Doing Mathematics with 3D Pens: Five Years of Research on 3D Printing Integration in Mathematics Classrooms

Oi-Lam Ng and Huiyan Ye

1 Motivation

Our work has been inspired by Seymour Papert, who put forward a new conception of learning known as *constructionism* or "learning-by-making" (Papert & Harel, 1991, p. 1), which holds that knowledge is constructed during human contact with external materials. In turn, while hands-on making, students simultaneously construct their knowledge as well as build meaningful products in the physical world. In addition, the artefacts are shareable in the community to facilitate peer learning. Among his contributions, Papert has shown that technology-based constructions can change students' ways of thinking about mathematics. While we were interested in educational programming languages such as *Logo*, which enables students to express their ideas in a computational manner, we also wanted to explore new forms of multimodal technologies, such as touchscreens, 3D printing, and the like, which provide a more direct way of interacting with and expressing mathematical ideas (Hegedus & Tall, 2016). These hands-on technologies have created more opportunities for learners to interact with technologies or products, thereby facilitating learners' knowledge construction processes in the mathematics classroom.

O.-L. Ng (✉) · H. Ye
Department of Curriculum and Instruction, The Chinese University of Hong Kong,
Shatin, Hong Kong SAR
e-mail: oilamn@cuhk.edu.hk

H. Ye
e-mail: huiyanye@cuhk.edu.hk

F. Dilling et al. (eds.), *Learning Mathematics in the Context of 3D Printing*,
MINTUS – Beiträge zur mathematisch-naturwissenschaftlichen Bildung,
https://doi.org/10.1007/978-3-658-38867-6_7

Grounded in the theory of constructionism, we believe that 3D printing is a powerful technology for “learning-by-making.” 3D printing can extend 2D products to the 3D environment; moreover, unlike traditional manipulatives preselected by teachers, 3D printing can provide students with hands-on opportunities to generate 3D models. Such learning experiences correspond to the “learning-by-making” approach advocated by constructionism. Furthermore, considering the unique characteristics of hands-on and “embodied making” (Ng & Ye, 2022), in our research, we used 3D pens as a form of 3D printing in mathematics education to explore the potential transformations that 3D printing can induce in mathematics teaching and learning (Ng & Ferrara, 2020; Ng & Sinclair, 2018; Ng & Tsang, 2021; Ng et al., 2018). In this chapter, we review the first authors’ five years of research on the use of 3D pens in mathematics education. First, we describe the practice of mathematical diagramming and discuss the potential possibilities of 3D diagramming in engendering students’ mathematical thinking as a way to introduce three affordances of diagramming with a 3D pen. Then, we illustrate the theoretical background that underpins our research and describe some lesson designs with 3D pen. Finally, we present both quantitative and qualitative results to discuss the role of 3D pens in mathematics learning and suggest future research direction.

2 Affordances of Diagramming with a 3D Pen

2.1 Support Visualization of 3D Geometrical Objects

The first unique feature of using a 3D pen is the ability to draw in 3D, which overcomes the limitations of paper and pencil and improves the visualization of 3D geometrical objects. For example, one way to draw a cube (Fig. 1), is firstly to draw four straight “segments” on a surface to form a square, then four vertical “segments” that join the four vertices of the square, and four more “segments” in the air, while drawing an identical square parallel to the base. Note that the constructionist practice mentioned here parallels the process of drawing, for example, a square in Papert’s Turtle Geometry in the sense that a sequence of actions is taken to construct a figure, which is also the very process that defines the figure itself. The difference between the two constructionist practices lies in the tools used, which facilitated different (programmable and non-programmable) artefacts, where 3D pens afforded a mode of making merely by moving one’s hands. We note that in the process of drawing such a 3D object, one can visualize vertices, segments, and planes and observe how these 0D, 1D, and 2D objects

