6. The Origin of Fuzzy Extensions

Humberto Bustince, Edurne Barrenechea, Javier Fernández, Miguel Pagola, Javier Montero

Many different kinds of sets have been defined within the framework of fuzzy sets. This paper focusses on those fuzzy set extensions that address the difficulties that experts find in order to build the membership values. In particular, we analyze type-2 fuzzy sets, interval-valued fuzzy sets, Atanassov's intuitionistic fuzzy sets, or bipolar sets of type-2 and Atanassov's interval-valued fuzzy sets. After stating a general approach to these extensions, we remark some structural problems in the extension problem and stress some applications for which the results obtained with extensions are better than those obtained with Zadeh's fuzzy sets.

Many different types of fuzzy sets have appeared in the literature since *Zadeh* introduced the concept of fuzzy set (or type-1 fuzzy set) [6.1]. Roughly speaking, the basic characteristics of all those definitions are the following:

i) They are particular instances of the L-fuzzy sets defined by *Goguen* [6.2].
ii) They arise from theoretical problems and are very efficient to solve such theoretical problems.
iii) The specific characteristics of the new definitions do not use to play a formal role, quite often becoming an easy adaptation of Zadeh's fuzzy sets.
iv) It is not always shown to what extent the new proposal implies a practical advantage when compared to Zadeh's fuzzy sets.

The last point gives rise to a key criticism when additional information is needed for the management of a new kind of fuzzy sets, but the improvement we obtain in practice cannot be justified by the effort required to obtain such an information. But more important than that is the previous criticism (iii), about the difficulty of building the best family of sets for the application we are considering. Surprisingly, this key issue has not captured the attention of too many researchers.

In this paper, we shall focuss on those sets conceived to address the problem stated by Zadeh in 1971 in order to address the difficulty of finding the membership degree of each element (we shall refer to these sets as *extensions of the fuzzy sets*), and then we shall point out applications that can be found in the literature in which the use of some extensions provides better results than the use of type-1 fuzzy sets, according to the comparison carried out in the papers where this improvement is shown. Once the definition of extension of fuzzy sets has been introduced, we shall describe some of its properties and remark the structural problems of the different types of these extensions. Among those extensions we shall consider type-2 fuzzy sets, interval-valued fuzzy sets, Atanassov's intuitionistic fuzzy sets or type-2 bipolar fuzzy sets and Atanassov's interval-valued fuzzy sets.

We have organized this chapter as follows. In Sect. 6.2 we start recalling the reasons that led Zadeh to introduce fuzzy sets. We also remind the basic notions in Brouwer's intuitionistic theory to later justify the terminological problems linked to the sets defined by Atanassov. In Sect. 6.3 we present the origin of the extensions of fuzzy sets as well as the definitions. Section 6.4 is devoted to type-2 fuzzy sets. We stress the problems related to the definition of the basic operations and the terminology. In Sect. 6.5 we analyze a particular case of the previous sets, namely, interval-valued fuzzy sets. We present their properties and different construction methods, depending on the application that we are dealing with. We also refer to the papers in which it is shown that the results that we obtain with these sets are better than those obtained with other techniques. In Sects. 6.6 and 6.7 we describe the sets defined by Atanassov. Section 6.8 explains the links between the considered extensions. In Sect. 6.9 we exhibit some other definitions of fuzzy sets in the literature that do not fall into the scope of our notion of extension. We finish with some conclusions and references.

6.1 Considerations Prior to the Concept of Extension of Fuzzy Sets

In classical logic, propositions can only be either true or false. Aristotle formulated the basic principles of this logic: the noncontradiction principle (a statement cannot be true and false at the same time) and the middle-excluded principle (every statement is either true or false).

It is easy to note that there are many situations for which more than two truth values are needed. This fact led C.S. Peirce to say that Aristotle's formulation is the *simplest hypothesis* we can work with. In fact, meanwhile human knowledge representation is based upon concepts [6.3], and these concepts are not crisp in nature, we should not expect that human beings use binary logic so often in their daily life.

Everyday situations such as taste, meaning of adjectives, etc., can only be studied precisely if gradings more complex than true or false are considered. Even very widely used mathematical models can lead to paradoxes. For instance, quite often we are forced to establish arbitrary cuts in order to make reality fit our binary model.

These considerations led to propose different logical formulations which allowed for more than two truth values, like Brouwer's intuitionistic logic (partially caught by the so-called intuitionistic propositional calculus modeled by Heyting algebras), multivalued logics presented by Lukasiewicz, or Zadeh's fuzzy logic (which replaces the set $\{0,1\}$ by the set $[0,1]$), for example.

6.1.1 Brouwer's Intuitionistic Logic

In 1907, the Dutch mathematician L.E.J. Brouwer (1881–1966) introduced the intuitionistic logic. Between the precursors of intuitionistic logic, we can include Kronecker, Poincare, Borel, or Weyl.

For intuitionistic researchers, the objects of study in Mathematics are just some intuitions of the mind and the constructions that can be made with them. Hence, the intuitionistic mathematics only handles built objects and only recognizes the properties assigned to these objects in their construction. In particular, the negation of the impossibility of a fact is not a construction of such a fact, and so both the double negation principle and the reduction ad absurdum method are not acceptable for the intuitionist. In the same way, it may happen that it is impossible to build both a fact and its negation, so also the middle-excluded principle is excluded by intuitionism.

In 1930 Heyting, a Brouwer's disciple, went one step ahead and defined a propositional calculus in terms of axioms and rules in Hilbert's style. This calculus is known as intuitionistic propositional calculus (intuitionistic logic). For several decades, the research in intuitionism was almost stopped. But it has reappeared with strength in the logic of categories and topos [6.4, 5]. In this sense, the studies by *Takeuti* and *Titani* in 1984 [6.6] on intuitionistic fuzzy logic and intuitionistic fuzzy set theory are of special interest for us. In [6.7], it is settled that

> *Takeuti and Titani's intuitionitic fuzzy logic is simply an extension of intuitionistic logic, i. e., all formulas provable in the intuitionistic logic are provable in their logic. They give a sequent calculus which extends Heyting intuitionistic logic, an extension that does not collapse to classical logic and keeps the flavor of intuitionism.*

6.1.2 Lukasiewicz's Multivalued Logics

In 1920s, Jan Lukasiewicz (1878–1956) along with Lesniewski founded a school of logic in Warsaw that became one of the most important mathematical teams in the world, and among whose members was Alfred Tarski.

Lukasiewicz's idea consists in distributing the truth values uniformly on the $[0,1]$ interval: if n values are considered, they should be $0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1$; if they are infinite, we should take $Q \cap [0,1]$. Negation is defined as $n(x) = 1 - x$, and the following operation is also defined: $x \oplus y = \min(1, x+y)$.

6.1.3 Zadeh's Fuzzy Logic. First Generalization by Goguen

Consistently to Lukasiewicz's studies, *Zadeh* [6.1] introduced fuzzy logic in his 1965 paper, *Fuzzy Sets*. Born in Azerbaijan in 1921, he moved to the University of California at Berkeley in 1959. His ideas on fuzzy sets were soon applied to different areas such as artificial intelligence, natural language, decision making, expert systems, neural networks, control theory, etc.

In mathematics, every subset of a given referential universe U can be identified with its *characteristic function* f; that is, the function $f: U \to \{0,1\}$ which takes the value 1 if the element belongs to the considered subset and 0 in other case. In contrast, a fuzzy set is a mapping from the universe U to $[0,1]$; that is,

Definition 6.1
A fuzzy set (or type-1 fuzzy set) A over a referential set U is an object

$$A = \{(u_i, \mu_A(u_i)) | u_i \in U\} ,$$

where $\mu_A : U \to [0,1]$.

$\mu_A(u_i)$ represents the degree of membership of the element $u_i \in U$ to the set A. The elements for which $\mu_A(u_i) = 1$ belong to the set A; those for which $\mu_A(u_i) = 0$ do not belong to A and there are elements with a greater or smaller degree of membership to A depending on $\mu_A(u_i)$.

We are going to denote by $FS(U)$ the class of fuzzy sets defined over U; that is, $FS(U) \equiv [0,1]^U$. The membership degree of an element $u_i \in U$ to the fuzzy set A is usually denoted by $A(u_i)$ instead of $\mu_A(u_i)$.

From the classical definition of union and intersection for crisp sets, Zadeh proposes the following definitions:

$$\begin{aligned} A \cup B(u_i) &= \max(A(u_i), B(u_i)) , \\ A \cap B(u_i) &= \min(A(u_i), B(u_i)) . \end{aligned} \tag{6.1}$$

A key concept in the following developments is that of lattice. We review now its definition, that can be found for instance in [6.8].

Recall that an order relationship over a set L is a relation $\leq_L$ such that

i) $x \leq_L x$ for all $x \in L$ (reflexivity);
ii) if $x \leq_L y$ and $y \leq_L z$ then $x \leq_L z$ for any $x, y, z \in L$ (transitivity);
iii) if $x \leq_L y$ and $y \leq_L x$, then $x = y$, for any $x, y \in L$ (antisymmetry).

If $\leq_L$ is an order relationship over L then $(L, \leq_L)$ is called a partially ordered set. Now, in order to define a lattice we need first to introduce the following definition.

