5. Monotone Measures-Based Integrals

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The theory of classical measures and integral reflects the genuine property of several quantities in standard physics and/or geometry, namely the σ-additivity. Though monotone measure not assuming σ-additivity appeared naturally in models extending the classical ones (for example, inner and outer measures, where the related integral was considered by Vitali already in 1925), their intensive research was initiated in the past 40 years by the computer science applications in areas reflecting human decisions, such as economy, psychology, multicriteria decision support, etc. In this chapter, we summarize basic types of monotone measures together with the basic monotone measures-based integrals, including the Choquet and Sugeno integrals, and we introduce the concept of universal integrals proposed by Klement et al. to give a common roof for all mentioned integrals. Benvenuti's integrals linked to semicopulas are shown to be a special class of universal integrals. Up to several other integrals, we also introduce decomposition integrals due to Even and Lehrer, and show which decomposition integrals are inside the framework of universal integrals.

Before Cauchy, there was no definition of the integral in the actual sense of the word *definition*, though the integration was already well established and in many areas applied method. Recall that constructive approaches to integration can be traced as far back as the ancient Egypt around 1850 BC; the *Moscow Mathematical Papyrus* (Problem 14) contains a formula of a frustum of a square pyramid [5.1]. The first documented systematic technique, capable of determining integrals, is the method of exhaustion of the ancient Greek astronomer Eudoxus of Cnidos (ca. 370 BC) [5.2] who tried to find areas and volumes by approximating them by a (large) number of shapes for which the area or volume was known. This method was further developed by Archimedes in third-century BC who calculated the area of parabolas and gave an approximation to the area of a circle. Similar methods were independently developed in China around third-century AD by Liu Hui, who used it to find the area of the circle. This was further developed in the fifth century by the Chinese mathematicians Zu Chongzhi and Zu Geng to find the volume of a sphere. In the same century, the Indian mathematician Aryabhata used a similar method in order to find the circumference of a circle. More than 1000 years later, *Johannes Kepler* invented the *Kepler'sche Fassregel* [5.3] (also known as *Simpson rule*) in order to compute the (approximative) volume of (wine) barrels.

Based on the fundamental work of *Isaac Newton* and *Gottfried Wilhelm Leibniz* in the 18th century (see [5.4, 5]), the first indubitable access to integration was given by *Bernhard Riemann* in his Habilitation Thesis at the University of Göttingen [5.6]. Note that Riemann has generalized the Cauchy definition of integral defined for continuous real functions (of one variable) defined on a closed interval $[a, b]$.

Among several other developments of the integration theory, recall the Lebesgue approach covering measurable functions defined on a measurable space and general σ-additive measures. Here we recall the final words of H. Lebesgue from his lecture held at a conference in Copenhagen on May 8, 1926, entitled *The Development of the Notion of the Integral* (for the full text see [5.7]):

> *... if you will, that a generalization made not for the vain pleasure of generalizing, but rather for the solution of problems previously posed, is always a fruitful generalization. The diverse applications which have already taken the concepts which we have just examine prove this super-abundantly.*

All till now mentioned approaches to integration are related to measurable spaces, measurable real functions and (σ-)additive real-valued measures. Though there are many generalizations and modifications concerning the range and domain of considered functions and measures (and thus integrals), in this chapter we will stay in the above-mentioned framework, with the only exception that the (σ-)additivity of measures is relaxed into their monotonicity, thus covering many natural generalizations of (σ-)additive measures, such as outer or inner measures, lower or upper envelopes of systems of measures, etc.

Maybe the first approach to integration not dealing with the additivity was due to *Vitali* [5.8]. Vitali was looking for integration with respect to lower/upper measures and his approach is completely covered by the later, more general, approach of *Choquet* [5.9], see Sect. 5.1. Note that the Choquet integral is a generalization of the Lebesgue integral in the sense that they coincide whenever the considered measure is σ-additive (i. e., when the Lebesgue integral is meaningful).

A completely different approach, influenced by the starting development of fuzzy set theory [5.10], is due to *Sugeno* [5.11]. Sugeno even called his integral as fuzzy integral (and considered set functions as fuzzy measures), though there is no fuzziness in this concept (Sect. 5.1). Later, several approaches generalizing or modifying the above-mentioned integrals were introduced. In this chapter, we give a brief overview of these integrals, i. e., integrals based on monotone measures. In the next section, some preliminaries and basic notions are recalled, as well as the Choquet and Sugeno integrals. Section 5.2 brings a generalization of both Choquet and Sugeno integrals, now known as the Benvenuti integral. In Sect. 5.3, universal integrals as a rather general framework for monotone measures-based integral is given and discussed, including copula-based integrals, among others. In Sect. 5.4, we bring some integrals not giving back the underlying measure. Finally, some possible applications are indicated and some concluding remarks are added. Note that we will not discuss integrals defined only for some special subclasses of monotone measures, such as pseudoadditive integrals [5.12, 13] or t-conorms-based integrals of *Weber* [5.14]. Moreover, we restrict our considerations to normed measures satisfying $m(X) = 1$, and to functions with range contained in $[0, 1]$. This is done for the sake of higher transparentness and the generalizations for $m(X) \in\,]0, \infty]$ and functions with different ranges will be covered by the relevant quotations only.

5.1 Preliminaries, Choquet, and Sugeno Integrals

For a fixed measurable space $(X, \mathcal{A})$, where $\mathcal{A}$ is a σ-algebra of subsets of the universe X, we denote by $\mathcal{F}_{(X,\mathcal{A})}$ the set of all $\mathcal{A}$-measurable functions $f : X \to [0, 1]$, and by $\mathcal{M}_{(X,\mathcal{A})}$ the set of all monotone measures $m : \mathcal{A} \to [0, 1]$ (i. e., $m(\emptyset) = 0$, $m(X) = 1$ and $m(A) \leq m(B)$ whenever $A \subset B \subset X$). Note that functions f from $\mathcal{F}_{(X,\mathcal{A})}$ can be seen as membership functions of fuzzy events on $(X, \mathcal{A})$, and that monotone measures are in different references also called fuzzy measures, capacities, pre-measures, etc. Moreover, if X is finite, we will always consider $\mathcal{A} = 2^X$ only. In such case, any monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$ is determined by $2^{|X|} - 2$ weights from $[0, 1]$ (measures of proper subsets of X) constraint by the monotonicity condition only, and to each monotone measure $m : \mathcal{A} \to [0, 1]$ we can assign its Möbius transform $M_m : \mathcal{A} \to \mathbb{R}$ given by

$$M_m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \cdot m(B). \tag{5.1}$$

Then

$$m(A) = \sum_{B \subseteq A} M_m(B). \tag{5.2}$$

Moreover, dual monotone measure $m^d : \mathcal{A} \to [0, 1]$ is given by $m^d(A) = 1 - m(A^c)$.