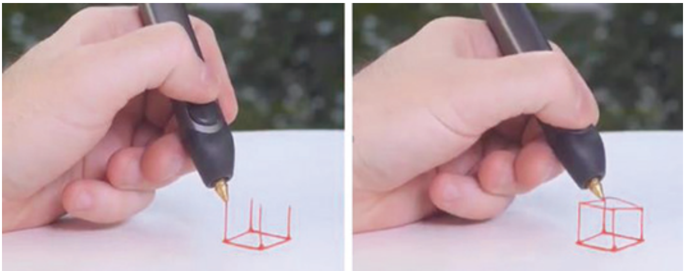

Fig. 1 Drawing a cube with 3D pen

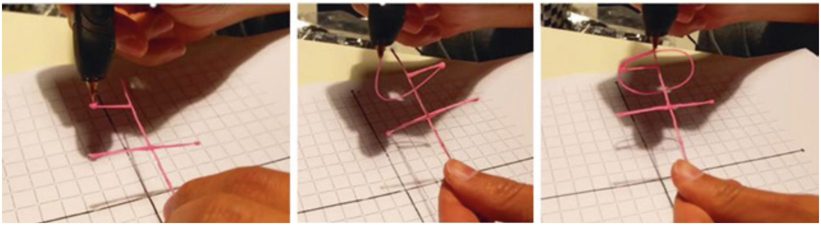

Fig. 2 Thinking while drawing with a 3D pen

compose the 3D object. The hand movements involved in the process of drawing are a significant learning component, one that is not present in screen-based media and different from operating with premade objects, as the movements themselves (moving one's hands to make a square) imply geometrical meanings.

2.2 New Modes of Thinking

Besides drawing in 3D, the diagramming process with a 3D pen can uniquely stimulate new modes of thinking. This is because the process of 3D drawing simulates the very process of gesturing; as the hand moves with the 3D pen, a 3D model is generated. The unique nature of interacting with the 3D model with one's hands affords some interesting movement that could not be possible on the flat surface of a paper-and-pencil environment. For example, as shown in Fig. 2, the drawing of a circle can involve the experience of spinning an axis of rotation with one's left hand while the 3D material continues to be added to a point at a distance from the axis at which the right hand remains still. Such material interactions stimulate new modes of thinking and facilitate meaningful bodily experiences, i.e., physically picking up and rotating an axis (for 360 degrees) to locate the trajectory of a point at a fixed distance from the axis, resulting in a circular trajectory with the help of two hands that are equidistance from each other.

2.3 The Dual Nature of Diagramming and Manipulating

Finally, 3D pens provide additional tactile experience of drawing 2D figures by affording students the ability to touch and feel 2D models drawn. Diagrams that would have been drawn using paper and pencil, such as a triangle, can be recreated and become physical objects that can be held, moved, and turned when drawn by 3D pens (Fig. 3). This enables one to interact with 2D figures in ways that one could not when using the traditional paper-and-pencil medium, as they can be physically transformed or manipulated during the meaning-making process. For example, our most recent research suggests that students conceive triangles as flat figures that could not be, in any way, manipulated in the third dimension. In particular, they no longer recognize it as a triangle if it was drawn by a 3D pen and made "standing up" rather than "lying flat on the table". This indicates that students' meaning making are constrained by the paper-and-pencil medium, where the triangle will stay dormant on paper once drawn. As such, students often lack the tangible experience of manipulating with geometric shapes after drawing them. In response, 3D drawings have a dual nature: they are both a *diagram* and a physical *manipulative*.

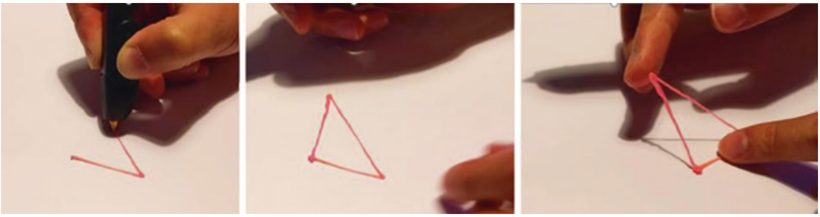

Fig. 3 Drawing a triangle with a 3D pen

3 Theoretical Background

Our research interest is framed from the perspective of embodied cognition, which focuses on exploring some salient aspects or less easily observable features that occur in the teaching and learning of mathematics until recent years. These aspects include language, the body, gestures, non-verbal communication, which reflect the movement from static, individualistic learning to a broader level and more dynamic features of learning that considers multimodality and sensorimotor experiences, as well as the role of the body in mathematical cognition. Embodied cognition is a study in cognitive science that frames a deep understanding of what human ideas are, and how they are organized in vast (mostly unconscious) conceptual systems grounded in physical, lived reality (Núñez et al., 1999). Further, Núñez et al. (1999, p. 50) defines embodiment as follows:

> Embodiment is not simply about an individual's conscious experience of some bodily aspects of being or acting in the world. Embodiment does not necessarily involve conscious awareness of its influence. Nor does embodiment refer to the physical manipulation of tangible objects, or to virtual manipulation of graphical images and objects […] an embodied perspective does not constitute a prescription for teaching in a 'concrete' way.