Definition 6.2

Let $(L, \leq_L)$ be a partially ordered set and $A \subset L$ (in the sense of the usual set theory). The greatest lower bound of A (if it exists) is the element $x_{\inf} \in L$ such that:

i) $x_{\inf} \leq_L z$ for all $z \in A$ and
ii) for any $y \in L$ such that $y \leq_L z$ for all $z \in A$ it follows that $y \leq_L x_{\inf}$.

Analogously, the least upper bound of A (if it exists) is the element $x_{\sup} \in L$ such that

i) $z \leq_L x_{\sup}$ for all $z \in A$ and
ii) for any $y \in L$ such that $z \leq_L y$ for all $z \in A$ it follows that $x_{\sup} \leq_L y$.

Now we can introduce the notion of lattice.

Definition 6.3

A lattice is a partially ordered set $(L, \leq_L)$ such that any two elements $x, y \in L$ have the greatest lower bound or meet, denoted by $x \wedge y$ and the lowest upper bound or join, denoted by $x \vee y$. A lattice L is called complete if any subset of L has the lowest upper bound and the greatest lower bound.

Given a lattice $(L \leq_L)$, we will call supremum of L and denote by 1_L the lowest upper bound of L (if it exists). Analogously, we will call the infimum of L and denote by 0_L the greatest lower bound of L. In case both 1_L and 0_L exist, L is called a bounded lattice.

Observe that if we know how the join and meet operations are defined for any two elements of a set L, we can recover the ordering $\leq_L$ just by defining for any $x, y \in L$

$$x \leq_L y \text{ if and only if } x \wedge y = x$$
$$\text{if and only if } x \vee y = y$$

Taking into account (6.1) and Definition 6.3, it is easy to prove the following theorem.

Theorem 6.1

$(FS(U), \cup, \cap)$ is a complete lattice.

From Theorem 6.1 and the concept of lattice, we can define the following partial order relation: For $A, B \in FS(U)$

$$A \leq_{FS} B \text{ if and only if } A(u_i) \leq B(u_i)$$
$$\text{for every } u_i \in U\,.$$

The first criticism to fuzzy sets theory arises from this order relation $\leq_{FS}$. Since Zadeh presented fuzzy sets to represent uncertainty, it comes out that $\leq_{FS}$ is a crisp relation. Note that the following may happen: Let U be a referential set with 1000 elements and let A and B be two fuzzy sets over U such that for every element except for one $A(u_i) \leq B(u_i)$. Then, from the previous relation, A is not less than B. This fact led *Willmott* [6.9], *Bandler* and *Kohout* [6.10] and others to consider the concept of inclusion measure. These measures have been widely used in fuzzy morphologic mathematics [6.11], in image processing [6.12], etc.

It is easy to see that with the operations defined in (6.1) and the standard negation, $n(x) = 1 - x$ for all $x \in [0, 1]$, neither the noncontradiction principle nor the middle excluded principle hold. Nowadays, operations in (6.1) are given in terms of t-norms and t-conorms [6.13–16].

Definition 6.1 can be clearly extended to consider mappings valued over any kind of set. In particular, for our future developments and following Goguen's work [6.2], it is interesting to consider the case of mappings that take values over a lattice L. In this case, we speak of L-fuzzy sets.

Taking into account Definition 6.3 Goguen presents the concept of L-fuzzy set as follows:

Definition 6.4

Let $(L, \vee, \wedge)$ be a lattice. An L-fuzzy set over the referential set U is a mapping

$$A \colon U \to L\,.$$

For a given lattice L, we will denote by L-$FS(U)$, the space of L-fuzzy sets over the referential U. That is, L-$FS(U) \equiv L^U$.

Union and intersection of L-fuzzy sets can be easily defined as follows.

Definition 6.5
Let L be a lattice, and let $\vee$ and $\wedge$ be its join and meet operators respectively. Then intersection and union are defined, respectively, by:

i)
$$\cap_L : L\text{-}FS(U) \times L\text{-}FS(U) \to L\text{-}FS(U) \text{ given by}$$
$$\cap_L (A,B)(u_i) = A(u_i) \wedge B(u_i) \ .$$

In order to recover the usual notation for fuzzy sets, we will write $\cap_L(A,B)$ as $A \cap_L B$;

ii)
$$\cup_L : L\text{-}FS(U) \times L\text{-}FS(U) \to L\text{-}FS(U) \text{ given by}$$
$$\cup_L (A,B)(u_i) = A(u_i) \vee B(u_i) \ .$$

In order to recover the usual notation for fuzzy sets, we will write $\cup_L(A,B)$ as $A \cup_L B$.

We can state the following result for L-fuzzy sets.

Proposition 6.1
Let L be a bounded lattice with a supremum given by 1_L and an infimum given by 0_L. Let $\vee$ and $\wedge$ be the join and meet operators of L, respectively. Then, the set $(L\text{-}FS(U), \le_{L\text{-}FS(U)})$ is a bounded lattice, where the order is defined as

$$A \le_{L\text{-}FS(U)} B \text{ if and only if } A \cup_L B = B$$

or equivalently

$$A \le_{L\text{-}FS(U)} B \text{ if and only if } A \cap_L B = A \ .$$

That is

$$A \le_{L\text{-}FS(U)} B \text{ if and only if } A(u_i) \vee B(u_i) = B(u_i) \text{ for all } u_i \in U$$

or equivalently

$$A \le_{L\text{-}FS(U)} B \text{ if and only if } A(u_i) \wedge B(u_i) = A(u_i) \text{ for all } u_i \in U \ .$$

The supremum of this lattice is given by

$$1_{L\text{-}FS(U)} : U \to L \ , \quad u_i \to 1_L$$

and the infimum is given by

$$0_{L\text{-}FS(U)} : U \to L \quad u_i \to 0_L \ .$$

6.2 Origin of the Extensions

In 1971, *Zadeh* in his paper [6.17] settled that the construction of the fuzzy sets, that is, the determination of the membership degree of each element to the set, is the biggest problem for using fuzzy sets theory in applications. This fact led him to introduce the concept of type-2 fuzzy set.

Later, in December 11, 2008, in the *bisc-group* mail list Zadeh proposes the following definitions.

Definition 6.6
Fuzzy logic is a precise system of reasoning, deduction, and computation in which the objects of discourse and analysis are associated with information which is, or is allowed to be, imperfect.

Definition 6.7
Imperfect information is defined as information which in one or more respects is imprecise, uncertain, vague, incomplete, partially true, or partially possible.

On the same date and place, Zadeh made the following remarks:

1. In fuzzy logic everything is or is allowed to be a matter of degree. Degrees are allowed to be fuzzy.
2. Fuzzy logic is not a replacement for bivalent logic or bivalent-logic-based probability theory. Fuzzy logic adds to bivalent logic and bivalent-logic-based probability theory a wide range of concepts and techniques for dealing with imperfect information.
3. Fuzzy logic is designed to address problems in reasoning, deduction, and computation with imperfect information which are beyond the reach of traditional methods based on bivalent logic and bivalent-logic-based probability theory.
4. In fuzzy logic the writing instrument is a spray pen (Fig. 6.1) with a precisely known adjustable spray pattern. In bivalent logic the writing instrument is a ballpoint pen.

5. The importance of fuzzy logic derives from the fact that in much of the real-world imperfect information is the norm rather than exception.

All these considerations justify the use of fuzzy sets theory whenever objects are linked to soft concepts, those that do not show clear boundaries. Of course, applications might require tools other than fuzzy [6.18]. In any case, if we decide to use fuzzy sets and it is hard for us to build the characteristic functions of the involved sets, then we must use set representations that take into account these difficulties, and focus on those fuzzy sets that we call *extensions*.

So the origin of the concept of extension of fuzzy sets is directly associated with the idea of building fuzzy sets that allow us to represent objects that are described through imperfect information, and that also allow us to represent the lack of knowledge or uncertainty associated with the membership degrees that are given by the experts.

It is clear that working with extensions implies that we need to use more information than in the basic model of Zadeh. As already pointed out, in order to justify the use of these extensions in practice, the results obtained with them must be better than those obtained with usual fuzzy sets.

6.3 Type-2 Fuzzy Sets

The idea of taking into account the experts' uncertainty when they build the membership degrees of the elements to a given fuzzy sets led *Zadeh* to present in 1971 the notion of type-2 fuzzy set [6.17] as follows: A type-2 fuzzy set is a fuzzy set over a referential set U for which the membership degrees of the elements are given by fuzzy sets defined over the referential set $[0,1]$.

The mathematical formalization of this concept was made in 1976 by *Mizumoto* and *Tanaka* in [6.19] and in 1979 by *Dubois* and *Prade* in [6.20] as follows:

Definition 6.8
A type-2 fuzzy set is a mapping $A\colon U \to FS([0,1])$.

In Fig. 6.2 we show an example of type-2 fuzzy set.

We denote by $T2FS(U)$ the set of all type-2 fuzzy sets over U. That is

$$T2FS(U) \equiv (FS([0,1]))^U .$$

6.3.1 Type-2 Fuzzy Sets as a Lattice

From Definition 6.8, the following result is obvious.

Corollary 6.1
Type-2 fuzzy sets are a particular type of Goguen's L-fuzzy sets.

Taking into account Corollary 6.1, it is clear that we can define the following operations over type-2 fuzzy sets [6.21].

