4. Aggregation Functions on [0,1]

Radko Mesiar, Anna Kolesárová, Magda Komorníková

After a brief presentation of the history of aggregation, we recall the concept of aggregation functions on $[0,1]$ and on a general interval $I \subseteq [-\infty, \infty]$. We give a list of basic examples as well as some peculiar examples of aggregation functions. After discussing the classification of aggregation functions on $[0,1]$ and presenting the prototypical examples for each introduced class, we also recall several construction methods for aggregation functions, including optimization methods, extension methods, constructions based on given aggregation functions, and introduction of weights. Finally, a remark on aggregation of more general inputs, such as intervals, distribution functions, or fuzzy sets, is added.

Aggregation (fusion, joining) of several input values into one, in some sense the most informative value, is a basic processing method in any field dealing with quantitative information. We only recall mathematics, physics, economy, sociology or finance, among others. Basic arithmetical operations of addition and multiplication on $[0,\infty]$ are typical examples of aggregation functions. As another example let us recall integration and its application to geometry allowing us to compute areas, surfaces, volumes, etc.

4.1 Historical and Introductory Remarks

Just in the field of integration one can find the first historical traces of aggregation known in the written form. Recall the Moscow *mathematical* papyrus and its problem no. 14, dating back to 1850 BC, concerning the computation of the volume of a pyramidal frustum [4.1], or the exhaustive method allowing to compute several types of areas proposed by Eudoxus of Cnidos around 370 BC [4.2]. The roots of a recent penalty-based method of constructing aggregation functions [4.3] can be found in books of Appolonius of Perga (living in the period about 262–190 BC) who (motivated by the center of gravity problems) proposed an approach leading to the centroid, i. e., to the arithmetic mean, minimizing the sum of squares of the Euclidean distances of the given n points from an unknown but fixed one. Generalization of the Appolonius of Perga method based on a general norm is known as the Fréchet mean, or also as the *Karcher* mean, and it was deeply discussed in [4.4].

Another type of mean, the Heronian mean of two nonnegative numbers x and y is given by the formula

$$\mathrm{He}(x,y) = \frac{1}{3}\left(x + \sqrt{xy} + y\right) . \quad (4.1)$$

It is named after Hero of Alexandria (10–70 AD) who used this aggregation function for finding the volume of a conical or pyramidal frustum. He showed that this volume is equal to the product of the height of the frustum and the Heronian mean of areas of parallel bases.

Another interesting historical example can be found in multivalued logic. Already Aristotle (384–322 BC) was a classical logician who did not fully accept the law of excluded middle, but he did not create a system of multivalued logic to explain this isolated remark (in the work De Interpretatione, chapter IX). Systems of multivalued logics considering 3, n (finitely many), and later also infinitely many truth degrees were introduced by *Łukasiewicz* [4.5], *Post* [4.6], *Gödel* [4.7], respectively, and in each of these systems the aggregation of truth values was considered (conjunction, disjunction).

Though several particular aggregation functions (or classes of aggregation functions) were discussed in many earlier works (we only recall means discussed around 1930 by *Kolmogorov* [4.8] and *Nagumo* [4.9], or later by *Aczél* [4.10], triangular norms and copulas studied by *Schweizer* and *Sklar* in 1960s of the previous century and summarized in [4.11]), an independent theory of aggregation can be dated only about 20 years back and the roots of its axiomatization can be found in [4.12–14]. Probably the first monograph devoted purely to aggregation is the monograph by *Calvo* et al. [4.15]. As a basic literature for any scientist interested in aggregation we recommend the monographs [4.16–18].

In this chapter, not only we summarize some earlier, but also some recent results concerning aggregation, including classification, construction methods, and several examples. We will deal with inputs and outputs from the unit interval $[0,1]$. Note that though, in general, we can consider an arbitrary interval $I \subseteq [-\infty,\infty]$, there is no loss of generality (up to the isomorphism) when restricting our considerations to $I=[0,1]$. As an example, consider the aggregation of nonnegative inputs, i. e., fix $I=[0,\infty[$. Then any aggregation function A on $[0,\infty[$ can be seen as an isomorphic transform of some aggregation function B on $[0,1]$, restricted to $[0,1[$ and satisfying two constraints:

i) $B(\boldsymbol{x})=1$ if and only if $\boldsymbol{x}=(1,\dots,1)$,
ii) $\sup\{B(\boldsymbol{x}) \mid \boldsymbol{x} \in [0,1[^n\}=1,\ n\in\mathbb{N}$.

Note that any increasing bijection $\varphi : [0,1[\to [0,\infty[$ can be applied as the considered isomorphism. For more details about aggregation on a general interval $I \subseteq [-\infty,\infty]$ refer to [4.17].

We can consider either aggregation functions with a fixed number $n\in\mathbb{N}$, $n\ge 2$, of inputs or extended aggregation functions defined for any number $n\in\mathbb{N}$ of inputs. The number n is called the arity of the aggregation function.

Definition 4.1
For a fixed $n\in\mathbb{N}$, $n\ge 2$, a function $A:[0,1]^n\to[0,1]$ is called an (n-ary) aggregation function whenever it is increasing in each variable and satisfies the boundary conditions

$$A(0,\dots,0)=0 \quad\text{and}\quad A(1,\dots,1)=1\,.$$

A mapping $A:\bigcup_{n\in\mathbb{N}}[0,1]^n\to[0,1]$ is called an extended aggregation function whenever $A(x)=x$ for each $x\in[0,1]$, and for each $n\in\mathbb{N}$, $n\ge 2$, $A\mid[0,1]^n$ is an n-ary aggregation function.

The framework of extended aggregation functions is rather general, not relating different arities, and thus some additional constraints are often considered, such as associativity, decomposability, neutral element, etc.

The Heronian mean He given in (4.1) is an example of a binary aggregation function. Prototypical examples of extended aggregation functions on $[0,1]$ are:

- The smallest extended aggregation function A_s given by
$$A_s(\boldsymbol{x})=\begin{cases}1 & \text{if } \boldsymbol{x}=(1,\dots,1)\\ 0 & \text{else}\,.\end{cases}$$
- The greatest extended aggregation function A_g given by
$$A_g(\boldsymbol{x})=\begin{cases}0 & \text{if } \boldsymbol{x}=(0,\dots,0)\\ 1 & \text{else}\,.\end{cases}$$
- The arithmetic mean M given by
$$M(x_1,\dots,x_n)=\frac{1}{n}\sum_{i=1}^{n}x_i\,.$$
- The geometric mean G given by
$$G(x_1,\dots,x_n)=\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}\,.$$
- The product Π given by
$$\Pi(x_1,\dots,x_n)=\prod_{i=1}^{n}x_i\,.$$
- The minimum Min given by
$$\mathrm{Min}(x_1,\dots,x_n)=\min\{x_1,\dots,x_n\}\,.$$

- The maximum Max given by

$$\mathrm{Max}(x_1,\ldots,x_n) = \max\{x_1,\ldots,x_n\} .$$

- The truncated sum S_{L} (also known as the Łukasiewicz t-conorm) given by

$$S_{\mathrm{L}}(x_1,\ldots,x_n) = \min\left\{1, \sum_{i=1}^{n} x_i\right\} ,$$

- The 3-Π-operator E introduced in [4.19] and given by

$$E(x_1,\ldots,x_n) = \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n x_i + \prod_{i=1}^n (1-x_i)} ,$$

with some convention covering the case $\frac{0}{0}$,
- The Pascal weighted arithmetic mean W_{P} given by

$$W_{\mathrm{P}}(x_1,\ldots,x_n) = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} x_i .$$

As distinguished examples of n-ary aggregation functions for a fixed arity $n \geq 2$, recall the projections P_i and order statistics OS_i, $i = 1,\ldots,n$, given by

$$P_i(x_1,\ldots,x_n) = x_i$$

and

$$\mathrm{OS}_i(x_1,\ldots,x_n) = x_{\sigma(i)} ,$$

where σ is an arbitrary permutation of $(1,\ldots,n)$ such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$. Observe that the first projection $P_{\mathrm{F}} = P_1$ and the last projection $P_{\mathrm{L}} = P_n$ can be seen as instances of extended aggregation functions P_{F} and P_{L}, respectively. On the other hand, for any fixed $n \geq 2$, OS_1 is just $\mathrm{Min} \mid [0,1]^n$ and $\mathrm{OS}_n = \mathrm{Max} \mid [0,1]^n$.

