\chi)^{c/24} Z_V^{(1)}(f(\chi)).$$

This follows from (69) and the isomorphism between the original and square bracket formalisms. Taking $w \to 0$ for fixed $\chi$ the result follows. □

## 5 The Heisenberg VOA

In this section we compute the genus two partition function in the $\rho$-formalism for the rank $l = 1$ Heisenberg VOA $M$. We also compute the genus two $n$-point function for $n$ copies of the Heisenberg vector $a$ and the genus two one-point function for the Virasoro vector $\omega$. The main results mirror those obtained in the $\epsilon$-formalism in Sect. 6 of [20].

### 5.1 The Genus Two Partition Function $Z_M^{(2)}(\tau, w, \rho)$

We begin by establishing a formula for $Z_M^{(2)}(\tau, w, \rho)$ in terms of the infinite matrix $R$ (19). Recalling that the genus zero partition function is $Z_M^{(1)}(\tau) = 1/\eta(\tau)$ where $\eta(\tau) = q^{1/24} \prod_{n\geq 1}(1 - q^n)$ is the Dedekind $\eta$-function, we find

**Theorem 8.** *We have*

$$Z_M^{(2)}(\tau, w, \rho) = \frac{Z_M^{(1)}(\tau)}{\det(1 - R)^{1/2}}. \quad (75)$$

*Remark 1.* From Remark 2 of [20] it follows that the genus two partition function for $l$ free bosons $M^l$ is just the $l$th power of (75).

*Proof.* The proof is similar in structure to that of Theorem 5 of [20]. From (72) we have

$$Z_M^{(2)}(\tau, w, \rho) = \sum_{n\geq 0} \sum_{u\in M_{[n]}} Z_M^{(1)}(u, \overline{u}, w, \tau)\,\rho^n, \tag{76}$$

where $u$ ranges over any basis of $M_{[n]}$ and $\overline{u}$ is the dual state with respect to the square-bracket LiZ metric. $Z_M^{(1)}(u, v, w, \tau)$ is a genus one Heisenberg 2-point function (62). We choose the square bracket Fock basis:

$$v = a[-1]^{e_1} \ldots a[-p]^{e_p}\mathbf{1}. \tag{77}$$

The Fock state $v$ naturally corresponds to an unrestricted partition $\lambda = \{1^{e_1} \ldots p^{e_p}\}$ of $n = \sum_{1\leq i\leq p} i e_i$. We write $v = v(\lambda)$ to indicate this correspondence. The Fock vectors form an orthogonal set from (54) with

$$\overline{v}(\lambda) = \frac{1}{\prod_{1\leq i\leq p}(-i)^{e_i} e_i!} v(\lambda).$$

The 2-point function $Z_M^{(1)}(v(\lambda), v(\lambda), w, \tau)$ is given in Corollary 1 of [17] where it is denoted by $F_M(v, w_1, v, w_2; \tau)$. In order to describe this explicitly we introduce the set $\Phi_{\lambda,2}$ which is the disjoint union of two isomorphic label sets $\Phi_\lambda^{(1)}$, $\Phi_\lambda^{(2)}$ each with $e_i$ elements labelled $i$ determined by $\lambda$. Let $\iota : \Phi_\lambda^{(1)} \leftrightarrow \Phi_\lambda^{(2)}$ denote the canonical label identification. Then we have (loc. cit.)

$$Z_M^{(1)}(v(\lambda), v(\lambda), w, \tau) = Z_M^{(1)}(\tau) \sum_{\phi\in F(\Phi_{\lambda,2})} \Gamma(\phi), \tag{78}$$

where

$$\Gamma(\phi, w, \tau) = \Gamma(\phi) = \prod_{\{r,s\}} \xi(r, s, w, \tau), \tag{79}$$

and $\phi$ ranges over the elements of $F(\Phi_{\lambda,2})$, the fixed-point-free involutions in $\Sigma(\Phi_{\lambda,2})$ and where $\{r, s\}$ ranges over the orbits of $\phi$ on $\Phi_{\lambda,2}$. Finally

$$\xi(r, s) = \xi(r, s, w, \tau) = \begin{cases} C(r, s, \tau), & \text{if } \{r, s\} \subseteq \Phi_\lambda^{(a)}, a = 1 \text{ or } 2, \\ D(r, s, w_{ab}, \tau) & \text{if } r \in \Phi_\lambda^{(a)}, s \in \Phi_\lambda^{(b)}, a \neq b. \end{cases},$$

where $w_{12} = w_1 - w_2 = w$ and $w_{21} = w_2 - w_1 = -w$.

*Remark 2.* Note that $\xi$ is well-defined since $D(r, s, w_{ab}, \tau) = D(s, r, w_{ba}, \tau)$.


**Fig. 3** A doubly-indexed edge

Using the expression (78), it follows that the genus two partition function (76) can be expressed as

$$Z^{(2)}_M(\tau, w, \rho) = Z^{(1)}_M(\tau) \sum_{\lambda=\{i^{e_i}\}} \frac{E(\lambda)}{\prod_i (-i)^{e_i} e_i!} \rho^{\sum i e_i}, \tag{80}$$

where $\lambda$ runs over all unrestricted partitions and

$$E(\lambda) = \sum_{\phi \in F(\Phi_{\lambda,2})} \Gamma(\phi). \tag{81}$$

We employ the doubly-indexed diagrams of Sect. 2.3. Consider the 'canonical' matching defined by $\iota$ as a fixed-point-free involution. We may then compose $\iota$ with each fixed-point-free involution $\phi \in F(\Phi_{\lambda,2})$ to define a 1-1 mapping $\iota\phi$ on the underlying labelled set $\Phi_{\lambda,2}$. For each $\phi$ we define a doubly-indexed diagram $D$ whose nodes are labelled by $k, a$ for an element $k \in \Phi^{(a)}_\lambda$ for $a = 1, 2$ and with cycles corresponding to the orbits of the cyclic group $\langle \iota\phi \rangle$. Thus, if $l = \phi(k)$ for $k \in \Phi^{(a)}_\lambda$ and $l \in \Phi^{(\bar{b})}_\lambda$ and $\iota : \bar{b} \mapsto b$ with convention (16) then the corresponding doubly-indexed diagram contains the edge (Fig. 3).
Consider the permutations of $\Phi_{\lambda,2}$ that commute with $\iota$ and preserve both $\Phi^{(1)}_\lambda$ and $\Phi^{(2)}_\lambda$. We denote this group, which is plainly isomorphic to $\Sigma(\Phi_\lambda)$, by $\Delta_\lambda$. By definition, an automorphism of a doubly-indexed diagram $D$ in the above sense is an element of $\Delta_\lambda$ which preserves edges and node labels.

For a doubly-indexed diagram $D$ corresponding to the partition $\lambda = \{1^{e_1} \ldots p^{e_p}\}$ we set

$$\gamma(D) = \frac{\prod_{\{k,l\}} \xi(k, l, w, \tau)}{\prod (-i)^{e_i}} \rho^{\sum i e_i} \tag{82}$$

where $\{k, l\}$ ranges over the edges of $D$. We now have all the pieces assembled to copy the arguments used to prove Theorem 5 of [20]. First we find

$$\sum_{\lambda=\{i^{e_i}\}} \frac{E(\lambda)}{\prod_i (-i)^{e_i} e_i!} \rho^{\sum i e_i} = \sum_D \frac{\gamma(D)}{|\mathrm{Aut}(D)|}, \tag{83}$$

the sum ranging over all doubly-indexed diagrams.

