12. Fuzzy Implications: Past, Present, and Future

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Fuzzy implications are a generalization of the classical two-valued implication to the multi-valued setting. They play a very important role both in the theory and applications, as can be seen from their use in, among others, multivalued mathematical logic, approximate reasoning, fuzzy control, image processing, and data analysis. The goal of this chapter is to present the evolution of fuzzy implications from their beginnings to the current days. From the theoretical point of view, we present the basic facts, as well as the main topics and lines of research around fuzzy implications. We also devote a specific section to state and recall a list of main application fields where fuzzy implications are employed, as well as another one to the main open problems on the topic.

Fuzzy logic connectives play a fundamental role in the theory of fuzzy sets and fuzzy logic. The basic fuzzy connectives that perform the role of generalized *And*, *Or*, and *Not* are *t*-norms, *t*-conorms, and negations, respectively, whereas fuzzy conditionals are usually managed through fuzzy implications. Fuzzy implications play a very important role both in theory and applications, as can be seen from their use in, among others, multivalued mathematical logic, approximate reasoning, fuzzy control, image processing, and data analysis. Thus, it is hardly surprising that many researchers have devoted their efforts to the study of implication functions. This interest has become more evident in the last decade when many works have appeared and have led to some surveys [12.1, 2] and even some research monographs entirely devoted to this topic [12.3, 4]. Thus, most of the known results and applications of fuzzy implications until the publication date were collected in [12.3], and very recently the edited volume [12.4] has been published complimenting the earlier monograph with the most recent lines of investigation on fuzzy implications.

In this regard, we have decided to devote this chapter, as the title suggests, to present the evolution of fuzzy implications from their beginnings to the present time. The idea is not to focus on a list of results already collected in other works, but unraveling the relations and highlighting the importance in the development and progress that fuzzy implications have experienced along the time. From the theoretical point of view we present the basic facts, as well as the main topics and lines of research around fuzzy implications, recalling in most of the cases where the corresponding results can be found, instead of listing them. Of course, we also devote a specific section to state and recall a list of the main application fields where fuzzy implications are employed. A final section looks ahead to the future by listing some of the main open-problem-solutions of which are certain to enrich the existing literature on the topic.

12.1 Fuzzy Implications: Examples, Properties, and Classes

Fuzzy implications are a generalization of the classical implication to fuzzy logic. It is a well-established fact that fuzzy concepts have to generalize the corresponding crisp one, and consequently fuzzy implications restricted to $\{0,1\}^2$ must coincide with the classical implication. Currently, the most accepted definition of a fuzzy implication is the following one.

Definition 12.1 [12.3, Definition 1.1.1]
A function $I\colon [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies the following conditions:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$
(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$
(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

This definition is flexible enough to allow uncountably many fuzzy implications. This great repertoire of fuzzy implications allows a researcher to pick out, depending on the context, that fuzzy implication which satisfies some desired additional properties. Many additional properties, all of them arising from tautologies in classical logic, have been postulated in many works. The most important of them are collected below:

- *(NP)*: The *left neutrality principle*,
$$I(1,y) = y\,, \quad y \in [0,1]\,.$$
- *(EP)*: The *exchange principle*,
$$I(x, I(y,z)) = I(y, I(x,z))\,, \quad x,y,z \in [0,1]\,.$$
- *(OP)*: The *ordering property*,
$$x \leq y \Longleftrightarrow I(x,y) = 1\,, \quad x,y \in [0,1]\,.$$
- *(IP)*: The *identity principle*,
$$I(x,x) = 1\,, \quad x \in [0,1]\,.$$
- *(CP(N))*: The *contrapositive symmetry* with respect to a fuzzy negation N,
$$I(x,y) = I(N(y), N(x))\,, \quad x,y \in [0,1]\,.$$

Given a fuzzy implication I, its *natural negation* is defined as $N_I(x) = I(x,0)$ for all $x \in [0,1]$. This function is always a fuzzy negation. For the definitions of basic fuzzy logic connectives like fuzzy negations, t-norms and t-conorms please see [12.5]. Moreover, N_I can be continuous, strict, or strong and these are also additional properties usually required of a fuzzy implication I.

Table 12.1 lists the most well-known fuzzy implications along with the additional properties they satisfy [12.3, Chap.1]. In addition, the following

Table 12.1 Basic fuzzy implications and the additional properties they satisfy where $N_{\mathbf{C}}$, $N_{\mathbf{D}_1}$, and $N_{\mathbf{D}_2}$ stand for the classical, the least and the greatest fuzzy negations, respectively

Name	Formula	(NP)	(EP)	(IP)	(OP)	(CP(N))	N_I
Łukasiewicz	$I_{\mathbf{LK}}(x,y) = \min\{1, 1-x+y\}$	✓	✓	✓	✓	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$
Gödel	$I_{\mathbf{GD}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$	✓	✓	✓	✓	X	$N_{\mathbf{D}_1}$
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$	✓	✓	X	X	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$
Kleene–Dienes	$I_{\mathbf{KD}}(x,y) = \max\{1-x, y\}$	✓	✓	X	X	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$
Goguen	$I_{\mathbf{GG}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{if } x > y \end{cases}$	✓	✓	✓	✓	X	$N_{\mathbf{D}_1}$
Rescher	$I_{\mathbf{RS}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases}$	X	X	✓	✓	$N_{\mathbf{C}}$	$N_{\mathbf{D}_1}$
Yager	$I_{\mathbf{YG}}(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ y^x & \text{if } (x,y) \neq (0,0) \end{cases}$	✓	✓	X	X	X	$N_{\mathbf{D}_1}$
Weber	$I_{\mathbf{WB}}(x,y) = \begin{cases} 1 & \text{if } x < 1 \\ y & \text{if } x = 1 \end{cases}$	✓	✓	✓	X	X	$N_{\mathbf{D}_2}$
Fodor	$I_{\mathbf{FD}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max\{1-x, y\} & \text{if } x > y \end{cases}$	✓	✓	✓	✓	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$

two implications

$$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1\,, \\ 0, & \text{if } x > 0 \text{ and } y < 1\,, \end{cases}$$

$$I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0\,, \\ 0, & \text{if } x = 1 \text{ and } y = 0\,, \end{cases}$$

are the least and the greatest fuzzy implications, respectively, of the family of all fuzzy implications.

Beyond these examples of fuzzy implications, several families of these operations have been proposed and deeply studied. There exist basically two strategies in order to define classes of fuzzy implications. The most usual strategy is based on some combinations of aggregation functions. In this way, t-norms and t-conorms [12.5] were the first classes of aggregation functions used to generate fuzzy implications. Thus, the following are the three most important classes of fuzzy implications of this type:

1) (S,N)-*implications* defined as

$$I_{S,N}(x,y) = S(N(x),y)\,, \quad x,y \in [0,1]\,,$$

where S is a t-conorm and N a fuzzy negation. They are the immediate generalization of the classical boolean material implication $p \to q \equiv \neg p \vee q$. If N is involutive, they are called strong or S-implications.

2) *Residual* or *R-implications* defined by

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}\,,\ x,y \in [0,1]\,,$$

where T is a t-norm. When they are obtained from left-continuous t-norms, they come from residuated lattices based on the residuation property

$$T(x,y) \le z \Leftrightarrow I(x,z) \ge y\,, \quad \text{for all } x,y,z \in [0,1]\,.$$

3) *QL-operations* defined by

$$I_{T,S,N}(x,y) = S(N(x),T(x,y))\,, \quad x,y \in [0,1]\,,$$

where S is a t-conorm, T is a t-norm and N is a fuzzy negation. Their origin is the quantum mechanic logic.

Note that R- and (S,N)-implications are always implications in the sense of Definition 12.1, whereas QL-operations are not implications in general (they are called QL-implications when they actually are). A characterization of those QL-operations which are also implications is still open (Sect. 12.4), but a common necessary condition is $S(N(x),x) = 1$ for all $x \in [0,1]$. Yet another class of fuzzy implications is that of Dishkant or D-operations [12.6] which are the contraposition of QL-operations with respect to a strong fuzzy negation.

These initial classes were successfully generalized considering more general classes of aggregation functions, mainly uninorms, generating new classes of fuzzy implications with interesting properties. In this way, (U,N), RU-implications and QLU-operations have been deeply analyzed [12.3, Chap. 5], [12.6].