Among several distinguished subclasses of monotone measures from $\mathcal{M}_{(X,\mathcal{A})}$ we recall these classes, supposing the finiteness of X:

- *Additive* measures, $m(A \cup B) = m(A) + m(B)$ whenever $A \cap B = \emptyset$;
- *Maxitive* measures, $m(A \cup B) = m(A) \vee m(B)$ (these measures are called also possibility measures [5.15, 16]);
- *k-additive* measures, $M_m(A) = 0$ whenever $|A| > k$ (hence additive measures are 1-additive);
- *Belief* measures, $M_m(A) \geq 0$ for all $A \subset X$;
- *Plausibility* measures, m^d is a belief measure;
- *Symmetric* measures, $M_m(A)$ depends on $|A|$ only.

For more details on monotone measures, we recommend [5.17–19] and [5.20].

Concerning the functions, for any $c \in [0,1], A \in \mathcal{A}$ we define a basic function $b(c,A) : X \to [0,1]$ by

$$b(c,A)(x) = \begin{cases} c & \text{if } x \in A\,, \\ 0 & \text{else}\,. \end{cases}$$

Obviously, basic functions can be related to the characteristic functions, $1_A = b(1,A)$ and $b(x,A) = c \cdot 1_A$. However, as we are considering more general types of multiplication as the standard product, in general, we prefer not to depend in our consideration on the standard product.

The first integral introduced for monotone measures was proposed by *Choquet* [5.9] in 1953.

Definition 5.1

For a fixed monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$, a functional $\mathrm{Ch}_m : \mathcal{F}_{(X,\mathcal{A})} \to [0,1]$ given by

$$\mathrm{Ch}_m(f) = \int_0^1 m(f \geq t)\,\mathrm{d}t \qquad (5.3)$$

is called the *Choquet* integral (with respect to m), where the right-hand side of (5.3) is the classical Riemann integral.

Note that the Choquet integral is well defined because of the monotonicity of m. Observe that if m is σ-additive, i.e., if it is a probability measure on $(X,\mathcal{A})$, then the function $h : [0,1] \to [0,1]$ given by $h(t) = m(f \geq t)$ is the standard survival function of the random variable f, and then $\mathrm{Ch}_m(f) = \int_0^1 h(t)\mathrm{d}t = \int_X f\,\mathrm{d}m$ is the standard expectation of f (i.e., Lebesgue integral of f with respect to m).

Due to *Schmeidler* [5.21, 22], we have the following axiomatization of the Choquet integral.

Theorem 5.1

A functional $I : \mathcal{F}_{(X,\mathcal{A})} \to [0,1]$, $I(1_X) = 1$, is the Choquet integral with respect to monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$ given by $m(A) = I(1_A)$ if and only if I is comonotone additive, i.e., if $I(f+g) = I(f) + I(g)$ for all $f, g \in \mathcal{F}_{(X,\mathcal{A})}$ such that $f + g \in \mathcal{F}_{(X,\mathcal{A})}$ and f and g are comonotone, $(f(x) - f(y)) \cdot (g(x) - g(y)) \geq 0$ for any $x, y \in X$.

We recall some properties of the Choquet integral.

It is evident that the Choquet integral Ch_m is an increasing functional, $\mathrm{Ch}_m(f) \leq \mathrm{Ch}_m(g)$ for any $m \in \mathcal{M}_{(X,\mathcal{A})}$, $f, g \in \mathcal{F}_{(X,\mathcal{A})}$ such that $f \leq g$. Moreover, for each $A \in \mathcal{A}$ it holds $\mathrm{Ch}_m(b(c,A)) = c \cdot m(A)$, and especially $\mathrm{Ch}_m(1_A) = m(A)$.

Remark 5.1

i) Due to results of *Šipoš* [5.23], see also [5.24], the comonotone additivity of the functional I in Theorem 5.1, which implies its positive homogeneity, $I(cf) = c \cdot I(f)$ for all $c > 0$ and $f \in \mathcal{F}_{(X,\mathcal{A})}$ such that $cf \in \mathcal{F}_{(X,\mathcal{A})}$ can be replaced by the positive homogeneity of I and its horizontal additivity, i.e.,

$$I(f) = I(f \wedge a) + I(f - f \wedge a)$$

for all $f \in \mathcal{F}_{(X,\mathcal{A})}$ and $a \in [0,1]$.

ii) Choquet integral $\mathrm{Ch}_m : \mathcal{F}_{(X,\mathcal{A})} \to [0,1]$ is continuous from below,

$$\lim_{n\to\infty} \mathrm{Ch}_m(f_n) = \mathrm{Ch}_m(f)$$

whenever for $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}}_{(X,\mathcal{A})}$ we have $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $f = \lim_{n\to\infty} f_n$, if and only if m is continuous from below,

$$\lim_{n\to\infty} m(A_n) = m(A)$$

whenever for $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ we have $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $A = \bigcup_{n\in\mathbb{N}} A_n$.

iii) Choquet integral $\mathrm{Ch}_m : \mathcal{F}_{(X,\mathcal{A})} \to [0,1]$ is subadditive (superadditive),

$$I(f+g) \leq I(f) + I(g) \qquad (I(f+g) \geq I(f) + I(g))$$

for all $f, g, f+g \in \mathcal{F}_{(X,\mathcal{A})}$, if and only if m is submodular (supermodular),

$$m(A \cup B) + m(A \cap B) \le m(A) + m(B)\,,$$
$$(m(A \cup B) + m(A \cap B) \ge m(A) + m(B))$$

for all $A, B \in \mathcal{A}$.

iv) For any $m \in \mathcal{M}_{(X,\mathcal{A})}$ and $f \in \mathcal{F}_{(X,\mathcal{A})}$ it holds

$$\mathrm{Ch}_{m^d}(f) = 1 - \mathrm{Ch}_m(1-f)\,,$$

i. e., in the framework of aggregation functions [5.25] the dual to a Choquet integral (with respect to a monotone measure m) is again the Choquet integral (with respect to the dual monotone measure m^d).

For the proofs and more details about the above results on Choquet integral, we recommend [5.18, 19, 24, 26].

Restricting our considerations to finite universes, we have also the next evaluation formula due to *Chateauneuf* and *Jaffray* [5.27]

$$\mathrm{Ch}_m(f) = \sum_{A \subseteq X} M_m(A) \cdot \min\left(f(x) \mid x \in A\right)\,. \tag{5.4}$$

In the Dempster–Shafer theory of evidence [5.28, 29], belief measures are considered, and then the Möbius transform $M_m : 2^X \setminus \{\emptyset\} \to [0,1]$ of a belief measure m is called a *basic probability assignment*. Evidently, M_m can be seen as a probability measure (of singletons) on the finite space $2^X \setminus \{\emptyset\}$ (with cardinality $2^{|X|} - 1$), and defining a function $F : 2^X \setminus \{\emptyset\} \to [0,1]$ by $F(A) = \min(f(x) | x \in A)$, the formula (5.4) can be seen as the Lebesgue integral of F with respect to M_m (i. e., it is the standard expectation of variable F)

$$\mathrm{Ch}_m(f) = \sum_{A \in 2^X \setminus \{\emptyset\}} F(A) \cdot M_m(A)\,.$$

Another genuine relationship of Choquet and Lebesgue integrals in the framework of the Dempster–Shafer theory is based on the fact that each belief measure m can be seen as a lower envelope of the class of dominating probability measures, i. e., for each $A \subseteq X$ (X is finite)

$$m(A) = \inf\{P(A) | P \ge m\}\,.$$

Then

$$\mathrm{Ch}_m(f) = \inf\left\{\int_X f\,\mathrm{d}P | P \ge m\right\}\,.$$

Similarly, for the related plausibility measure m^d, it holds

$$\mathrm{Ch}_{m^d}(f) = \sup\left\{\int_X f\,\mathrm{d}P | P \le m^d\right\} = \sup\left\{\int_X f\,\mathrm{d}P | P \ge m\right\}\,.$$

For interested readers, we recommend the collection [5.30].