Therefore, an embodied cognitive approach to learning is not simply learning in a physical way but to draw on the individual's bodily experience and conceptual system in sense-making. In addition to theoretical considerations around embodied cognition, our work is also informed by research evidence that our hands contribute significantly to cognitive processes: a) there is a connection between spatial reasoning and gestures (Ehrlich et al., 2006)—for example, gestures accompany spatial information in speech; b) there is evidence of finger perception (Penner-Wilger, 2013), in the sense that college students' finger perception predicts calculation scores and that finger perception in Grade 1 is a better predictor of mathematics achievements in Grade 2 than test scores; and c) there is evidence that gestures contribute to effective communication (Alibali & Nathan, 2012). Therefore, gestures are highly important in thinking and learning. At the same time, mathematical cognition is deeply rooted in embodied interactions with the environment and materials (e.g., tools). As Nemirovsky et al. (2013) proposed by the term *mathematical instruments*, 3D pens serve as "material and semiotic device[s] together with a set of embodied practices, enabling the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) as acknowledged within the culture of mathematics" (p. 376). Overall, as reviewed in Ng et al. (2020), approaches to learning that take on the perspective of embodied cognition predict that sensorimotor experiences, including visual perceptions and bodily actions, strengthen students' sensemaking processes, especially their visualization capacity and spatial reasoning in the science, technology, engineering, and mathematics (STEM) disciplines (Weisberg & Newcombe, 2017). These empirical results are further extended by research showing that the transition from action to abstraction in mathematics and science learning can be supported via gestures (Novack et al., 2014). In relation to embodied learning, the use of 3D pens can facilitate hand movements that support gestural forms of thinking about mathematical concepts.

We are also interested in de Freitas and Sinclair's (2014) theoretical approach of *inclusive materialism*, which sheds light on the ontologies of the body and mathematics. This framework aims to redefine the boundaries of the body and claims, based on the interactionist perspective, that materials are constantly interacting with one another and with the human body rather than being inert. This approach reflects the relatively more social aspects of meaning making compared to the conceptualist tradition, which "ultimately demotes activity to simulation rather than full-body Making" (Ng & Ferrara, 2020, p. 928). This perspective offers a re-conceptualisation of classroom learning as assemblage of mathematical knowledge, teacher, students and material surrounding. Since materials do not have confined properties of their own, boundaries of materials, mathematics and the human body are re-defined, and learning is redistributed across the situation amongst the players and their material surrounding.

Given that the use of 3D pens constitutes a material interaction and the diagrammatic nature 3D drawing, the work of Châtelet (2000) is important, as it considers diagramming (the making of diagrams) and gesturing to be inseparable processes as well as creative embodied acts that constitute new relationships between mathematics and the material activity. In other words,

> [G]estures and diagrams are sources of mathematical meaning, which presuppose each other. They are never complete and share similar mobility and potentiality: gestures give rise to the possibility of diagramming, while diagrams give rise to new possibilities for gesturing. (Ng & Ferrara, 2020, p. 926)

As mentioned in the previous section, the use of 3D pens offers new gestural forms of thinking (Ng & Sinclair, 2018; Ng et al., 2018). This means that 3D diagramming is not only iconic representations of mental operations, they also affect the individuation of mathematical meaning. As de Freitas and Sinclair

(2014) state: “Does mathematics really just stand there, silently waiting for the breakthrough insight or shift in attention? Or might it somehow be much more implicated in the moving hands and the configuration of [materials]?” (p. 30). Due to the close resemblance of 3D diagramming and gesturing, we find it useful to adopt a materialist perspective when exploring the unique prospect of using 3D pens in mathematical activities, as embodied diagramming/gesturing during material creation which engenders new possibilities for encounters with mathematical concepts. Consequently, the hands-on production of artifacts with 3D pens is not only a form of making but also a kind of assemblage or emerging intra-action among the learner(s), concept, and tool.