A second approach to obtain fuzzy implications is based on the direct use of unary monotonic functions. In this way, the most important families are Yager's f- and g-generated fuzzy implications which can be seen as implications generated from additive generators of continuous Archimedean t-norms and t-conorms, respectively [12.3, Chap. 3]:

1) *Yager's f-generated implications* are defined as

$$I_f(x,y) = f^{-1}(x \cdot f(y))\,, \quad x,y \in [0,1]\,,$$

with the understanding $0 \cdot \infty = 0$, where $f\colon [0,1] \to [0,\infty]$ is a strictly decreasing and continuous function with $f(1) = 0$.

2) *Yager's g-generated implications* are defined as

$$I_g(x,y) = g^{-1}\left(\min\left\{\frac{1}{x} \cdot g(y), g(1)\right\}\right)\,,$$
$$x,y \in [0,1]\,,$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$ where $g\colon [0,1] \to [0,\infty]$, is a strictly increasing and continuous function with $g(0) = 0$.

The above classes give rise to fuzzy implications with different additional properties which are collected in Table 12.2. All the results referred in Table 12.2 are from [12.3, Chaps. 2 and 3].

One of the main topics in this field is the characterization of each of these families of fuzzy implications through algebraic properties. This is an essential step in order to understand the behavior of these families. The available characterization results of the above families of implications are collected below.

Theorem 12.1 [12.3, Theorem 2.4.10]
For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:

Table 12.2 Classes of fuzzy implications and the additional properties they satisfy

Class / Properties	**(NP)**	**(EP)**	**(IP)**	**(OP)**	**(CP(N))**	N_I
(S,N)-imp.	✓	✓	Thm. 2.4.17	Thm. 2.4.19	Prop. 2.4.3	N
R-imp. with l-c. T	✓	✓	✓	✓	Prop. 2.5.28	N_T
QL-imp.	✓	Thm. 2.6.19	Sect. 2.6.3	Sect. 2.6.4	Sect. 2.6.5	N
f-gen.	✓	✓	×	×	Thm. 3.1.7	Prop. 3.1.6
g-gen.	✓	✓	Thm. 3.2.8	Thm. 3.2.9	×	$N_{\mathbf{D}_1}$

i) I is an (S,N)-implication with a continuous (strict, strong) fuzzy negation N.
ii) I satisfies (I1), (EP), and N_I is a continuous (strict, strong) fuzzy negation.

Moreover, in this case the representation $I(x,y) = S(N(x),y)$ is unique with $N = N_I$ and $S(x,y) = I(\Re_N(x),y)$ (for the definition of $\Re_N$ see [12.3, Lemma 1.4.10]).

Theorem 12.2 [12.3, Theorem 2.5.17]
For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:

i) I is an R-implication generated from a left-continuous t-norm.
ii) I satisfies (I2), (EP), (OP) and it is right continuous with respect to the second variable.

Moreover, the representation

$$I(x,y) = \max\{t \in [0,1] | T(x,t) \le y\}$$

is unique with

$$T(x,y) = \min\{t \in [0,1] | I(x,t) \ge y\}\ .$$

As already said, it is still an open question when QL-operations are fuzzy implications. However, in the continuous case, when S and N are the φ-conjugates of the Łukasiewicz t-conorm $S_{\mathbf{LK}}$ and the classical negation $N_{\mathbf{C}}$, respectively, for some order automorphism φ on the unit interval, the QL-operation has the following expression

$$\begin{aligned} I_{T,S,N}(x,y) &= I_{\varphi,T}(x,y) \\ &= \varphi^{-1}(1-\varphi(x)+\varphi(T(x,y)))\ , \\ &\quad x,y \in [0,1]\ , \end{aligned}$$

and we have the following characterization result.

Theorem 12.3 [12.3, Theorem 2.6.12]
For a QL-operation $I_{\varphi,T}$, where T is a t-norm and φ is an automorphism on the unit interval, the following statements are equivalent:

i) $I_{\varphi,T}$ is a QL-implication.
ii) $T_{\varphi^{-1}}$ satisfies the Lipschitz condition, i. e.,

$$\begin{aligned} &|T_{\varphi^{-1}}(x_1,y_1) - T_{\varphi^{-1}}(x_2,y_2)| \\ &\quad \le |x_1-x_2| + |y_1-y_2|\ , \quad x_1,x_2,y_1,y_2 \in [0,1]\ . \end{aligned}$$

In addition, (U,N)-implications are characterized in [12.3, Theorem 5.3.12] and more recently, Yager's f and g-generated [12.7] and RU-implications [12.8] have been also characterized. Finally, due to its importance in many results, we recall the characterization of the family of the conjugates of the Łukasiewicz implication.

Theorem 12.4 [12.3, Theorem 7.5.1]
For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:

i) I is continuous and satisfies both (EP) and (OP).
ii) I is a φ-conjugate with the Łukasiewicz implication $I_{\mathbf{LK}}$, i. e., there exists an automorphism φ on the unit interval, which is uniquely determined, such that I has the form

$$\begin{aligned} I(x,y) &= (I_{\mathbf{LK}})_\varphi(x,y) \\ &= \varphi^{-1}(\min\{1-\varphi(x)+\varphi(y),1\})\ , \\ &\quad x,y \in [0,1]\ . \end{aligned}$$

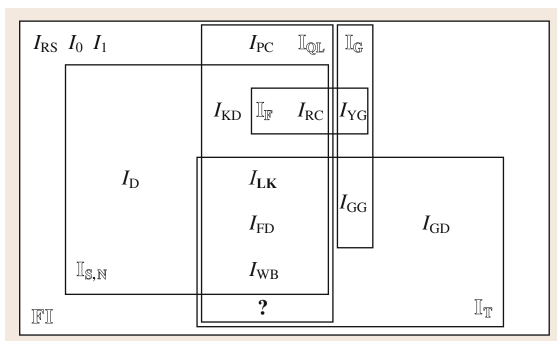

Fig. 12.1 Intersections between the main classes of fuzzy implications

For the conjugates of the other basic implications in Table 12.1, see the characterization results in [12.3, Sect. 7.5].

The great number of classes of fuzzy implications induces the study of the intersection between the different classes which brings out both the unity that exists among this diversity of classes and where the basic implications from Table 12.1 are located. The intersections among the main classes of fuzzy implications were studied in [12.3, Chap. 4] and are graphically displayed in Fig. 12.1 (note that $\mathbb{FI}$, $\mathbb{I}_{\mathbb{S},\mathbb{N}}$, $\mathbb{I}_{\mathbb{T}}$, $\mathbb{I}_{\mathbb{QL}}$, $\mathbb{I}_{\mathbb{F}}$ and $\mathbb{I}_{\mathbb{G}}$ denote the families of all fuzzy implications, (S,N)-implications, R-implications, QL-implications, Yager's f-generated implications and Yager's g-generated implications, respectively). In this figure, we have included the fuzzy implications of Table 12.1 and the following fuzzy implications which are examples of implications lying in some intersection between some families

$$I_{g^{\lambda}}(x,y) = \min\left\{1, \frac{y}{x^{\frac{1}{\lambda}}}\right\} ,$$

$$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 , \\ y, & \text{if } x > 0 , \end{cases}$$

$$I_{\mathbf{PC}}(x,y) = 1 - (\max\{x(x + xy^2 - 2y), 0\})^{\frac{1}{2}} .$$

Also note that it is still an open problem to prove if $(\mathbb{I}_{\mathbb{QL}} \cap \mathbb{I}_{\mathbb{T}}) \setminus \mathbb{I}_{\mathbb{S},\mathbb{N}} = \emptyset$.

12.2 Current Research on Fuzzy Implications

In the previous sections, we have seen some functional equations, namely, the exchange property (EP), the contrapositive symmetry (CP(N)) and the like. In this section, we deal with a few functional equations (or inequalities) involving fuzzy implications. These equations, once again, arise as the generalizations of the corresponding tautologies in classical logic involving boolean implications.

12.2.1 Functional Equations and Properties

A study of such equations stems from their applicability. The need for a plethora of fuzzy implications possessing various properties is quite obvious. On the one hand, they allow us to clearly classify and characterize different fuzzy implications, while on the other hand, they make themselves appealing to different applications. Thus, the functional equations presented in this section are chosen to reflect this dichotomy.