We next introduce a weight function $\zeta$ as follows: for a doubly-indexed diagram $D$ we set $\zeta(D) = \prod \zeta(E)$, the product running over all edges. Moreover for an edge $E$ with nodes labelled $(k, a)$ and $(l, b)$ as in Fig. 3, we set

$$\zeta(E) = R_{ab}(k,l),$$

for $R$ of (19). We then find

**Lemma 4.** $\zeta(D) = \gamma(D)$.

*Proof.* From (82) it follows that for a doubly-indexed diagram $D$ we have

$$\gamma(D) = \prod_{\{k,l\}} -\frac{\xi(k,l,w,\tau)\rho^{(k+l)/2}}{\sqrt{kl}}, \tag{84}$$

the product ranging over the edges $\{k,l\}$ of $D$. So to prove the lemma it suffices to show that if $k,l$ lie in $\Phi_\lambda^{(a)}, \Phi_\lambda^{(\bar{b})}$ respectively then the $(a,b)$-entry of $R(k,l)$ coincides with the corresponding factor of (84). This follows from our previous discussion together with Remark 2. □

From Lemma 4 and following similar arguments to the proof of Theorem 5 of [20] we find

$$\sum_D \frac{\gamma(D)}{|\mathrm{Aut}(D)|} = \sum_D \frac{\zeta(D)}{|\mathrm{Aut}(D)|}$$
$$= \exp\left(\sum_L \frac{\zeta(L)}{|\mathrm{Aut}(L)|}\right),$$

where $L$ denotes the set of non-isomorphic unoriented doubly indexed cycles. Orient these cycles, say in a clockwise direction. Let $\{M\}$ denote the set of non-isomorphic oriented doubly indexed cycles and $\{M_n\}$ the oriented cycles with $n$ nodes. Then we find (cf. [20, Lemma 2]) that

$$\frac{1}{n}\mathrm{Tr}R^n = \sum_{M_n} \frac{\zeta(M_n)}{|\mathrm{Aut}(M_n)|}.$$

It follows that

$$\sum_L \frac{\zeta(L)}{|\mathrm{Aut}(L)|} = \frac{1}{2}\sum_M \frac{\zeta(M)}{|\mathrm{Aut}(M)|}$$
$$= \frac{1}{2}\mathrm{Tr}\left(\sum_{n\geq 1}\frac{1}{n}R^n\right)$$
$$= -\frac{1}{2}\mathrm{Tr}\log(I-R)$$
$$= -\frac{1}{2}\log\det(I-R).$$

This completes the proof of Theorem 8. □

We may also find a product formula analogous to Theorem 6 of [20]. Let $\mathcal{R}$ denote the rotationless doubly-indexed oriented cycles i.e. cycles with trivial automorphism group. Then we find

**Theorem 9.**

$$Z_M^{(2)}(\tau, w, \rho) = \frac{Z_M^{(1)}(\tau)}{\prod_{\mathcal{R}}(1-\zeta(N))^{1/2}}. \quad \square \tag{85}$$

## 5.2 *Holomorphic and Modular-Invariance Properties*

In Sect. 2.2 we reviewed the genus two $\rho$-sewing formalism and introduced the domain $\mathcal{D}^\rho$ which parametrizes the genus two surface. An immediate consequence of Theorem 1 is the following.

**Theorem 10.** *$Z_M^{(2)}(\tau, w, \rho)$ is holomorphic in $\mathcal{D}^\rho$.* $\square$

We next consider the invariance properties of the genus two partition function with respect to the action of the $\mathcal{D}^\rho$-preserving group $\Gamma_1$ reviewed in Sect. 2.2. Let $\chi$ be the character of $\mathrm{SL}(2,\mathbb{Z})$ defined by its action on $\eta(\tau)^{-2}$, i.e.

$$\eta(\gamma\tau)^{-2} = \chi(\gamma)\eta(\tau)^{-2}(c\tau+d)^{-1}, \tag{86}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. Recall (e.g. [30]) that $\chi(\gamma)$ is a twelfth root of unity. For a function $f(\tau)$ on $\mathbb{H}_1$, $k \in \mathbb{Z}$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, we define

$$f(\tau)|_k\gamma = f(\gamma\tau)\,(c\tau+d)^{-k}, \tag{87}$$

so that

$$Z_{M^2}^{(1)}(\tau)|_{-1}\gamma = \chi(\gamma) Z_{M^2}^{(1)}(\tau). \tag{88}$$

At genus two, analogously to (87), we define

$$f(\tau, w, \rho)|_k\gamma = f(\gamma(\tau, w, \rho))\det(C\Omega + D)^{-k}. \tag{89}$$

Here, the action of $\gamma$ on the right-hand-side is as in (18). We have abused notation by adopting the following conventions in (89), which we continue to use below:

$$\Omega = F^\rho(\tau, w, \rho),\ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \tag{90}$$

where $F^{\rho}$ is as in Theorem 4, and $\gamma$ is identified with an element of $\mathrm{Sp}(4,\mathbb{Z})$ via (32) and (33). Note that (89) defines a right action of $G$ on functions $f(\tau,w,\rho)$. We then have a natural analog of Theorem 8 of [20]

**Theorem 11.** *If $\gamma \in \Gamma_1$ then*

$$Z^{(2)}_{M^2}(\tau,w,\rho)|_{-1}\gamma = \chi(\gamma) Z^{(2)}_{M^2}(\tau,w,\rho).$$

**Corollary 2.** *If $\gamma \in \Gamma_1$ with $Z^{(2)}_{M^{24}} = (Z^{(2)}_{M^2})^{12}$ then*

$$Z^{(2)}_{M^{24}}(\tau,w,\rho)|_{-12}\gamma = Z^{(2)}_{M^{24}}(\tau,w,\rho).$$

*Proof.* The proof is similar to that of Theorem 8 of [20]. We have to show that

$$Z^{(2)}_{M^2}(\gamma.(\tau,w,\rho)) \det(C\Omega + D) = \chi(\gamma) Z^{(2)}_{M^2}(\tau,w,\rho) \tag{91}$$

for $\gamma \in \Gamma_1$ where $\det(C\Omega_{11} + D) = c_1\Omega_{11} + d_1$. Consider the determinant formula (75). For $\gamma \in \Gamma_1$ define

$$R'_{ab}(k,l,\tau,w,\rho) = R_{ab}\left(k,l,\frac{a_1\tau+b_1}{c_1\tau+d_1},\frac{w}{c_1\tau+d_1},\frac{\rho}{(c_1\tau+d_1)^2}\right)$$

following (31). We find from Sect. 6.3 of [18] that

$$\begin{aligned} 1 - R' &= 1 - R - \kappa\Delta \\ &= (1-\kappa S).(1-R), \end{aligned}$$

where

$$\begin{aligned} \Delta_{ab}(k,l) &= \delta_{k1}\delta_{l1}, \\ \kappa &= \frac{\rho}{2\pi \mathrm{i}}\frac{c_1}{c_1\tau+d_1}, \\ S_{ab}(k,l) &= \delta_{k1}\sum_{c\in\{1,2\}}\left((1-R)^{-1}\right)_{cb}(1,l). \end{aligned}$$

Since $\det(1-R)$ and $\det(1-R')$ are convergent on $\mathcal{D}^{\rho}$ we find

$$\det(1-R') = \det(1-\kappa S).\det(1-R).$$

Indexing the columns and rows by $(a,k) = (1,1),(2,1),\ldots(1,k),(2,k)\ldots$ and noting that $S_{1b}(k,l) = S_{2b}(k,l)$ we find that

$$\det(1-\kappa S) = \begin{vmatrix} 1-\kappa S_{11}(1,1) & -\kappa S_{12}(1,1) & -\kappa S_{11}(1,2) & \cdots \\ -\kappa S_{11}(1,1) & 1-\kappa S_{12}(1,1) & -\kappa S_{11}(1,2) & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

$$= 1-\kappa S_{11}(1,1)-\kappa S_{12}(1,1),$$

$$= 1-\kappa\sigma\left((1-R)^{-1}\right)(1,1)),$$

where $\sigma(M)$ denotes the finite sum over the block labels for a $2\times 2$ block matrix $M$. Applying (27), it is clear that

$$\det(1-\kappa S) = \frac{c_1\Omega_{11}+d_1}{c_1\tau+d_1}.$$

The theorem follows from (88). □

*Remark 3.* $Z^{(2)}_{M^2}(\tau,w,\rho)$ can be trivially considered as function on the covering space $\hat{\mathcal{D}}^{\rho}$ discussed in [18, Sect. 6.3]. Then $Z^{(2)}_{M^2}(\tau,w,\rho)$ is modular with respect to $L=\hat{H}\Gamma_1$ with trivial invariance under the action of the Heisenberg group $\hat{H}$ (loc. cit.).

## 5.3 Some Genus Two n-Point Functions

In this section we calculate some examples of genus two $n$-point functions for the rank one Heisenberg VOA $M$. We consider here the examples of the $n$-point function for the Heisenberg vector $a$ and the 1-point function for the Virasoro vector $\tilde{\omega}$. We find that, up to an overall factor of the partition function, the formal differential form associated with the Heisenberg $n$-point function is described in terms of the global symmetric two form $\omega$ [33] whereas the Virasoro 1-point function is described by the genus two projective connection [11]. These results agree with those found in [20] in the $\epsilon$-formalism up to an overall $\epsilon$-formalism partition function factor.