In general, for any monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$ and any measurable (continuous from below) function $f \in \mathcal{F}_{(X,\mathcal{A})}$ there is a probability measure $P_{m,f}$ on $(X, \mathcal{A})$ so that

$$\mathrm{Ch}_m(f) = \int_X f\,\mathrm{d}P_{m,f}\,, \tag{5.5}$$

see, e.g., [5.24, Theorem 2.6], where the right-hand side of (5.5) is the standard Lebesgue integral. Moreover, if $f, g \in \mathcal{F}_{(X,\mathcal{A})}$ are comonotone, one can find unique probability measure P allowing to express the Choquet integral of f and g with respect to m as the Lebesgue integral of f and g with respect to P, respectively. As an immediate consequence of (5.5), Jensen's inequality for Choquet integral can be shown to be valid. Similarly, if f and g are comonotone, based on the above observations, one can prove the Minkowski and Chebyshev inequality. For more details, see [5.31].

For $k \in \mathbb{N}$, consider a probability measure P on the product space $(X, \mathcal{A})^k$, and define a set function $m : \mathcal{A} \to [0,1]$ by $m(A) = P(A^k)$. Then $m \in \mathcal{M}_{(X,\mathcal{A})}$ is a k-additive monotone measure (and belief measure, as well), and for all $f \in \mathcal{F}_{(X,\mathcal{A})}$ it holds

$$\mathrm{Ch}_m(f) = \int_{X^k} F\,\mathrm{d}P\,, \tag{5.6}$$

where $F : X^k \to [0,1]$ is given by $F(x_1, \ldots, x_k) = \min(f(x_1), \ldots, f(x_k))$. For more details see [5.32].

The Sugeno integral (in the original sources called fuzzy integral) was introduced by *Sugeno* in 1972 in Japanese in [5.33] and in English in 1974 in [5.11]. Inspired by the fuzzy set theory introduced by *Zadeh* [5.10], Sugeno has proposed a way how to formalize human subjectivity in spirit similar to the randomness but based only on ordinal scales. His concept is not fuzzy, though both fuzzy set theory and Sugeno's integral theory exploit the same aggregation functions (sup and inf), and considering functions $f \in \mathcal{F}_{(X,\mathcal{A})}$ as membership functions of fuzzy subsets of X, the corresponding Sugeno integral can be seen as a version of expectation of fuzzy sets.

Definition 5.2
For a fixed monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$, a functional $\mathrm{Su}_m : \mathcal{F}_{(X,\mathcal{A})} \to [0,1]$ given by

$$\mathrm{Su}_m(f) = \sup\{\min(t, m(f \geq t)) \,|\, t \in [0,1]\} \tag{5.7}$$

is called the *Sugeno integral* (with respect to m).

There is an equivalent formula for the Sugeno integral, compare ([5.11, Definition 3.1]),

$$\mathrm{Su}_m(f) = \sup\{\min(m(A), \inf\{f(x)|x \in A\}) \,|\, A \in \mathcal{A}\} \ , \tag{5.8}$$

which in the case of finite X (and using also lattice notation sup $= \vee$, min $= \wedge$) can be rewritten as

$$\mathrm{Su}_m(f) = \bigvee_{A \subset X} \left(M_m^{\vee}(A) \wedge \min(f(x)|x \in A)\right) \ , \tag{5.9}$$

showing the striking similarity with the evaluation formula (5.4) for the Choquet integral. Here the set function $M_m^{\vee} : 2^X \setminus \{\emptyset\} \to [0,1]$ is the so-called possibilistic Möbius transform introduced by *Mesiar* in [5.34] and given by

$$M_m^{\vee}(A) = \begin{cases} 0 & \text{if } m(A) = m(B) \text{ for some } B \subsetneq A \ , \\ m(A) & \text{else} \ . \end{cases}$$

Sugeno integral has properties similar to the Choquet integral. Indeed, it is nondecreasing functional such that $\mathrm{Su}_m(b(c,A)) = c \wedge m(A)$, and in particular $\mathrm{Su}_m(1_A) = m(A)$. Moreover, Su_m is comonotone maxitive, i. e., $\mathrm{Su}_m(f \vee g) = \mathrm{Su}_m(f) \vee \mathrm{Su}_m(g)$ for any comonotone $f, g \in \mathcal{F}_{(X,\mathcal{A})}$, and min-homogeneous, $\mathrm{Su}_m(c \wedge f) = c \wedge \mathrm{Su}_m(f)$. We have the next axiomatization of the Sugeno integral due to *Marichal* [5.35] (compare with Theorem 5.1 for the Choquet integral).

Theorem 5.2
A functional $I : \mathcal{F}_{(X,\mathcal{A})} \to [0,1]$, $I(1_X) = 1$, is the Sugeno integral with respect to monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$ given by $m(A) = I(1_A)$ if and only if I is comonotone maxitive and min-homogeneous.

For alternative axiomatizations see [5.24].

Choquet and Sugeno integrals with respect to a monotone measure m may differ not more than $\frac{1}{4}$, i. e., for all $f \in \mathcal{F}_{(X,\mathcal{A})}$ it holds

$$|\mathrm{Ch}_m(f) - \mathrm{Su}_m(f)| \leq \frac{1}{4} \ .$$

Moreover, $\mathrm{Ch}_m(f) = \mathrm{Su}_m(f)$ for all $f \in \mathcal{F}_{(X,\mathcal{A})}$ if and only if $m(A) \in \{0,1\}$ for all $A \in \mathcal{A}$, and then

$$\mathrm{Ch}_m(f) = \mathrm{Su}_m(f) = \sup\{\inf\{f(x)|x \in A\} \,|m(A) = 1\} \ ,$$

which in case X is finite turns out to be a lattice polynomial.

Note that if X has cardinality n and $m(A) \in \{0,1\}$ for all $A \subseteq X$, then $\mathrm{Ch}_m = \mathrm{Su}_m : [0,1]^n \to [0,1]$ are the only n-ary continuous aggregation functions invariant under each automorphism $\phi : [0,1] \to [0,1]$, i. e., $\phi \circ \mathrm{Ch}_m(f) = \mathrm{Ch}_m(f \circ \phi)$ for each $f[0,1]^n$ (for $f = (a_1, \ldots, a_n), f \circ \phi = (\phi(a_1), \ldots, \phi(a_n))$). For more details see [5.36].