As a peculiar example of an extended aggregation function we can introduce the mapping $V : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ given by

$$V(x_1,\ldots,x_n) = \min\left\{\left(\sum_{i=1}^{n} x_i^n\right)^{\frac{1}{n}}, 1\right\} . \tag{4.2}$$

4.2 Classification of Aggregation Functions

Let us denote by $\mathcal{A}$ the class of all extended aggregation functions, and by $\mathcal{A}_n$ (for a fixed $n \geq 2$) the class of all n-ary aggregation functions. Several classifications of n-ary aggregation functions can be straightforwardly extended to the class $\mathcal{A}$. The basic classification proposed by *Dubois* and *Prade* [4.20] distinguishes (both for n-ary and extended aggregation functions):

- *Conjunctive* aggregation functions,
  $\mathcal{C} = \{A \in \mathcal{A} \mid A \leq \mathrm{Min}\}$,
- *Disjunctive* aggregation functions,
  $\mathcal{D} = \{A \in \mathcal{A} \mid A \geq \mathrm{Max}\}$,
- *Averaging* aggregation functions,
  $\mathcal{A}v = \{A \in \mathcal{A} \mid \mathrm{Min} \leq A \leq \mathrm{Max}\}$,
- *Mixed* aggregation functions,
  $\mathcal{M} = \mathcal{A} \setminus (\mathcal{C} \cup \mathcal{D} \cup \mathcal{A}v)$.

Considering purely averaging aggregation functions $\mathcal{A}v^p = \mathcal{A}v \setminus \{\mathrm{Min},\mathrm{Max}\}$, we can see that the set $\{\mathcal{C},\mathcal{D},\mathcal{A}v^p,\mathcal{M}\}$ forms a partition of $\mathcal{A}$. Note that the classes $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{A}v$, $\mathcal{A}v^p$ are convex, which is not the case of the class $\mathcal{M}$. For the previously introduced examples it holds:

- $M, G, W_P, P_F, P_L \in \mathcal{A}v^p$,
- $\Pi \in \mathcal{C}$,
- $S_L, V \in \mathcal{D}$,
- $E \in \mathcal{M}$.

Observe that n-ary aggregation functions P_i and OS_i, $i = 1,\ldots,n$, are averaging, so are their convex sums, i. e., weighted arithmetic means

$$W = \sum_{i=1}^{n} w_i P_i ,$$

and ordered weighted averages (OWA operators) [4.21],

$$\mathrm{OWA} = \sum_{i=1}^{n} w_i \mathrm{OS}_i ,$$

with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. The binary Heronian mean He given in (4.1) is a convex combination of

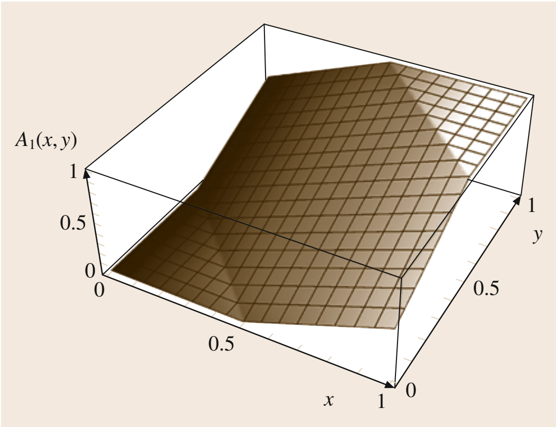

Fig. 4.1 3D plot of the aggregation function A_1 defined by (4.3)

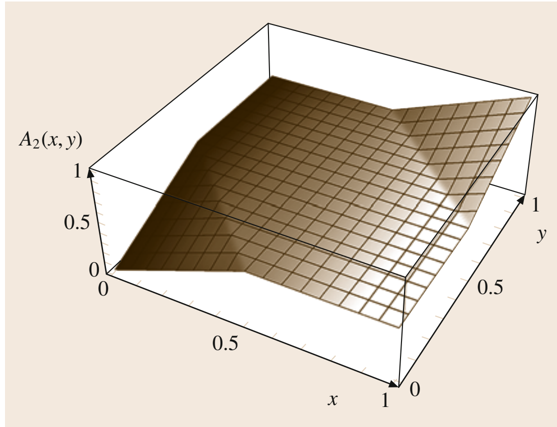

Fig. 4.2 3D plot of the aggregation function A_2 defined by (4.4)

the arithmetic mean M and the geometric mean G, $\mathrm{He} = \frac{2}{3}M + \frac{1}{3}G$, and thus it is also averaging.

Consider two binary aggregation functions $A_1, A_2 : [0,1]^2 \to [0,1]$ given by

$$A_1(x, y) = \mathrm{Med}(0, 1, x + y - 0.5) \tag{4.3}$$

and

$$A_2(x, y) = \mathrm{Med}(x + y, 0.5, x + y - 1)\ , \tag{4.4}$$

where Med is the standard median operator. Then $A_1, A_2 \in \mathcal{M}$ but $\frac{1}{2}A_1 + \frac{1}{2}A_2 = M \in \mathcal{A}v$. The 3D plots of aggregation functions A_1, A_2 and M are depicted in Figs. 4.1–4.3.

More refined classifications of n-ary aggregation functions are related to order statistics OS_i, $i = 1, \ldots, n$. The conjunctive classification [4.22] deals with the partition of the class $\mathcal{A}_n$ given by $\{C_1, \ldots, C_n, R_C\}$, where the class of i-conjunctive aggregation functions, $i = 1, \ldots, n$, is defined by

$$\begin{aligned} C_i &= \{A \in \mathcal{A}_n \mid \min\{\mathrm{card}\{j \mid \boldsymbol{x}_j \ge A(\boldsymbol{x})\} \\ &\qquad \mid x \in [0,1]^n\} = i\} \\ &= \{A \in \mathcal{A}_n \mid A \le \mathrm{OS}_{n-i+1} \text{ but not } A \le \mathrm{OS}_{n-i}\}\ , \end{aligned}$$

where formally $\mathrm{OS}_0 \equiv 0$.

In other words, A is i-conjunctive if and only if the aggregated value $A(\boldsymbol{x})$ is dominated by at least i input values independently of $\boldsymbol{x} \in [0,1]^n$, but not by $(i+1)$ values, in general.

Clearly, the classes $C_1, \ldots, C_n$ are pairwise disjoint and the remaining aggregation functions are members of the class $R_C = \mathcal{A}_n \setminus \bigcup_{i=1}^{n} C_i$. If we come back to the above-mentioned basic classification of aggregation functions (applied to $\mathcal{A}_n$), we obtain $C = C_n$ and $\mathcal{W}_C = \bigcup_{i=1}^{n-1} C_i = \mathcal{A}v \setminus \{\mathrm{Min}\}$. The class $\mathcal{W}_C$ is called *weakly conjunctive* [4.22].

Similarly, we have a disjunctive type of classification of $\mathcal{A}_n$ related to the partition $\{\mathcal{D}_1, \ldots, \mathcal{D}_n, R_{\mathcal{D}}\}$, with

$$\begin{aligned} \mathcal{D}_i &= \{A \in \mathcal{A}_n \mid A \ge \mathrm{OS}_i \text{ but not } A \ge \mathrm{OS}_{i+1}\}\ , \\ &\quad i = 1, \ldots, n\ . \end{aligned}$$

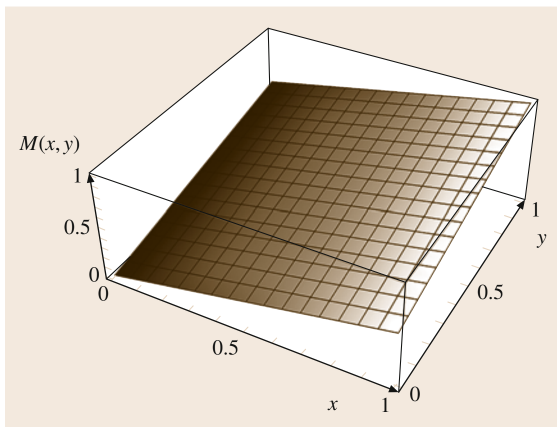

Fig. 4.3 3D plot of the aggregation function $\frac{1}{2}A_1 + \frac{1}{2}A_2 = M$

Then $\mathcal{D}_n = \mathcal{D}$ and for the class of *weakly disjunctive* aggregation functions $\mathcal{W}_\mathcal{D} = \bigcup_{i=1}^{n-1} \mathcal{D}_i$ we have $\mathcal{W}_\mathcal{D} = \mathcal{A}v \setminus \{\text{Max}\}$. Hence $\mathcal{W}_C \bigcap \mathcal{W}_\mathcal{D} = \mathcal{A}v^p$, and $A \in \bigcup_{i=1}^{n} C_i$ if and only if $A \leq \text{Max}$, while $A \in \bigcup_{i=1}^{n} \mathcal{D}_i$ if and only if $A \geq \text{Min}$.