The genus two Heisenberg vector 1-point function with the Heisenberg vector $a$ inserted at $x$ is $Z^{(2)}_M(a,x;\tau,w,\rho)=0$ since $Z^{(1)}_M(Y[a,x]Y[v,w]v,\tau)=0$ from [17]. The 2-point function for two Heisenberg vectors inserted at $x_1,x_2$ is

$$Z^{(2)}_M(a,x_1;a,x_2;\tau,w,\rho)=\sum_{r\geq 0}\rho^r\sum_{v\in M_{[r]}}Z^{(1)}_M\left(a,x_1;a,x_2;v,w_1,\bar{v},w_2;\tau\right). \quad (92)$$

We consider the associated formal differential form

$$\mathcal{F}^{(2)}_M(a,a;\tau,w,\rho)=Z^{(2)}_M\left(a,x_1;a,x_2;\tau,w,\rho\right)dx_1dx_2, \quad (93)$$

and find that it is determined by the bilinear form $\omega$ (12):

**Theorem 12.** *The genus two Heisenberg vector 2-point function is given by*

$$\mathcal{F}_M^{(2)}(a,a;\tau,w,\rho) = \omega(x_1,x_2)Z_M^{(2)}(\tau,w,\rho). \tag{94}$$

*Proof.* The proof proceeds along the same lines as Theorem 8. As before, we let $v(\lambda)$ denote a Heisenberg Fock vector (77) determined by an unrestricted partition $\lambda = \{1^{e_1}\ldots p^{e_p}\}$ with label set $\Phi_\lambda$. Define a label set for the four vectors $a,a,v(\lambda),v(\lambda)$ given by $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_\lambda^{(1)} \cup \Phi_\lambda^{(2)}$ for $\Phi_1,\Phi_2 = \{1\}$ and let $F(\Phi)$ denote the set of fixed point free involutions on $\Phi$. For $\phi = \ldots(rs)\ldots \in F(\Phi)$ let $\Gamma(x_1,x_2,\phi) = \prod_{(r,s)}\xi(r,s)$ as defined in (80) for $r,s \in \Phi_\lambda^{(2)} = \Phi_\lambda^{(1)} \cup \Phi_\lambda^{(2)}$ and

$$\xi(r,s) = \begin{cases} D(1,1,x_i-x_j,\tau) = P_2(\tau,x_i-x_j), & r,s\in\Phi_i,\ i\neq j,\\ D(1,s,x_i-w_a,\tau) = sP_{s+1}(\tau,x_i-w_a), & r\in\Phi_i,\ s\in\Phi_\lambda^{(a)},\end{cases} \tag{95}$$

for $i,j,a \in \{1,2\}$ with $D$ of (11). Then following Corollary 1 of [17] we have

$$Z_M^{(1)}(a,x_1;a,x_2;v(\lambda),w_1;v(\lambda),w_2;\tau) = Z_M^{(1)}(\tau)\sum_{\phi\in F(\Phi)}\Gamma(x_1,x_2,\phi).$$

We then obtain the following analog of (80)

$$\mathcal{F}^{(2)}(a,a;\tau,w,\rho) = Z_M^{(1)}(\tau)\sum_{\lambda=\{i^{e_i}\}}\frac{E(x_1,x_2,\lambda)}{\prod_i(-i)^{e_i}e_i!}\rho^{\sum ie_i}dx_1dx_2, \tag{96}$$

where

$$E(x_1,x_2,\lambda) = \sum_{\phi\in F(\Phi)}\Gamma(x_1,x_2,\phi).$$

The sum in (96) can be re-expressed as the sum of weights $\zeta(D)$ for isomorphism classes of doubly-indexed configurations $D$ where here $D$ includes two distinguished valency one nodes labelled $x_i$ (see Sect. 2.3) corresponding to the label sets $\Phi_1,\Phi_2 = \{1\}$. As before, $\zeta(D) = \prod_E \zeta(E)$ for standard doubly-indexed edges $E$ augmented by the contributions from edges connected to the two valency one nodes with weights as in (37). Thus we find

$$\mathcal{F}^{(2)}(a,a;\tau,w,\rho) = Z_M^{(1)}(\tau)\sum_D\frac{\zeta(D)}{\prod_i e_i!}dx_1dx_2,$$

Each $D$ can be decomposed into *exactly* one necklace configuration $N$ of type $\mathcal{N}(x;y)$ of (38) connecting the two distinguished nodes and a standard configuration $\hat{D}$ of the type appearing in the proof of Theorem 8 with $\zeta(D) = \zeta(N)\zeta(\hat{D})$. Since $|\mathrm{Aut}(N)| = 1$ we obtain

$$\begin{aligned}\mathcal{F}^{(2)}(a,a;\tau,w,\rho) &= Z_M^{(1)}(\tau)\sum_{\hat{D}}\frac{\zeta(\hat{D})}{|\mathrm{Aut}(\hat{D})|}\sum_{N\in\mathcal{N}(x;y)}\zeta(N)\\ &= Z_M^{(2)}(\tau,w,\rho)\zeta(x_1;x_2)\\ &= Z_M^{(2)}(\tau,w,\rho)\omega(x_1,x_2),\end{aligned}$$

using (41) of Proposition 2. □

Theorem 12 can be generalized to compute the $n$-point function corresponding to the insertion of $n$ Heisenberg vectors. We find that it vanishes for $n$ odd, and for $n$ even is determined by the symmetric tensor

$$\mathrm{Sym}_n\,\omega = \sum_{\psi}\prod_{(r,s)}\omega(x_r,x_s), \tag{97}$$

where the sum is taken over the set of fixed point free involutions $\psi = \ldots(rs)\ldots$ of the labels $\{1,\ldots,n\}$. We then have

**Theorem 13.** *The genus two Heisenberg vector $n$-point function vanishes for odd $n$ even; for even $n$ it is given by the global symmetric meromorphic $n$-form:*

$$\frac{\mathcal{F}_M^{(2)}(a,\ldots,a;\tau,w,\rho)}{Z_M^{(2)}(\tau,w,\rho)} = \mathrm{Sym}_n\,\omega. \tag{98}$$

□

This agrees with the corresponding ratio in Theorem 10 of [20] in the $\epsilon$-formalism, and also with earlier results in [33] which assume an analytic structure for the $n$-point function.

Using this result and the associativity of vertex operators, we can compute all $n$-point functions. In particular, the 1-point function for the Virasoro vector $\tilde{\omega} = \frac{1}{2}a[-1]a$ is as follows (cf. [20], Proposition 8):

**Proposition 3.** *The genus two 1-point function for the Virasoro vector $\tilde{\omega}$ inserted at $x$ is given by*

$$\frac{\mathcal{F}_M^{(2)}(\tilde{\omega};\tau,w,\rho)}{Z_M^{(2)}(\tau,w,\rho)} = \frac{1}{12}s^{(2)}(x), \tag{99}$$

*where $s^{(2)}(x) = 6\lim_{x\to y}\left(\omega(x,y) - \frac{dxdy}{(x-y)^2}\right)$ is the genus two projective connection [11].* □

# 6 Lattice VOAs

## 6.1 The Genus Two Partition Function $Z^{(2)}_{V_L}(\tau, w, \rho)$

Let $L$ be an even lattice with $V_L$ the corresponding lattice theory vertex operator algebra. The underlying Fock space is

$$V_L = M^l \otimes C[L] = \oplus_{\beta \in L} M^l \otimes \mathrm{e}^{\beta}, \tag{100}$$

where $M^l$ is the corresponding Heisenberg free boson theory of rank $l = \dim L$ based on $H = C \otimes_Z L$. We follow Sect. 3.1 and [17] concerning further notation for lattice theories. We utilize the Fock basis $\{u \otimes \mathrm{e}^{\beta}\}$ where $\beta$ ranges over $L$ and $u$ ranges over the usual orthogonal basis for $M^l$. From Lemma 1 and Corollary 1 we see that

$$Z^{(2)}_{V_L}(\tau, w, \rho) = \sum_{\alpha, \beta \in L} Z^{(2)}_{\alpha,\beta}(\tau, w, \rho), \tag{101}$$

$$Z^{(2)}_{\alpha,\beta}(\tau, w, \rho) = \sum_{n \geq 0} \sum_{u \in M^l_{[n]}} Z^{(1)}_{M^l \otimes \mathrm{e}^{\alpha}}\left(u \otimes \mathrm{e}^{\beta}, \overline{u} \otimes \mathrm{e}^{-\beta}, w, \tau\right) \rho^{n+(\beta,\beta)/2}. \tag{102}$$