Example 5.1

i) Let $X = \{1,2,3\}$ and define $m : 2^X \to [0,1]$ by

$$m(A) = \begin{cases} 0 & \text{if card } A \leq 1 \ , \\ 1 & \text{otherwise} \ . \end{cases}$$

Then, for each $f = (x,y,z) \in [0,1]^3$,

$$\begin{aligned} \mathrm{Ch}_m(f) = \mathrm{Su}_m(f) &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \\ &= \mathrm{med}(x,y,z) \end{aligned}$$

brings the classical median.

ii) Let $X = \{1,2\}$ and define $m : 2^X \to [0,1]$ by $m(A) = \frac{\text{card } A}{2}$. Then, for each $f = (x,y) \in [0,1]^2$,

$$\mathrm{Ch}_m(f) = \frac{x+y}{2}$$

(i. e., Ch_m is the standard arithmetic mean), while

$$\mathrm{Su}_m(f) = (x \wedge y) \vee \left((x \vee y) \wedge \frac{1}{2}\right) \ .$$

For $f_1 = \left(\frac{1}{2}, 1\right)$, $\mathrm{Ch}_m(f_1) = \frac{3}{4}$ and $\mathrm{Su}_m(f_1) = \frac{1}{2}$.
For $f_2 = \left(0, \frac{1}{2}\right)$, $\mathrm{Ch}_m(f_2) = \frac{1}{4}$ and $\mathrm{Su}_m(f_2) = \frac{1}{2}$.

In general,

$$|\mathrm{Ch}_m(f)-\mathrm{Su}_m(f)| = \frac{1}{2}\left(|x-y| \wedge |x+y-1|\right) \le \frac{1}{4}\,.$$

iii) Let $X=[0,1]$, $\mathcal{A}=\mathcal{B}([0,1])$ and let $m:\mathcal{A}\to[0,1]$ be given by $m(A)=\lambda^p(A)$, where $p\in\,]0,\infty[$ is a fixed constant and $\lambda:\mathcal{A}\to[0,1]$ is the standard Lebesgue measure. For any Lebesgue measure preserving function $f:X\to[0,1]$, such as $f(x)=x$, $f(x)=1-x$, or $f(x)=|2x-1|$, we have

$$\mathrm{Ch}_m(f)=\int_0^1 m(f\ge t)\,\mathrm{d}t=\int_0^1 (1-t)^p\mathrm{d}t=\frac{1}{p+1}$$

and

$$\mathrm{Su}_m(f)=\sup\{\min(t,(1-t)^p)\,|\,t\in[0,1]\}=c_p\,,$$

where c_p is the unique solution of the equation $t=(1-t)^p$, $t\in\,]0,1[$. Hence,

$$\text{if } p=1,\ \mathrm{Ch}_m(f)=\mathrm{Su}_m(f)=\frac{1}{2}\,;$$

$$\text{if } p=2,\ \mathrm{Ch}_m(f)=\frac{1}{3}$$

$$\text{and } \mathrm{Su}_m(f)=\frac{3-\sqrt{5}}{2}\doteq 0.382\,;$$

$$\text{if } p=3,\ \mathrm{Ch}_m(f)=\frac{2}{3}$$

$$\text{and } \mathrm{Su}_m(f)=\frac{\sqrt{5}-1}{2}\doteq 0.618\,.$$

5.2 Benvenuti Integral

Comparing Theorems 5.1 and 5.2, we see a striking similarity in the axiomatic characterization of the Choquet and Sugeno integrals. This similarity was generalized under a common roof by *Benvenuti* et al. [5.24], calling there introduced integral *general fuzzy integral*. This integral is now also known as Benvenuti integral (compare [5.25]).

Choquet integral is linked to the standard arithmetic operations $+$ and $\cdot$ on $[0,\infty]$, while the Sugeno integral deals with lattice operations $\wedge$ and $\vee$ on $[0,1]$. To generalize these two couples of operations, pseudoaddition $\oplus$ and pseudomultiplication $\odot$ was introduced in [5.24].

Definition 5.3

Let $u\in[1,\infty]$ be a fixed constant. An operation $\oplus:[0,u]^2\to[0,u]$ is called a *pseudoaddition* on $[0,u]$ whenever it is associative, nondecreasing in both components, 0 is its neutral element, and $\oplus$ is continuous.

Observe that the structure $([0,u],\oplus)$ with $\oplus$ a pseudoaddition on $[0,u]$ is just an I-semigroup of *Mostert* and *Shields* [5.37] and hence $\oplus$ is also commutative. Moreover, considering the principles of Galois connections, we can introduce a *pseudodifference* $\ominus$ related to $\oplus$ satisfying, for all $a,b,c\in[0,u]$, $(a\ominus b)\le c$ if and only if $a\le b\ominus c$.

It is not difficult to see the link to the pseudodifference considered already by *Weber* [5.14].

Lemma 5.1

Let $\oplus:[0,u]^2\to[0,u]$ be a given pseudoaddition on $[0,u]$. The related pseudo difference $\ominus:[0,u]^2\to[0,u]$ is given by

$$a\ominus b=\inf\{c\in[0,u]|b\oplus c\ge a\}\ .$$

Considering the standard addition $+$ on $[0,\infty]$, and $a\ge b$, then the corresponding (pseudo-) difference is the standard difference $a-b$. On the other hand, $\vee$ is a pseudoaddition on any interval $[0,u]$, and its corresponding pseudodifference $\ominus_\vee$ is given by

$$a\ominus_\vee b=\begin{cases}0 & \text{if } a\le b\,,\\ a & \text{otherwise}\,.\end{cases}$$

Due to [5.37], each pseudoaddition $\oplus$ on $[0,u]$ can be represented as an ordinal sum,

$$a\oplus b=\begin{cases} g_k^{-1}(g_k(\beta_k)\wedge(g_k(a)+g_k(b))) \\ \qquad \text{if } (a,b)\in\,]\alpha_k,\beta_k[^2\,,\\ a\vee b \qquad \text{otherwise}\,,\end{cases}$$

where $(]\alpha_k,\beta_k[)_{k\in\mathcal{K}}$ is a disjoint system of open subintervals of $[0,u]$, and $g_k:[\alpha_k,\beta_k]\to[0,\infty]$ is a continuous strictly increasing function such that $g_k(\alpha_k)=0, k\in\mathcal{K}$ ($\mathcal{K}$ can be also empty). Two extremal cases correspond to $\oplus=\vee$ (when $\mathcal{K}$ is empty) and

Archimedean pseudoaddition $\oplus$ on $[0,u]$ generated by $g:[0,u]\to[0,\infty]$ (when $\mathcal{K}$ is singleton, say $\mathcal{K}=\{1\}$, and $\alpha_1=0$, $\beta_1=u$),

$$a\oplus b=g^{-1}\left(g(u)\wedge(g(a)+g(b))\right) .$$

Then g is called an *additive generator* of $\oplus$ and it is unique up to a positive multiplicative constant.

Note that if g is a bijection, i. e., $g(u)=\infty$, then $a\oplus b=g^{-1}(g(a)+g(b))$ and $\oplus$ is called a *strict pseudoaddition*.

For a fixed pseudoaddition $\oplus$ on $[0,u]$, *Benvenuti* et al. [5.24] have introduced a $\oplus$-fitting pseudomultiplication $\odot$.