4 Lesson Designs with 3D Pens

In view of the aforementioned theoretical framings and the three affordances of 3D diagramming, our research team has engaged in research contextualized in the mathematics classroom to improve our understanding of 3D pens’ impact on learning mathematics. In this section, we will describe some examples of lessons designed for elementary mathematics topics (e.g., geometry) and secondary mathematics topics (e.g., functions and calculus).

4.1 Example 1: Primary School Geometry (Ng & Ferrara, 2020)

In Ng and Ferrara (2020), we developed a lesson design for teaching and learning properties of prisms, pyramids, and cross-sections of 3D solids. Using an inquiry-based and student-centered approach, in the lesson on the properties (faces, vertices, and edges) of prisms and pyramids, students used 3D pens to draw different prisms and pyramids and to investigate the target properties (Fig. 4). The students worked in pairs and exhibited high engagement with the inquiry-based learning activities (Fig. 5). Among the results, we found that the students construct triangular prism with different strategies which also indicated they have visualized a triangular prism differently. For instance, many students started off with drawing a triangular base and then constructed three vertical pillars in the air, followed by moving the 3D pen from the top of one vertex to another in a triangular path to form another triangular base at the top. In another construction, a student began by drawing a rectangular base, then created two facing triangles perpendicular to the rectangular base on both sides and finally drew a line connecting the two

Fig. 4 Students used 3D pens to draw different prisms and pyramids


[redacted] [redacted]

Fig. 5 Students worked in pairs on 3D pen activities

triangles in the air. From the perspective of embodied cognition, the combination of students' linguistic expressions about 1D ("lines") and 2D shapes ("triangles", "rectangle", etc.) together with their gestures (e.g., perpendicularly moving, rotating, making four right-angled turns, etc.) suggest that "3D making did not only yield a product that was physical and sharable, but it was also a material process of thinking mathematically that outline differences in how to make a triangular prism" (Ng & Ferrara, 2020, p. 936).

In another lesson, students drew cross-sections using different cutting methods for different kinds of prisms and pyramids when learning the cross-sections of 3D solids. The 3D pen provides an alternative "cutting" method, which allows students to visualize a cross-section of the 3D solids. Students can hold a 3D pen and draw the outlines of the cross-sections of prisms and pyramids on the physical models themselves. When they pull their drawing out of the physical models, they could see the corresponding geometrical shape (e.g., circle, rectangle, trapezoid), which is the very shape of the cross-section (Fig. 6). From the perspective of inclusive materialism, when students were making an oblique or perpendicular

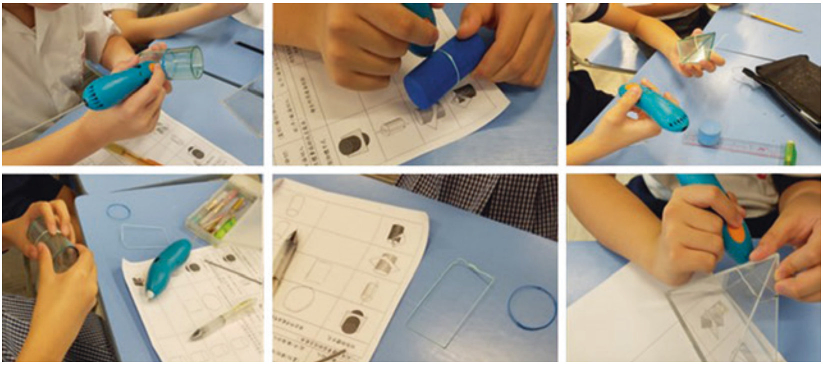

Fig. 6 Students draw cross-sections on different models of prisms and pyramids (Ng & Ferrara, 2020, pp. 932, 939)

cut to the cylinder, the notion of time, distance, angle and turns were made manifest in students' learning assemblage because of the human-material interaction of using a 3D pen. In so doing, the mathematical meaning of cross-sections, such as ellipse, rectangle, squares and trapezoids, were co-implicated due to the movement of the hand holding the 3D pen. This shows that 3D drawing provides opportunities for students to explore and create new mathematical meanings.