The general shape of the 2-point function occurring in (102) is discussed extensively in [17]. By Proposition 1 (op. cit.) it splits as a product

$$Z^{(1)}_{M^l \otimes \mathrm{e}^{\alpha}}\left(u \otimes \mathrm{e}^{\beta}, u \otimes \mathrm{e}^{-\beta}, w, \tau\right) = Q^{\beta}_{M^l \otimes \mathrm{e}^{\alpha}}(u, u, w, \tau)\, Z^{(1)}_{M^l \otimes \mathrm{e}^{\alpha}}\left(\mathrm{e}^{\beta}, \mathrm{e}^{-\beta}, w, \tau\right), \tag{103}$$

where we have identified $\mathrm{e}^{\beta}$ with $\mathbf{1} \otimes \mathrm{e}^{\beta}$, and where $Q^{\beta}_{M^l \otimes \mathrm{e}^{\alpha}}$ is a function[6] that we will shortly discuss in greater detail. In [17, Corollary 5] (cf. the Appendix to the present paper) we established also that

$$Z^{(1)}_{M^l \otimes \mathrm{e}^{\alpha}}\left(\mathrm{e}^{\beta}, \mathrm{e}^{-\beta}, w, \tau\right) = \epsilon(\beta, -\beta) q^{(\alpha,\alpha)/2} \frac{\exp((\alpha, \beta)w)}{K(w, \tau)^{(\beta,\beta)}} Z^{(1)}_{M^l}(\tau), \tag{104}$$

where, as usual, we are taking $w$ in place of $z_{12} = z_1 - z_2$. With cocycle choice $\epsilon(\beta, -\beta) = (-1)^{(\beta,\beta)/2}$ (cf. Appendix) we may then rewrite (102) as

[6]Note: in [17] the functional dependence on $\beta$, here denoted by a superscript, was omitted.

$$Z^{(2)}_{\alpha,\beta}(\tau,w,\rho) = Z^{(1)}_{M^l}(\tau)\exp\left\{\pi\mathrm{i}\Big((\alpha,\alpha)\tau + 2(\alpha,\beta)\frac{w}{2\pi\mathrm{i}} + \frac{(\beta,\beta)}{2\pi\mathrm{i}}\log\left(\frac{-\rho}{K(w,\tau)^2}\right)\Big)\right\}$$
$$\cdot\sum_{n\geq 0}\sum_{u\in M^l_{[n]}} Q^{\beta}_{M^l\otimes\mathrm{e}^{\alpha}}(u,\overline{u},w,\tau)\rho^n. \quad (105)$$

We note that this expression is, as it should be, independent of the choice of branch for the logarithm function. We are going to establish the *precise* analog of Theorem 14 of [20] as follows:

**Theorem 14.** *We have*

$$Z^{(2)}_{V_L}(\tau,w,\rho) = Z^{(2)}_{M^l}(\tau,w,\rho)\theta^{(2)}_L(\Omega), \quad (106)$$

*where $\theta^{(2)}_L(\Omega)$ is the genus two theta function of L [7].*

*Proof.* We note that

$$\theta^{(2)}_L(\Omega) = \sum_{\alpha,\beta\in L}\exp\left(\pi\mathrm{i}((\alpha,\alpha)\Omega_{11} + 2(\alpha,\beta)\Omega_{12} + (\beta,\beta)\Omega_{22})\right). \quad (107)$$

We first handle the case of rank 1 lattices and then consider the general case. The inner double sum in (105) is the object which requires attention, and we can begin to deal with it along the lines of previous sections. Namely, arguments that we have already used several times show that the double sum may be written in the form

$$\sum_D \frac{\gamma(D)}{|\mathrm{Aut}(D)|} = \exp\left(\frac{1}{2}\sum_{N\in\mathcal{N}}\gamma(N)\right).$$

Here, $D$ ranges over the oriented doubly indexed cycles of Sect. 5, while $N$ ranges over oriented doubly-indexed necklaces $\mathcal{N} = \{\mathcal{N}(k,a;l,b)\}$ of (35). Leaving aside the definition of $\gamma(N)$ for now, we recognize as before that the piece involving only connected diagrams with no end nodes splits off as a factor. Apart from a $Z^{(1)}_M(\tau)$ term this factor is, of course, precisely the expression (83) for $M$. With these observations, we see from (105) that the following holds:

$$\frac{Z^{(2)}_{\alpha,\beta}(\tau,w,\rho)}{Z^{(2)}_M(\tau,w,\rho)} = \exp\left\{\mathrm{i}\pi\left((\alpha,\alpha)\tau + 2(\alpha,\beta)\frac{w}{2\pi\mathrm{i}} + \frac{(\beta,\beta)}{2\pi\mathrm{i}}\log\left(\frac{-\rho}{K(w,\tau)^2}\right)\right.\right.$$
$$\left.\left.+\frac{1}{2\pi\mathrm{i}}\sum_{N\in\mathcal{N}}\gamma(N)\right)\right\}. \quad (108)$$

To prove Theorem 14, we see from (107) and (108) that it is sufficient to establish that for each pair of lattice elements $\alpha,\beta\in L$, we have

$$\frac{Z^{(2)}_{\alpha,\beta}(\tau,w,\rho)}{Z^{(2)}_{M}(\tau,w,\rho)} = \exp\left(\pi\mathrm{i}\left((\alpha,\alpha)\Omega_{11} + 2(\alpha,\beta)\Omega_{12} + (\beta,\beta)\Omega_{22}\right)\right). \quad (109)$$

Recall the formula for $\Omega$ in Proposition 1. In order to reconcile (109) with the formula for $\Omega$, we must carefully consider the expression $\sum_{N\in\mathcal{N}}\gamma(N)$. The function $\gamma$ is essentially (82), except that we also get contributions from the end nodes which are now present. Suppose that an end node has label $k \in \Phi^{(a)}, a \in \{1,2\}$. Then according to Proposition 1 and display (45) of [17] (cf. (133) of the Appendix to the present paper), the contribution of the end node is equal to

$$\xi_{\alpha,\beta}(k,a) = \xi_{\alpha,\beta}(k,a,\tau,w,\rho) =$$

$$\begin{cases} \frac{\rho^{k/2}}{\sqrt{k}}\left(a, \delta_{k1}\alpha + C(k,0,\tau)\beta + D(k,0,w,\tau)(-\beta)\right), & a=1 \\ \frac{\rho^{k/2}}{\sqrt{k}}\left(a, \delta_{k1}\alpha + C(k,0,\tau)(-\beta) + D(k,0,-w,\tau)\beta\right), & a=2 \end{cases} \quad (110)$$

together with a contribution arising from the $-1$ in the denominator of (82) (we will come back to this point later). Using (cf. [17], displays (6), (11) and (12))

$$D(k,0,-w,\tau) = (-1)^{k+1}P_k(-w,\tau) = -P_k(w,\tau),$$
$$C(k,0,\tau) = (-1)^{k+1}E_k(\tau),$$

we can combine the two possibilities in (110) as follows (recalling that $E_k = 0$ for odd $k$):

$$\xi_{\alpha,\beta}(k,a) = (a,\alpha)\rho^{1/2}\delta_{k1} + (a,\beta)d_{\overline{a}}(k), \quad (111)$$

where $d_a(k)$ is given by (25). We may then compute the weight for an oriented doubly-indexed necklace $N \in \mathcal{N}(k,a;l,b)$ (35). Let $N'$ denote the oriented necklace from which the two end nodes and edges have been *removed* (we refer to these as *shortened* necklaces). From (111) we see that the total contribution to $\gamma(N)$ is

$$-\xi_{\alpha,\beta}(k,a)\xi_{\alpha,\beta}(l,b)\gamma(N') = -\Big[(\alpha,\alpha)\rho\delta_{k1}\delta_{l1} + (\beta,\beta)d_{\overline{a}}(k)d_{\overline{b}}(l) + (\alpha,\beta)\rho^{1/2}\left(d_{\overline{a}}(k)\delta_{l,1} + d_{\overline{b}}(l)\delta_{k,1}\right)\Big]\gamma(N'), \quad (112)$$

where we note that a sign $-1$ arises from each *pair* of nodes, as follows from (82).