Definition 6.9
The operations of union $\cup_{T2}$ and intersection $\cap_{T2}$ of $A, B \in T2FS(U)$ (in the sense of lattices) are defined, respectively, as

$$\cup_{T2}(A,B)\colon U \to FS([0,1]) \text{ given by}$$
$$A \cup_{T2} B(u_i) = A(u_i) \cup B(u_i)$$

and

$$\cap_{T2}(A,B)\colon U \to FS([0,1]) ,$$
$$A \cap_{T2} B(u_i) = A(u_i) \cap B(u_i) .$$

Proposition 6.2
The set $(T2FS(U), \cup_{T2}, \cap_{T2})$ is a bounded lattice with respect to the order

$$A \leq_{T2FS(U)} B \text{ if and only if } A \cup_{T2} B = B$$

or equivalently

$$A \leq_{T2FS(U)} B \text{ if and only if } A \cap_{T2} B = A .$$

That is

$$A \leq_{T2FS(U)} B \text{ if and only if } A(u_i) \cup B(u_i) = B(u_i) \text{ for all } u_i \in U$$

or equivalently

$$A \leq_{T2FS(U)} B \text{ if and only if } A(u_i) \cap B(u_i) = A(u_i) \text{ for all } u_i \in U .$$

The supremum of this lattice is given by $1_{T2FS(U)}\colon U \to FS(U)$ where, for every $u_i \in U$, $1_{T2FS(U)}(u_i)$ is

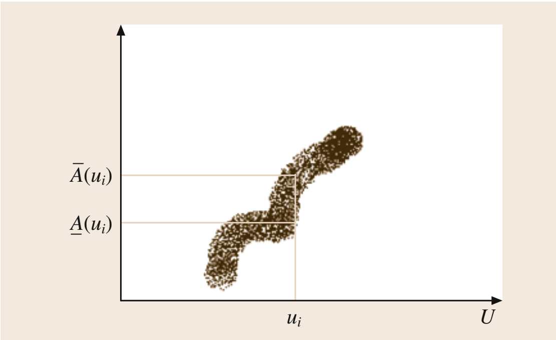

Fig. 6.1 The writing instrument is a spray pen

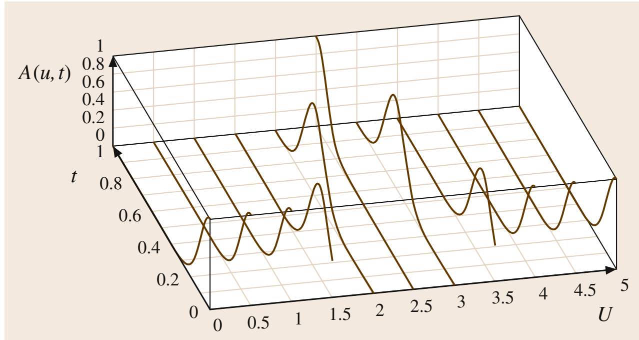

Fig. 6.2 Example of a type-2 fuzzy set

the fuzzy sets that assigns to every $t \in [0,1]$ membership equal to 1. The infimum is given by $0_{T2FS(U)}: U \to FS(U)$ where, for every $u_i \in U$, $0_{T2FS(U)}(u_i)$ is the fuzzy sets that assigns to every $t \in [0,1]$ membership equal to 0.

6.3.2 Remarks on the Notation

Mizumoto and *Tanaka* in 1976 [6.19] and *Mendel* and *John* in 2000 [6.22] used the following notation:

$$\int_{u\in U}\int_{t\in J_u} \frac{A(u,t)}{(u,t)}, \qquad J_u \subset [0,1] ,$$

where J_u is the primary membership of $u \in U$ and, for each fixed $u = u_0$, the fuzzy set $\int_{t\in J_{u_0}} A(u_0,t)/t$ is the secondary membership of u_0.

From our point of view, this notation is not the most appropriate one, so now we try to introduce a more clarifying notation. Observe that a type-2 fuzzy set assigns to an element in the referential U a mapping $A(u): [0,1] \to [0,1]$. To represent fuzzy sets (or type-1 fuzzy sets) defined by a mapping A it is quite usual the notation

$$\{(u_i, A(u_i)) \mid u \in U\} . \qquad (6.2)$$

In this type-1 case, $A(u)$ is a real number in $[0,1]$ for every $u_i \in U$. In the case of type-2 fuzzy sets, if we imitate this notation, we formally lead to $\{(u_i, A(u_i)) \mid u_i \in U\}$. But now for each $u_i \in U$, we have that $A(u_i)$ is not a real number but a mapping (a type-1 fuzzy set)

$$\begin{aligned} A(u)&:[0,1] \to [0,1] ,\\ t &\to A(u)(t) . \end{aligned}$$

Taking into account these considerations *Harding* et al. [6.21] and *Aisbett* et al. [6.23] suggested the following notation for a type-2 fuzzy set A:

$$A = \{(u_i, (t, A(u_i)(t)) | u_i \in U , t \in [0,1]\} .$$

But an easier one to use one could be the following.

Definition 6.10

Let $A: U \to FS([0,1])$ be a type-2 fuzzy set. Then A is denoted as

$$\{(u_i, A(u_i,t)) | u_i \in U , t \in [0,1]\} .$$

where $A(u_i,\cdot): [0,1] \to [0,1]$ is defined as $A(u_i,t) = A(u_i)(t)$.

6.3.3 A First Definition of Operations Between Type-2 Fuzzy Sets: Lattice-Based Approach

With Definition 6.10, if we have two type-2 fuzzy sets

$$A = \{(u_i, (A(u_i,t)) \mid u_i \in U , \ t \in [0,1]\}$$

and

$$B = \{(u_i, (B(u_i,t)) \mid u_i \in U , \ t \in [0,1]\}$$

we have (Fig. 6.3)

$$A \cup_{T2FS} B = \{(u_i, A \cup B(u_i,t)) \mid u_i \in U , \ t \in [0,1]\} ,$$

where, for each $u_i \in U$ and each $t \in [0,1]$, we have

$$\begin{aligned} A \cup B(u_i,t) &= \max(A(u_i,t), B(u_i,t)) \\ &= \max(A(u_i)(t), B(u_i)(t)) \qquad (6.3) \end{aligned}$$

Analogously,

$$A \cap_{T2FS} B = \{(u_i, A \cap B(u_i, t)) \mid u_i \in U,\ t \in [0,1]\}\ ,$$

where, for each $u_i \in U$ and each $t \in [0,1]$, we have

$$\begin{aligned} A \cap B(u_i, t) &= \min(A(u_i, t), B(u_i, t)) \\ &= \min(A(u_i)(t), B(u_i)(t)) \end{aligned} \tag{6.4}$$

Observe that this notation is very similar to that proposed by *Deschrijver* and *Kerre* [6.24, 25].

6.3.4 Problems with the Lattice-Based Definitions. Operations Based on Zadeh's Extension Principle

Although meaningful from a mathematical point of view, as pointed out by *Dubois* and *Prade* in [6.26], from these definitions we do not recover the usual ones for fuzzy sets. To see it, just consider a finite referential set $U = \{u_1, u_2, u_3\}$ with three elements, and consider the following two fuzzy sets over U. We use the notation of (6.2) for the sake of brevity.

$$A = \left\{\left(u_1, \frac{1}{2}\right), \left(u_2, \frac{1}{3}\right), (u_3, 1)\right\}$$

and

$$B = \left\{\left(u_1, \frac{1}{4}\right), \left(u_2, \frac{1}{2}\right), \left(u_3, \frac{1}{7}\right)\right\}\ .$$

Then we have, for instance,

$$A \cup B = \left\{\left(u_1, \frac{1}{2}\right), \left(u_2, \frac{1}{2}\right), (u_3, 1)\right\}$$

On the other hand, we can also see A and B as type-2 fuzzy sets, that we denote by A_2 and B_{T2}, respectively, just taking

$$A_{T2}(u_1)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{2} \\ 0 & \text{in other case} \end{cases}\ ,$$

$$A_{T2}(u_2)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{3} \\ 0 & \text{in other case} \end{cases}\ ,$$

$$A_{T2}(u_3)(t) = \begin{cases} 1 & \text{if } t = 1 \\ 0 & \text{in other case} \end{cases}\ ,$$

and

$$B_{T2}(u_1)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{4} \\ 0 & \text{in other case} \end{cases}\ ,$$

$$B_{T2}(u_2)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{2} \\ 0 & \text{in other case} \end{cases}\ ,$$

$$B_{T2}(u_3)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{7} \\ 0 & \text{in other case} \end{cases}\ .$$

Then we have

$$A_{T2} \cup_{T2FS} B_{T2}(u_1)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{4} \text{ or } t = \frac{1}{2} \\ 0 & \text{in other case} \end{cases}\ ,$$

$$A_{T2} \cup_{T2FS} B_{T2}(u_2)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{2} \text{ or } t = \frac{1}{3} \\ 0 & \text{in other case} \end{cases}\ ,$$

and

$$A_{T2} \cup_{T2FS} B_{T2}(u_3)(t) = \begin{cases} 1 & \text{if } t = \frac{1}{7} \text{ or } t = 1 \\ 0 & \text{in other case} \end{cases}\ ,$$

which does not coincide with our previous result. Moreover, observe that we do not even recover a fuzzy set but a *true* type-2 fuzzy set.