Distributivity over other Fuzzy Logic Operations

The distributivity of fuzzy implications over different fuzzy logic connectives, like t-norms, t-conorms, and uninorms is reduced to four equations

$$I(x, C_1(y,z)) = C_2(I(x,y), I(x,z)) , \tag{12.1}$$

$$I(x, D_1(y,z)) = D_2(I(x,y), I(x,z)) , \tag{12.2}$$

$$I(C(x,y), z) = D(I(x,z), I(y,z)) , \tag{12.3}$$

$$I(D(x,y), z) = C(I(x,z), I(y,z)) , \tag{12.4}$$

satisfied for all $x, y, z \in [0,1]$, where I is some generalization of classical implication, C, C_1, C_2 are some generalizations of classical conjunction and D, D_1, D_2 are some generalizations of classical disjunction.

All the above equations can be investigated in two different ways. On the one hand, one can assume that function I belongs to some known class of fuzzy implications and investigate the connectives C_i, D_i that satisfy (12.1)–(12.4), as is done in the following works, for e.g., *Trillas* and *Alsina* [12.9], *Balasubramaniam* and *Rao* [12.10], *Ruiz-Aguilera* and *Torrens* [12.11, 12] and *Massanet* and *Torrens* [12.13]. On the other hand, one can assume that the connectives C_i, D_i come from the known classes of functions and investigate the fuzzy implications I that satisfy (12.1)–(12.4). See the works of *Baczyński* [12.14, 15], *Baczyński* and *Jayaram* [12.16], *Baczyński* and *Qin* [12.17, 18] for such an approach.

The above distributive equations play an important role in reducing the complexity of fuzzy systems, since the number of rules directly affects the computational duration of the overall application (we will discuss this problem again in Sect. 12.3.2).

Law of Importation

One of the desirable properties of a fuzzy implication is the law of importation as given below

$$I(x, I(y,z)) = I(T(x,y), z) , \quad x, y, z \in [0,1] , \tag{12.5}$$

where T is a t-norm (or, in general, some conjunction). It generalizes the classical tautology $(p \wedge q) \to r \equiv (p \to (q \to r))$ into fuzzy logic context. This equation has been investigated for many different families of fuzzy implications (for results connected with main classes see [12.3, Sect. 7.3]). Fuzzy implications satisfying (12.5) have been found extremely useful in fuzzy relational inference mechanisms, since one can obtain an equivalent hierarchical scheme which significantly decreases the computational complexity of the system without compromising on the approximation capability of the inference scheme. For more on this, we refer the readers to the following works [12.19, 20]. Related with (12.5) is its equivalence with (EP) that has been an open problem till the recent paper [12.21], where it is proved that (12.5) is stronger than (EP) and equivalent when N_I is continuous.

T-Conditionality or Modus Ponens

Another property investigated in the scientific literature, which is of great practical importance (see also Sect. 12.3.1), is the so-called T-conditionality, defined in the following way. If I is a fuzzy implication and T is a t-norm, then I is called an *MP-fuzzy implication* for T, if

$$T(x, I(x, y)) \leq y\,, \quad x, y \in [0, 1]\,. \qquad (12.6)$$

Investigations of (12.6) have been done for the three main families of fuzzy implications, namely, (S, N)-, R-, and QL-implications [12.3, Sect. 7.4].

Nonsaturating Fuzzy Implications

Investigations connected with subsethood measures (see Sect. 12.3.3) and constructing strong equality functions by aggregation of implication functions by the formula $\Psi(x, y) = M(I(x, y), I(y, x))$, where M is some symmetric function, have led researchers to consider under which properties a fuzzy implication I satisfies the following conditions:

(P1) $I(x, y) = 1$ if and only if $x = 0$ or $y = 1$;
(P2) $I(x, y) = 0$ if and only if $x = 1$ and $y = 0$.

In [12.22], the authors considered the possible relationships between these two properties and the properties usually required of implication operations. Moreover, they developed different construction methods of strong equality indexes using fuzzy implications that satisfy these two additional properties.

Special Fuzzy Implications

Special implications were introduced by *Hájek* and *Kohout* [12.23] in their investigations on some statistics on marginals. The authors further have shown that they are related to special GUHA-implicative quantifiers (see, for instance, [12.24–26]). Thus, special fuzzy implications are related to data mining. In their quest to obtain some many-valued connectives as extremal values of some statistics on contingency tables with fixed marginals, they especially focussed on special homogenous implicational quantifiers and showed that:

> *Each special implicational quantifier determines a special implication. Conversely, each special implication is given by a special implicational quantifier.*

Definition 12.2

A fuzzy implication I is said to be special, if for any $\varepsilon > 0$ and for all $x, y \in [0, 1]$ such that $x+\varepsilon, y+\varepsilon \in [0, 1]$ the following condition is satisfied

$$I(x, y) \leq I(x + \varepsilon, y + \varepsilon)\,. \qquad (12.7)$$

Recently, *Jayaram* and *Mesiar* [12.27] have investigated the above functional equation. Their study shows that among the main classes of fuzzy implications, no f-implication is a special implication, while the Goguen implication $I_{\mathbf{GG}}$ is the only special g-implication. Based on the available results, they have conjectured that the (S, N)-implications that are special also turn out to be R-implications. However, in the case of R-implications (generated from any t-norm) they have obtained the following result.

Theorem 12.5 [12.27, Theorem 4.6]

Let T be any t-norm and I_T be the R-implication obtained from T. Then the following statements are equivalent:

i) I_T satisfies (12.7).
ii) T satisfies the 1-Lipschitz condition.
iii) T has an ordinal sum representation $(\langle e_\alpha, a_\alpha, T_\alpha\rangle)_{\alpha\in A}$ where each t-norm T_α, $\alpha \in A$ is generated by a convex additive generator (for the definition of ordinal sum, see [12.5]).

Having shown that the families of (S, N)-, f-, and g-implications do not lead to any new special implications, *Jayaram* and *Mesiar* [12.27] turned to the most natural question: *Are there any other special implications, than those that could be obtained as residuals of t-norms?* This led them to propose some interesting constructions of fuzzy implications which were also special – one such construction is given in Definition 12.4 in Sect. 12.2.2.

12.2.2 New Classes and Generalizations

Another current research line on fuzzy implications is devoted to the study of new classes and generalizations of the already known families. The research in this direction has been extensively developed in recent years. Among many generalizations of already known classes of implications that have been dealt with in the literature, we highlight the following ones.

Generalizations of R-implications

The family of residual implications is one of the most commonly selected families for generalization. As already mentioned in Sect. 12.1, the RU-implications were the first generalization obtained via residuation from uninorms instead of from t-norms. In the same line, many other families of aggregation functions have been used to derive residual implications:

1. Copulas, quasi-copulas, and semicopulas were used in [12.28]. The main results in this work relate to the axiomatic characterizations of those functions I that are the residual implications of left-continuous commutative semicopulas, the residuals of quasi-copulas, and the residuals of associative copulas. For details on these characterizations, that involve up to ten different axioms, see [12.28].
2. Representable aggregation functions (RAFs) were used in [12.29]. These are aggregation functions constructed from additive generators of continuous Archimedean t-conorms and strong negations. The interest in the residual implications obtained from them lies in the fact that they are always continuous and in many cases they also satisfy the modus ponens with a nilpotent t-conorm. In particular, residual implications that depend only on a strong negation N are deduced from the general method just by considering specific generators of continuous Archimedean t-conorms.
3. A more general situation is studied in [12.30] where residual implications derived from binary functions $F\colon[0,1]^2 \to [0,1]$ are studied. In this case, the paper deals with the minimal conditions that F must satisfy in order to obtain an implication by residuation. The same is done in order to obtain residual implications satisfying each one of the most usual properties.
4. It is well known that residual implications derived from continuous Archimedean t-norms can be expressed directly from the additive generator of the t-norm. A generalization of this idea is presented in [12.31], where strictly decreasing functions $f\colon[0,1] \to [0,+\infty]$ with $f(1)=0$ are used to derive implications as follows

$$I(x,y) = \begin{cases} 1\,, & \text{if } x \le y\,,\\ f^{(-1)}(f(y^+) - f(x))\,, & \text{if } x > y\,, \end{cases}$$

where $f(y^+) = \lim_{y \to y^+} f(y)$ and $f(1^+) = f(1)$. Properties of these implications are studied and many new examples are also derived in [12.31].

Generalizations of (S,N)-Implications

Once again a first generalization of this class of implications has been done using uninorms leading to the (U,N)-implications mentioned in Sect. 12.1, but recently many other aggregation functions were also employed.