We next consider the terms in (112) corresponding to $(\alpha,\alpha)$, $(\alpha,\beta)$ and $(\beta,\beta)$ separately, and show that they are precisely the corresponding terms on each side of (109). This will complete the proof of Theorem 14 in the case of rank 1 lattices. From (112), an $(\alpha,\alpha)$ term arises only if the end node weights $k, l$ are both equal to 1. Hence $\sum\gamma(N') = \zeta(1;1)$ (cf. (36)), where the sum ranges over shortened necklaces with end nodes of weight $1 \in \Phi^{(a)}$ and $1 \in \Phi^{(b)}$. Thus using Proposition 1, the total contribution to the right-hand-side of (109) is equal to

$$2\pi i\tau - \rho\zeta(1;1) = 2\pi i\Omega_{11}. \tag{113}$$

Next, from (112) we see that an $(\alpha,\beta)$-contribution arises whenever at least one of the end nodes has label 1. If the labels of the end nodes are unequal then the shortened necklace with the *opposite* orientation makes an equal contribution. The upshot is that we may assume that the end node to the right of the shortened necklace has label $l = 1 \in \Phi^{(\bar{b})}$, as long as we count accordingly. We thus find $\sum \gamma(N') = \zeta(d;1)$ (cf. (36)), where the sum ranges over shortened necklaces with end nodes of weight $k \in \Phi^{(a)}$ and $1 \in \Phi^{(b)}$. Then Proposition 1 implies that the total contribution to the $(\alpha,\beta)$ term on the right-hand-side of (109) is

$$2w - 2\rho^{1/2}\zeta(d;1) = 2\Omega_{12},$$

as required.

It remains to deal with the $(\beta,\beta)$ term, the details of which are very much along the lines as the case $(\alpha,\beta)$ just handled. A similar argument shows that the contribution to the $(\beta,\beta)$-term from (112) is equal to the expression $-\zeta(d;\overline{d})$ of (36). Thus the total contribution to the $(\beta,\beta)$ term on the right-hand-side of (109) is

$$\log\left(\frac{-\rho}{K(w,\tau)^2}\right) - \zeta(d;\overline{d}) = \Omega_{22},$$

as in (29). This completes the proof of Theorem 14 in the rank 1 case.

As for the general case—we adopt the mercy rule and omit details! The reader who has progressed this far will have no difficulty in dealing with the general case, which follows by generalizing the calculations in the rank 1 case just considered.

□

The analytic and automorphic properties of $Z^{(2)}_{V_L}(\tau,w,\rho)$ can be deduced from Theorem 14 using the known behaviour of $\theta^{(2)}_L(\Omega)$ and the analogous results for $Z^{(2)}_{M^l}(\tau,w,\rho)$ established in Sect. 5. We simply record

**Theorem 15.** *$Z^{(2)}_{V_L}(\tau,w,\rho)$ is holomorphic on the domain $\mathcal{D}^{\rho}$.* □

## 6.2 Some Genus Two n-Point Functions

In this section we consider the genus two $n$-point functions for $n$ Heisenberg vectors and the 1-point function for the Virasoro vector $\tilde{\omega}$ for a rank $l$ lattice VOA. The results are similar to those of Sect. 5.3 so that detailed proofs will not be given.

Consider the 1-point function for a Heisenberg vector $a_i$ inserted at $x$. We define the differential 1-form

$$\mathcal{F}^{(2)}_{\alpha,\beta}(a_i;\tau,w,\rho)=\sum_{n\geq 0}\sum_{u\in M^l_{[n]}} Z^{(1)}_{M^l\otimes e^{\alpha}}\left(a_i,x;u\otimes e^{\beta},w;\overline{u}\otimes e^{-\beta},0,\tau\right)\rho^{n+(\beta,\beta)/2}dx. \tag{114}$$

This can be expressed in terms of the genus two holomorphic 1-forms $\nu_1,\nu_2$ of (24) in a similar way to Theorem 12 of [20]. Defining

$$\nu_{i,\alpha,\beta}(x)=(a_i,\alpha)\nu_1(x)+(a_i,\beta)\nu_2(x),$$

we find

**Theorem 16.**

$$\mathcal{F}^{(2)}_{\alpha,\beta}(a_i;\tau,w,\rho)=\nu_{i,\alpha,\beta}(x)Z^{(2)}_{\alpha,\beta}(\tau,w,\rho). \tag{115}$$

*Proof.* The proof proceeds along the same lines as Theorems 12 and 14 and Theorem 12 of (op. cit.). We find that

$$\mathcal{F}^{(2)}_{\alpha,\beta}(a_i;\tau,w,\rho)=Z^{(1)}_M(\tau)\sum_D\frac{\gamma(D)}{|\mathrm{Aut}(D)|}dx,$$

where the sum is taken over isomorphism classes of doubly-indexed configurations $D$ where, in this case, each configuration includes one distinguished valence one node labelled by $x$ as in (37). Each $D$ can be decomposed into exactly one necklace configuration of type $\mathcal{N}(x;k,a)$ of (39), standard configurations of the type appearing in Theorem 12 and necklace contributions as in Theorem 106. The result then follows on applying (111) and the graphical expansion for $\nu_1(x),\nu_2(x)$ of (42) and (43). □

Summing over all lattice vectors, we find that the Heisenberg 1-point function vanishes for $V_L$. Similarly, one can generalize Theorem 13 concerning the $n$-point function for $n$ Heisenberg vectors $a_{i_1},\ldots,a_{i_n}$. Defining

$$\mathcal{F}^{(2)}_{\alpha,\beta}(a_{i_1},\ldots,a_{i_n};\tau,w,\rho)=\prod_{t=1}^{n}dx_t\sum_{n\geq 0}\sum_{u\in M^l_{[n]}}\rho^{n+(\beta,\beta)/2}\cdot$$
$$Z^{(1)}_{M^l\otimes e^{\alpha}}\left(a_{i_1},x_1;\ldots;a_{i_n},x_n;u\otimes e^{\beta},w;\overline{u}\otimes e^{-\beta},0,\tau\right),$$

we obtain the analogue of Theorem 13 of (op. cit.):

**Theorem 17.**

$$\mathcal{F}^{(2)}_{\alpha,\beta}(a_{i_1},\ldots,a_{i_n};\tau,w,\rho)=\mathrm{Sym}_n\left(\omega,\nu_{i_t,\alpha,\beta}\right)Z^{(2)}_{\alpha,\beta}(\tau,w,\rho), \tag{116}$$

*the symmetric product of* $\omega(x_r, x_s)$ *and* $\nu_{i_t,\alpha,\beta}(x_t)$ *defined by*

$$\mathrm{Sym}_n\left(\omega, \nu_{i_t,\alpha,\beta}\right) = \sum_{\psi} \prod_{(r,s)} \omega(x_s, x_s) \prod_{(t)} \nu_{i_t,\alpha,\beta}(x_t), \tag{117}$$

*where the sum is taken over the set of involutions* $\psi = \ldots(ij)\ldots(k)\ldots$ *of the labels* $\{1,\ldots,n\}$. □

We may also compute the genus two 1-point function for the Virasoro vector $\tilde{\omega} = \frac{1}{2}\sum_{i=1}^{l} a_i[-1]a_i$ using associativity of vertex operators as in Proposition 3. We find that for a rank $l$ lattice,

$$\begin{aligned}\mathcal{F}^{(2)}_{\alpha,\beta}(\tilde{\omega};\tau,w,\rho) &\equiv \sum_{n\geq 0}\sum_{u\in M^l_{[n]}} Z^{(1)}_{M^l\otimes \mathrm{e}^{\alpha}}\left(\tilde{\omega},x;u\otimes \mathrm{e}^{\beta},w;\overline{u}\otimes \mathrm{e}^{-\beta},0,\tau\right)\rho^{n+(\beta,\beta)/2}dx^2\\ &= \left(\frac{1}{2}\sum_i \nu_{i,\alpha,\beta}(x)^2 + \frac{l}{2}s^{(2)}(x)\right) Z^{(2)}_{\alpha,\beta}(\tau,w,\rho)\\ &= Z^{(2)}_{M^l}(\tau,w,\rho)\left(\mathcal{D} + \frac{l}{12}s^{(2)}\right)\mathrm{e}^{\mathrm{i}\pi((\alpha,\alpha)\Omega_{11}+2(\alpha,\beta)\Omega_{12}+(\beta,\beta)\Omega_{22})}.\end{aligned}$$