In order to solve this problem, several authors [6.19, 22, 26] proposed the following definitions of the operations of union and intersection.

6.3.5 Second Definition of the Operations: Zadeh's Extension Principle Approach

Definition 6.11
Given two type-2 fuzzy sets

$$A = \{(u_i, A(u_i, t)) \mid u_i \in U\ ,\ t \in [0,1]\}$$

and

$$B = \{(u_i, B(u_i, t)) \mid u_i \in U\ ,\ t \in [0,1]\}$$

we can define (Fig. 6.4)

$$A \sqcap B = \{(u_i, A \sqcap B(u_i, t)) \mid u_i \in U\ ,\ t \in [0,1]\}$$

with

$$A \sqcap B(u_i, t) = \sup_{\min(z,w)=t} \min(A(u_i, z), B(u_i, w))$$

and

$$A \sqcup B = \{(u_i, A \sqcup B(u_i, t)) \mid u_i \in U \ , \ t \in [0,1]\}$$

with

$$A \sqcup B(u_i, t) = \sup_{\max(z,w)=t} \min(A(u_i, z), B(u_i, w)) \ .$$

For instance, let us recover our previous example. Consider the type-2 fuzzy sets A_{T2} and B_{T2}. Then we have that

$$A_{T2} \sqcup B_{T2}(u_1, t) = \begin{cases} 0 & \text{if } t \notin \{\frac{1}{4}, \frac{1}{2}\} \\ \sup\limits_{\max(z,w)=t} & (\min(A_{T2}(u_1, z), B_{T2}(u_1, w))) \ . \\ & \text{in other case} \end{cases}$$

But if $t = \frac{1}{4}$, then, as $\frac{1}{2} > \frac{1}{4}$ and since $A_{T2}(u_1, z) = 0$ for all $z \leq \frac{1}{4}$, it follows that $\min(A_{T2}(u_1, z), B_{T2}(u_1, w)) = 0$ whenever $\max(z, w) = \frac{1}{4}$. Finally, if $t = \frac{1}{2}$, then $\min(A_{T2}(u_1, \frac{1}{2}), B_{T2}(u_1, \frac{1}{4})) = 1$, so we finally arrive at

$$A_{T2} \sqcup B_{T2}(u_1, t) = \begin{cases} 0 & \text{if } t \neq \frac{1}{2} \\ 1 & \text{if } t = \frac{1}{2} \end{cases} \ .$$

Since for u_2 and u_3 the same arguments work, we see that we indeed recover the fuzzy case. In particular, with respect to these new operations, we have the following result [6.21].

Proposition 6.3

Let U be a referential set. $(T2FS(U), \sqcup, \sqcap)$ is not a lattice.

In fact, the problem is that the absorption laws

$$A \sqcap (A \sqcup B) = A$$

and

$$A \sqcup (A \sqcap B) = A$$

do not hold. Nevertheless, it is also possible to provide a positive result [6.21].

Proposition 6.4

Let U be a referential set. Then for any $A, B, C \in T2FS(U)$ the following properties hold:

i) $A \sqcup A = A$ and $A \sqcap A = A$;
ii) $A \sqcup B = B \sqcup A$ and $A \sqcap B = B \sqcap A$;
iii) $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$.

That is, $(T2FS(U), \sqcup, \sqcap)$ is a bisemilattice.

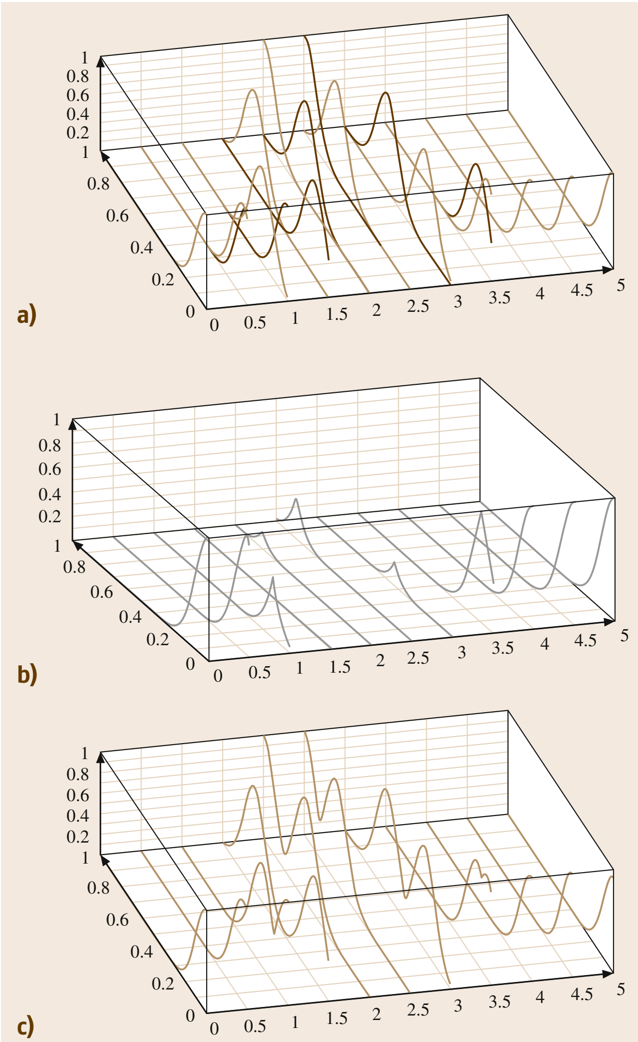

Fig. 6.3a–c Two different type 2 fuzzy sets **(a)** $A \cup_{T2FS} B$ **(b)** $A \cap_{T2FS} B$ **(c)**

Remark 6.1

We should remark the following:

1. If we work with the operations defined in Eqs. (6.3) and (6.4), and consider fuzzy sets as particular instances of type-2 fuzzy sets, then we do not recover the classical operations defined by Zadeh.
2. If we use the operations in Definition 6.11, then we recover Zadeh's classical operations for fuzzy sets, but we do not have a lattice structure. This fact makes that the use of type-2 fuzzy sets in many

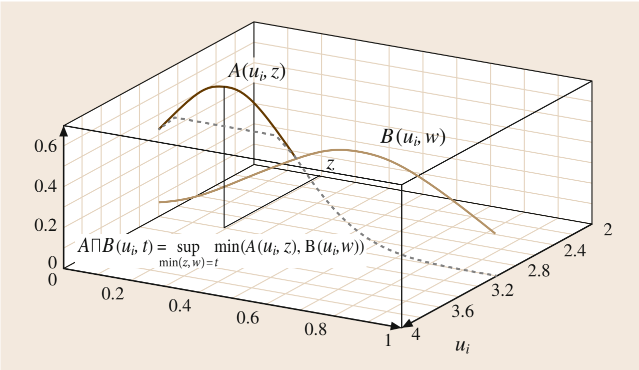

Fig. 6.4 Example of intersection of two membership sets $A(u_i, t)$ and $B(u_i, t)$. *Green line* is the set obtained

applications, such as decision making, is very complicate.

Obviously, an interesting problem is to analyze which further conclusions and results can be obtained from this new formulation of the operations between type-2 fuzzy sets.

6.3.6 About Computational Efficiency

Note also that although the computational complexity and the efficiency in time of type-2 fuzzy sets are not as high as used to be a few years ago, it is clear that the use of these kinds of sets introduces additional complexity in any given problem. For this reason, many times the possible improvement of results is not as big as replacing type-1 fuzzy sets by type-2 fuzzy sets in many applications.

On the other hand, we can also define type-3 fuzzy sets as those fuzzy sets whose membership of each element is given by a type-2 fuzzy set [6.27]. Even more, it is possible to define recursively type-n fuzzy sets as those fuzzy sets whose membership values are type-$(n-1)$ fuzzy sets. The computational efficiency of these sets decreases as the complexity level of the building increases. From a theoretical point of view, we consider that it is necessary to carry out a complete analysis of type-n fuzzy sets structures and operations. But up to now no applications has been developed on the basis of a type-n fuzzy sets.

6.3.7 Applications

It is worth to mention the works by *Mendel* in computing with words and perceptual computing [6.28–31], of *Hagras* [6.32, 33], of *Sepulveda* et al. [6.34] in control, of *Xia* et al. in mobiles [6.35] and of *Wang* in neural networks [6.36]. We will see in the next section that the advantage of using these kinds of sets versus usual fuzzy sets has been shown only for a particular type of them, namely, the so-called interval-valued fuzzy sets.

6.4 Interval-Valued Fuzzy Sets

These sets were introduced in the 1970s. In May 1975, *Sambuc* [6.37] presented, in his doctoral thesis, the concept of an interval-valued fuzzy set named a Φ-fuzzy set. That same year, *Jahn* [6.38] wrote about these sets. One year later, *Grattan-Guinness* [6.39] established a definition of an interval-valued membership function. In that decade interval-valued fuzzy sets appeared in the literature in various guises and it was not until the 1980s, with the work of *Gorzalczany* and *Türksen* [6.40–45], that the importance of these sets, as well as their name, was definitely established.