This is the case for instance in [12.32], where the authors make use of TS-functions obtained from a t-norm T, a t-conorm S and a continuous, strictly monotone function $f\colon[0,1] \to [-\infty,+\infty]$ through the expression

$$TS_{\lambda,f}(x,y) = f^{-1}((1-\lambda)f(T(x,y)) + \lambda f(S(x,y)))$$

for $x,y \in [0,1]$, where $\lambda \in (0,1)$. Operators defined by $I(x,y) = TS_{\lambda,f}(N(x),y)$ are studied in [12.32] giving the conditions under which they are fuzzy implications.

Another approach is based on the use of dual representable aggregation functions G, that are simply the N-dual of RAFs, introduced earlier. In this case, the corresponding (G,N)-operator is always a fuzzy implication and several examples and properties of this class can be found in [12.33]. See also [12.34] where it is proven that they satisfy (EP) (or (12.5)) if and only if G is in fact a nilpotent t-conorm.

Generalizations of Yager's Implications

In this case, the generalizations usually deal with the possibility of varying the generator used in the definition of the implication. A first step in this line was taken in [12.35] by considering multiplicative generators of t-conorms, but it was proven in [12.36] that this new class is included in the family of all (S,N)-implications obtained from t-conorms and continuous fuzzy negations.

Another approach was given in [12.37] introducing *(f,g)-implications*. In this case, the idea is to generalize f-generated Yager's implications by substituting the factor x by $g(x)$ where $g\colon[0,1] \to [0,1]$ is an increasing function satisfying $g(0)=0$ and $g(1)=1$.

In the same direction, a generalization of f- and g-generated Yager's implications based on aggregation operators is presented and studied in [12.38], where the implications are constructed by replacing the product t-norm in Yager's implications by any aggregation function.

Finally, *h-implications* were introduced in [12.39] and are constructed from additive generators of representable uninorms as follows.

Definition 12.3 ([12.39])
Let $h\colon [0,1] \to [-\infty, \infty]$ be a strictly increasing and continuous function with $h(0) = -\infty$, $h(e) = 0$ for an $e \in (0,1)$ and $h(1) = +\infty$. The function $I^h\colon [0,1]^2 \to [0,1]$ defined by

$$I^h(x,y) = \begin{cases} 1\,, & \text{if } x = 0\,,\\ h^{-1}(x \cdot h(y))\,, & \text{if } x > 0 \text{ and } y \le e\,,\\ h^{-1}\left(\frac{1}{x} \cdot h(y)\right)\,, & \text{if } x > 0 \text{ and } y > e\,, \end{cases}$$

is called an h-implication.

This kind of implications maintains several properties of those satisfied by Yager's implications, like (EP) and (12.5) with the product t-norm, but at the same time they satisfy other interesting ones. For more details on this kind of implications, as well as some generalizations of them, see [12.39].

12.2.3 New Construction Methods

In this section, we recall some construction methods of fuzzy implications. The relevance of these methods is based on their capability of preserving the additional properties satisfied by the initial implication(s). First, note that some of them were already collected in [12.3, Chaps. 6 and 7], like:

- The φ-conjugation of a fuzzy implication I

$$I_\varphi(x,y) = \varphi^{-1}(I(\varphi(x), \varphi(y)))\,, \quad x, y \in [0,1]\,,$$

  where φ is an order automorphism on $[0,1]$.
- The min and max operations from two given fuzzy implications

$$(I \vee J)(x,y) = \max\{I(x,y), J(x,y)\}\,,\ x, y \in [0,1]\,,$$
$$(I \wedge J)(x,y) = \min\{I(x,y), J(x,y)\}\,,\ x, y \in [0,1]\,.$$

- The convex combinations of two fuzzy implications, where $\lambda \in [0,1]$

$$I_{I,J}^{\lambda}(x,y) = \lambda \cdot I(x,y) + (1-\lambda) \cdot J(x,y)\,,$$
$$x, y \in [0,1]\,.$$

- The N-reciprocation of a fuzzy implication I

$$I_N(x,y) = I(N(y), N(x))\,, \quad x, y \in [0,1]\,,$$

  where N is a fuzzy negation.
- The upper, lower, and medium contrapositivization of a fuzzy implication I defined, respectively, as

$$\begin{aligned} I_N^u(x,y) &= \max\{I(x,y), I_N(x,y)\}\\ &= (I \vee I_N)(x,y)\,,\\ I_N^l(x,y) &= \min\{I(x,y), I_N(x,y)\}\\ &= (I \wedge I_N)(x,y)\,,\\ I_N^m(x,y) &= \min\{I(x,y) \vee N(x), I_N(x,y) \vee y\}\,, \end{aligned}$$

  where N is a fuzzy negation and $x, y \in [0,1]$. Please note that the lower (upper) contrapositivization is based on applying the min (max) method to a fuzzy implication I and its N-reciprocal.

It should be emphasized that the first major work to explore contrapositivization in detail, in its own right, was that of *Fodor* [12.40], where he discusses the contrapositive symmetry of fuzzy implications for the three main families, namely, S-, R-, and QL-implications. In fact, during this study Fodor discovered the nilpotent minimum t-norm $T_{\mathbf{nM}}$, which is by far the first left-continuous but noncontinuous t-norm known in the literature. This study had a major impact on the development of left-continuous t-norms with strong natural negation, for instance, see the early works of *Jenei* [12.41, and references therein].

The above fact clearly illustrates how the study of functional equations involving fuzzy implications have also had interesting spin-offs and have immensely benefited other areas and topics in fuzzy logic connectives.

Among the new construction methods proposed in the recent literature, we can roughly divide them into the following categories.

Implications Generated from Negations

The first method was introduced by *Jayaram* and *Mesiar* in [12.42], while they were studying special implications (see Definition 12.2). From this study, they introduced the neutral special implications with a given negation and they studied the main properties of this new class.

Definition 12.4 [12.42]
Let N be a fuzzy negation such that $N \leq N_{\mathbf{C}}$. Then the function $I_{[N]}\colon [0,1]^2 \to [0,1]$ given by

$$I_{[N]}(x,y) = \begin{cases} 1, & \text{if } x \leq y\,, \\ y + \dfrac{N(x-y)(1-x)}{1-x+y}, & \text{if } x > y\,, \end{cases}$$

with the understanding $\frac{0}{0} = 0$, is called the neutral special implication generated from N.

The second method of generation of fuzzy implications from fuzzy negations was introduced in [12.43].

Definition 12.5 [12.43]
Let N be a fuzzy negation. The function $I^{[N]}\colon [0,1]^2 \to [0,1]$ is defined by

$$I^{[N]}(x,y) = \begin{cases} 1, & \text{if } x \leq y\,, \\ \dfrac{(1-N(x))y}{x} + N(x), & \text{if } x > y\,. \end{cases}$$

Again, several properties of these new implications can be derived, specially when the following classes of fuzzy negations are considered

$$N_A(x) = \begin{cases} 1, & \text{if } x \in A\,, \\ 0, & \text{if } x \notin A\,, \end{cases}$$

$$N_{A,\beta}(x) = \begin{cases} 1, & \text{if } x \in A\,, \\ \dfrac{1-x}{1+\beta x}, & \text{if } x \notin A\,, \end{cases}$$

where $A = [0,\alpha)$ with $\alpha \in (0,1)$ or $A = [0,\alpha]$ with $\alpha \in [0,1]$. Note that $N_{\{0\}} = N_{\mathbf{D}_1}$ and $N_{\{0\},\beta}$ is the Sugeno class of negations. Note also that $I^{[N]}$ can be expressed as $I^{[N]}(x,y) = S_{\mathbf{P}}(N(x), I_{\mathbf{GG}}(x,y))$ for all $x,y \in [0,1]$. From this observation, replacing $S_{\mathbf{P}}$ for any t-conorm S and $I_{\mathbf{GG}}$ for any implication I, the function

$$I^{[N,S,I]}(x,y) = S(N(x), I(x,y))\,, \quad x,y \in [0,1]\,,$$

is always a fuzzy implication.

Implications Constructed from Two Given Implications

In this section, we present methods that generate a fuzzy implication from two given ones.

The first method is based on an adequate scaling of the second variable of the two initial implications and it is called the *threshold generation method* [12.44].

Definition 12.6 [12.44]
Let I_1 and I_2 be two fuzzy implications and $e \in (0,1)$. The function $I_{I_1-I_2}\colon [0,1]^2 \to [0,1]$ defined by

$$I_{I_1-I_2}(x,y) = \begin{cases} 1, & \text{if } x = 0\,, \\ e \cdot I_1\left(x, \dfrac{y}{e}\right), & \text{if } x > 0 \text{ and } y \leq e\,, \\ e + (1-e)\cdot I_2\left(x, \dfrac{y-e}{1-e}\right), & \\ \qquad \text{if } x > 0 \text{ and } y > e\,, \end{cases}$$

is called the e-threshold generated implication from I_1 and I_2.