Here, we used (109) and the differential operator [5, 20, 35]

$$\mathcal{D} = \frac{1}{2\pi \mathrm{i}}\sum_{1\leq a\leq b\leq 2} \nu_a\nu_b \frac{\partial}{\partial \Omega_{ab}}. \tag{118}$$

Defining the normalized Virasoro 1-point form

$$\hat{\mathcal{F}}^{(2)}_{V_L}(\tilde{\omega};\tau,w,\rho) = \frac{\mathcal{F}^{(2)}_{V_L}(\tilde{\omega};\tau,w,\rho)}{Z^{(2)}_{M^l}(\tau,w,\rho)}, \tag{119}$$

we obtain

**Proposition 4.** *The normalized Virasoro 1-point function for the lattice theory* $V_L$ *satisfies*

$$\hat{\mathcal{F}}^{(2)}_{V_L}(\tilde{\omega};\tau,w,\rho) = \left(\mathcal{D} + \frac{l}{12}s^{(2)}\right)\theta^{(2)}_L(\Omega). \tag{120}$$

□

(The Ward identity (120) is similar to Proposition 11 in [20] in the $\epsilon$-sewing formalism.)

Finally, we can obtain the analogue of Proposition 12 (op. cit.), where we find that $\hat{\mathcal{F}}^{(2)}_{V_L}$ enjoys the same modular properties as $\hat{Z}^{(2)}_{V_L} = \theta^{(2)}_L(\Omega(\tau,w,\rho))$. That is,

**Proposition 5.** *The normalized Virasoro 1-point function for a lattice VOA obeys*

$$\hat{\mathcal{F}}^{(2)}_{V_L}(\tilde{\omega};\tau,w,\rho)|_{l/2}\gamma=\left(\mathcal{D}+\frac{l}{12}s^{(2)}\right)\left(\hat{Z}^{(2)}_{V_L}(\tau,w,\rho)|_{l/2}\gamma\right), \tag{121}$$

*for* $\gamma\in\Gamma_1$. □

## 7 Comparison Between the $\epsilon$ and $\rho$-Formalisms

In this section we consider the relationship between the genus two boson and lattice partition functions computed in the $\epsilon$-formalism of [20] (based on a sewing construction with two separate tori with modular parameters $\tau_1,\tau_2$ and a sewing parameter $\epsilon$) and the $\rho$-formalism developed in this paper. We write

$$Z^{(2)}_{V,\epsilon}=Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)=\sum_{n\geq 0}\epsilon^n\sum_{u\in V_{[n]}}Z^{(1)}_V(u,\tau_1)\,Z^{(1)}_V(\overline{u},\tau_2),$$

$$Z^{(2)}_{V,\rho}=Z^{(2)}_{V,\rho}(\tau,w,\rho)=\sum_{n\geq 0}\rho^n\sum_{u\in V_{[n]}}Z^{(1)}_V(\overline{u},u,w,\tau)\,.$$

Although, for a given VOA $V$, the partition functions enjoy many similar properties, we show below that the partition functions are *not equal* in the two formalisms. This result follows from an explicit computation of the partition functions for two free bosons in the neighborhood of a two-tori degeneration points where $\Omega_{12}=0$. It then follows that there is likewise no equality between the partition functions in the $\epsilon$- and $\rho$-formalisms for a lattice VOA.

As shown in Theorem 12 of [18], we may relate the $\epsilon$- and $\rho$-formalisms in certain open neighborhoods of the two-tori degeneration point, where $\Omega_{12}=0$. In the $\epsilon$-formalism, the genus two Riemann surface is parameterized by the domain

$$\mathcal{D}^{\epsilon}=\left\{(\tau_1,\tau_2,\epsilon)\in\mathbb{H}_1\times\mathbb{H}_1\times\mathbb{C}\mid|\epsilon|<\frac{1}{4}D(q_1)D(q_2)\right\}, \tag{122}$$

with $q_a=\exp(2\pi i\tau_a)$ and $D(q)$ as in (15). In this case the two-tori degeneration is, by definition, given by $\epsilon\to 0$. In the $\rho$-formalism, the two torus degeneration is described by the limit (74). In order to understand this more precisely we introduce the domain [18]

$$\mathcal{D}^{\chi}=\left\{(\tau,w,\chi)\in\mathbb{H}_1\times\mathbb{C}\times\mathbb{C}\mid(\tau,w,-w^2\chi)\in\mathcal{D}^{\rho},0<|\chi|<\frac{1}{4}\right\}, \tag{123}$$

for $\mathcal{D}^{\rho}$ of (18) and $\chi=-\frac{\rho}{w^2}$ of (73). The period matrix is determined by a $\Gamma_1$-equivariant holomorphic map

$$\begin{aligned}F^{\chi}&:\mathcal{D}^{\chi}\to\mathbb{H}_2,\\(\tau,w,\chi)&\mapsto\Omega^{(2)}(\tau,w,-w^2\chi).\end{aligned}\tag{124}$$

Then

$$\mathcal{D}_0^{\chi}=\left\{(\tau,0,\chi)\in\mathbb{H}_1\times\mathbb{C}\times\mathbb{C}\mid 0<|\chi|<\frac{1}{4}\right\},\tag{125}$$

is the space of two-tori degeneration limit points of the domain $\mathcal{D}^{\chi}$. We may compare the two parameterizations on certain $\Gamma_1$-invariant neighborhoods of a two-tori degeneration point in both parameterizations to obtain:

**Theorem 18 (op. cit., Theorem 12).** *There exists a 1-1 holomorphic mapping between $\Gamma_1$-invariant open domains $\mathcal{I}^{\chi}\subset(\mathcal{D}^{\chi}\cup\mathcal{D}_0^{\chi})$ and $\mathcal{I}^{\epsilon}\subset\mathcal{D}^{\epsilon}$ where $\mathcal{I}^{\chi}$ and $\mathcal{I}^{\epsilon}$ are open neighborhoods of a two-tori degeneration point.* □

We next describe the explicit relationship between $(\tau_1,\tau_2,\epsilon)$ and $(\tau,w,\chi)$ in more detail. Firstly, from Theorem 4 of [18] we obtain

$$\begin{aligned}2\pi\mathrm{i}\Omega_{11}&=2\pi\mathrm{i}\tau_1+E_2(\tau_2)\epsilon^2+E_2(\tau_1)E_2(\tau_2)^2\epsilon^4+O(\epsilon^6),\\2\pi\mathrm{i}\Omega_{12}&=-\epsilon-E_2(\tau_1)E_2(\tau_2)\epsilon^3+O(\epsilon^5),\\2\pi\mathrm{i}\Omega_{22}&=2\pi\mathrm{i}\tau_2+E_2(\tau_1)\epsilon^2+E_2(\tau_1)^2E_2(\tau_2)\epsilon^4+O(\epsilon^6).\end{aligned}$$

Making use of the identity

$$\frac{1}{2\pi\mathrm{i}}\frac{d}{d\tau}E_2(\tau)=5E_4(\tau)-E_2(\tau)^2,\tag{126}$$

it is straightforward to invert $\Omega_{ij}(\tau_1,\tau_2,\epsilon)$ to find

**Lemma 5.** *In the neighborhood of the two-tori degeneration point $r=2\pi\mathrm{i}\Omega_{12}=0$ of $\Omega\in\mathbb{H}_2$ we have*

$$\begin{aligned}2\pi\mathrm{i}\tau_1&=2\pi\mathrm{i}\Omega_{11}-E_2(\Omega_{22})r^2+5E_2(\Omega_{11})E_4(\Omega_{22})r^4+O(r^6),\\\epsilon&=-r+E_2(\Omega_{11})E_2(\Omega_{22})r^3+O(r^5),\\2\pi\mathrm{i}\tau_2&=2\pi\mathrm{i}\Omega_{22}-E_2(\Omega_{11})r^2+5E_2(\Omega_{22})E_4(\Omega_{11})r^4+O(r^6).\end{aligned}$$