Note that the conjunctive and disjunctive classifications can be applied to aggregation functions defined on posets, too [4.22], and that this approach to the classification of aggregation functions on [0, 1] was already proposed by *Marichal* in [4.23] as *i*-tolerant and *i*-intolerant aggregation functions (Marichal's approach based on order statistics is applicable when considering chains only).

Observe that this approach to classification has no direct extension to extended aggregation functions. On the other hand, we have the next classification valid for extended aggregation functions only. We distinguish:

- *Dimension decreasing* aggregation functions forming the class $\mathcal{A}_\searrow$, satisfying $A(x_1,\dots,x_n,x_{n+1}) \leq A(x_1,\dots,x_n)$ for any $n \in \mathbb{N}$, $x_1,\dots,x_{n+1} \in [0,1]$, but violating the equality, in general.
- *Dimension increasing* aggregation functions forming the class $\mathcal{A}_\nearrow$, satisfying $A(x_1,\dots,x_n,x_{n+1}) \geq A(x_1,\dots,x_n)$ for any $n \in \mathbb{N}$, $x_1,\dots,x_{n+1} \in [0,1]$, but violating the equality, in general.
- *Dimension averaging* aggregation functions forming the class $\overset{\leftrightarrow}{\mathcal{A}}$, satisfying $A(x_1,\dots,x_n,0) \leq A(x_1,\dots,x_n) \leq A(x_1,\dots,x_n,1)$ for any $n \in \mathbb{N}$, $x_1,\dots,x_n \in [0,1]$, and attaining strict inequalities for at least one $\boldsymbol{x} \in [0,1]^n$.

Evidently, the classes $\mathcal{A}_\searrow, \mathcal{A}_\nearrow$, and $\overset{\leftrightarrow}{\mathcal{A}}$ are disjoint and they, together with their reminder $\mathcal{A} \setminus (\mathcal{A}_\searrow \cup \mathcal{A}_\nearrow \cup \overset{\leftrightarrow}{\mathcal{A}})$, form a partition of $\mathcal{A}$. Let us note that each associative conjunctive aggregation function is dimension decreasing, and thus, $\Pi, \text{Min} \in \mathcal{A}_\searrow$. Similarly, each associative disjunctive aggregation function is dimension increasing, so, $S_\text{L}, \text{Max} \in \mathcal{A}_\nearrow$.

Recently, *Yager* has introduced extended aggregation functions with the self-identity property [4.24] characterized by the equality

$$A\left(x_1,\dots,x_n,A(x_1,\dots,x_n)\right) = A(x_1,\dots,x_n)$$

for any $n \in \mathbb{N}$ and $x_1,\dots,x_n \in [0,1]$ (e.g., the arithmetic mean M or the geometric mean G satisfy this property). Evidently, each such aggregation function satisfies

$$\begin{aligned} A(x_1,\dots,x_n,0) &\leq A(x_1,\dots,x_n,x_{n+1}) \\ &\leq A(x_1,\dots,x_n,1) \end{aligned}$$

for all $n \in \mathbb{N}, x_1,\dots,x_{n+1} \in [0,1]$ and thus, if the strict inequalities are attained for some $n \in \mathbb{N}$ and

$$x_1,\dots,x_{n+1} \in [0,1]\,,$$

A belongs to $\overset{\leftrightarrow}{\mathcal{A}}$. So, for example, $M, G \in \overset{\leftrightarrow}{\mathcal{A}}$. The extended aggregation function V (4.2) also belongs to $\overset{\leftrightarrow}{\mathcal{A}}$. On the other hand, the first projection P_F does not belong to

$$\mathcal{A}_\searrow \cup \mathcal{A}_\nearrow \cup \overset{\leftrightarrow}{\mathcal{A}}\,,$$

and the last projection P_L belongs to $\overset{\leftrightarrow}{\mathcal{A}}$. Recall that if $A \in \mathcal{A}_\searrow$, it is also said to have the *downward attitude property* [4.24]. Similarly, the upward attitude property introduced in [4.24] corresponds to the class $\mathcal{A}_\nearrow$. Dimension increasing aggregation functions were also considered in [4.25].

Let us return to the basic classification of aggregation functions and recall several distinguished types of aggregation functions belonging to the classes C, $\mathcal{D}$, $\mathcal{A}v^p$, and $\mathcal{M}$:

- *Conjunctive aggregation functions*: Triangular norms [4.26, 27], copulas [4.27, 28], quasi-copulas [4.29, 30], and semicopulas [4.31].
- *Disjunctive aggregation functions*: Triangular conorms [4.26, 27], dual copulas [4.28].
- *Averaging aggregation functions*: (Weighted) quasi-arithmetic means [4.10], idempotent uninorms [4.32], integrals based on capacities, including the Choquet and Sugeno integrals [4.18, 33–36], also covering OWA [4.21], ordered weighted maximum (OWMax) [4.37] and ordered modular average (OMA) [4.38] operators, as well as lattice polynomials [4.39].
- *Mixed aggregation functions*: nonidempotent uninorms [4.40], gamma-operators [4.41], special convex sums in fuzzy linear programming [4.42].

For more details concerning these aggregation functions see [4.17] or references given above.

4.3 Properties and Construction Methods

Properties of aggregation functions are mostly related to the field of their application, such as multicriteria decision aid, multivalued logics, or probability theory, for example. Besides the standard analytical properties of functions, such as continuity, the Lipschitz property, and (perhaps adapted) algebraic properties, such as symmetry, associativity, bisymmetry, neutral element, annihilator, cancellativity, or idempotency [4.17, Chapter 2], the above-mentioned applied fields have brought into aggregation theory properties as decomposability, conjunctivity, or n-increasigness. Each of the mentioned properties can be introduced for n-ary aggregation functions (excepting decomposability), and thus also for extended aggregation functions. However, in the case of extended aggregation functions, some properties can be introduced in a stronger form, involving different arities in a single formula.

For example, the (weak) idempotency of $A \in \mathcal{A}$ means the idempotency of each $A \mid [0,1]^n$, which means that for each $n \in \mathbb{N}$ and

$$x \in [0,1]\,, \quad A(\underbrace{x,\dots,x}_{n\text{-times}}) = x\,.$$

Note that an extended aggregation function A is idempotent if and only if it is averaging, i. e., $A \in \mathcal{A}v$. The strong idempotency [4.15] of an extended aggregation function $A \in \mathcal{A}$ means that

$$A(\underbrace{\boldsymbol{x},\dots,\boldsymbol{x}}_{k\text{-times}}) = A(\boldsymbol{x})$$

for each $k \in \mathbb{N}$ and $\boldsymbol{x} \in \bigcup_{n\in\mathbb{N}}[0,1]^n$. For example, the extended aggregation function W_{P} is idempotent but not strongly idempotent.

Similarly, $e \in [0,1]$ is a (weak) neutral element of an extended aggregation function $A \in \mathcal{A}$ if and only if for each $n \geq 2$ and $\boldsymbol{x} \in [0,1]^n$ such that $x_j = e$ for $j \neq i$ it holds $A(\boldsymbol{x}) = x_i$. On the other hand, e is a strong neutral element of an extended aggregation function $A \in \mathcal{A}$ if and only if for any $n \geq 2$, $\boldsymbol{x} \in [0,1]^n$ with $x_i = e$, it holds

$$A(x_1,\dots,x_{i-1},e,x_{i+1},\dots,x_n) \\ = A(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\,.$$

Obviously, if e is a strong neutral element of $A \in \mathcal{A}$ then it is also a (weak) neutral element of A. As an example, consider the extended copula $D \in \mathcal{A}$ given by

$$D(x_1,\dots,x_n) = x_1 \cdot \min\{x_2,\dots,x_n\}\,.$$

Obviously, $e = 1$ is a weak neutral element of D. However $D\left(1,\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2} \neq \frac{1}{4} = D\left(\frac{1}{2},\frac{1}{2}\right)$, i. e., $e = 1$ is not a strong neutral element of D. For a deeper discussion and exemplification of properties of aggregation functions we recommend [4.17].