This method allows for a certain degree of control over the rate of increase in the second variable of the generated implication. Furthermore, the importance of this method derives from the fact that it allows us to characterize h-implications as the threshold generated implications of an f-generated and a g-generated implication [12.13, Theorem 2 and Remark 30]. Further, in contrast to many other generation methods of fuzzy implications from two given ones, it preserves (EP) and (12.5) if the initial implications possess them. Moreover, for an $e \in (0,1)$, the e-threshold generated implications can be characterized as those implications that satisfy $I(x,e) = e$ for all $x > 0$.

The threshold generation method given above is based on splitting the domain of the implication with a horizontal line and then scaling the two initial implications in order to be well defined in those two regions. An alternate but analogous method can be proposed by using a vertical line instead of a horizontal line. This is the idea behind the *vertical threshold generation* method of fuzzy implications. This method does not preserve as many properties as the horizontal threshold method, but some results can still be proven. In particular, they are characterized as those fuzzy implications such that $I(e,y) = e$ for all $y < 1$ [12.45].

The following two construction methods were presented in [12.46]. Given two implications I, J, the following operations are introduced

$$(I \nabla J)(x,y) = I(J(y,x), J(x,y))\,,$$
$$(I \otimes J)(x,y) = I(x, J(x,y))\,,$$

for all $x,y \in [0,1]$. The properties of these new operations as well as the structure of the set of all implications $\mathbb{FI}$ equipped with each one of these operations is studied in [12.46].

Other Construction Methods

In addition to the above methods, we would like to recall the following interesting method based on conditional probability and conditional distribution functions presented by *Grzegorzewski* in [12.47].

Definition 12.7 [12.47, 48]
The function $I_C\colon [0,1]^2 \to [0,1]$ given by

$$I_C(x,y) = \begin{cases} 1, & \text{if } x = 0\,, \\ \dfrac{C(x,y)}{x}, & \text{if } x > 0\,, \end{cases}$$

where C is a copula, is called a *probabilistic implication* based on copula C.

Conditions on copula C ensuring that the corresponding I_C is an implication, as well as properties of these implications are detailed in [12.48]. The main interest on this kind of implications lies in the fact that they are a powerful link between probability theory and fuzzy implications theory that can be useful in approximate reasoning. Moreover, results on these probabilistic implications can also be useful for examining and interpreting the behavior of some stochastic events. Some early results in this direction have appeared in [12.49, 50], where some generalizations of the previous idea are considered. In particular in [12.51], *survival implications* based on the probability that a given object will survive a fixed time into a population are studied. In this case, the survival implications are defined by

$$I_C^*(x,y) = \begin{cases} 1, & \text{if } x = 0\,, \\ \dfrac{x+y-1+C(1-x,1-y)}{x}, & \text{if } x > 0\,, \end{cases}$$

where C is again a copula.

Finally, we only briefly mention that there exist other construction methods. For instance, *Massanet* and *Torrens* [12.13, 44, 45] have proposed methods of constructing implications derived from a given implication I and a fuzzy negation N as part of their study on some properties of horizontal and vertical threshold generated implications.

12.2.4 Fuzzy Implications in Nonclassical Settings

When we deal with uncertainty through fuzzy sets and fuzzy logic the natural framework is the unit interval $[0,1]$ and hence the logical connectives to be used are interpreted as operators on this interval. However, there are many different tools that have been proposed for managing uncertainty. In this context, some extensions of fuzzy logic and fuzzy sets have also been developed. One can list at least the following extensions: interval-valued fuzzy sets, Atanassov intuitionistic fuzzy sets (that are equivalent to the interval-valued approach, [12.52]), interval-valued intuitionistic fuzzy sets, type-2 fuzzy sets, fuzzy multisets, n-dimensional fuzzy sets, and hesitant fuzzy sets.

For all these extensions, the usual logical connectives like fuzzy conjunctions and fuzzy disjunctions need to be studied to develop a comprehensive theory, and especially fuzzy implications in order to make inferences in each one of these extensions. Due to space constraints, we only recall some aspects of interval-valued (or intuitionistic) fuzzy implications and the references where they can be found.

Interval-Valued Approach

A good compilation of the known results related to fuzzy implications (and other operations) in the interval-valued framework, can be found in [12.53] or [12.54] wherein, interval-valued or intuitionistic (S,N)- and R-implications are developed and some of their properties are presented. Works that deal with the construction of these classes of interval-valued implications can also be found in the literature. For instance, in [12.55] a construction method for the residual implication associated with a representable t-norm (constructed from two standard t-norms T_1 and T_2 with $T_1 \le T_2$) is presented. Similarly, (S,N)- and R-implications generated from:

i) Aggregation functions and a standard fuzzy negation are presented in [12.56].
ii) Some classes of interval-valued aggregation functions based on t-norms and t-conorms are dealt with in [12.57].
iii) The so-called K_α-operators have been proposed in [12.58].

Discrete Approach

Note that all the above mentioned tools are mainly used in the management of imprecise quantitative information. However, experts deal with many problems where qualitative information is usually expressed through linguistic terms. Qualitative information is often interpreted to take values in a totally ordered finite scale like

$$\{\textit{Extremely Bad, Very Bad, Bad, Fair,} \\ \textit{Good, Very Good, Extremely Good}\}. \tag{12.8}$$

In these cases, the representative finite chain $L_n = \{0, 1, \ldots, n\}$ is usually considered to model these linguistic hedges and several researchers have developed an extensive study of operations on L_n, usually called *discrete operations*. This approach allows avoiding numerical interpretations and consequently, the fuzzification and defuzzification steps become unnecessary. In this framework, the smoothness condition is usually considered as the discrete counterpart of continuity. In fact, in the discrete framework this property is equivalent to the divisibility property as well as to the Lipschitz condition. In this way, smooth discrete t-norms and t-conorms were studied and characterized in [12.59] and also discrete fuzzy implications derived from them have been introduced.

As in the case of $[0,1]$, the four most usual ways to construct discrete implications from t-norms and t-conorms on L_n are (S,N)-, R-, QL-, and D-implications. The first two classes derived from smooth t-norms and t-conorms and the only strong negation on L_n (given by $N_0(x) = n - x$) were studied in [12.60]. In the smooth case, it is proven that the intersection between (S,N)- and R-implications contains only the Łukasiewicz implication [12.60, Proposition 10]. Further, the nonsmooth case has also been investigated showing a parameterized family of nonsmooth t-norms T for which the corresponding R-implication coincides with the (S,N)-implication derived from the N_0-dual of T. The case of discrete QL- and D-operators is studied in [12.61], where characterization results on when such operators are in fact implications are given and, moreover, it is proven that both these classes coincide in the smooth case.

However, the modeling of linguistic information is limited because the information provided by experts for each variable must be expressed by a simple linguistic term. In most cases, this is a problem for experts because their opinion does not agree with a concrete term. On the contrary, experts' values are usually expressions like *better than Good*, *between Fair and Very Good*, or other even more complex expressions.

To avoid the limitation above, an approach has recently appeared trying to increase the flexibility of the elicitation of linguistic information. This approach deals with the possibility of extending monotonic operations on L_n to operations on the set of *discrete fuzzy numbers* whose support is a subinterval of L_n, usually denoted by $\mathcal{A}_1^{L_n}$. The idea lies in the fact that any discrete fuzzy number $A \in \mathcal{A}_1^{L_n}$ can be considered (identifying the scale $\mathcal{L}$ given in (12.8) with the chain L_6) as an assignment of a $[0,1]$-value to each term in our linguistic scale. As an example, the above mentioned expression *between Fair and Very Good* can be performed, for instance, by a discrete fuzzy number $A \in \mathcal{A}_1^{L_6}$, with support given by the subinterval

$$[\text{Fair, Very Good}] = \{\text{Fair, Good, Very Good}\}\,,$$

(that corresponds to the subinterval $[3,5]$ in L_6). The values of A in its support should be described by experts, allowing in this way a complete flexibility of the qualitative valuation. Usual operations like t-norms, t-conorms, strong negations, aggregation functions, and also fuzzy implications have been introduced in this framework. The case of (S,N)-, QL- and D-implications can be found in [12.62, 63] and the case of R-implications in [12.64].