□

From Theorem 4 we may also determine $\Omega_{ij}(\tau,w,\chi)$ to $O(w^4)$ in a neighborhood of a two-tori degeneration point to find

**Proposition 6.** *For $(\tau,w,\chi)\in\mathcal{D}^{\chi}\cup\mathcal{D}_0^{\chi}$ we have*

$$2\pi i\Omega_{11} = 2\pi i\tau + G(\chi)\sigma^2 + G(\chi)^2 E_2(\tau)\sigma^4 + O(w^6),$$
$$2\pi i\Omega_{22} = \log f(\chi) + E_2(\tau)\sigma^2 + \left(G(\chi)E_2(\tau)^2 + \frac{1}{2}E_4(\tau)\right)\sigma^4 + O(w^6)$$
$$2\pi i\Omega_{12} = \sigma + G(\chi)E_2(\tau)\sigma^3 + O(w^5),$$

*where* $\sigma = w\sqrt{1-4\chi}$, $G(\chi) = \frac{1}{12} + E_2(q = f(\chi)) = O(\chi)$ *and* $f(\chi)$ *is the Catalan series (66).*

This result is an extension of [18, Proposition 13] and the general proof proceeds along the same lines. For our purposes, it is sufficient to expand the non-logarithmic terms to $O(w^4, \chi^0)$. Since $R(k,l) = O(\chi)$ and $d_a(k) = O(\chi^{1/2})$ then Theorem 4 implies

$$2\pi i\Omega_{11} = 2\pi i\tau + O(\chi), \tag{127}$$
$$2\pi i\Omega_{22} = \log\chi + 2\sum_{k\geq 2}\frac{1}{k}E_k(\tau)w^k + O(\chi), \tag{128}$$
$$2\pi i\Omega_{12} = w + O(\chi), \tag{129}$$

to all orders in $w$. In particular, we can readily confirm Proposition 6 to $O(w^4, \chi^0)$. Substituting (127)–(129) into Lemma 5 and using (126) and (136) we obtain

**Proposition 7.** *For* $(\tau, w, \chi) \in \mathcal{D}^\chi \cup \mathcal{D}^\chi_0$ *we have*

$$2\pi i\tau_1 = 2\pi i\tau + \frac{1}{12}w^2 + \frac{1}{144}E_2(\tau)w^4 + O(w^6, \chi),$$
$$2\pi i\tau_2 = \log(\chi) + \frac{1}{12}E_4(\tau)w^4 + O(w^6, \chi),$$
$$\epsilon = -w - \frac{1}{12}E_2(\tau)w^3 + O(w^5, \chi).$$

□

Define the ratio

$$T_{\epsilon,\rho}(\tau, w, \chi) = \frac{Z^{(2)}_{M^2,\epsilon}(\tau_1, \tau_2, \epsilon)}{Z^{(2)}_{M^2,\rho}(\tau, w, -w^2\chi)}, \tag{130}$$

for $\tau_1, \tau_2, \epsilon$ as given in Proposition 7. From Theorems 8 of [20] and Theorems 11 and 18 above we see that $T_{\epsilon,\rho}$ is $\Gamma_1$-invariant. From Theorem 7 for $V = M^2$, we find in the two tori degeneration limit that

$$\lim_{w\to 0} T_{\epsilon,\rho}(\tau, w, \chi) = f(\chi)^{-1/12},$$

i.e., the two partition functions do not even agree in this limit! The origin of this discrepancy may be thought to arise from the central charge dependent factors of $q^{-c/24}$ and $q_1^{-c/24}q_2^{-c/24}$ present in the definitions of $Z^{(2)}_{V,\rho}$ and $Z^{(2)}_{V,\epsilon}$ respectively (which, of course, are necessary for any modular invariance). One modification of the definition of the genus two partition functions compatible with the two tori degeneration limit might be:

$$Z^{\mathrm{new}(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)=\epsilon^{-c/12}Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon),\quad Z^{\mathrm{new}(2)}_{V,\rho}(\tau,w,\rho)=\rho^{-c/24}Z^{(2)}_{V,\rho}(\tau,w,\rho).$$

However, for $V=M^2$, we immediately observe that the ratio cannot be unity due to the incompatible $\Gamma_1$ actions arising from

$$\epsilon\to\frac{\epsilon}{c_1\tau_1+d_1},\quad \rho\to\frac{\rho}{(c_1\tau+d_1)^2},$$

as given in Lemmas 8 and 15 of [18] (cf. (31)).

Consider instead a further $\Gamma_1$-invariant factor of $f(\chi)^{-c/24}$ in the definition of the genus two partition function in the $\rho$-formalism. Once again, we find that the partition functions do not agree in the neighborhood of a two-tori degeneration point:

**Proposition 8.**

$$f(\chi)^{1/12}T_{\epsilon,\rho}(\tau,w,\chi)=1-\frac{1}{288}E_4(\tau)w^4+O(w^6,\chi).$$

*Proof.* As noted earlier, $R(k,l)=O(\chi)$ so that we immediately obtain

$$f(\chi)^{-1/12}Z^{(2)}_{M^2,\rho}(\tau,w,-w^2\chi)=\frac{1}{\eta(\tau)^2\eta(f(\chi))^2}+O(\chi),$$

to all orders in $w$. On the other hand, $Z^{(2)}_{M^2,\epsilon}(\tau_1,\tau_2,\epsilon)$ of Theorem 5 of [20] to $O(\epsilon^4)$ is given by

$$\frac{1}{\eta(\tau_1)^2\eta(\tau_2)^2}\left[1+E_2(\tau_1)E_2(\tau_2)\epsilon^2+\left(E_2(\tau_1)^2E_2(\tau_2)^2+15E_4(\tau_1)E_4(\tau_2)\right)\epsilon^4\right]+O(\epsilon^6). \tag{131}$$

We expand this to $O(w^4,\chi)$ using Proposition 7, (126) and

$$\frac{1}{2\pi\mathrm{i}}\frac{d}{d\tau}\eta(\tau)=-\frac{1}{2}E_2(\tau)\eta(\tau),$$

to eventually find that

$$Z^{(2)}_{M^2,\epsilon}(\tau_1, \tau_2, \epsilon) = \frac{1}{\eta(q)^2\eta(f(\chi))^2}\left(1 - \frac{1}{288}E_4(\tau)w^4\right) + O(w^6, \chi). \quad \square$$

## 8 Final Remarks

Let us briefly and heuristically sketch how our results compare to some related ideas in the physics and mathematics literature. There is a wealth of literature concerning the bosonic string e.g. [10, 29]. In particular, the conformal anomaly implies that the physically defined path integral partition function $Z_{\text{string}}$ cannot be reduced to an integral over the moduli space $\mathcal{M}_g$ of a Riemann surface of genus $g$ except for the 26 dimensional critical string where the anomaly vanishes. Furthermore, for the critical string, Belavin and Knizhnik argue that

$$Z_{\text{string}} = \int_{\mathcal{M}_g} |F|^2 \, d\mu,$$

where $d\mu$ denotes a natural volume form on $\mathcal{M}_g$ and $F$ is holomorphic and non-vanishing on $\mathcal{M}_g$ [1, 14]. They also claim that for $g \geq 2$, $F$ is a global section for the line bundle $K \otimes \lambda^{-13}$ (where $K$ is the canonical bundle and $\lambda$ the Hodge bundle) on $\mathcal{M}_g$ which is trivial by Mumford's theorem [26]. In this identification, the $\lambda^{-13}$ section is associated with 26 bosons, the $K$ section with a $c = -26$ ghost system and the vanishing conformal anomaly to the vanishing first Chern class for $K \otimes \lambda^{-13}$ [28]. More recently, some of these ideas have also been rigorously proved for a zeta function regularized determinant of an appropriate Laplacian operator $\Delta_n$ [24]. The genus two partition functions $Z^{(2)}_{M^2,\epsilon}(\tau_1, \tau_2, \epsilon)$ and $Z^{(2)}_{M^2,\rho}(\tau, w, \rho)$ constructed in [20] and the present paper for a rank 2 Heisenberg VOA should correspond in these approaches to a local description of the holomorphic part of $\left(\frac{\det' \Delta_1}{\det N_1}\right)^{-1}$ of [14, 24], giving a local section of the line bundle $\lambda^{-1}$. Given these assumptions, it follows that $T_{\epsilon,\rho} = Z^{(2)}_{M^2,\epsilon}/Z^{(2)}_{M^2,\rho} \neq 1$ in the neighborhood of a two-tori degeneration point where the ratio of the two sections is a non-trivial transition function $T_{\epsilon,\rho}$.