Aggregation functions in many fields are constrained by the required properties – axioms in each considered field. As a typical example recall multivalued logics (fuzzy logics) with truth values domain $[0,1]$, where conjunction is modeled by means of triangular norms [4.26, 43, 44]. Recall that a binary aggregation function $T : [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm for short) whenever it is symmetric, associative and $e = 1$ is its neutral element. Due to associativity, there is a genuine extension of a t-norm T into an extended aggregation function (we will also use the same notation T in this case). Then $e = 1$ is a strong neutral element for the extended T. However, without some additional properties we still cannot determine a t-norm convenient for our purposes. Requiring, for example, the idempotency of T, we obtain that the only solution is $T = \mathrm{Min}$, the strongest triangular norm. Considering continuous triangular norms satisfying the diagonal inequalities $0 < T(x,x) < x$ for all $x \in]0,1[$, we can show that T is isomorphic to the product Π, i. e., there is an automorphism $\varphi : [0,1] \to [0,1]$ such that $T(x,y) = \varphi^{-1}(\Pi(\varphi(x),\varphi(y)))$, and in the extended form, $T(x_1,\dots,x_n) = \varphi^{-1}(\Pi(\varphi(x_1),\dots,\varphi(x_n)))$. For more details and several other results we recommend [4.26].

As another example consider probability theory, namely the relationship between the joint distribution function F_Z of a random vector $Z = (X_1,\dots,X_n)$, and the corresponding marginal one-dimensional distribution functions $F_{X_1},\dots,F_{X_n}$. By the *Sklar* theorem [4.45], for all $(x_1,\dots,x_n) \in \overline{\mathbb{R}}$ we have

$$F_Z(x_1,\dots,x_n) = C\left(F_{X_1}(x_1),\dots,F_{X_n}(x_n)\right)$$

for some n-ary aggregation function C. Obviously, constrained by the basic properties of probabilities, C should possess a neutral element $e = 1$ and annihilator (zero element) $a = 0$, and the function C should be n-increasing (i. e., probability $P(Z \in [u_1,v_1] \times \dots \times [u_n,v_n]) \geq 0$ for any n-dimensional box $[u_1,v_1] \times \dots \times [u_n,v_n]$), which yields an axiomatic definition of copulas. More details for interested readers can be found in [4.28]. Considering some additional constraints, we obtain special subclasses of copulas. For example, if we fix $n = 2$ and consider the stability of copulas with respect to positive powers, i. e., the property $C(x^\lambda, y^\lambda) = (C(x, y))^\lambda$ for each $\lambda \in]0, \infty[$ and each $(x, y) \in [0, 1]^2$, then we obtain *extreme value copulas* (EV copulas) [4.46, 47]. Recall that a copula $C : [0, 1]^2 \to [0, 1]$ is an EV copula if and only if there is a convex function $d : [0, 1] \to [0, 1]$ such that for each $t \in [0, 1]$, $\max\{t, 1-t\} \le d(t) \le 1$ and for all $(x, y) \in]0, 1[^2$,

$$C(x, y) = (xy)^{d\left(\frac{\log x}{\log xy}\right)}$$

(observe that on $[0, 1]^2 \setminus]0, 1[^2$ for each copula it holds $C(x, y) = \min\{x, y\}$).

Our third example comes from economics. In multicriteria decision problems, we often meet the requirement of the comonotone additivity of the considered n-ary (extended) aggregation function A, i. e., we expect that $A(\boldsymbol{x} + \boldsymbol{y}) = A(\boldsymbol{x}) + A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in [0, 1]^n$ such that $\boldsymbol{x} + \boldsymbol{y} \in [0, 1]^n$ and $(x_i - x_j)(y_i - y_j) \ge 0$ for any $i, j \in \{1, \ldots, n\}$. The comonotonicity of $\boldsymbol{x}$ and $\boldsymbol{y}$ means that the ordering on $\{1, \ldots, n\}$ induced by $\boldsymbol{x}$ is not contradictory to that one induced by $\boldsymbol{y}$. Due to *Schmeidler* [4.48], we know that then A is necessarily the Choquet integral based on the fuzzy measure $m : 2^{\{1,\ldots,n\}} \to [0, 1]$, $m(E) = A(1_E)$, given by (4.6).

The axiomatic approach to aggregation characterizes some special classes of aggregation functions. Another important look at aggregation involves construction methods. We can roughly divide them into the next four groups:

- Optimization methods,
- Extension methods,
- Constructions based on the given aggregation functions,
- Introduction of weights.

An exhaustive overview of construction methods for aggregation functions can be found in [4.17, Chapter 6]. Here we briefly recall the most distinguished ones.

A typical *optimization* method is the penalty-based approach proposed in [4.49] and generalized in [4.3], where dissimilarity functions were introduced, see also [4.50].

Definition 4.2

A function $D : [0, 1]^2 \to [0, \infty[$ given by

$$D(x, y) = K(f(x) - f(y)) ,$$

where $f : [0, 1] \to \mathbb{R}$ is a continuous strictly monotone function and $K : \mathbb{R} \to [0, \infty[$ is a convex function attaining the unique minimum $K(0) = 0$, is called a *dissimilarity* function.

Theorem 4.1

Let $D : [0, 1]^2 \to [0, \infty]$ be a dissimilarity function. Then for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in [0, 1]$, the function $h : [0, 1] \to \mathbb{R}$ given by $h(t) = \sum_{i=1}^n D(x_i, t)$ attains its minimal value exactly on a closed interval $[a, b]$ and the formula

$$A(x_1, \ldots, x_n) = \frac{a + b}{2}$$

defines a strongly idempotent symmetric extended aggregation function A on $[0, 1]$.

Construction given in Theorem 4.1 covers:

- the arithmetic mean $\left(D(x, y) = (x - y)^2\right)$,
- quasi-arithmetic means $\left(D(x, y) = (f(x) - f(y))^2\right)$,
- the median $(D(x, y) = |x - y|)$,

among others. This method is a generalization of the Appolonius of Perga method. Note that in general, a function D need not be symmetric, i. e., K need not be an even function (compare with the symmetry of metrics). As a typical example, let us recall the dissimilarity function $D_c : [0, 1]^2 \to [0, \infty], c \in]0, \infty[$, given by

$$D_c(x, y) = \begin{cases} x - y & \text{if } x \ge y\,, \\ c(y - x) & \text{if } x < y\,, \end{cases}$$

yielding by means of Theorem 4.1 the α-quantile of a sample $(x_1, \ldots, x_n)$ with $\alpha = \frac{1}{1+c}$.

As a possible generalization of Theorem 4.1, one can consider different dissimilarity functions D_i (which violates the symmetry of the constructed aggregation function A). Consider, for example, $D_1(x, y) = |x - y|$ and $D_2(x, y) = \cdots = D_n(x, y) = \cdots = (x - y)^2$. Then the minimization of the sum $\sum_{i=1}^n D_i(x_i, t)$ results in the extended aggregation function $A : \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$ given by

$$A(x_1, \ldots, x_n) = \operatorname{Med}\left(x_1, M(x_2, \ldots, x_n) - 0.5, M(x_2, \ldots, x_n) + 0.5\right) , \quad (4.5)$$

whenever $n > 1$.

Some other generalizations based on a generalized approach to dissimilarity (penalty) functions can be found in [4.16].

Extension methods are based on a partial information that is available about an aggregation function. As a typical example, we recall integral-based aggregation functions. Suppose that for a fixed arity n the values of an aggregation function A are known at Boolean inputs only, i. e., we know $A \mid \{0,1\}^n$ only. Identifying subsets of the space $X = \{1,\ldots,n\}$ with the corresponding characteristic functions, we get the set function $m : 2^X \to [0,1]$ given by $m(E) = A(1_E)$. Obviously, m is monotone, i. e., $m(E_1) \le m(E_2)$ whenever $E_1 \subseteq E_2 \subseteq X$, and $m(\emptyset) = A(0,\ldots,0) = 0$, $m(X) = A(1,\ldots,1) = 1$. Note that m is often called a *fuzzy measure* [4.51, 52] or a *capacity* [4.17].