Let us denote by $L([0,1])$ the set of all closed subintervals in $[0,1]$, that is,

$$L([0,1]) = \left\{\mathbf{x} = \left[\underline{x}, \overline{x}\right] \mid \left(\underline{x}, \overline{x}\right) \in [0,1]^2 \text{ and } \underline{x} \leq \overline{x}\right\} . \tag{6.5}$$

Definition 6.12
An interval-valued fuzzy set (or interval type-2 fuzzy set) A on the universe $U \neq \emptyset$ is a mapping

$$A\colon U \to L([0,1]) ,$$

such that the membership degree of $u \in U$ is given by $A(u) = [\underline{A}(u), \overline{A}(u)] \in L([0,1])$, where $\underline{A}\colon U \to [0,1]$ and $\overline{A}\colon U \to [0,1]$ are mappings defining the lower and the upper bounds of the membership interval $A(u)$, respectively (Fig. 6.5).

From Definition 6.12, it is clear that for these sets the membership degree of each element $u_i \in U$ to A is given by a closed subinterval in $[0,1]$; that is, $A(u_i) = [\underline{A}(u_i), \overline{A}(u_i)]$. Obviously, if for every $u_i \in U$, we have $\underline{A}(u_i) = \overline{A}(u_i)$, then the considered set is a fuzzy set. So

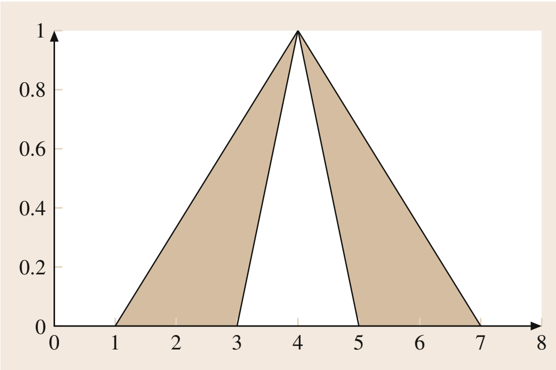

Fig. 6.5 Example of interval valued fuzzy set

fuzzy sets are particular cases of interval-valued fuzzy sets.

In 1989, *Deng* [6.46] presented the concept of Grey sets. Later Dubois proved that these are also interval-valued fuzzy sets.

We denote by $IVFS(U)$ the class of all interval-valued fuzzy sets over U; that is, $IVFS(U) \equiv L([0,1])^U$. From Zadeh's definitions of union and intersections, Sambuc proposed the following definition:

Definition 6.13
Given $A, B \in IVFS(U)$.

$$\begin{aligned} A \cup_{L([0,1])} B(u_i) = [\,&\max(\underline{A}(u_i), \underline{B}(u_i)),\\ &\max(\overline{A}(u_i), \overline{B}(u_i))] \\ A \cap_{L([0,1])} B(u_i) = [\,&\min(\underline{A}(u_i), \underline{B}(u_i)),\\ &\min(\overline{A}(u_i), \overline{B}(u_i))] \end{aligned}$$

These operations can be generalized by the use of the widely analyzed concepts of IV t-conorm and IV t-norm [6.47–49].

Corollary 6.2
Interval valued fuzzy sets are a particular case of L-fuzzy sets.

Proof: Just note that $L([0,1])$ with the operations in Definition 6.13 is a lattice. ■

Proposition 6.5
The set $(IVFS(U), \cup_{L([0,1])}, \cap_{L([0,1])})$ is a bounded lattice, where the order is defined as

$$A \leq_{IVFS(U)} B \text{ if and only if } A \cup_{L([0,1])} B = B$$

or equivalently

$$A \leq_{IVFS(U)} B \text{ if and only if } A \cap_{L([0,1])} B = A\,.$$

That is

$$\begin{aligned} &A \leq_{IVFS(U)} B \text{ if and only if}\\ &\max(\underline{A}(u_i), \underline{B}(u_i)) = \underline{B}(u_i) \text{ and}\\ &\max(\overline{A}(u_i), \overline{B}(u_i)) = \overline{B}(u_i) \end{aligned}$$

for all $u_i \in U$, or equivalently

$$\begin{aligned} &A \leq_{IVFS(U)} B \text{ if and only if}\\ &\min(\underline{A}(u_i), \underline{B}(u_i)) = \underline{A}(u_i) \text{ and}\\ &\min(\overline{A}(u_i), \overline{B}(u_i)) = \overline{A}(u_i) \end{aligned}$$

for all $u_i \in U$.

From Proposition 6.5, we deduce that the order $A \leq_{IVFS(U)} B$ if and only if $\underline{A}(u_i) \leq \underline{B}(u_i)$ and $\overline{A}(u_i) \leq \overline{B}(u_i)$ for all $u_i \in U$ is not linear. The use of these sets in decision making has led several authors to consider the problem of defining total orders between intervals [6.50]. In this sense, in [6.51] a construction method for such orders by means of aggregation functions can be found.

6.4.1 Two Interpretations of Interval-Valued Fuzzy Sets

From our point of view, interval-valued fuzzy sets can be understood in two different ways [6.52]:

1. The membership degree of an element to the set is a value that belongs to the considered interval. The interval representation is used since we cannot say precisely which that number is. For this reason, we provide bounds for that number. We think this is the correct interpretation for these sets.
2. The membership degree of each element is the whole closed subinterval provided as membership. From a mathematical point of view, this interpretation is very interesting, but, in our opinion, it is very difficult to understand it in the applied field. Moreover, in this case, we find the following paradox [6.53]:
   For fuzzy sets and with the standard negation it holds that $\min(A(u_i), 1 - A(u_i)) \leq 0.5$ for all $u_i \in U$. But for interval-valued fuzzy sets, if we use the stan-

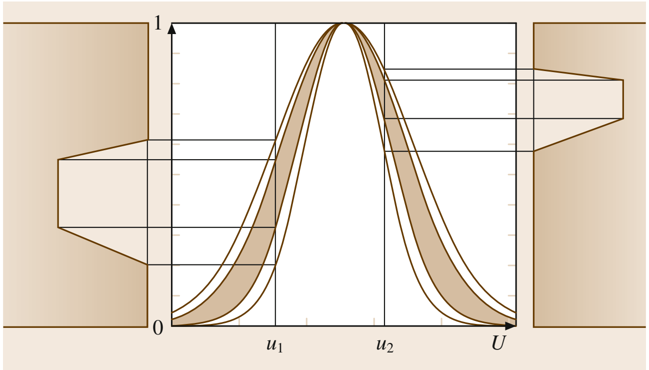

Fig. 6.6 Construction of type-2 fuzzy sets from interval-valued fuzzy sets

dard negation $N(A(u_i)) = [1-\overline{A}(u_i), 1-\underline{A}(u_i)]$, we have that there is no equivalent bound for

$$\min\left[\underline{A}(u_i), \overline{A}(u_i)\right], \left[1-\overline{A}(u_i), 1-\underline{A}(u_i)\right] .$$

6.4.2 Shadowed Sets Are a Particular Case of Interval-Valued Fuzzy Sets

The so-called shadow sets were suggested by *Pedrycz* [6.54] and developed later together with *Vukovic* [6.55, 56]. A shadowed set B induced by a given fuzzy set A defined in U is an interval-valued fuzzy set in U that maps elements of U into 0,1 and the unit interval [0, 1], i.e., B is a mapping $B\colon U \to \{0, 1, [0, 1]\}$, where 0, 1, [0, 1] denote complete exclusion from B, complete inclusion in B and complete ignorance, respectively. Shadow sets are isomorphic with a three-valued logic.

6.4.3 Interval-Valued Fuzzy Sets Are a Particular Case of Type-2 Fuzzy Sets

In 1995, *Klir* and *Yuan* proved in [6.27] that from an interval-valued fuzzy set, we can build a type-2 fuzzy set as pointed out in Fig. 6.6.

Later in 2007 *Deschrijver* and *Kerre* [6.24, 25] and *Mendel* [6.57], proved that interval-valued fuzzy sets are particular cases of type-2 fuzzy sets.

6.4.4 Some Problems with Interval-Valued Fuzzy Sets

1. Taking into account the definition of interval-valued fuzzy sets, we follow *Gorzalczany* [6.41] and define the compatibility degree between two interval-valued fuzzy sets as an element in $L([0,1])$. The other information measures [6.58–62] (interval-valued entropy, interval-valued similarity, etc.) should also be given by an interval. However, in most of the works about these measures, the results are given by a number, and not by an interval. This consideration leads us to settle that, from a theoretical point of view, we should distinguish between two different types of information measures: those which give rise to a number and those which give rise to an interval. Obviously, the problem of interpreting both types of measures arises. Moreover, if the result of the measure is an interval, we should consider its amplitude as a measure of the lack of knowledge [6.63] linked to the considered measure.
2. In [6.57], *Mendel* writes:

   It turns out that an interval type-2 fuzzy set is the same as an interval-valued fuzzy set for which there is a very extensive literature. These two seemingly different kinds of fuzzy sets were historically approached from very different starting points, which as we shall explain next has turned out to be a very good thing.

   Nonetheless, we consider that interval-valued fuzzy sets are a particular case of interval type-2 fuzzy sets and therefore they are not the same thing.
3. Due to the current characteristics of computers, we can say that the computational cost of working with these sets is not much higher than the cost of working with type-1 fuzzy sets [6.64].
4. We have already said that the commonly used order is not linear. This is a problem for some applications, such as decision making. In [6.65], it is shown that the choice of the order should depend on the considered application. Often experts do not have enough information to choose a total order. This is a big problem since the choice of the order influences strongly the final outcome.