In the case of a general rational conformal field theory, the conformal anomaly continues to obstruct the existence of a global partition function on moduli space for $g \geq 2$. However, *all* CFTs of a given central charge $c$ are believed to share the same conformal anomaly e.g. [6]. Thus, the identification of the normalized lattice partition and $n$-point functions of Sect. 6 reflect the equality of the first Chern class of some bundle associated to a rank $c$ lattice VOA to that for $\lambda^{-c}$ with transition function $T^{c/2}_{\epsilon,\rho}$. It is interesting to note that even in the case of a unimodular lattice VOA with a unique conformal block [25, 33] the genus two partition function can therefore only be described locally. It would obviously be extremely valuable to find a rigorous description of the relationship between the VOA approach described here and these related ideas in conformal field theory and algebraic geometry.

## Appendix

We list here some corrections to [17] and [18] that we needed above.

(a) Display (27) of [17] should read

$$\epsilon(\alpha, -\alpha) = \epsilon(\alpha, \alpha) = (-1)^{(\alpha,\alpha)/2}. \tag{132}$$

(b) Display (45) of [17] should read

$$\gamma(\Xi) = (a, \delta_{r,1}\beta + C(r, 0, \tau)\alpha_k + \sum_{l \neq k} D(r, 0, z_{kl}, \tau)\alpha_l). \tag{133}$$

(c) As a result of (a), displays (79) and (80) of [17] are modified and now read

$$F_N(\mathrm{e}^{\alpha}, z_1; \mathrm{e}^{-\alpha}, z_2; q) = \epsilon(\alpha, -\alpha)\frac{q^{(\beta,\beta)/2}}{\eta^l(\tau)}\frac{\exp((\beta, \alpha)z_{12})}{K(z_{12}, \tau)^{(\alpha,\alpha)}}, \tag{134}$$

$$F_{V_L}(\mathrm{e}^{\alpha}, z_1; \mathrm{e}^{-\alpha}, z_2; q) = \epsilon(\alpha, -\alpha)\frac{1}{\eta^l(\tau)}\frac{\Theta_{\alpha,L}(\tau, z_{12}/2\pi\mathrm{i})}{K(z_{12}, \tau)^{(\alpha,\alpha)}}. \tag{135}$$

(d) The expression for $\epsilon(\tau, w, \chi)$ of display (172) of [18] should read

$$\epsilon(\tau, w, \chi) = -w\sqrt{1 - 4\chi}\left(1 + \frac{1}{24}w^2 E_2(\tau)(1 - 4\chi) + O(w^4)\right) \tag{136}$$

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## References

1. Belavin, A.A., Knizhnik, V.G.: Algebraic geometry and the geometry of quantum strings. Phys. Lett. **168B**, 202–206 (1986)
2. Borcherds, R.E.: Vertex algebras. Kac-Moody algebras and the Monster. Proc. Natl. Acad. Sci. **83**, 3068–3071 (1986)
3. Dolan, L., Goddard, P., Montague, P.: Conformal field theories, representations and lattice constructions. Commun. Math. Phys. **179**, 61–120 (1996)
4. Farkas, H.M., Kra, I.: Riemann Surfaces. Springer, New York (1980)
5. Fay, J.: Theta Functions on Riemann Surfaces. Lecture Notes in Mathematics, vol. 352. Springer, Berlin/New York (1973)
6. Freidan, D., Shenker, S.: The analytic geometry of two dimensional conformal field theory. Nucl. Phys. **B281**, 509–545 (1987)
7. Freitag, E.: Siegelische Modulfunktionen. Springer, Berlin/New York (1983)
8. Frenkel, I., Lepowsky, J., Meurman, A.: Vertex Operator Algebras and the Monster. Academic, New York (1988)
9. Frenkel, I., Huang, Y., Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules. Mem. Am. Math. Soc. **104** (1993)

10. Green, M., Schwartz, J., Witten, E.: Superstring Theory I. Cambridge University Press, Cambridge (1987)
11. Gunning, R.C.: Lectures on Riemann Surfaces. Princeton University Press, Princeton (1966)
12. Hurley, D., Tuite, M.P.: On the torus degeneration of the genus two partition function. Int. J. Math. **24**, 1350056 (2013)
13. Kac, V.: Vertex Operator Algebras for Beginners. University Lecture Series, vol. 10. American Mathematical Society, Providence (1998)
14. Knizhnik, V.G.: Multiloop amplitudes in the theory of quantum strings and complex geometry. Sov. Phys. Usp. **32**, 945–971 (1989); Sov. Sci. Rev. A **10**, 1–76 (1989)
15. Lepowsky, J., Li, H.: Introduction to Vertex Operator Algebras and Their Representations. Birkhäuser, Boston (2004)
16. Li, H.: Symmetric invariant bilinear forms on vertex operator algebras. J. Pure Appl. Alg. **96**, 279–297 (1994)
17. Mason, G., Tuite, M.P.: Torus chiral $n$-point functions for free boson and lattice vertex operator algebras. Commun. Math. Phys. **235**, 47–68 (2003)
18. Mason, G., Tuite, M.P.: On genus two Riemann surfaces formed from sewn tori. Commun. Math. Phys. **270**, 587–634 (2007)
19. Mason, G., Tuite, M.P.: Partition functions and chiral algebras. Contemp. Math. **442**, 401–410 (2007)
20. Mason, G., Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces I. Commun. Math. Phys. **300**, 673–713 (2010)
21. Mason, G., Tuite, M.P.: The genus two partition function for free bosonic and lattice vertex operator algebras. arXiv:0712.0628 (unpublished)
22. Mason, G., Tuite, M.P.: Vertex operators and modular forms. In: Kirsten, K., Williams, F. (eds.) A Window into Zeta and Modular Physics. MSRI Publications, vol. 57, pp. 183–278. Cambridge University Press, Cambridge (2010)
23. Matsuo, A., Nagatomo, K.: Axioms for a vertex algebra and the locality of quantum fields. Math. Soc. Jpn. Mem. **4** (1999)
24. McIntyre, A., Takhtajan, L.A.: Holomorphic factorization of determinants of Laplacians on Riemann surfaces and higher genus generalization of Kronecker's first limit formula. GAFA, Geom. Funct. Anal. **16**, 1291–1323 (2006)
25. Moore, G., Seiberg, N.: Classical and quantum conformal field theory. Commun. Math. Phys. **123**, 177–254 (1989)
26. Mumford, D.: Stability of projective varieties. L. Ens. Math. **23**, 39–110 (1977)
27. Mumford, D.: Tata Lectures on Theta I and II. Birkhäuser, Boston (1983)
28. Nelson, P.: Lectures on strings and moduli space. Phys. Rep. **149**, 337–375 (1987)
29. Polchinski, J.: String Theory I. Cambridge University Press, Cambridge (1998)
30. Serre, J.-P.: A Course in Arithmetic. Springer, Berlin (1978)
31. Sonoda, H.: Sewing conformal field theories I. Nucl. Phys. **B311**, 401–416 (1988)
32. Sonoda, H.: Sewing conformal field theories II. Nucl. Phys. **B311**, 417–432 (1988)
33. Tsuchiya, A., Ueno, K., Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries. Adv. Stud. Pure Math. **19**, 459–566 (1989)
34. Tuite, M.P: Genus two meromorphic conformal field theory. CRM Proc. Lect. Notes **30**, 231–251 (2001)
35. Ueno, K.: Introduction to conformal field theory with gauge symmetries. In: Geometry and Physics - Proceedings of the Conference at Aarhus University, Aaarhus. Marcel Dekker, New York (1997)
36. Yamada, A.: Precise variational formulas for abelian differentials. Kodai Math. J. **3**, 114–143 (1980)
37. Zhu, Y.: Modular invariance of characters of vertex operator algebras. J. Am. Math. Soc. **9**, 237–302 (1996)