12.3 Fuzzy Implications in Applications

So far, we have discussed the theoretical aspects of fuzzy implications, namely, analytical and algebraic. In this section, we discuss their applicational value which shows a wide spectrum of areas wherein they are employed and how the gamut of properties that a fuzzy implication possesses plays an important role in its employability.

12.3.1 FL_n–Fuzzy Logic in the Narrow Sense

Boolean implications are employed in inference schemas like *modus ponens*, *modus tollens*, etc., where the reasoning is done with statements or propositions whose truth-values are two valued. Fuzzy implications play a similar role in the generalizations of the above inference schemas, where reasoning is done with fuzzy statements whose truth-value lies in $[0,1]$ instead of $\{0,1\}$.

Fuzzy Propositions

An expression of the form **x** *is* **A** where A is a fuzzy set on an appropriate domain U, with reference to the context, is termed as a *Fuzzy Statement* or a *Fuzzy Proposition*. (The above two interpretations bear a close resemblance to the *Adjunctive* and *Connective* interpretations as given in [12.65, pp. 331], though they are originally given for a binary operator. For other views and interpretation of the above statement, see, for instance, *Bezdek* et al., [12.66].)

Let it be given that $\mathbf{x}$ *is* $\mathbf{A}$ and also that x assumes the precise value, let us say, $x = u$, where $u \in U$, the domain of A. Then the truth value of the above fuzzy statement is obtained as follows

$$t(\mathbf{x} \textit{ is } \mathbf{A}) = A(u) ,$$

i. e., the truth value of the above fuzzy statement, given that x is precisely known, is equal to the degree to which u – the value x assumes – is itself compatible with the fuzzy set A. Thus greater the membership degree of u in the concept A, higher is the truth value of the fuzzy statement.

Consider the statement *John is Tall* and that x – the height of John – is precisely given to be $5'10'' \in U$. Now, $A(5'10'')$ gives the membership degree of $5'10''$ in the concept $A = \textit{Tall}$, which can be interpreted as how much *John* belongs to the set of all *Tall* men, or equivalently, how much *John is Tall* is true, which is nothing but the truth-value $t(\textit{John is Tall})$.

Fuzzy Conditionals or Fuzzy *IF–THEN* Rules

A fuzzy statement of the type discussed above $\widetilde{\mathbf{x}}$ *is* $\mathbf{A}$ can be interpreted in yet another way, namely, as a linguistic statement, i. e., as an assignment of a fuzzy set to a variable.

Let $A\colon U \to [0,1]$ be a fuzzy set on a suitable domain U. Then A can be taken to represent a concept. A *linguistic variable* of U is a symbol $\widetilde{x}$ that can assume or be assigned any fuzzy subset of U. Then a linguistic statement $\widetilde{x}$ *is* A is interpreted as the linguistic variable $\widetilde{x}$ taking the *linguistic value* A.

For example, let U denote the set of all values in degrees centigrade. If the linguistic variable $\widetilde{x}$ denotes *Temperature*, then it can assume the following linguistic values A, namely, *high, more or less high, medium, cool, very cold*, etc. Each of the linguistic values (say A = cool) is represented by a fuzzy set on the domain U of the linguistic variable $\widetilde{x}$, i. e., $A\colon U \to [0,1]$.

The shape of the graph of the function represents the concept (say high temperature). The concept of high temperature is itself again context dependent. For example, high temperature (fever) for a human being is different from the high temperature in a blast furnace, and accordingly the domain of the linguistic variable is selected.

A fuzzy *IF–THEN* rule is of the form

$$\text{IF } \widetilde{x} \text{ is } A \text{ THEN } \widetilde{y} \text{ is } B \,, \tag{12.9}$$

where A, B are linguistic expressions/values assumed by the linguistic variables $\widetilde{x}, \widetilde{y}$. For example,

IF $\widetilde{x}$ *(temperature) is A (high)*
THEN $\widetilde{y}$ *(pressure) is B (low).*

Generalized Modus Ponens

Let α, β be two fuzzy propositions as given above and let $\alpha \longrightarrow \beta$ be the fuzzy conditional which is a fuzzy IF–THEN rule as above. In classical logic, one uses rules of deduction, like modus ponens and modus tollens to deduce new knowledge from a given set of propositions. For instance, modus ponens states that $\alpha \wedge (\alpha \longrightarrow \beta) \vdash \beta$.

In fuzzy logic, since we deal with fuzzy propositions whose truth values vary over the entire $[0,1]$ interval we employ fuzzy logic operations. Typically $\wedge$ is interpreted as a t-norm T and for the $\longrightarrow$ a fuzzy implication is used.

Unlike with classical propositions, when we deal with fuzzy propositions it is not always given that from $\alpha \wedge (\alpha \longrightarrow \beta)$ one obtains β. This type of deduction is known as *generalized modus ponens* (GMP) and the study of pairs of operators $(\wedge, \longrightarrow)$, or alternately, a t-norm and fuzzy implication (T, I), that can be employed in GMP becomes important. It can be shown that this property translates to studying pairs (T, I) that satisfy the functional equation $T(x, I(x,y)) \leq y$ for $x, y \in [0,1]$, which is nothing but T-conditionality as dealt with in Sect. 12.2.1.

Proof by Contradiction

In classical logic, many a time one proves a statement of the form $\alpha \longrightarrow \beta$ by proving its contrapositive, i. e., $\neg\beta \longrightarrow \neg\alpha$. However, in the setting of fuzzy logic, often the negation $\neg$ used is noninvolutive, i. e., $\neg\neg\alpha \neq \alpha$.

For instance, when the underlying fuzzy logic operations come from the Gödel residuated lattice $([0,1], T_{\mathbf{M}}, I_{\mathbf{GD}}, \wedge, \vee)$, the natural negation of the fuzzy implication $I_{\mathbf{GD}}$ is not involutive and $I_{\mathbf{GD}}$ is not contrapositive w.r.t. any fuzzy negation. This led to the study of contrapositivization of fuzzy implications which was begun by *Fodor* [12.40] and is dealt with in Sect. 12.2.3 above.

12.3.2 Approximate Reasoning

One of the best known application areas of fuzzy logic is *approximate reasoning* (AR), wherein from imprecise inputs and fuzzy premises or rules we obtain, often, imprecise conclusions [12.67]. AR with fuzzy sets encompasses a wide variety of inference schemes and have been readily embraced in many fields, especially among others: decision making, expert systems, and control. Fuzzy implications play a vital role in many of these inference mechanisms, a brief discussion of which is presented below.

Inference Mechanisms in AR

Let us be given a set of n fuzzy IF–THEN rules of the form given in (12.10)

$$\textit{If}\, \widetilde{x}\ \textit{is}\ A_i\ \textit{Then}\, \widetilde{y}\ \textit{is}\ B_i\ , \quad i = 1, 2, \ldots, n\ , \tag{12.10}$$

where A_i, B_i are fuzzy sets on input and output domains. Now, given a fuzzy input, i. e., a fuzzy proposition or a statement of the form $\widetilde{x}$ *is* A', the role of an inference mechanism is to obtain a fuzzy output B' that satisfies some desirable properties [12.68, 69].

Note that, if we denote the fuzzy rules as $A_i \longrightarrow B_i$, $i = 1, 2, \ldots, n$, as is typically done, then these are exactly the fuzzy conditionals discussed above in Sect. 12.3.1. Further, if we denote the input as A' then an inference mechanism implements the generalized modus ponens by *composing* the fuzzy input A' with all the rules $A_i \longrightarrow B_i$ to obtain the fuzzy output B'.

There are two established ways to accomplish the above, namely, *fuzzy relational inference* (FRI) and *similarity based reasoning* (SBR). Fuzzy implications play a major role in both the types of inference mechanisms as detailed below.

Fuzzy Relational Inference (FRI)

In a fuzzy relational inference, all the rules $A_i \longrightarrow B_i$ are combined into a single fuzzy relation R and the output B' is obtained as an image of the input A' composed with R.