4.2 Example 2: Functions and Calculus (Ng & Sinclair, 2018)

In this subsection, we provide another example of 3D pens being integrated in the teaching and learning of functions and calculus. There were several aspects of 3D drawing that were significant to the lesson design. First, the "ink" of a 3D pen is extruded continuously, which makes 3D drawing into a process of continuous construction. Such continuous construction facilitates the understanding of functions as processes and objects. As we know, when students draw functions with paper and pencil, it is simply a drawing process that does not create an artifact. However, when they draw with 3D pens, students can produce physical objects. They can pick up a 3D drawing object (the graph of the function) and manipulate (translate and reflect on) it to touch and feel the function. Our observations show that, linguistically, student always say nouns "slope", "point" and "tangent line" as a singular which is different from the plural in the textbook. This indicates

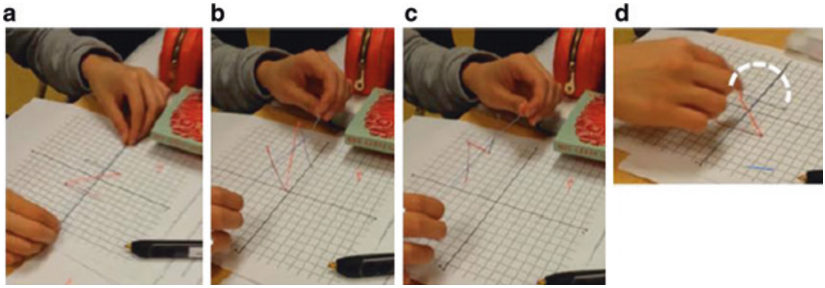

Fig. 7 (a–c) Picking up the drawn graph and rotating the axis to visualize the solid formed; (d) gesturing a semi-circle above the diagram (Ng & Sinclair, 2018, p. 307)

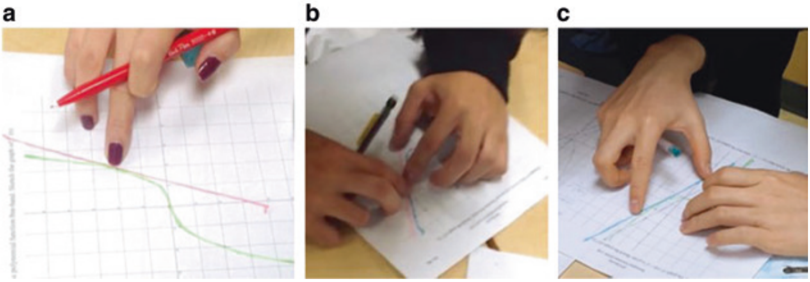

Fig. 8 (a) Physically feeling tangent to a curve with one or (b) two fingers. (c) Gesture-diagram interaction facilitated by using two fingers (or two points) to anchor the line tangent to the function, i.e., making a gesture expressing the slope of tangent on the diagram (Ng & Sinclair, 2018, p. 309)

that students considered the tangent line as a continuously moving object along the function graph rather than a discrete change of “slopes”. When asked by the teacher to draw “solids of revolution”, students leveraged the affordance of 3D drawing (as both a diagram and a manipulative) to draw a curve along with the coordinate axes and then spinning the x-axis (Fig. 7), thereby improving their visualization of how the solid is formed.

In the lesson on derivative functions, we designed a task in such a way that drawing with a 3D pen would offer new gestural forms of thinking. More specifically, when learning about slope of lines tangent to a curve, students used their fingers to push drawn 3D “lines” to be tangent to a curve, that is, to embody the tangent line (Fig. 8). The study showed that 3D drawing enables creating a physical instantiation of mathematical ideas, which can make students physically "feel" the idea of local linearity and the point of tangency via the sense of touch. As shown in Fig. 8, this was evident in students' gesture-diagram interaction which included students using their fingers to push the tangent line toward the curve and try to re-orient the tangent line to make it locally linear or "parallel" to the curve (Fig. 8a–c). Thus, 3D drawing enables what Tall (2003) referred to as the embodied mode of thinking about tangents as the "changing slope of the graph" and the idea of local linearity (p. 10). To restate, we can see that in 3D drawing environments, students could draw mathematical objects and manipulate these objects to construct meaning.

5 Empirical Studies with 3D Pens

Over the past five years, the research team led by the first author has conducted quantitative and qualitative studies on the role of 3D pens in mathematics learning. Adopting different research methods has allowed us to better investigate the potential benefits of using 3D pens in mathematics education; some of these benefits are illustrated in this chapter.