Among several integral-based extension methods we recall:

- The *Choquet* integral [4.53], $\mathrm{Ch}_m : [0,1]^n \to [0,1]$,

$$\mathrm{Ch}_m(\boldsymbol{x}) = \sum_{i=1}^{n} x_{\sigma(i)} \cdot (m(E_{\sigma,i}) - m(E_{\sigma,i+1})) \ , \tag{4.6}$$

  where $\sigma : X \to X$ is a permutation such that $x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(n)}$, $E_{\sigma,i} = \{\sigma(i),\ldots,\sigma(n)\}$ for $i = 1,\ldots,n$, and $E_{\sigma,n+1} = \emptyset$. Note that the Choquet integral can be seen as a weighted arithmetic mean with the weights dependent on the ordinal structure of the input vector $\boldsymbol{x}$. If the capacity m is additive, i. e., $m(E) = \sum_{i\in E} m(\{i\})$, then

$$\mathrm{Ch}_m(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_i \ ,$$

  where for the weights it holds $w_i = m(\{i\})$, $i \in X$ (hence $\sum_{i=1}^{n} w_i = 1$).
- The *Sugeno* integral [4.51], $\mathrm{Su}_m : [0,1]^n \to [0,1]$,

$$\mathrm{Su}_m(\boldsymbol{x}) = \max\{\min\{x_{\sigma(i)}, m(E_{\sigma,i})\} \mid i \in X\} \ .$$

  If m is maxitive, i. e., $m(E) = \max\{m(\{i\}) \mid i \in E\}$, then we recognize the weighted maximum $\mathrm{Su}_m(\boldsymbol{x}) = \max\{\min\{x_i, v_i\} \mid i \in X\}$, with weights $v_i = m(\{i\})$ (hence $\max\{v_i \mid i \in X\} = 1$).
- The copula-based integral [4.34], $I_{C,m} : [0,1]^n \to [0,1]$, where $C : [0,1]^2 \to [0,1]$ is a binary copula,

$$I_{C,m}(\boldsymbol{x}) = \sum_{i=1}^{n} \left(C\left(\boldsymbol{x}_{\sigma(i)}, m(E_{\sigma,i})\right) - C\left(x_{\sigma(i)}, m(E_{\sigma,i+1})\right)\right) .$$

This integral covers the Choquet integral if C is equal to the product copula Π, $I_{\Pi,m} = \mathrm{Ch}_m$, as well as the Sugeno integral in the case of the greatest copula Min, $I_{\mathrm{Min},m} = \mathrm{Su}_m$. Observe that if the capacity m is symmetric, i. e., $m(E) = v_{\mathrm{card}\,E}$, where $0 = v_0 \le v_1 \le \cdots \le v_n = 1$, then $I_{C,m}$ turns to OMA operator introduced in [4.38]. Its special instances are the OWA operators [4.21] based on the Choquet integral,

$$\mathrm{OWA}(\boldsymbol{x}) = \sum_{i=1}^{n} x_{\sigma(i)} \cdot w_i \ ,$$

with $w_i = v_i - v_{i-1}$, and the OWMax operator [4.37],

$$\mathrm{OWMax}(\boldsymbol{x}) = \max\{\min\{x_{\sigma(i)}, v_i\} \mid i \in X\} \ .$$

For better understanding, fix $n = 2$, i. e., consider $X = \{1,2\}$. Then $m(\{1\}) = a$ and $m(\{2\}) = b$ are any constants from $[0,1]$, and $m(\emptyset) = 0$, $m(X) = 1$ due to the boundary conditions. The following equalities hold:

- $\mathrm{Ch}_m(x,y) = \begin{cases} ax + (1-a)y & \text{if } x \ge y, \\ (1-b)x + by & \text{else,} \end{cases}$
- $\mathrm{Su}_m(x,y) = \max\{\min\{a,x\}, \min\{b,y\}, \min\{x,y\}\}$,
- $I_{C,m}(x,y) = \begin{cases} C(x,a) + y - C(y,a) & \text{if } x \ge y, \\ C(y,b) + x - C(x,b) & \text{else.} \end{cases}$

The considered capacity m is symmetric if and only if $a = b$, and then:

- $\mathrm{Ch}_m(x,y) = \mathrm{OWA}(x,y) = (1-a)\cdot\min\{x,y\} + a \cdot \max\{x,y\}$,
- $\mathrm{Su}_m(x,y) = \mathrm{OW\,Max}(x,y) = \mathrm{Med}(x,a,y)$ is the so-called a-median [4.54, 55],
- $I_{C,m}(x,y) = \mathrm{OMA}(x,y) = f_1(\min\{x,y\}) + f_2(\max\{x,y\})$, where $f_1, f_2 : [0,1] \to [0,1]$ are given by $f_1(t) = t - C(t,a)$ and $f_2(t) = C(t,a)$.

For more details concerning integral-based constructions of aggregation functions we recommend [4.34, 36, 56] or [4.34] by *Klement*, *Mesiar*, and *Pap*.

Another kind of extension methods exploiting capacities is based on the Möbius transform. Recall that for a capacity $m : 2^X \to [0,1]$, its Möbius transform $\mu : 2^X \to \mathbb{R}$ is given by

$$\mu(E) = \sum_{L\subseteq E} (-1)^{\mathrm{card}(E\setminus L)} m(L) \ .$$

Theorem 4.2
[4.57] Let $C:[0,1]^n \to [0,1]$ be an n-ary copula, and $m:2^X \to [0,1]$ a capacity. Then the function $A_{C,m}:[0,1]^n \to [0,1]$ given by

$$A_{C,m}(\boldsymbol{x}) = \sum_{E \subseteq X} \mu(E) \cdot C(\boldsymbol{x} \vee 1_{E^c})$$

is an aggregation function.

Special instances of Theorem 4.2 are the *Lovász* extension [4.58] corresponding to the strongest copula Min ($A_{\text{Min},m} = I_{\Pi,m} = \text{Ch}_m$ is just the Choquet integral), and the Owen extension [4.59] corresponding to the product copula Π ($A_{\Pi,m}(x) = \sum_{E \subseteq X} \left(\mu(E) \prod_{i \in E} x_i\right)$).

Several extension methods were introduced for binary copulas, for example, in the case when only the information about their diagonal section $\delta_C:[0,1] \to [0,1]$, $\delta_C(x) = C(x,x)$ is available. If $\delta:[0,1] \to [0,1]$ is any increasing 2-Lipschitz function such that $\delta(0) = 0, \delta(1) = 1$, and $\delta(x) \le x$ for each $x \in [0,1]$, then the formula

$$D(x,y) = \min\left\{x, y, \frac{\delta(x)+\delta(y)}{2}\right\}, \quad (x,y) \in [0,1]^2 ,$$

defines a binary copula with $\delta_D = \delta$. Note that D is the greatest symmetric copula with the given diagonal section. Among numerous papers dealing with such types of extensions we recommend the overview paper [4.60]. Similarly, one can extend horizontal or vertical sections to copulas [4.61]. An overview of extension methods for triangular norms can be found in [4.26].

The third group of construction methods involves methods creating new aggregation functions from the given ones. These methods are applied either to aggregation functions with a fixed arity n, or to extended aggregation functions. Some of them can be applied to any kind of aggregation functions. As a typical example, recall transformation of aggregation functions by means of an automorphism $\varphi:[0,1] \to [0,1]$ (i. e., an isomorphic transformation) given by

$$A_\varphi(x_1,\dots,x_n) = \varphi^{-1}\left(A\left(\varphi(x_1),\dots,\varphi(x_n)\right)\right) . \tag{4.7}$$

Transformation (4.7) preserves all algebraic properties as well as the classification of aggregation functions. However, some analytical properties can be broken, for example, the Lipschitz property or n-increasigness. Some special classes of aggregation functions can be characterized by a unique member and its isomorphic transforms. Consider, for example, triangular norms. Then strict triangular norms are isomorphic to the product t-norm Π, nilpotent t-norms are isomorphic to the Łukasiewicz t-norm T_L. Similarly, quasi-arithmetic means with no annihilator are isomorphic to the arithmetic mean M. The only n-ary aggregation functions invariant under isomorphic transformations are the lattice polynomials [4.62], i. e., the Choquet integrals with respect to $\{0,1\}$-valued capacities. So, for $n=2$, only Min, Max, P_F and P_L are invariant under isomorphic transformations. There are several generalizations of (4.7). One can consider, for example, decreasing bijections $\eta:[0,1] \to [0,1]$ and define A_η via (4.7). This type of transformations reverses the conjunctivity of aggregation function into disjunctivity, and vice versa. It preserves the existence of a neutral element (annihilator), however, if e is a neutral element of A (a is an annihilator of A) then $\eta^{-1}(e)$ is a neutral element of A_η ($\eta^{-1}(a)$ is an annihilator of A_η). If η is involutive, i. e., if $\eta \circ \eta = \text{id}_{[0,1]}$, then $(A_\eta)_\eta = A$, so there is a duality between A and A_η. The most applied duality is based on the standard (or Zadeh's) negation $\eta:[0,1] \to [0,1]$ given by $\eta(x) = 1-x$. In that case, we use the notation $A^d = A_\eta$ and $A^d(x_1,\dots,x_n) = 1 - A(1-x_1,\dots,1-x_n)$. As a distinguished example recall the class of triangular conorms which are just the dual aggregation functions to triangular norms, i. e., S is a triangular conorm [4.26] if and only if there is a triangular norm T such that $S = T^d$.