Definition 5.4

Fix $u,v\in[1,\infty]$ and let $\oplus$ be a given pseudoaddition on $[0,u]$. A mapping $\odot:[0,u]\times[0,v]\to[0,u]$ is called a $\oplus$-*fitting pseudomultiplication* whenever it is nondecreasing in both components, 0 is its annihilator, i. e., $0\odot b=a\odot 0=0$ for all $a\in[0,u]$, $b\in[0,v]$, it is left distributive over $\oplus$, i. e., $(a\oplus b)\odot c=(a\odot c)\oplus(b\odot c)$ for all $a,b\in[0,u]$, $c\in[0,v]$, and it is lower semicontinuous, i. e.,

$$(\vee_{n\in\mathbb{N}}a_n)\odot(\vee_{m\in\mathbb{N}}b_m)=\vee_{n,m\in\mathbb{N}}(a_n\odot b_m) .$$

The left distributivity of a pseudomultiplication $\odot$ over $\vee$ simply means the nondecreasingness of $\odot$ in the first coordinate, and thus there are several kinds of $\vee$-fitting pseudomultiplication $\odot$. On the other hand, this is a rather restrictive constraint when $\oplus$ is Archimedean, i. e., generated by an additive generator $g:[0,u]\to[0,\infty]$.

Proposition 5.1

Let $\oplus:[0,u]^2\to[0,u]$ be an Archimedean pseudoaddition generated by an additive generator $g:[0,u]\to[0,\infty]$. A mapping $\odot:[0,u]\times[0,v]\to[0,u]$ is a $\oplus$-fitting pseudomultiplication if and only if there is a lower semicontinuous nondecreasing function $h:[0,v]\to[0,\infty]$ such that $h(w)=0$ for some $w\in[0,v]$, and $g(u)\cdot h(a)\geq g(u)$ for all $a\in]w,v]$, so that

$$a\odot b=g^{-1}\left(g(u)\wedge g(a)\cdot h(b)\right) .$$

In particular, if $\oplus$ is a strict pseudoaddition, then $h:[0,v]\to[0,\infty]$ is a lower semicontinuous nondecreasing function, satisfying $h(0)=0$, and $a\odot b=g^{-1}(g(a)\cdot h(b))$.

Definition 5.5

Let $u,v\in[1,\infty]$ be fixed given constants and let $\oplus:[0,u]^2\to[0,u]$ be a given pseudoaddition, and $\odot:[0,u]\times[0,v]\to[0,u]$ be a given $\oplus$-fitting pseudomultiplication such that $1\odot 1\leq 1$. For a fixed monotone measure $m\in\mathcal{M}_{(X,\mathcal{A})}$, a functional $\mathcal{B}_m^{\oplus,\odot}:\mathcal{F}_{(X,\mathcal{A})}\to[0,1]$ given by

$$\mathcal{B}_m^{\oplus,\odot}(f)=\sup\left\{\bigoplus_{i=1}^{n}(a_i\odot m(A_i))\,|n\in\mathbb{N},\right.$$
$$\left.\bigoplus_{i=1}^{n}b(a_i,A_i)\leq f,\,(A_i)_{i=1}^{n}\text{ is a chain}\right\}$$

is called Benvenuti integral (with respect to m, based on $\oplus$ and $\odot$).

Observe that if $s\in\mathcal{F}_{(X,\mathcal{A})}$ is a simple function, range $s=\{b_1,\dots,b_n\}$, $b_1<b_2<\dots<b_n$, then

$$\mathcal{B}_m^{\oplus,\odot}(s)=\bigoplus_{i=1}^{n}\left((b_i\ominus b_{i-1})\odot m(s\geq b_i)\right) ,$$

with the convention $b_0=0$. Then for any $f\in\mathcal{F}_{(X,\mathcal{A})}$,

$$\mathcal{B}_m^{\oplus,\odot}(f)$$
$$=\sup\left\{\mathcal{B}_m^{\oplus,\odot}(s)|s\in\mathcal{F}_{(X,\mathcal{A})}\text{ is simple, }s\leq f\right\} .$$

Evidently,

$$\mathcal{B}_m^{\oplus,\odot}(b(a,A))=a\odot m(A)$$

and hence

$$\mathcal{B}_m^{\oplus,\odot}(1_A))=m(A)$$

for all $m\in\mathcal{M}_{(X,\mathcal{A})}$, $A\in\mathcal{A}$ only if $1\odot b=b$ for all $b\in[0,1]$.

If $\oplus$ is a strict pseudoaddition on $[0,u]$ generated by an additive generator g, this means that $\odot$ restricted to $[0,1]^2$ is given by

$$a\odot b=g^{-1}\left(\frac{g(a)\cdot g(b)}{g(1)}\right) .$$

If $\oplus$ is a nonstrict pseudoaddition, then there is no $\oplus$-fitting pseudomultiplication $\odot$ such that $1\odot b=b$ for all $b\in[0,1]$.

Note that for the standard arithmetic operations $+$ and $\cdot$ on $[0,\infty]$, $\mathcal{B}_m^{+,\cdot}=\mathrm{Ch}_m$, i. e., the Choquet integral is recovered. Similarly, $\mathcal{B}_m^{\vee,\wedge}=\mathrm{Su}_m$.

Example 5.2

i) Let $u = v = 1$, $\oplus = \vee$ and $\odot : [0,1]^2 \to [0,1]$ be given by $a \odot b = a^p \cdot b^q$, $p, q \in]0, \infty[$. Then $\mathcal{B}_m^{\oplus,\odot}(f)) = \sup\{t^p \cdot (m(f \geq t))^q \,|\, t \in [0,1]\}$ for any $m \in \mathcal{M}_{(X,\mathcal{A})}$ and $f \in \mathcal{F}_{(X,\mathcal{A})}$, and $\mathcal{B}_m^{\oplus,\odot}(1_A) = (m(A))^q$. Note that if $p = q = 1$, the Shilkret integral $\mathrm{Sh}_m = \mathcal{B}_m^{\oplus,\odot}$ is recovered, see [5.19, 38]. In general, $\mathcal{B}_m^{\oplus,\odot}(f) = \mathrm{Sh}_{m^q}(f^p)$ for any $f \in \mathcal{F}_{(X,\mathcal{A})}$.
ii) For a strict pseudoaddition $\oplus$ on $[0,u]$ and a $\oplus$-fitting pseudomultiplication $\odot$ on $[0,u] \times [0,v]$, see Proposition 5.1, the constraint $1 \odot 1 \leq 1$ means $h(b) \leq 1$, and then $\mathcal{B}_m^{\oplus,\odot}(f) = g^{-1}\left(\mathrm{Ch}_{h(m)}\left(g(f)\right)\right)$, i. e., $\mathcal{B}_m^{\oplus,\odot}$ is obtained as a transformation of the Choquet integral.

For more details, we recommend the original source [5.24], but also [5.25, 39].