A fuzzy IF–THEN rule base of the form (12.10) is modeled as a fuzzy relation $\hat{R}(x, y)\colon X \times Y \to [0, 1]$ as follows

$$\begin{aligned}\hat{R}(x, y) &= \wedge_{i=1}^{n} \left(A_i(x) \to B_i(y)\right) \\ &= \wedge_{i=1}^{n} \left(I(A_i(x), B_i(y))\right) ,\end{aligned} \tag{12.11}$$

which reflects the conditional nature of the rules and where I is usually a fuzzy implication. Then given a fact $\widetilde{x}$ is A', the inferred output B' is obtained either as:

i) sup-T composition, as in the compositional rule of inference (CRI) of *Zadeh* [12.70], or
ii) An inf-I composition, as in the Bandler–Kohout subproduct (BKS) [12.71],

of $A'(x)$ and $\hat{R}(x, y)$, i. e.,

$$B'(y) = A'(x) \overset{T}{\circ} \hat{R}(x, y) = \sup_{x \in X} T(A'(x), \hat{R}(x, y))\ , \tag{12.12}$$

$$B'(y) = A'(x) \overset{I}{\lhd} \hat{R}(x, y) = \inf_{x \in X} I(A'(x), \hat{R}(x, y))\ , \tag{12.13}$$

where T can be any t-norm and I is any fuzzy implication.

It is clear from (12.12) and (12.13) that the important role fuzzy implications and their properties play in the goodness of an inference scheme. In the following subsection, we present a few issues where this role is highlighted.

Issues in FRI

While the rule base is an example of a single input single output (SISO) case, in practice we need multi-input single-output (MISO) rules of the form given below, with m input domains X_j, $j = 1, 2, \ldots, m$,

$$\begin{aligned}&R_i:\ \text{IF}\, \widetilde{x}_1\ \text{is}\ A_{i1}\ \text{AND}\, \widetilde{x}_2\ \text{is}\ A_{i2}\ \text{AND} \\ &\ \ldots\ \text{AND}\, \widetilde{x}_n\ \text{is}\ A_{in}\ \text{THEN}\, \widetilde{y}\ \text{is}\ B_i\ .\end{aligned}$$

While MISO rule bases are of great practical necessity, they spring up some new issues when they are employed in FRIs.

Combinatorial Explosion of Rules and Distributivity of Fuzzy Implications

Let there be k_j fuzzy sets defined on each of the domains X_j, $j = 1, 2, \ldots, m$. Then in a complete MISO rule base, we will have $n = k_1 \times k_2 \times \cdots k_m$ number of rules. Clearly, as m or k_j increases n increases and we have a combinatorial explosion of rules.

In a seminal work on studying this issue, *Combs* and *Andrews* [12.72] proposed an equivalent transformation of the CRI to mitigate the computational cost. The authors showed that the distributivity of fuzzy implications over t-norms play a major role in this transformation. This was further studied by *Balasubramaniam* and *Rao* [12.10] and its use in SBR was also demonstrated later by *Jayaram* [12.73].

Computational Complexity, Hierarchical Systems, and the Law of Importation

Let us consider an MISO rule base. From (12.11), it is clear that the relation $\hat{R}$ obtained is a multidimensional matrix, with $\hat{R}: X_1 \times X_2 \times \cdots \times X_m \times Y \to [0,1]$. In fact, when one uses the First-Infer-Then-Aggregate mechanism in an FRI, either CRI or BKS, one needs to store n such m-dimensional matrices. Further, the input A' is also an m-dimensional matrix and the computation of the output gets costlier.

To overcome this, *Jayaram* [12.19] proposed an alternate hierarchical inference scheme which can be shown to be equivalent both in the CRI [12.19] and BKS [12.20] setting, when the underlying operators are such that the t-norm T and the fuzzy implication I satisfy the law of importation (12.5).

12.3.3 Fuzzy Subsethood Measures

Inclusion or subsethood of sets is an important concept. The first such definition of inclusion of a fuzzy set A over X in another fuzzy set B, was given by *Zadeh* [12.74] as follows

$$A \subset_Z B \Longleftrightarrow A(x) \le B(x)\,, \quad \text{for all } x \in X\,.$$

Note that this definition was more or less *crisp*, since an A was either contained in B or not. A more general notion of degree of inclusion was missing in the above definition. Subsequently many fuzzy subsethood measures, denoted (usually) *Inc*, were proposed.

Axiomatic Studies on Fuzzy Subsethood Measures

From the isomorphism that exists between classical set theory and classical logic, we know that $A \subseteq B$ is equivalent to $\chi_A \Longrightarrow \chi_B$, where χ_X is the characteristic function of the set X. Thus, early fuzzy subsethood measures also mimicked this equivalence by defining them based on fuzzy implications. Many researchers, in particular, *Sinha* and *Dougherty* [12.75], *Kitainik* [12.76], *Bandler* and *Kohout* [12.77] proposed sets of axioms for an *Inc* to satisfy.

It is easy to see that all of the above axiomatic approaches, eventually lead to employing implications as the underlying operators to define the corresponding *Inc* measure, as given below

$$\begin{aligned}
Inc_{\mathbf{SD}}(A,B) &= \inf_{x\in X} \min\left(1, \lambda(A(x)) + \lambda(1 - B(x))\right)\,,\\
Inc_{\mathbf{K}}(A,B) &= \inf_{x\in X} \varphi(I_{\mathbf{KD}}(B(x),A(x))\,,\\
&\qquad\qquad 1 - I_{\mathbf{KD}}(A(x),B(x)))\,,\\
Inc_{\mathbf{BK}}(A,B) &= \inf_{x\in X} (I(A(x),B(x)))\,,
\end{aligned}$$

where $\lambda\colon [0,1] \to [0,1]$ is a decreasing function with some additional properties, $\varphi\colon \mathcal{A} \to [0,1]$ a function with additional properties where $\mathcal{A} = \{(x,y) \in [0,1]^2 | x \ge y\}$ and I is any fuzzy implication.

From the above formulae the important position a fuzzy implication I holds in measuring fuzzy subsethood is apparent. Note that the *Inc* measure is used extensively in similarity based reasoning (SBR) and in fuzzy mathematical morphology (FMM) which are discussed below.

12.3.4 Fuzzy Control

While Sect. 12.3.2 dealt with FRIs which are largely used in the context of decision making and expert systems, in this section we deal with another type of fuzzy inference mechanism (FIM) that is used in fuzzy control, where the approximation properties of the FIM are important.

Similarity-Based Reasoning (SBR)

Let us once again consider a fuzzy IF–THEN rule base of the form (12.10) and a fuzzy input A'. In an SBR inference scheme, the following steps are employed to produce the output:

- *Matching*: The input A' is matched against each of the antecedents A_i of the rules (12.10) using a matching function M to obtain the corresponding similarity values $s_i = M(A', A_i) \in [0,1]$ for $i = 1, 2, \ldots, n$.
- *Modification*: Each of the similarity values s_i is used to modify the corresponding consequent B_i of the rule (12.10) using a modification J to obtain the modified output $B'_i = J(s_i, B_i)$.
- *Aggregation*: Finally all the modified outputs B'_i are aggregated to obtain an overall output $B = G(B'_1, \ldots, B'_n)$.

In notations, we can write the above as

$$B'(y) = G_{i=1}^{n}\left(J(M(A', A_i), B_i(y))\right)\,, \quad y \in Y\,. \tag{12.14}$$

Fuzzy Implications and Matching Functions

Clearly, since A, $A'_i \in \mathcal{F}(X)$, we see that the matching function $M : \mathcal{F}(X) \times \mathcal{F}(X) \to [0,1]$. Typically, a fuzzy subsethood measure *Inc* is employed as an M. While there exist M that are not based on fuzzy implications, it is seen that those that are based on fuzzy implications often satisfy many of the desirable properties required on the matching function M in different contexts, for instance, when the SBR is required to be interpolative, monotonic or for the SBR to possess good approximation properties. For more on this topic, see the works of Jayaram [12.73] or *Mandal* and *Jayaram* [12.78].

Fuzzy Implications and Modification Functions

From (12.14), it is clear that the modification function J can be seen simply as a binary function on [0, 1]. While any fuzzy logic operation could be used for J, fuzzy implications are preferred either due to their properties or due to the conditional nature of the underlying rules. For instance, when $J = I$ a fuzzy implication, if the original output B_i is normal then the modified output B'_i is also normal, which is usually not the case when one uses, say, a t-norm. In fact, different properties of I like (OP), (IP) and the nature of its natural negation N_I all play a role in the reasonableness of the final output of an SBR.

In real-life systems, the input and output domains X, Y are subsets of $\mathbb{R}$. Now, let the consequents B_i be of bounded support, i. e., $\{y \in Y \subset \mathbb{R} | B_i(y) > 0\} = [a,b] \subsetneq Y$ for some finite $a, b \in \mathbb{R}$. When an I whose N_I is not the Gödel least negation $N_{\mathbf{D1}}$ is employed, the support of B'_i becomes larger and in the case N_I is involutive then the support of the modified output sets B'_i become the whole of the set Y. This often makes the modified output sets B'_i to be nonconvex (and of larger support) and makes it difficult to apply standard defuzzification methods. For more on these see the works of *Štěpnička* and *De Baets* [12.79]. The above discussion brings out an interesting aspect of fuzzy implications. While fuzzy implications I whose N_I are strong are to be preferred in the setting of fuzzy logic FL_n for inferencing as noted in Sect. 12.3.1 above, an I with an N_I that is not even continuous is to be preferred in inference mechanisms used in fuzzy control.