Further generalizations of (4.7) consider different automorphisms $\varphi, \varphi_1, \dots, \varphi_n:[0,1] \to [0,1]$,

$$A_{\varphi,\varphi_1,\dots,\varphi_n}(x_1,\dots,x_n) = \varphi\left(A\left(\varphi_1(x_1),\dots,\varphi_n(x_n)\right)\right) . \tag{4.8}$$

Moreover, it is enough to suppose that $\varphi_1,\dots,\varphi_n$ are monotone (not necessarily strictly) and satisfy $\varphi_i(0) = 0$, $\varphi_i(1) = 1$, $i = 1,\dots,n$, as in such case it also holds that for any aggregation function A, $A_{\varphi,\varphi_1,\dots,\varphi_n}$ given by (4.8) is an aggregation function.

Another construction well known from functional theory is linked to the composition of functions. We have two kinds of composition methods. In the first one, considering a k-ary aggregation function $B:[0,1]^k \to [0,1]$, we can choose arbitrary k aggregation functions $C_1,\dots,C_k$ (either all of them are extended aggregation functions, or all of them are n-ary aggregation functions for some fixed $n > 1$), and then we can introduce a new aggregation function A (either extended, with the convention $A(x) = x$, $x \in [0,1]$; or n-ary) such that

$$A(\boldsymbol{x}) = B\left(C_1(\boldsymbol{x}), \dots, C_k(\boldsymbol{x})\right) . \quad (4.9)$$

As a typical example of construction (4.9), consider B to be a weighted arithmetic mean W, $W(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i x_i$. Then

$$A(\boldsymbol{x}) = \sum_{i=1}^{k} w_i \cdot C_i(\boldsymbol{x}) ,$$

i. e., A is a convex combination of aggregation functions $C_1, \dots, C_k$.

The second method is based on a partition of the space of coordinates $\{1, \dots, n\}$ into subspaces

$$\{1, \dots, n_1\}, \{n_1 + 1, \dots, n_1 + n_2\}, \dots, \{n_1 + \dots + n_{k-1} + 1, n\} .$$

Then, considering a k-ary aggregation function $B : [0,1]^k \to [0,1]$ and aggregation functions $C_i : [0,1]^{n_i} \to [0,1]$, $i = 1, \dots, k$, we can define a composite aggregation function $A : [0,1]^n \to [0,1]$ by

$$\begin{aligned} &A(x_1, \dots, x_n) \\ &\quad = B\left(C_1(x_1, \dots, x_{n_1}), C_2(x_{n_1+1}, \dots, x_{n_1+n_2}), \right. \\ &\qquad \left. \dots, C_k(x_{n_1+\dots+n_{k-1}}, \dots, x_n)\right) . \end{aligned} \quad (4.10)$$

This method can be generalized by considering an arbitrary partition of $\{1, \dots, n\}$ into $\{I_1, \dots, I_k\}$. As an example, consider the n-ary copula $C : [0,1]^n \to [0,1]$ defined for a fixed partition $\{I_1, \dots, I_k\}$ of $\{1, \dots, n\}$ by

$$C(x_1, \dots, x_n) = \prod_{i=1}^{k} \min\left\{x_j \mid j \in I_i\right\} .$$

For more details, see [4.63].

The third group containing constructions based on some given aggregation functions can be seen as a group of patchwork methods. As typical examples, we can recall several types of ordinal sums. Besides the well-known Min-based ordinal sums for conjunctive aggregation functions (especially for triangular norms and copulas) [4.26, 64], W-ordinal sums for copulas (or quasi-copulas) [4.65], as well as g-ordinal sums for copulas [4.66], we recall one kind of ordinal sums introduced in [4.67] which is applicable to arbitrary aggregation functions.

Theorem 4.3

Let $f : [0,1] \to [-\infty, \infty]$ be a continuous strictly monotone function, and let $0 = a_0 < a_1 < \dots < a_k = 1$ be a given sequence of real constants. Then for any system $(A_i)_{i=1}^{k}$ of n-ary (extended) aggregation functions the function $A : [0,1]^n \to [0,1]$ ($A : \cup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$) given by

$$\begin{aligned} A(\boldsymbol{x}) = f^{-1}\Bigg(\Bigg(&\sum_{j=1}^{k} f(a_{j-i} + (a_j - a_{j-1}) A_j(\boldsymbol{x}^{(j)})\Bigg) \\ &- \sum_{j=1}^{k-1} f(a_j)\Bigg) , \end{aligned} \quad (4.11)$$

where

$$\boldsymbol{x}^{(j)} = \left(x_1^{(j)}, \dots, x_n^{(j)}\right)$$

and

$$x_i^{(j)} = \max\left\{0, \min\left\{1, \frac{x_i - a_{j-1}}{a_j - a_{j-1}}\right\}\right\} ,$$

is an n-ary (extended) aggregation function.

Observe that if all A_i's are triangular norms (copulas, quasi-copulas, triangular conorms, continuous aggregation functions, idempotent aggregation functions, symmetric aggregation functions) then so is the newly constructed aggregation function A.

The fourth group contains construction methods allowing one to introduce weights into the aggregation procedure. The *quantitative* look at weights can be seen as the corresponding *repetition* of inputs, and the weights roughly correspond to the occurrence of single input arguments. For example, when considering a strongly idempotent (symmetric) aggregation function constructed by means of a dissimilarity function D (see Theorem 4.1) and weights $w_1, \dots, w_n$ (at least one of them should be positive, and all of them are nonnegative), we look for minimizers of the sum $\sum_{i=1}^{n} w_i D(x_i, t)$. For example, if $D(x, y) = (x - y)^2$, then we obtain the weighted arithmetic mean

$$W(x_1, \dots, x_n) = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} .$$

This approach can also be introduced in the case when different dissimilarity functions are applied. As an example, consider the aggregation function $A : [0,1]^n \to$

Part A | 4.3

$[0,1]$ given by (4.5). We look for minimizers of the expression $w_1|x_1-t|+\sum_{i=2}^n w_i(x_i-t)^2$ and the resulting weighted aggregation function $A_{\mathbf{w}}:[0,1]^n\to[0,1]$ is given by

$$A_{\mathbf{w}}(x_1,\dots,x_n) = \mathrm{Med}\left(x_1, M(x_2,\dots,x_n)-\frac{w_1}{2\sum_{i=2}^n w_i}, M(x_2,\dots,x_n)+\frac{w_1}{2\sum_{i=2}^n w_i}\right) .$$

Considering the integer weights $\boldsymbol{w}=(w_1,\dots,w_n)$, for an extended aggregation function A which is symmetric and strongly idempotent, we obtain the weighted aggregation function $A_{\mathbf{w}}:[0,1]^n\to[0,1]$ given by

$$A_{\mathbf{w}}(x_1,\dots,x_n) = A\left(\underbrace{x_1,\dots,x_1}_{w_1\text{-times}},\underbrace{x_2,\dots,x_2}_{w_2\text{-times}},\dots,\underbrace{x_n,\dots,x_n}_{w_n\text{-times}}\right) .$$

The strong idempotency of A also allows one to introduce rational weights into aggregation. Observe that for each $k\in\mathbb{N}$, the weights $k\cdot\boldsymbol{w}$ result in the same weighted aggregation function as when considering the weights $\boldsymbol{w}$ only. For general weights the limit approach described in [4.17, Proposition 6.27] should be applied.