6.4.5 Applications

We can say that there already exist applications of interval-valued fuzzy sets that provide results which are better than those obtained with fuzzy sets. For instance:

1. *In classification problems*. Specifically, in [6.66–69] a methodology to enhance the performance of fuzzy rule-based classification systems (FRBCSs)

is presented. The methodology used in these papers has the following structure:

1) An initial FRBCS is generated by using a fuzzy rule learning algorithm.
2) The linguistic labels of the learned fuzzy rules are modeled with interval-valued fuzzy sets in order to take into account the ignorance degree associated with the assignment of a number as the membership degree of the elements to the sets. These sets are constructed starting from the fuzzy sets used in the learning process and their shape is determined by the value of one or two parameters.
3) The fuzzy reasoning method is extended so as to take into account the ignorance represented by the interval-valued fuzzy sets throughout the inference process.
4) The values of the system's parameters, for instance the ones determining the shape of the interval-valued fuzzy sets, are tuned applying evolutionary algorithms. See [6.66–69] for details about the specific features of each proposal.

The methodology allows us to statistically outperforming the performance of the following approaches:

a) In [6.66], the performance of the initial FRBCS generated by the *Chi* et al. algorithm [6.70] and the fuzzy hybrid genetics-based machine learning method [6.71] are outperformed. In addition, the results of the GAGRAD (genetic algorithm gradient) approach [6.72] are notably improved.
b) A new tuning approach is defined in [6.67], where the results obtained by the tuning of the lateral position of the linguistic labels ([6.73]) and the performance provided by the tuning approach based on the linguistic 3-tuples representation [6.74] are outperformed.
c) Fuzzy decision trees (FDTs) are used as the learning method in [6.68]. In this contribution, numerous decision trees are enhanced, including crisp decision trees, FDTs, and FDTs constructed using genetic algorithms. For instance, the well-known C4.5 decision tree ([6.75]) or the fuzzy decision tree proposed by *Janikow* [6.76] is outperformed.
d) The proposal presented in [6.69] is the most remarkable one, since it allows outperforming two state-of-the-art fuzzy classifiers, namely, the FARC-HD method [6.77] and the unordered fuzzy rule induction algorithm (FURIA) [6.78]. Furthermore, the performance of the fuzzy counterpart of the presented approach is outperformed as well.

2. *Image processing*. In [6.63, 79–85], it has been shown that if we use interval-valued fuzzy sets to represent those areas of an image for which the experts have problems to build the fuzzy membership degrees, then edges, segmentation, etc., are much better.
3. In some decision-making problems, it has also been shown that the results obtained with interval-valued fuzzy sets are better than the ones obtained with fuzzy sets [6.86]. They have also been used in Web problems [6.87], pattern recognition [6.88], medicine [6.89], etc., see also [6.90, 91].

Construction of Interval-Valued Fuzzy Sets

In many cases, it is easier for experts to give the membership degrees by means of numbers instead of by means of intervals. In this case it may happen that the obtained results are not the best ones. If this is so, we should build intervals from the numerical values provided by the experts. For this reason, we study methods to build intervals from real numbers. For any such methods, we require the following:

i) The numerical value provided by the expert should be interior to the considered interval. We require this property since we assume that the membership degree for the expert is a number but he or she is not able to fix it exactly so he or she provides two bounds for it.
ii) The amplitude of the built interval is going to represent the degree of ignorance of the expert to fix the numerical value he or she has provided us.

The previous considerations have led us to define in [6.63] the concept of ignorance degree G_i associated with the value given by an expert. In such definition, it is settled that if the degree of membership given by the expert is equal to 0 or 1, then the ignorance is equal to 0, since the expert is sure of the fact that the element belongs or does not belong to the considered set. However, if the provided membership degree is equal to 0.5, then ignorance is maximal, since the expert does not know at all whether the element belongs or not to the set. Different considerations and construction methods for such ignorance functions using overlap functions can be found in [6.92].

Taking into account the previous argumentation, in Fig. 6.7 we show the schema of construction of an interval from a membership degree μ given by the expert

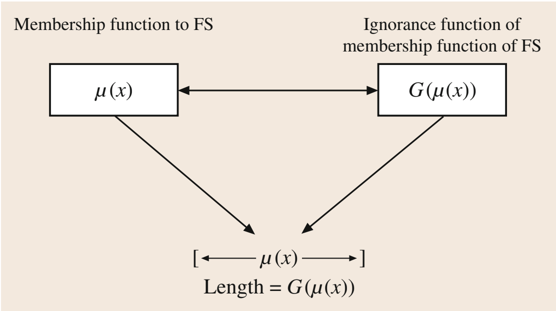

Fig. 6.7 Construction with ignorance functions

and from an ignorance function G_i chosen for the considered problem [6.63]:

There exist other methods for constructing interval-valued fuzzy sets. The choice of the method depends on the application we are working in. One of the most used methods in magnetic resonance image processing (for fuzzy theory) is the following: several doctors are asked for building, for an specific region of an image, a fuzzy set representing that region. At the end, we will have several fuzzy sets, and with them we build an interval-valued fuzzy set as follows. For each element's membership, we take as lower bound the minimum of the values provided by the doctors, and as the upper bound, the maximum. This method has shown itself very useful in particular images [6.83]. In Fig. 6.8, we represent the proposed construction.

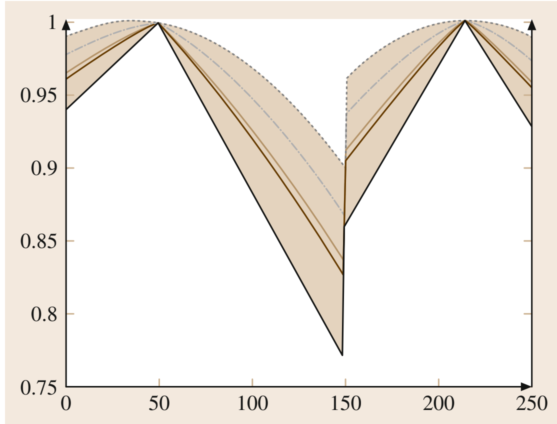

Fig. 6.8 Construction with different experts

In [6.63], it is shown that for some specific ultrasound images, if we use fuzzy theory to obtain the objects in the image, results are worse than if we use interval-valued fuzzy sets using the method proposed by *Tizhoosh* in [6.84]. Such method consists of the following (see Fig. 6.9): from the numerical membership degree μ_A given by the expert and from a numerical coefficient $\alpha > 1$, associated with the doubt of the expert when he or she constructs μ_A we generate the membership interval

$$\left[\mu_{A_t}^{\alpha}(q), \mu_{A_t}^{\frac{1}{\alpha}}(q)\right] .$$

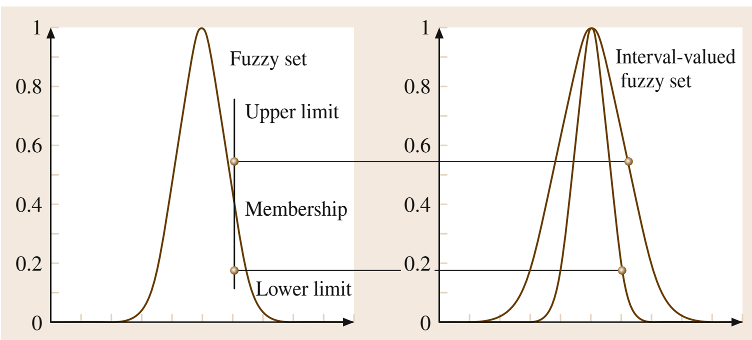

Fig. 6.9 Tizhoosh's construction

6.5 Atanasssov's Intuitionistic Fuzzy Sets or Bipolar Fuzzy Sets of Type 2 or IF Fuzzy Sets

In 1983, *Atanassov* presented his definition of intuitionistic fuzzy set [6.93]. This paper was written in Bulgarian, and in 1986 he presented his ideas in English in [6.94].

Definition 6.14
An intuitionistic fuzzy set over U is an expression A given by

$$A = \{(u_i, \mu_A(u_i), \nu_A(u_i))|u_i \in U\},$$

where $\mu_A\colon U \longrightarrow [0,1]$

$$\nu_A\colon U \longrightarrow [0,1]$$

such that $0 \leq \mu_A(u_i) + \nu_A(u_i) \leq 1$ for every $u_i \in U$.

Atanassov also introduced the following two essential characteristics of these sets:

1. The complementary of

$$A = \{(u_i, \mu_A(u_i), \nu_A(u_i))|u_i \in U\}$$

is

$$A_c = \{(u_i, \nu_A(u_i), \mu_A(u_i))|u_i \in U\} .$$

2. For each $u_i \in U$, the intuitionistic or hesitance index of such element in the considered set A is given by

$$\pi_A(u_i) = 1 - \mu_A(u_i) - \nu_A(u_i) .$$

$\pi_A(u_i)$ is a measure of the hesitance of the expert to assign a numerical value to $\mu_A(u_i)$ and $\nu_A(u_i)$. For this reason, we consider that these sets are an extension of fuzzy sets. It is clear that if for each $u_i \in U$ we take $\nu_A(u_i) = 1 - \mu_A(u_i)$, then the considered set A is a fuzzy set in Zadeh's sense. So fuzzy sets are a particular case of those defined by Atanassov.