Remark 5.2

When considering $u = 1$, a pseudoaddition $\oplus$ on $[0,1]$ becomes a (continuous) t-conorm. Integrals based on t-conorms closely related to Benvenuti integrals were discussed by *Murofushi* and *Sugeno* [5.40], resulting to the two classes of t-conorm based integrals. Those based on the smallest t-conorm $\vee$ coincide with Benvenuti integral based on $\vee$, with stronger requirements on the corresponding $\vee$-fitting pseudomultiplication $\odot$. The second one, based on continuous Archimedean t-conorms, is a special transform of the Choquet integral, compare Example 5.2 ii),

$$\mathrm{MS}_m(f) = k\left(\mathrm{Ch}_{h(m)}(g(f))\right),$$

with appropriately chosen functions $k, h, g : [0,1] \to [0,\infty]$. Note that the Murofushi–Sugeno integral covers also the integral of *Weber* [5.14] based on strict t-conorms. Another closely related approach to integration, fixing $u = v = \infty$, can be found in [5.41], where Choquet-like integrals were introduced and discussed. For more details on these types of integrals we refer to [5.42, 43].

5.3 Universal Integrals

The concept of universal integrals on $[0,\infty]$ was proposed and discussed in [5.44]. As already mentioned, we will restrict our considerations to the interval $[0,1]$.

Definition 5.6

Let S be the class of all measurable spaces. A mapping

$$I : \bigcup_{(X,\mathcal{A}) \in S} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,1]$$

is called a universal integral whenever it satisfies

UI1 I is nondecreasing in both components;
UI2 there is a semicopula $\otimes : [0,1]^2 \to [0,1]$ (i. e., $\otimes$ is nondecreasing in both components and $1 \otimes a = a \otimes 1$ for all $a \in [0,1]$) such that $I(m, b(a,E)) = a \otimes m(E)$ for all $a \in [0,1]$, any $(X,\mathcal{A}) \in S$, $m \in \mathcal{M}_{(X,\mathcal{A})}$ and $E \in \mathcal{A}$;
UI3 $I(m_1, f_1) = I(m_2, f_2)$ whenever $(m_i, f_i) \in (X_i, \mathcal{A}_i)$, $i = 1, 2$, and $m_1(f_1 \geq t) = m_2(f_2 \geq t)$ for all $t \in [0,1]$.

Observe that the axiom (UI1) reflects the standard monotonicity of integrals. On the other hand, (UI2) expresses the fact that an integral of a basic function $b(a,E)$ with respect to a monotone measure m depends on the values a and $m(E)$ only, independently of the considered measurable space $(X,\mathcal{A})$ and a monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$ (compare the truth values principle in the propositional logics). Finally, (UI3) generalizes the well-known fact from the probability theory that two random variables (defined possibly on two different probability spaces) have the same expectation whenever their distribution functions coincide (in fact, for a probability measure P, $P(f \geq t)$ defines a survival function which is complementary to the related distribution function).

There are several construction methods for universal integrals. First of all, for any given semicopula $\otimes : [0,1]^2 \to [0,1]$, one can introduce the smallest universal integral $I_\otimes$ and the greatest universal integral $I^\otimes$ related to $\otimes$ through (UI2):

$$I_\otimes(m,f) = \sup\{t \otimes m(f \geq t) \,|\, t \in [0,1]\}$$

and

$$I^\otimes(m,f) = \mathrm{essup}_m(f) \otimes m(\mathrm{supp}\, f),$$

where

$$\operatorname{essup}_m(f) = \sup\{t \in [0,1] | m(f \geq t) > 0\}$$

and

$$\operatorname{supp} f = \{x \in X | f(x) > 0\} .$$

Observe that $I_\wedge(m,\cdot) = \mathrm{Su}_m$ is the Sugeno integral, $I_\Pi(m,\cdot) = \mathrm{Sh}_m$ is the Shilkret integral (Π denotes the product semicopula), while I_T with T a strict t-norm is an integral introduced by *Weber* in [5.45].

Considering the Benvenuti integral based on a pseudoaddition $\oplus$ on $[0,u]$ and a $\oplus$-fitting pseudomultiplication $\odot : [0,u] \times [0,v] \to [0,u]$, $u, v, \in [1,\infty]$, such that $\otimes = \odot|[0,1]^2$ is a semicopula, one get a universal integral given by

$$I^{\oplus,\otimes}(m,f) = \mathcal{B}_m^{\oplus,\odot}(f) .$$

Note that $I^{+,\cdot}(m,f) = \mathrm{Ch}_m(f)$ and $I^{\vee,\wedge}(m,f) = \mathrm{Su}_m(f)$.

As an important class of universal integrals we introduce copula-based integrals. Recall that a semicopula $C : [0,1]^2 \to [0,1]$ is called a copula [5.46] whenever it is supermodular, i. e., for any $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^2$ it holds

$$C(\boldsymbol{x} \vee \boldsymbol{y}) + C(\boldsymbol{x} \wedge \boldsymbol{y}) \geq C(\boldsymbol{x}) + C(\boldsymbol{y}) .$$

Note that there is a one-to-one correspondence between copulas and probability measures on Borel subsets of $[0,1]^2$ with uniformly distributed margins, this relation is stated by the equality

$$P_C([0,a] \times [0,b]) = C(a,b),\ (a,b) \in [0,1]^2 .$$

The next result is extracted from [5.44], also compare [5.47, 48].

Proposition 5.2

Let $C : [0,1]^2 \to [0,1]$ be a fixed copula. Then the mapping

$$K_C : \bigcup_{(X,\mathcal{A}) \in \mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,1]$$

given by

$$K_C(m,f) = P_C\left(\left\{(u,v) \in [0,1]^2 | v \leq m(f \geq u)\right\}\right)$$

is a universal integral (with C being the corresponding semicopula).

Note that for the product copula Π, $K_\Pi(m,\cdot) = \mathrm{Ch}_m$ is the Choquet integral, while for the greatest copula $\wedge = \mathrm{Min}$, $K_\wedge(m,\cdot) = \mathrm{Su}_m$ is the Sugeno integral. For the smallest copula $W : [0,1]^2 \to [0,1]$ given by

$$W(a,b) = \max(0, a+b-1) ,$$

K_W was called *opposite Sugeno integral* in [5.49] and it is given by

$$K_W(m,f) = \lambda\left(\{t \in [0,1] | m(f \geq t) \geq 1-t\}\right) ,$$

where λ is the standard Lebesgue measure on Borel subsets of $[0,1]$.

Remark 5.3

The class of universal integrals is convex, i. e., for I_1, I_2 universal integrals and a constant $c \in [0,1]$, also

$$I = cI_1 + (1-c)I_2$$

is a universal integral (related to the semicopula $\odot = c \cdot \odot_1 + (1-c) \cdot \odot_2$).

Though the class of semicopulas is also convex, for the weakest universal integrals we can ensure only the inequality

$$I_{c \cdot \odot_1 + (1-c) \cdot \odot_2} \leq cI_{\odot_1} + (1-c)I_{\odot_2} .$$

On the other hand, for the convex class of copulas it holds

$$K_{cC_1 + (1-c)C_2} = cK_{C_1} + (1-c)K_{C_2} ,$$

i. e., the class of copula-based integrals is convex.

5.4 General Integrals Which Are Not Universal

There are several integrals defined on any measurable space $(X, \mathcal{A})$, for any monotone measure $m \in \mathcal{M}_{(X,\mathcal{A})}$ and any function $f \in \mathcal{F}_{(X,\mathcal{A})}$ which are not universal. We recall two of them based on the standard arithmetic operations $+$ and $\cdot$.