The *qualitative* approach to weights considers a transformation of inputs $x_1,\dots,x_n$ accordingly to the considered weights (importances) $w_1,\dots,w_n\in[0,1]$, with constraint that at least once it holds $w_i=1$. This approach is applied when we consider an extended aggregation function A with a strong neutral element $e\in[0,1]$. Then the weighted aggregation function $A_{\mathbf{w}}:[0,1]^n\to[0,1]$ is given by

$$A_{\mathbf{w}}(x_1,\dots,x_n)=A\left(h(w_1,x_1),\dots,h(w_n,x_n)\right) ,$$

where $h:[0,1]^2\to[0,1]$ is a relevancy transformation (RET) operator [4.24, 68] satisfying $h(0,x)=e$, $h(1,x)=x$, which is increasing in the second coordinate as well as in the first coordinate for all $x\geq e$, while $h(\cdot,x)$ is decreasing for all $x\leq e$. As an example, consider the RET operator h given by

$$h(w,x)=wx+(1-w)e .$$

For more details, we recommend [4.17, Chapter 6].

4.4 Concluding Remarks

As already mentioned, all introduced results (sometimes for special types of aggregation functions only) can be straightforwardly extended to any interval $I\subseteq[-\infty,\infty]$. Moreover, one can aggregate more general objects than real numbers. For example, a quite expanding field concerns interval mathematics. The aggregation of interval inputs can be done coordinatewise,

$$A([x_1,y_1],\dots,[x_n,y_n]) = [A_1(x_1,\dots,x_n),A_2(y_1,\dots,y_n)] ,$$

where A_1,A_2 are an arbitrary couple of classical aggregation functions such that $A_1\leq A_2$ (mostly $A_1=A_2$ is considered). However, there are also more sophisticated approaches [4.69].

Already in 1942, *Menger* [4.43] introduced the aggregation of distribution functions whose supports are contained in $[0,\infty]$ (distance functions), which led not only to the concept of triangular norms [4.44], but also to triangle functions directly aggregating such distribution functions [4.70]. Some triangle functions are derived from special aggregation functions (triangular norms), some of them have more complex background (as a distinguished example recall the standard convolution of distribution functions). For an overview and details we recommend [4.71, 72].

In 1965, *Zadeh* [4.73] introduced fuzzy sets. Their aggregation, in particular union and intersection, is again built by means of special aggregation functions on $[0,1]$, namely by means of triangular conorms and triangular norms [4.26]. Triangular norms also play an important role in the *Zadeh* extension principle [4.74–76] allowing to extend standard aggregation functions acting on real inputs to the generalized aggregation functions acting on fuzzy inputs. As a typical example recall the arithmetic of fuzzy numbers [4.77]. In some special fuzzy logics also uninorms have found the application in modeling conjunctions. Among recent generalizations of fuzzy set theory recall the type 2-fuzzy sets, including interval-valued fuzzy sets, or

n-fuzzy sets. In all these fields, a deep study of aggregation functions is one of the major theoretical tasks to build a sound background.

Observe that all mentioned particular domains are covered by the aggregation on posets, where up to now only some particular general results are known [4.22, 78]. We expect an enormous growth of interest in this field, as it can be seen, for example, in its special subdomain dealing with computing and aggregation with words [4.79–81].

References

4.1 R.C. Archibald: Mathematics before the Greeks Science, Science **71**(1831), 109–121 (1930)
4.2 D. Smith: *History of Mathematics* (Dover, New York 1958)
4.3 T. Calvo, R. Mesiar, R.R. Yager: Quantitative weights and aggregation, IEEE Trans. Fuzzy Syst. **12**(1), 62–69 (2004)
4.4 H. Karcher: Riemannian center of mass and mollifier smoothing, Commun. Pure Appl. Math. **30**(5), 509–541 (1977), published online: 13 October 2006
4.5 J. Łukasiewicz: O logice trójwartosciowej (in Polish), Ruch Filoz. **5**, 170–171 (1920); English translation: On three-valued logic. In: *Selected Works by Jan Lukasiewicz*, ed. by L. Borkowski (North-Holland, Amsterdam 1970) pp. 87–88
4.6 E.L. Post: Introduction to a general theory of elementary propositions, Am. J. Math. **43**, 163–185 (1921)
4.7 K. Gödel: Zum intuitionistischen Aussagenkalkül, Anz. Akad. Wiss. Wien **69**, 65–66 (1939)
4.8 A.N. Kolmogoroff: Sur la notion de la moyenne, Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. **12**, 388–391 (1930)
4.9 M. Nagumo: Über eine Klasse der Mittelwerte, Jpn. J. Math. **6**, 71–79 (1930)
4.10 J. Aczél: *Lectures on Functional Equations and Their Applications* (Academic, New York 1966)
4.11 B. Schweizer, A. Sklar: *Probabilistic Metric Spaces*, Ser. Probab. Appl. Math, Vol. 5 (North-Holland, New York 1983)
4.12 G.J. Klir, T.A. Folger: *Fuzzy Sets, Uncertainty, and Information* (Prentice-Hall, Hemel Hempstead 1988)
4.13 A. Kolesárová, M. Komorníková: Triangular norm-based iterative aggregation and compensatory operators, Fuzzy Sets Syst. **104**, 109–120 (1999)
4.14 R. Mesiar, M. Komorníková: Triangular norm-based aggregation of evidence under fuzziness. In: *Studies in Fuzziness and Soft Computing, Aggregation and Fusion of Imperfect Information*, Vol. 12, ed. by B. Bouchon-Meunier (Physica, Heidelberg 1998) pp. 11–35
4.15 T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar: *A Review of Aggregation Operators* (Univ. of Alcalá Press, Alcalá de Henares, Madrid 2001)
4.16 G. Beliakov, A. Pradera, T. Calvo: *Aggregation Functions: A Guide for Practitioners* (Springer, Berlin 2007)
4.17 M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap: *Aggregation Functions*, Encyclopedia of Mathematics and Its Applications, Vol. 127 (Cambridge Univ. Press, Cambridge 2009)
4.18 Y. Narukawa (Ed.): *Modeling Decisions: Information Fusion and Aggregation Operators*, Cognitive Technologies (Springer, Berlin, Heidelberg 2007)
4.19 R.R. Yager, D.P. Filev: *Essentials of Fuzzy Modelling and Control* (Wiley, New York 1994)
4.20 D. Dubois, H. Prade: On the use of aggregation operations in information fusion processes, Fuzzy Sets Syst. **142**, 143–161 (2004)
4.21 R.R. Yager: On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man. Cybern. **18**, 183–190 (1988)
4.22 M. Komorníková, R. Mesiar: Aggregation functions on bounded partially ordered sets and their classification, Fuzzy Sets Syst. **175**(1), 48–56 (2011)
4.23 J.-L. Marichal: k-intolerant capacities and Choquet integrals, Eur. J. Oper. Res. **177**(3), 1453–1468 (2007)
4.24 R.R. Yager: Aggregation operators and fuzzy systems modeling, Fuzzy Sets Syst. **67**(2), 129–146 (1995)
4.25 M. Gagolewski, P. Grzegorzewski: Arity-monotonic extended aggregation operators, Commun. Comput. Inform. Sci. **80**, 693–702 (2010)
4.26 E.P. Klement, R. Mesiar, E. Pap: *Triangular Norms* (Kluwer, Dordrecht 2000)
4.27 C. Alsina, M.J. Frank, B. Schweizer: *Associative Functions, Triangular Norms and Copulas* (World Scientific, Hackensack 2006)
4.28 R.B. Nelsen: *An Introduction to Copulas*, Lecture Notes in Statistics, Vol. 139, 2nd edn. (Springer, New York 2006)
4.29 C. Alsina, R.B. Nelsen, B. Schweizer: On the characterization of a class of binary operations on distribution functions, Stat. Probab. Lett. **17**(2), 85–89 (1993)
4.30 C. Genest, J.J. Quesada Molina, J.A. Rodriguez Lallena, C. Sempi: A characterization of quasi-copulas, J. Multivar. Anal. **69**, 193–205 (1999)
4.31 B. Bassano, F. Spizzichino: Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes, J. Multivar. Anal. **93**, 313–339 (2005)
4.32 B. De Baets: Idempotent uninorms, Eur. J. Oper. Res. **180**, 631–642 (1999)
4.33 D. Denneberg: *Non-Additive Measure and Integral* (Kluwer, Dordrecht 1994)
4.34 E.P. Klement, R. Mesiar, E. Pap: A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. **18**, 178–187 (2010)