In 1993, *Gau* and *Buehre* [6.95] introduced the concept of vague set and later in 1994 it was shown that these are the same as those introduced by Atanassov in 1983 [6.96].

We denote by $A - IFS(U)$ the class of all intuitionistic sets (in the sense of Atanassov) defined over the referential U. Atanassov also gave the following definition:

Definition 6.15
Given $A, B \in A - IFS(U)$.

$$\begin{aligned} A \cup_{A-IFS} B = \{(u_i, &\max(\mu_A(u_i), \mu_B(u_i)),\\ &\min(\nu_A(u_i), \nu_B(u_i)))|u_i \in U\}\\ A \cap_{A-IFS} B = \{(u_i, &\min(\mu_A(u_i), \mu_B(u_i)),\\ &\max(\nu_A(u_i), \nu_B(u_i)))|u_i \in U\} . \end{aligned}$$

Definitions of connectives for Atanassov's sets in terms of t-norms, etc. can be found in [6.49, 97].

Corollary 6.3
Atanassov's intuitionistic fuzzy sets are a particular case of L-fuzzy sets.

Proof: Just note that $L = \{(x_1, x_2)|x_1 + x_2 \leq 1$ with $x_1, x_2 \in [0,1]\}$ with the operations in Definition 6.15 is a lattice. ■

Proposition 6.6
The set $(A - IFS(U), \cup_{A-IFS}, \cap_{A-IFS})$ is a bounded lattice, where the order is defined as

$$A \leq_{A-IFS} B \text{ if and only if } A \cup_{A-IFS} B = B$$

or equivalently

$$A \leq_{A-IFS} B \text{ if and only if } A \cap_{A-IFS} B = A .$$

From Proposition 6.6, we see that the order

$$\begin{aligned} A \leq_{A-IFS} B \text{ if and only if } &\mu_A(u_i) \leq \mu_B(u_i) \text{ and}\\ &\nu_A(u_i) \geq \nu_B(u_i) \text{ for all } u_i \in U \end{aligned}$$

is not linear. Different methods to get linear orders for these sets can be found in [6.50, 51].

6.5.1 Relation Between Interval-Valued Fuzzy Sets and Atanassov's Intuitionistic Fuzzy Sets: Two Different Concepts

In 1989, *Atanassov* and *Gargov* [6.98] and later *Deschrijver* and *Kerre* [6.24] proved that from an interval-valued fuzzy set we can build an intuitionistic fuzzy set and vice-versa.

Theorem 6.2
The mapping

$$\Phi: IVFS(U) \to A-IFS(U)\,,$$
$$A \to A'\,,$$

where $A' = \{(u_i, \underline{A}(u_i), 1-\overline{A}(u_i))|u_i \in U\}$, is a bijection.

Theorem 6.2 shows that interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets, are equivalent from a mathematical point of view. But, as pointed out in [6.52], the absence of a structural component in their description might explain this result, since from a conceptual point of view they very different models:

a) The representation of the membership of an element to a set using an interval means that the expert doubts about the exact value of such membership, so such an expert provides two bounds, and we never consider the representation of the nonmembership to a set.
b) By means of the intuitionistic index we, represent the hesitance of the expert in simultaneously building the membership and the nonmembership degrees.

From an applied point of view, the conceptual difference between both concepts has also been clearly displayed in [6.99]. On page 204 of this paper, Ye adapts an example by *Herrera* and *Herrera-Viedma* appeared in 2000 [6.100]. Ye's example runs as follows: n experts are asked about a money investment in four different companies. Ye considers that the membership to the set that represents each company is given by the number of experts who would invest their money in that company (normalized by n), and the nonmembership is given by the number of experts who would not invest their money in that company. Clearly, the intuitionistic index corresponds to the experts that do not provide either a positive or a negative answer about investing in that company. In this way, Ye proves that:

1. The results obtained with this representation are more realistic than those obtained in [6.100] using Zadeh's fuzzy sets.
2. In the considered problem, the interval interpretation does not make much sense besides its use as a mathematical tool.

6.5.2 Some Problems with the Intuitionistic Sets Defined by Atanassov

Besides the missed structural component pointed out in [6.52]:

1. In these sets, each element has two associated values. For this reason, we consider that the information measures as entropy [6.59, 61], similarity [6.101, 102], etc. should also be given by two numerical values. That is, in our opinion, we should distinguish between those measures that provide a single number and those others that provide two numbers. This fact is discussed in [6.103] where the two concepts of entropy given in [6.59] and [6.61] are jointly used to represent the uncertainty linked to Atanassov's intuitionistic fuzzy set. So we think that it is necessary to carry out a conceptual revision of the definitions of similarity, dissimilarity, entropy, comparability, etc., given for these sets. Even more since nowadays working with two numbers instead of a single one does not imply a much larger computational cost.
2. As in the case of interval-valued fuzzy sets, in many applications, there is a problem to choose the most appropriate linear order associated with that application [6.50, 51]. We should remark that the chosen order directly influences the final outcome, so it is necessary to study the conditions that determine the choice of one order or another [6.65].

6.5.3 Applications

Extensions have shown themselves very useful in problems of decision making [6.99, 104–108]. In general, they work very well in problems for which we have to represent the difference between the positive and the negative representation of something [6.109], in particular in cognitive psychology and medicine [6.110]. Also in image processing they have been used often, as in [6.111, 112]. We should remark that the mathematical equivalence between these sets and interval-valued fuzzy sets makes that in many applications in which interval-valued fuzzy sets are useful, so are Atanassov's intuitionistic fuzzy sets [6.113].

6.5.4 The Problem of the Name

From Sect. 6.1.1, it is clear that the term *intuitionistic* was used in 1907 by Brouwer, in 1930 by Heyting, etc. So, 75 years before Atanassov used it, it already had a specific meaning in logic. Moreover, one year after Atanassov first used it in Bulgarian, Takeuti and Titani (1984) presented a set representation for Heyting ideas, using the expression *intuitionistic fuzzy sets*. From our point of view, this means that in fact the correct terminology is that of Takeuti and Titani. Nevertheless, all these facts have originated a serious notation problem in the literature about the subject.

In 2005, in order to solve these problems, *Dubois* et al. published a paper [6.7] on the subject and, they proposed to replace the name intuitionistic fuzzy sets by *bipolar fuzzy sets*, justifying this change. Later, *Atanassov* has answered in [6.114], where he defends the reasons he had to choose the name intuitionistic and states a clear fact: the sets he defined are much more cited and used than those defined by Takeuti and Titani, so in his opinion the name must not change.

In *Dubois* and *Prade*'s works about bipolarity types [6.115, 116], these authors stated that Atanassov's sets are included in the type-2 bipolar sets, so they call these sets fuzzy bipolar sets of type-2.

But we must say that nine years before Dubois et al.'s paper about the notation, *Zhang* in [6.117, 118] used the word bipolar in connection with the fuzzy sets theory and presented the concept of bipolar-valued set.

All these considerations have led some authors to propose the name *Atanassov's intuitionistic fuzzy sets*. However, Atanassov himself disagrees with this notation and asserts that his notation must be hold; that is, intuitionistic fuzzy sets. Other authors use the name IF-sets (intuitionistic fuzzy) [6.119].

In any case, only time will fix the appropriate names.

6.6 Atanassov's Interval-Valued Intuitionistic Fuzzy Sets

In 1989, *Atanassov* and *Gargov* presented the following definition [6.98]:

Definition 6.16
An Atanassov's interval-valued intuitionistic fuzzy set over U is an expression A given by

$$A = \{(u_i, M_A(u_i), N_A(u_i))|u_i \in U\}\ ,$$

where $M_A\colon U \longrightarrow L([0,1])$,

$$N_A\colon U \longrightarrow L([0,1])$$

such that $0 \leq \overline{M_A}(u_i) + \overline{N_A}(u_i) \leq 1$ for every $u_i \in U$.

In this definition, authors adapt Atanassov's intuitionistic sets to Zadeh's ideas on the problem of building the membership degrees of the elements to the fuzzy set. Moreover, if for every $u_i \in U$, we have that $\underline{M_A}(u_i) = \overline{M_A}(u_i)$ and $\underline{N_A}(u_i) = \overline{N_A}(u_i)$, then we recover an Atanassov's intuitionistic fuzzy set, so the latter are a particular case of Atanassov's interval-valued intuitionistic fuzzy sets. As in the case of Atanassov's intuitionistic fuzzy sets, the complementary of a set is obtained by interchanging the membership and nonmembership intervals.

We represent by $A-IVIFS(U)$ the class of all Atanassov's interval-valued intuitionistic fuzzy sets over a referential set U.