Definition 5.7

A mapping

$$G: \bigcup_{(X,\mathcal{A})\in S} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,\infty]$$

given by

$$G(m,f) = \sup\left\{\sum_{i=1}^{n} a_i \cdot m(A_i) | n \in \mathbb{N}, a_1, \dots, a_n \geq 0\,, \sum_{i=1}^{n} b(a_i, A_i) \leq f \text{ and } (A_i)_{i=1}^{n} \text{ is a disjoint subsystem of } \mathcal{A}\right\}$$

is called a PAN-integral.

Note that this integral was introduced by *Yang* [5.50], see also [5.51] in more general setting on $[0,\infty]$ involving operations $\oplus$ and $\odot$. Due to the results of [5.52], each PAN-integral on $[0,1]$ is either a transformation of integral given in Definition 5.7, $I(m,f) = g^{-1}\left(G(g(m), g(f))\right)$ for some automorphism $g: [0,1] \to [0,1]$, or if $\oplus = \vee$, it is a special instant of integrals $I_{\odot}$ discussed in Sect. 5.3. Also observe that a deep discussion on PAN-integral G can be found in [5.53].

PAN-integral allows one to recognize the underflying monotone measure m only if m is superadditive. Moreover, as a major defect of this integral we recall that it does not exclude the equality of integrals based on two different monotone measures, i.e., there are monotone measures $m_1, m_2 \in \mathcal{M}_{(X,\mathcal{A})}$, $m_1 \neq m_2$, such that $G(m_1,f) = G(m_2,f)$ for all $f \in \mathcal{F}_{(X,\mathcal{A})}$. Note that PAN-integral coincide with the Lebesgue integral whenever m is σ-additive. A similar situation is linked to the concave integral introduced by *Lehrer* [5.54], see also [5.55].

Definition 5.8

A mapping

$$L: \bigcup_{(X,\mathcal{A})\in S} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,\infty]$$

given by

$$L(m,f) = \sup\left\{\sum_{i=1}^{n} a_i \cdot m(A_i) | n \in \mathbb{N}, a_1, \dots, a_n \geq 0, \sum_{i=1}^{n} b(a_i, A_i) \leq f\right\}$$

is called a concave integral.

Observe that this integral is concave in the sense that for each $m \in \mathcal{M}_{(X,\mathcal{A})}, f, g \in \mathcal{F}_{(X,\mathcal{A})}$ and $c \in [0,1]$,

$$L(m, cf + (1-c)\,g) \geq c\,L(m,f) + (1-c)\,L(m,g)\,.$$

Concave integral coincides with the Choquet integral whenever m is supermodular. However, also here $L(m_1,f) = L(m_2,f)$ may hold for all $f \in \mathcal{F}_{(X,\mathcal{A})}$ for some monotone measures $m_1, m_2 \in \mathcal{M}_{(X,\mathcal{A})}$, $m_1 \neq m_2$. Finally, recall that it trivially holds

$$L(m,f) \geq G(m,f) \text{ and } L(m,f) \geq \mathrm{Ch}_m(f)$$

for all $m \in \mathcal{M}_{(X,\mathcal{A})}$ and $f \in \mathcal{F}_{(X,\mathcal{A})}$.

Example 5.3

i) Consider $X = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ and λ the standard Lebesgue measure on $\mathcal{A}$. Let $m = \lambda^p$, $p \in]0,1[$. Then for any $f \in \mathcal{F}_{(X,\mathcal{A})}$ with nonvanishing support (i.e., $m(f > 0) > 0$) it holds

$$G(m,f) = L(m,f) = +\infty\,.$$

On the other hand, for $m = \lambda^2$ (observe that m is supermodular, and thus also superadditive) we get, considering $f = \mathrm{id}_X$,

$$G(m,f) = \frac{2}{13} \text{ while } L(m,f) = \mathrm{Ch}_m(f) = \frac{1}{3}\,.$$

ii) For $X = \{1,2,3\}$ and $\mathcal{A} = 2^X$, let $m_a: \mathcal{A} \to \mathbb{R}$ be given by $m_a(\emptyset) = 0$, $m_a(A) = 0.1$ if card $A = 1$,

$m_a(A) = a$ if card $A = 2$ and $m_a(X) = 1$. Evidently, $m_a \in \mathcal{M}_{(X,\mathcal{A})}$ if and only if $a \in [0.1, 1]$. Let $f \in \mathcal{F}_{(X,\mathcal{A})}$ be given by $f(1) = \frac{1}{3}, f(2) = \frac{2}{3}, f(3) = 1$. Then

$$G(m_a,f) = \sup\left\{\frac{1}{3}\cdot 1, \frac{1}{3}\cdot 0.1 + \frac{2}{3}\cdot a\right\} = \begin{cases} \frac{1}{3} & \text{if } a \in [0.1, 0.45]\,, \\ \frac{0.1+2a}{3} & \text{if } a \in]0.45, 1]\,, \end{cases}$$

and

$$L(m_a,f) = \sup\left\{\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot a + \frac{1}{3}\cdot 0.1, \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0.1 + \frac{2}{3}\cdot 0.1, \frac{1}{3}\cdot a + \frac{2}{3}\cdot a\right\} = \begin{cases} \frac{1.3}{3} & \text{if } a \in [0.1, 0.2[\,, \\ \frac{1.1+a}{3} & \text{if } a \in [0.2, 0.55]\,, \\ a & \text{if } a \in]0.55, 1]\,. \end{cases}$$

Moreover,

$$\mathrm{Ch}_{m_a}(f) = \frac{1.1+a}{3}\,.$$

Observe that m_a is supermodular if and only if $a \in [0.2, 0.55]$ and then

$$L(m_a,f) = \mathrm{Ch}_{m_a}(f) = \frac{1.1+a}{3}\,.$$

iii) For X finite and $m \in \mathcal{M}_{(X,\mathcal{A})}$ such that $m(A) \in \{0, 1\}$ for all $A \subseteq X$, all universal integrals coincide, independently of the underlying semicopula $\otimes$, $I_m(f) = \sup\{\min(f(x)|x \in A)\,|m(A) = 1\}$. However, this does not hold for PAN-integral $G(m,\cdot)$ neither for the concave integral $L(m,\cdot)$. Consider as an example the greatest monotone measure $m^* \in \mathcal{M}_{(X,2^X)}$ given by

$$m^*(A) = \begin{cases} 0 & \text{if } A = \emptyset\,, \\ 1 & \text{else}\,. \end{cases}$$

Then for any universal integral I it holds $I(m^*,f) = \max(f(x)|x \in X)$, but $G(m^*,f) = L(m^*,f) = \sum_{x\in X} f(x)$.

iv) The only monotone measures $m \in \mathcal{M}_{(X,2^X)}$, X finite, such that all universal integrals as well as the PAN and concave integrals coincide, are so-called unanimity measures

$$m_B,\ B \subseteq X,\ B \neq \emptyset,\ m_B(A) = \begin{cases} 1 & \text{if } B \subseteq A\,, \\ 0 & \text{else}\,. \end{cases}$$

Then

$$I(m_B,f) = G(m_B,f) = L(m_B,f) = \min(f(x)|x \in B)\,.$$

Recently, a new concept of decomposition integrals was proposed in [5.56], unifying the PAN integral G, the concave integral L, the Choquet integral Ch, and the Shilkret integral Sh.