4.35 E. Pap: *Null-Additive Set Functions* (Kluwer, Dordrecht 1995)
4.36 Z. Wang, G.J. Klir: *Generalized Measure Theory* (Springer, New York 2009)
4.37 D. Dubois, H. Prade: A review of fuzzy set aggregation connectives, Inform. Sci. **36**, 85–121 (1985)
4.38 R. Mesiar, A. Mesiarová-Zemánková: The ordered modular averages, IEEE Trans. Fuzzy Syst. **19**, 42–50 (2011)
4.39 M. Couceiro, J.-L. Marichal: Representations and characterizations of polynomial functions on chains, J. Multiple-Valued Log. Soft Comput. **16**(1–2), 65–86 (2010)
4.40 J.C. Fodor, R.R. Yager, A. Rybalov: Structure of uninorms, Int. J. Uncertain. Fuzziness Knowledge-Based Syst. **5**, 411–427 (1997)
4.41 H.J. Zimmermann, P. Zysno: Latent connectives in human decision making, Fuzzy Sets Syst. **4**, 37–51 (1980)
4.42 M.K. Luhandjula: Compensatory operators in fuzzy linear programming with multiple objectives, Fuzzy Sets Syst. **8**(3), 245–252 (1982)
4.43 K. Menger: Statistical metrics, Proc. Natl. Acad. Sci. **28**, 535–537 (1942)
4.44 B. Schweizer, A. Sklar: Statistical metric spaces, Pac. J. Math. **10**(1), 313–334 (1960)
4.45 A. Sklar: *Fonctions de répartition à n dimensions et leurs marges*, Vol. 8 (Institut de Statistique, LUniversité de Paris, Paris 1959) pp. 229–231
4.46 J. Galambos: *The Asymptotic Theory of Extreme Order Statistics*, 2nd edn. (Krieger, Melbourne 1987)
4.47 J.A. Tawn: Bivariate extreme value theory: Models and estimation, Biometrika **75**, 397–415 (1988)
4.48 D. Schmeidler: Integral representation without additivity, Proc. Am. Math. Soc. **97**(2), 255–261 (1986)
4.49 R.R. Yager: Fusion od ordinal information using weighted median aggregation, Int. J. Approx. Reason. **18**, 35–52 (1998)
4.50 T. Calvo, G. Beliakov: Aggregation functions based on penalties, Fuzzy Sets Syst. **161**, 1420–1436 (2010)
4.51 M. Sugeno: Theory of fuzzy integrals and applications, Ph.D. Thesis (Tokyo Inst. of Technology, Tokyo 1974)
4.52 Z. Wang, G.J. Klir: *Fuzzy Measure Theory* (Plenum, New York 1992)
4.53 G. Choquet: Theory of capacities, Ann. Inst. Fourier **5**(54), 131–295 (1953)
4.54 J.C. Fodor: An extension of Fung-Fu's theorem, Int. J. Uncertain. Fuziness Knowledge-Based Syst. **4**, 235–243 (1996)
4.55 L.W. Fung, K.S. Fu: An axiomatic approach to rational decision making in a fuzzy environment. In: *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, ed. by L.A. Zadeh, K.S. Fu, K. Tanaka, M. Shimura (Academic, New York 1975) pp. 227–256
4.56 M. Grabisch, T. Murofushi, M. Sugeno (Eds.): *Fuzzy Measures and Integrals. Theory and Applications* (Physica, Heidelberg 2000)
4.57 A. Kolesárová, A. Stupňanová, J. Beganová: Aggregation-based extensions of fuzzy measures, Fuzzy Sets Syst. **194**, 1–14 (2012)
4.58 L. Lovász: Submodular functions and convexity. In: *Mathematical Programming: The State of the Art*, ed. by A. Bachem, M. Grotschel, B. Korte (Springer, Berlin, Heidelberg 1983) pp. 235–257
4.59 G. Owen: Multilinear extensions of games, Manag. Sci. **18**, 64–79 (1972)
4.60 F. Durante, A. Kolesárová, R. Mesiar, C. Sempi: Copulas with given diagonal sections: novel constructions and applications, Int. J. Uncertain. Fuzziness Knowlege-Based Syst. **15**(4), 397–410 (2007)
4.61 F. Durante, A. Kolesárová, R. Mesiar, C. Sempi: Copulas with given values on a horizontal and a vertical section, Kybernetika **43**(2), 209–220 (2007)
4.62 S. Ovchinnikov, A. Dukhovny: Integral representation of invariant functionals, J. Math. Anal. Appl. **244**, 228–232 (2000)
4.63 R. Mesiar, V. Jágr: *d*-dimensional dependence functions and Archimax copulas, Fuzzy Sets Syst. **228**, 78–87 (2013)
4.64 R. Mesiar, C. Sempi: Ordinal sums and idempotents of copulas, Aequ. Math. **79**(1–2), 39–52 (2010)
4.65 R. Mesiar, J. Szolgay: *W*-ordinal sums of copulas and quasi-copulas, Proc. MAGIA 2004 Conf. Kočovce (2004) pp. 78–83
4.66 R. Mesiar, V. Jágr, M. Juráňová, M. Komorníková: Univariate conditioning of copulas, Kybernetika **44**(6), 807–816 (2008)
4.67 R. Mesiar, B. De Baets: New construction methods for aggregation operators, IPMU'2000 Int. Conf. Madrid (Springer, Berlin, Heidelberg 2000) pp. 701–706
4.68 M. Šabo, A. Kolesárová, Š. Varga: RET operators generated by triangular norms and copulas, Int. J. Uncertain. Fuzziness Knowledge-Based Syst. **9**, 169–181 (2001)
4.69 G. Deschrijver, E.E. Kerre: Aggregation operators in interval-valued fuzzy and atanassov's intuitionistic fuzzy set theory. In: *Fuzzy Sets and Their Extensions: Representation, Aggregation and Models*, *Studies in Fuzziness and Soft Computing*, ed. by H. Bustince, F. Herrera, J. Montesa (Springer, Berlin, Heidelberg 2008) pp. 183–203
4.70 A.N. Šerstnev: On a probabilistic generalization of metric spaces, Kazan. Gos. Univ. Učen. Zap. **124**, 3–11 (1964)
4.71 S. Saminger-Platz, C. Sempi: A primer on triangle functions I, Aequ. Math. **76**(3), 201–240 (2008)
4.72 S. Saminger-Platz, C. Sempi: A primer on triangle functions II, Aequ. Math. **80**(3), 239–268 (2010)
4.73 L.A. Zadeh: Fuzzy sets, Inform. Control **8**, 338–353 (1965)

4.74 L.A. Zadeh: The concept of a linguistic variable and its application to approximate reasoning, Part I, Inform. Sci. **8**, 199–251 (1976)
4.75 L.A. Zadeh: The concept of a linguistic variable and its application to approximate reasoning, Part II, Inform. Sci. **8**, 301–357 (1975)
4.76 L.A. Zadeh: The concept of a linguistic variable and its application to approximate reasoning, Part III, Inform. Sci. **9**, 43–80 (1976)
4.77 D. Dubois, E.E. Kerre, R. Mesiar, H. Prade: Fuzzy interval analysis. In: *Fundamentals of Fuzzy Sets*, The Handbook of Fuzzy Sets Series, ed. by D. Dubois, H. Prade (Kluwer, Boston 2000) pp. 483–582
4.78 G. De Cooman, E.E. Kerre: Order norms on bounded partially ordered sets, J. Fuzzy Math. **2**, 281–310 (1994)
4.79 F. Herrera, S. Alonso, F. Chiclana, E. Herrera-Viedma: Computing with words in decision making: Foundations, trends and prospects, Fuzzy Optim. Decis. Mak. **8**(4), 337–364 (2009)
4.80 L.A. Zadeh: *Computing with Words – Principal Concepts and Ideas*, Stud. Fuzziness Soft Comput, Vol. 277 (Springer, Berlin, Heidelberg 2012)
4.81 L.A. Zadeh (Ed.): *Computing with Words in Information/Intelligent System 1: Foundations*, Stud. Fuzziness Soft Comput, Vol. 33 (Springer, Berlin, Heidelberg 2012) p. 1