5.1 Study 1: Ng et al. (2020)

We adopted a quasi-experimental research design to investigate the differences in geometry learning outcomes between a dynamic geometry environment (DGE) group (which used DGE technology for instruction) and a 3D-pen group (which used 3D pens for instruction) (Fig. 9). The participating students were in the sixth grade from two primary schools. The DGE group contained 65 students from School A, while the 3D-pen group contained 101 students from School B. The learning topic was "measure, shape, and space" and was meant to help students explore 3D shapes and understand the relationship between the number of sides of the base, the edges, and the vertices of prisms and pyramids.

The teaching intervention took place over two 70-min lessons, and the two groups shared an almost identical lesson procedure except for the technology used in the class. Every student pair in the 3D-pen group used one 3D pen to draw different prisms and pyramids, while in the DGE group, students used a pre-made DGE to explore the properties of prisms and pyramids.

To compare the differences between the DGE and the 3D-pen group, different tests were used to assess students' learning outcomes, including pre-tests (T0),

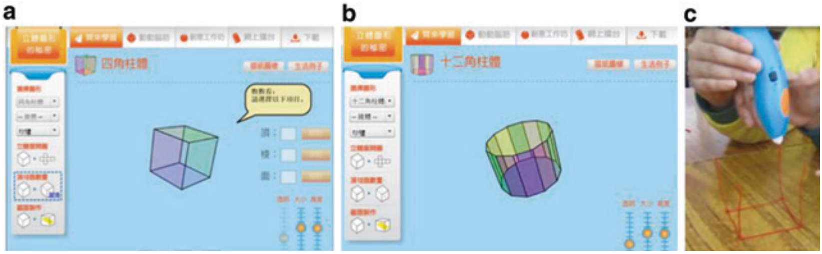

Fig. 9 (a–b) A computer applet that performs virtual transformations of various 3D figures; (c) constructing a physical artifact (i.e., a cube) with a 3D pen (Ng et al., 2020, p. 3)

post-tests (T1), and delayed post-tests (T2). The three tests used the same types of questions (Table 1) but had different assessment purposes. Pre-tests took place before the teaching experiment started, with the aim of assessing students' prior knowledge. Post-tests took place immediately after the interventions to assess students' learning outcomes, while the delayed post-tests, meant to assess students' knowledge retention, took place five months after the interventions. Students did not study the target concepts and relevant topics during this five-month period.

Using quantitative analysis, we obtained some findings on the effects of these two teaching interventions. First, the post-test indicated that after both technology-enhanced interventions, all students received a higher score in all (sub-)categories. Moreover, a higher improvement rate was obtained from the DGE group compared to the 3D-pen group regarding "vertices," "edges," "advanced," and "advanced questions on vertices (AV)." What's more, the delayed post-test showed that, compared to the DGE intervention, the 3D-pen intervention had a greater retention effect, particularly in relation to "vertices," "advanced," and

Table 1 A summary of the types of questions used in student assessments (Ng et al., 2020, p. 6)

Category	Sub-category	Number of items	Sample question	Total items
Simple: Determining the number of faces, vertices, or edges in a given prism or pyramid	Faces (SF)	5	How many faces does an 8-gonal pyramid have?	15
	Vertices (SV)	5	How many vertices does an 11-gonal prism have?	
	Edges (SE)	5	How many edges does a 9-gonal pyramid have?	
Advanced: Working backward given the number of faces, vertices, or edges in a prism or pyramid	Faces (AF)	4	A solid has 14 faces. This solid could be a _____-gonal prism or_____-gonal pyramid.	12
	Vertices (AV)	4	A solid has 9 vertices. This solid could be a _____-gonal prism or_____-gonal pyramid.	
	Edges (AE)	4	A solid has 18 edges. This solid could be a _____-gonal prism or_____-gonal pyramid	
				27

“AV” questions. Finally, we found the unexpected result that the 3D-pen group’s T2 scores were consistently higher than the T1 scores, and that five months after the teaching interventions, the T2 scores did not differ significantly in all the subcategories between the DGE and the 3D-pen group. This implies that over the long term, there was no significant difference in students’ geometric thinking levels between the two groups.