Definition 5.9

Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{H}$ be a system of some finite subsystems (i. e., of collections) from $\mathcal{A}$. Then the mapping

$$D_{\mathcal{H}} : \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})} \to [0, \infty]$$

given by

$$D_{\mathcal{H}}(m,f) = \sup\left\{\sum_{i\in I} a_i \cdot m(A_i) | a_i \geq 0, i \in I, \sum_{i\in I} b(a_i, A_i) \leq f,\ (A_i)_{i\in I} \in \mathcal{H}\right\}$$

is called a $\mathcal{H}$-decomposition integral.

Consider the next decomposition systems

$$\begin{aligned} \mathcal{H}^{(n)} &= \left\{(A_i)_{i=1}^n \text{ is a chain in } \mathcal{A}\right\},\ n \in \mathbb{N}\,; \\ \mathcal{H}_G &= \{(A_i)_{i\in I} \text{ is a finite measurable partition of } X\}\,; \\ \mathcal{H}_L &= \mathcal{A}\,; \\ \mathcal{H}_{\mathrm{Ch}} &= \{\mathcal{B}|\mathcal{B} \text{ is a finite chain in } \mathcal{A}\}\,. \end{aligned}$$

Then

$$\begin{aligned} D_{\mathcal{H}^{(1)}}(m,\cdot) &= Sh_m\,; \\ D_{\mathcal{H}_G} &= G\,; \\ D_{\mathcal{H}_L} &= L\,; \\ D_{\mathcal{H}_{\mathrm{Ch}}}(m,\cdot) &= \mathrm{Ch}_m\,. \end{aligned}$$

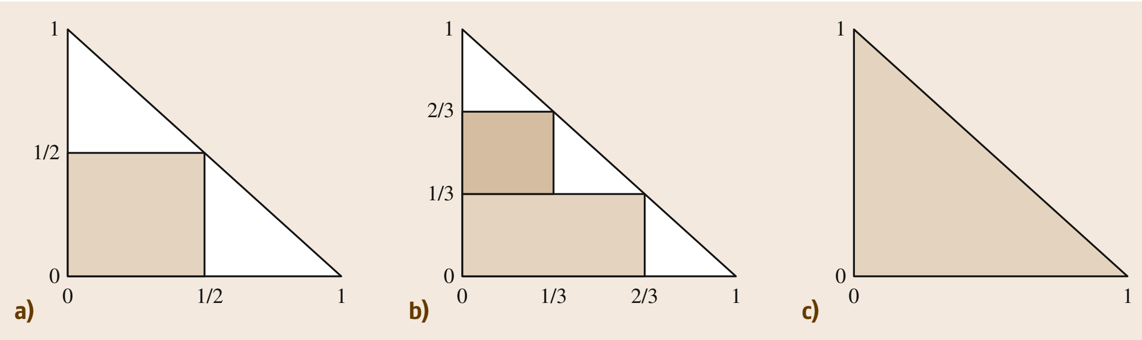

Fig. 5.1a–c The function $\lambda\,(\mathrm{id}_X \geq t)$ with *shaded areas* expressing the corresponding integrals $D_{\mathcal{H}^{(1)}}(\lambda, \mathrm{id}_X)$ **(a)**, $D_{\mathcal{H}^{(2)}}(\lambda, \mathrm{id}_X)$ **(b)**, $\mathrm{Ch}_\lambda(\mathrm{id}_X)$ **(c)**

Further, the only decomposable integrals which are also universal integrals are the Choquet integral and $\mathcal{H}^{(n)}$-decomposition integrals $D_{\mathcal{H}^{(n)}}$ and they satisfy

$$\mathrm{Sh} = D_{\mathcal{H}^{(1)}} \leq D_{\mathcal{H}^{(2)}} \leq \cdots \leq D_{\mathcal{H}^{(n)}} \leq \cdots \leq \mathrm{Ch}\,.$$

Observe that if X is finite, card $X = n$, then $D_{\mathcal{H}^{(n)}} = \mathrm{Ch}$ and that

$$\mathrm{Ch} = \lim_{n\to\infty} D_{\mathcal{H}^{(n)}} = \sup\{D_{\mathcal{H}^{(n)}} | n \in \mathbb{N}\}\ .$$

For more details and further discussion about decomposition integrals, we recommend [5.56–58].

Example 5.4
Using the notation from Example 5.3 i), it holds

$$D_{\mathcal{H}^{(n)}}(\lambda, \mathrm{id}_X) = \frac{n}{2(n+1)}$$

and

$$\lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2} = \mathrm{Ch}_\lambda(\mathrm{id}_X)\ .$$

For better understanding, see Fig. 5.1 with the graph of the function $\lambda\,(\mathrm{id}_X \geq t)$ and with shaded areas expressing the corresponding integrals.

5.5 Concluding Remarks, Application Fields

We have recalled and discussed several kinds of integrals defined on any measurable space for any monotone measure and any nonnegative measurable functions, restricting our considerations to the unit interval [0, 1]. There are several possible extensions of these integrals to the bipolar scale [−1, 1], i. e., for integrating functions with range in [−1, 1]. Recall only the case of the Choquet integral with bipolar extensions of different kinds, such as:

- *Asymmetric* Choquet integral,

$$\mathrm{Ch}_m^{\mathrm{as}}(f) = \mathrm{Ch}_m(f^+) - \mathrm{Ch}_{m^d}(f^-)\ ,$$

  where $f^+ : X \to [0,1]$ is given by $f^+(x) = \max(0, f(x))$, $f^- : X \to [0,1]$ is given by $f^-(x) = \max(0, -f(x))$, and $m^d : \mathcal{A} \to [0,1]$ is a monotone measure dual to m. For more details see [5.18, 19, 26];
- *Symmetric* (Šipoš) Choquet integral,

$$\mathrm{Ch}_m^{\mathrm{sym}}(f) = \mathrm{Ch}_m(f^+) - \mathrm{Ch}_m(f^-)\ ,$$

  see [5.18, 19, 23, 26];
- In the case when X is finite, two another extensions called a *balanced* Choquet integral [5.59] and a *merging* Choquet integral [5.60] reflecting (partial) compensation of positive and negative inputs were also introduced and discussed. Further generalizations yield the background of cummulative prospect theory CPT (Cummulative Prospect Theory) of *Tversky* and *Kahneman* [5.61, 62], however, then two monotone measures are considered,

$$\mathrm{Ch}_{m_1,m_2}(f) = \mathrm{Ch}_{m_1}(f^+) - \mathrm{Ch}_{m_2}(f^-)\ .$$

  Observe that economical applications of CPT have resulted into Nobel Prize for Tversky and Kahneman in 2002.

Some of introduced integrals were introduced because of solving some practical problems. For example, concave integral of *Lehrer* [5.54] is a solution of an optimization problem looking for a maximal global performance.

Among many fields where integrals discussed in this chapter are an important tool, we recall decision making under multiple criteria, multiobjective optimization, multiperson decision making, pattern recognition and classification, image analysis, etc. For more details, we recommend [5.25, Appendix B] or [5.19].

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