Definition 6.17
Given $A, B \in A-IVIFS(U)$.

$$A \cup_{A-IVIFS} B = \{(u_i, A \cup_{A-IVIFS} B(u_i))|u_i \in U\}$$
where $A \cup_{A-IVIFS} B(u_i)$
$$= \left[\left(\max\left(\underline{M_A}(u_i), \underline{M_B}(u_i)\right), \max\left(\overline{M_A}(u_i), \overline{M_B}(u_i)\right)\right)\right] \left[\min\left(\underline{N_A}(u_i), \underline{N_B}(u_i)\right), \min\left(\overline{N_A}(u_i), \overline{N_B}(u_i)\right)\right]\ ,$$
$$A \cap_{A-IVIFS} B = \{(u_i, A \cap_{A-IVIFS} B(u_i))|u_i \in U\}$$
where $A \cap_{A-IVIFS} B(u_i)$
$$= \left(\left[\min\left(\underline{M_A}(u_i), \underline{M_B}(u_i)\right), \min\left(\overline{M_A}(u_i), \overline{M_B}(u_i)\right)\right] \left[\max\left(\underline{N_A}(u_i), \underline{N_B}(u_i)\right), \max\left(\overline{N_A}(u_i), \overline{N_B}(u_i)\right)\right]\right)\ ,$$

Corollary 6.4
Atanassov's interval-valued intuitionistic fuzzy sets are a particular case of L-fuzzy sets.

Proof: Just note that $LL([0,1]) = \{(\mathbf{x}, \mathbf{y}) \in L([0,1])^2 | \overline{x} + \overline{y}\}$ with the operations in Definition 6.17 is a lattice. ∎

Proposition 6.7
The set $(A-IVIFS(U), \cup_{A-IVIFS}, \cap_{A-IVIFS})$ is a bounded lattice, where the order is defined as

$$A \leq_{A-IVIFS} B \text{ if and only if } A \cup_{A-IVIFS} B = B$$

or equivalently

$$A \leq_{A-IVIFS} B \text{ if and only if } A \cap_{A-IVIFS} B = A .$$

Note that $A \leq_{A-IVIFS} B$ if and only if $M_A(u_i) \leq_{IVFS} M_B(u_i)$ and $N_A(u_i) \geq_{IVFS} N_B(u_i)$ for all $u_i \in U$; that is, $A \leq_{A-IVIFS} B$ if and only if $\underline{M_A}(u_i) \leq \underline{M_B}(u_i)$, $\overline{M_A}(u_i) \leq \overline{M_B}(u_i)$, $\underline{N_A}(u_i) \geq \underline{N_B}(u_i)$, and $\overline{N_A}(u_i) \geq \overline{N_B}(u_i)$ for all $u_i \in U$ is not linear.

We make the following remarks regarding these extensions:

1. It is necessary to study two different types of information measures: those whose outcome is a single number [6.120] and those whose outcomes are two intervals in [0, 1] [6.120]. It is necessary a study of both types.
2. Nowadays, there are many works using these sets [6.121–123]. However none of them displays an example where the results obtained with these sets are better than those obtained with fuzzy sets or other techniques. As it happened until recent years with interval-valued fuzzy sets, it is necessary to find an application that provides better results using these extensions rather than using other sets. To do so, we should compare the results with those obtained with other techniques, which is something that it is not done for the moment in the papers that make use of these sets. From the moment, most of the studies are just theoretical [6.124–126].

6.7 Links Between the Extensions of Fuzzy Sets

Taking into account the study carried out in previous sections, we can describe the following links between the different extensions.

1. $FS \subset IVFS \equiv$ Grey Sets $\equiv A - IFS \equiv$ Vague sets $\subset A - IVIFS \subset L - FS$.
2. If we consider the operations in Definition 6.11, we have the sequence of inclusions:

$$\begin{aligned} FS \subset IVFS \equiv \text{ Grey Sets } &\equiv A - IFS \\ &\equiv \text{ Vague sets} \subset T2FS \subset L - FS . \end{aligned}$$

6.8 Other Types of Sets

In this section, we present the definition of other types of sets that have arisen from the idea of Zadeh's fuzzy set. However, for us none of them should be considered an extension of a fuzzy set, since we do not represent with them the degree of ignorance or uncertainty of the expert.

6.8.1 Probabilistic Sets

These sets were introduced in 1981 by *Hirota* [6.127].

Definition 6.18
Let (Ω, B, P) be a probability space and let $B(0,1)$ denote the family of Borel sets in [0, 1]. A probabilistic set A over the universe U is a function

$$A\colon U \times \Omega \to ([0,1], B(0,1)) ,$$

where $A(u_i, \cdot)$ is measurable for each $u_i \in U$.

6.8.2 Fuzzy Multisets and n-Dimensional Fuzzy Sets

The idea of multiset was given by *Yager* in 1986 [6.128] and later developed by *Miyamoto* [6.129]. In these multilevel sets, several degrees of membership are assigned to each element.

Definition 6.19
Let U be a nonempty set and $n \in N^+$. A fuzzy multiset A over U is given by

$$A = \{(u_i, \mu_{A_1}(u_i), \mu_{A_2}(u_i), \ldots, \mu_{A_n}(u_i)) | u_i \in U\} ,$$

where $\mu_{A_i}\colon U \to [0,1]$ is called the ith membership degree of A.

If in Definition 6.19 we require that: $\mu_{A_1} \leq \mu_{A_2} \leq \cdots \leq \mu_{A_n}$ we have an *n-Dimensional fuzzy set* [6.130, 131]. Nevertheless, it is worth to point out the relation of these families of fuzzy set with the classification model proposed in [6.132], and the particular model proposed in [6.133], where fuzzy preference intensity was arranged according to the basic preference attitudes.

6.8.3 Bipolar Valued Set or Bipolar Set

In 1996, *Zhang* presented the concept of bipolar set as follows [6.117]:

Definition 6.20
A bipolar-valued set or a bipolar set on U is an object

$$A = \{(u_i, \varphi^+(u_i), \varphi^-(u_i))|u_i \in U\}$$

with $\varphi^+ \colon U \to [0,1]$, $\varphi^- \colon U \to [-1,0]$.

In these sets, the value $\varphi^-(u_i)$ must be understood as how much the environment of the problem opposes to the fulfillment of $\varphi^+(u_i)$. Nowadays interesting studies exist about these sets [6.134–138].

6.8.4 Neutrosophic Sets or Symmetric Bipolar Sets

These sets were first studied by Smarandache in 2002 [6.139]. They arise from Atanassov's intuitionistic fuzzy sets ignoring the restriction on the sum of the membership and the nonmembership degrees.

Definition 6.21
A neutrosophic set or symmetric bipolar set on U is an object

$$A = \{(u_i, \mu_A(u_i), \nu_A(u_i))|u_i \in U\} ,$$

with $\mu_A \colon U \to [0,1]$, $\nu_A \colon U \to [0,1]$.

6.8.5 Hesitant Sets

These sets were introduced by *Torra* and *Narukawa* in 2009 to deal with decision-making problems [6.140, 141].

Definition 6.22
Let $\wp([0,1])$ be the set of all subsets of the unit interval and U be a nonempty set. Let $\mu_A \colon U \to \wp([0,1])$, then a *hesitant fuzzy set* (HFS in short) A defined over U is given by

$$A = \{(u_i, \mu_A(u_i))|u_i \in U\} . \quad (6.6)$$

6.8.6 Fuzzy Soft Sets

Based on the definition of soft set [6.142], *Maji* et al. present the following definition [6.143].

Definition 6.23
A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F \colon A \to FP(U)$.

Where $FP(U)$ denotes the set of all fuzzy subsets of U.

6.8.7 Fuzzy Rough Sets

From the concept of rough set given by *Pawlak* in [6.144], *Dubois* and *Prade* in 1990 proposed the following definition [6.145]. *From different point of views these sets could be considered as an extension of fuzzy sets in our sense*, besides these sets are being exhaustively studied, for this reason we consider that these sets need another chapter.

Definition 6.24
Let U be a referential set and R be a fuzzy similarity relation on U. Take $A \in \mathcal{F}S(U)$. A fuzzy rough set over U is a pair $(R\downarrow A, R\uparrow A) \in \mathcal{F}S(U) \times \mathcal{F}S(U)$, where

- $R\downarrow A \colon U \to [0,1]$ is given by
  $R\downarrow A(u) = \inf_{v\in U} \max(1 - R(v,u), A(v))$
- $R\uparrow A \colon U \to [0,1]$ is given by
  $R\uparrow A(u) = \sup_{v\in U} \min(R(v,u), A(v))$.

6.9 Conclusions

In this chapter, we have reviewed the main types of fuzzy sets defined since 1965. We have classified these sets in two groups: those that take into account the problem of building the membership functions, which we have included in the so-called extensions of fuzzy sets, and those that appear as an answer to such a key issue.

We have introduced the definitions and first properties of the extensions, that is, type-2 fuzzy sets, interval-valued fuzzy sets; Atanassov's intuitionistic fuzzy sets or type-2 bipolar fuzzy sets, and Atanassov's interval-valued fuzzy sets. We have described the properties and problems linked to type-2 fuzzy sets, and we have presented several construction methods for interval-valued fuzzy sets, depending on the application. We have also referred to some papers where it is shown that the use of interval-valued fuzzy sets improves the results obtained with fuzzy sets.

In general, we have stated the main problem in fuzzy sets extensions, namely, to find applications for which the results obtained with these sets are better than those obtained with other techniques. This has only been proved, up to now, for interval-valued fuzzy sets. We think that the great defy for some sets that are initially justified as a theoretical need is to prove their practical usefulness.

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