By the *core* of a fuzzy set B on Y, we mean the set $\{y \in Y | B(y) = 1\}$. Now, an I which possesses (OP) or (IP) is preferred in an SBR to ensure there is an overlap between the cores of the modified outputs B'_i – a property that is so important to ensure *coherence* in the system [12.80] and that, once again, standard defuzzification methods can be applied.

12.3.5 Fuzzy Mathematical Morphology

Consider a 2D binary image $\mathcal{P}$, i. e., the value at a pixel is either 0 or 1. $\mathcal{P}$ can be seen as a function from $X \subset \mathbb{R}^2 \to \{0,1\}$ or just a classical subset $X \subset \mathbb{R}^2$. Mathematical morphology (MM) is a set-theoretic method for the extraction of shape information from a scene. Here, a $Y \subset \mathbb{R}^2$ – which can be seen as another image $\mathcal{Q}$ and often referred to as the structuring element – is used to transform the original image $\mathcal{P}$ by some well-defined local operators termed *Dilation* and *Erosion* as defined below

$$D(\mathcal{P},\mathcal{Q}) = \{v \in \mathbb{R}^2 | \mathcal{A}_v(\mathcal{Q}) \cap \mathcal{P} \neq \emptyset\}, \tag{12.15}$$

$$E(\mathcal{P},\mathcal{Q}) = \{v \in \mathbb{R}^2 | \mathcal{A}_v(\mathcal{Q}) \subseteq \mathcal{P}\}, \tag{12.16}$$

where $\mathcal{A}_v(\mathcal{Q}) = \{u \in \mathbb{R}^2 | u - v \in \mathcal{Q}\}$ is the translation of $\mathcal{Q}$ by $v \in \mathbb{R}^2$.

FMM is the extension of MM to gray-level images by using fuzzy sets and possibility theory. Note that a gray-level image $\mathcal{P}$ can be interpreted as a fuzzy set $X \subset \mathbb{R}^2 \to [0,1]$ where the pixel value is interpreted as its membership degree to the original data set. This fuzzified image is then processed via morphological operators that are extensions of the boolean ones.

In the literature, one finds two approaches to this extension:

i) As a formal translation of crisp equations using t-norms and negations, by employing a fuzzy intersection for $\cap$ in (12.15) and a fuzzy subsethood measure *Inc* for $\subseteq$ in (12.16), and
ii) Using adjunction and residual implications.

While the first approach is based on the duality between dilation and erosion, the second approach stems more from an algebraic setting.

De Baets [12.81, 82] took the second approach, and defined the *fuzzy dilation* and *erosion* as follows

$$\tilde{D}(\mathcal{P},\mathcal{Q})(y) = \sup_{x \in \mathcal{A}_v(Y) \cap X} [C(\mathcal{P}(x-y), \mathcal{Q}(x))],$$

$$\tilde{E}(\mathcal{P},\mathcal{Q})(y) = \inf_{x \in \mathcal{A}_v(Y)} [I(\mathcal{P}(x-y), \mathcal{Q}(x))],$$

where C is any fuzzy conjunction and I is a fuzzy implication.

When the pair of operations (C, I) satisfy the adjunction property, or equivalently, I is a residual implication obtained from C, then many interesting aspects emerge. Firstly, it can be shown that *opening* and *closing* operations, which are some morphological operations obtained from the defined $\tilde{D}, \tilde{E}$ turn out to be idempotent, which is highly desirable [12.83]. Secondly, it can be shown, as was done by *Nachtegael* and *Kerre* [12.84], that this approach is more general and many other approaches become a specific case of it. Thirdly, recently, *Bloch* [12.85] showed that both the above approaches based on duality and adjunction are equivalent under some rather general and mild conditions, but those that often lead to highly desirable settings.

Recently, the approach initiated by De Baets has been enlarged by considering uninorms instead of t-norms and their residual implications with good results in edge detection, as well as in noise reduction [12.86, 87].

12.4 Future of Fuzzy Implications

Since the publication of [12.2, 3], the peak of interest in fuzzy implications has led to a rapid progress in attempts to solve open problems in this topic. Specially, in [12.3], many open problems were presented covering all the subtopics of this field: characterizations, intersections, additional properties, etc. Many of these problems have been already solved and the solutions have been collected in [12.88]. However, there still remain many open problems involving fuzzy implications. Thus, in this section, we will list some of them whose choice has been dictated either based on the importance of the problem or the significance of the solution.

The first subset corresponds to open problems dealing with the satisfaction of particular additional properties of fuzzy implications. The first one deals with the law of importation (LI). Recently, some works have dealt with this property and its equivalence to the exchange principle and from them, some new characterizations of (S,N)- and R-implications based on (12.5) have been proposed, see [12.21]. However, some questions are still open. Firstly, (12.5) with a t-norm (or a more general conjunction) and (EP) are equivalent when N_I is a continuous negation, but the equivalence in general is not fully determined.

Problem 12.1
Characterize all the cases when (LI) and (EP) are equivalent.

Secondly, it is not yet known which fuzzy implications satisfy (LI) when the conjunction operation is fixed.

Problem 12.2
Given a conjunction C (usually a t-norm or a conjunctive uninorm), characterize all fuzzy implications I that satisfy (LI) with this conjunction C. For instance, which implications I satisfy the following functional equation

$$I(xy,z) = I(x, I(y,z))$$

that comes from (LI) with $T = T_{\mathbf{P}}$?

Another problem now concerning only the exchange principle follows.

Problem 12.3
Give a necessary condition on a nonborder continuous t-norm T for the corresponding I_T to satisfy (EP).

It should be mentioned that some related work on the above problem appeared in [12.89].

Some other open problems with respect to the satisfaction of particular additional properties are based on the preservation of these properties from some initial fuzzy implications to the generated one using some construction methods like max, min, or the convex combination method.

Problem 12.4
Characterize all fuzzy implications I, J such that $I \vee J$, $I \wedge J$ and K^{λ} satisfy (EP) or (LI), where $\lambda \in [0,1]$.

The above problem is also related to the following one:

Problem 12.5
Characterize the convex closures of the following families of fuzzy implications: (S,N)-, R- and Yager's f- and g-generated implications.

Another open problem which has immense applicational value is the satisfaction of the T-conditionality by the Yager's families of fuzzy implications.

Problem 12.6
Characterize Yager's f-generated and g-generated implications satisfying the T-conditionality property with some t-norm T.

The following two open problems are related to the characterization of some particular classes of fuzzy implications.

Problem 12.7

What is the characterization of (S,N)-implications generated from noncontinuous negations?

Problem 12.8

Characterize triples (T,S,N) such that the corresponding QL-operation $I_{T,S,N}$ satisfies (I1).

Finally, a fruitful topic where many open problems are still to be solved is the study of the intersections among the classes of fuzzy implications (Fig. 12.1).

Problem 12.9

i) Is there a fuzzy implication I, other than the Weber implication $I_{\mathbf{WB}}$, which is both an (S,N)-implication and an R-implication which is obtained from a nonborder continuous t-norm and cannot be obtained as the residual of any other left-continuous t-norm?

ii) If the answer to the above question is affirmative, characterize the above nonempty intersection.

Problem 12.10

i) Characterize the nonempty intersection between (S,N)-implications and QL-implications, i.e., $\mathbb{I}_{\mathbb{S},\mathbb{N}} \cap \mathbb{I}_{\mathbb{QL}}$.

ii) Is the Weber implication $I_{\mathbf{WB}}$ the only QL-implication that is also an R-implication obtained from a nonleft continuous t-norm? If not, give other examples from the above intersection and hence, characterize the nonempty intersection between R-implications and QL-implications.

iii) Prove or disprove by giving an example: that there is no fuzzy implication which is both a QL- and an R-implication, but it is not an (S,N)-implication, i.e., $(\mathbb{I}_{\mathbb{QL}} \cap \mathbb{I}_{\mathbb{T}}) \setminus \mathbb{I}_{\mathbb{S},\mathbb{N}} = \emptyset$.

Part B | 12

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