According to the results of Study 1, it can be concluded that embodied interactions with 3D pens have a positive and sustained effect on geometry learning. From the persepctive of embodied cognition, the results can be explained by the consideration of the pen-hand movement as “concrete gestures that preserve the embodied nature of the interaction as found in physical manipulation” (Ng et al., 2020, p. 11). Corroborating with previous studies on gestures and memory, the results are consistent in showing the effects of gesture in promoting long-lasting learning (e.g., Cook et al., 2008). Moreover, the 3D Pen environment could make a much stronger “connection between mathematical and pedagogic dynamisms” (Jackiw & Sinclair, 2009, p. 418) due to the direct, hands-on interaction of mathematical representations as opposed to other tool-mediated devices such as using a 2D mouse to navigate a 3D scene. This study also raises the following question: *What role can 3D pens play in students’ mathematical meaning construction?* There seems to be abundant room for future research regarding the benefit of learning mathematics using 3D pens; consequently, we adopted qualitative methods to answer the aforementioned question which we elaborate in Study 2.

5.2 Study 2: Ng and Ye 2022

In Study 2, we further explored how students’ linguistic expressions and gestures produced while using 3D pens to construct 3D solids supported the students’ thinking and mathematical-meaning construction when studying the properties of prisms and pyramids.

(1) Constructions with 3D pens Supported Students’ Composition of 3D Solids by Their 0D, 1D, and 2D Parts, as well as Improved Students’ Visualization of the Relationships of these Parts in Embodied Ways.

Regarding the study of 3D solids using 3D pens, we asked a student who had completed a rectangular prism construction using a 3D pen how she might construct a pentagonal prism. When asked to count the number of “straight lines” that were required to complete the drawing, she responded “13” and was encouraged to anticipate the process of drawing a pentagonal prism using her finger

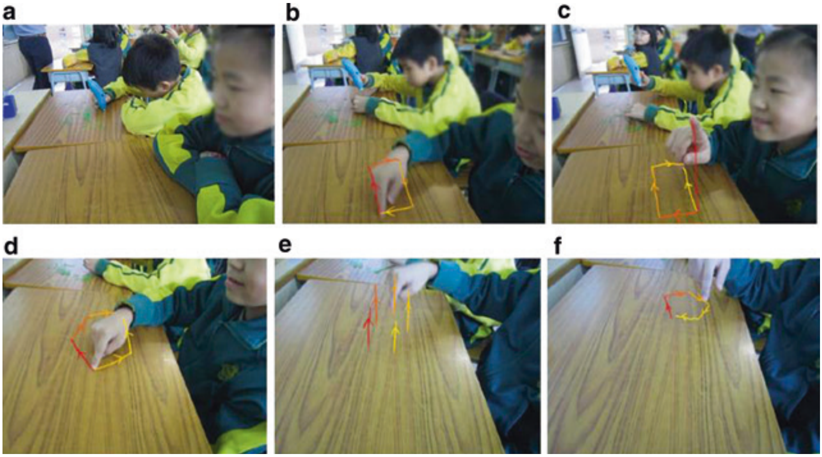

Fig. 10 (a–f) Snapshots of a series of gestures by a student (Ng & Ye, 2022)

(Fig. 10). At this point, she began to describe the drawing process while using her fingers on the table and in the air, imitating the method and process of drawing with the 3D pen previously. Moreover, the student used mathematical terms, such as base and sides, to describe her construction method and process, which indicates that the student thought of 3D solids using both 2D and 1D perspectives. Therefore, based on the linguistic expressions and gestures of students who use 3D pens to construct 3D solids, we can conclude that 3D pens can help students understand and reconstruct 3D solids from a low-dimensional level (0D, 1D, and 2D) and that such a visual construction process helps students understand the relationships between vertices (0D), edges (1D), and faces (2D).

(2) Constructions with 3D Printing Pens Gave Rise to Gestures Conducive to Learning the Properties of 3D Solids.

At the same time, drawing with 3D pens gave rise to gestures with geometrical meanings. When a student constructed 3D solids with a 3D pen, the hand holding the 3D pen moved constantly according to the shape of the constructed solid. Let us say that we are making cubes: students first need to draw a square with four congruent sides. The process of moving one's hand when drawing the square base reveals a property of squares: all sides must be of equal length, and they have to be perpendicular to one another. Or, when a student draws pyramids and prims with parallel or vertical lines and planes, the hand movement changes depending

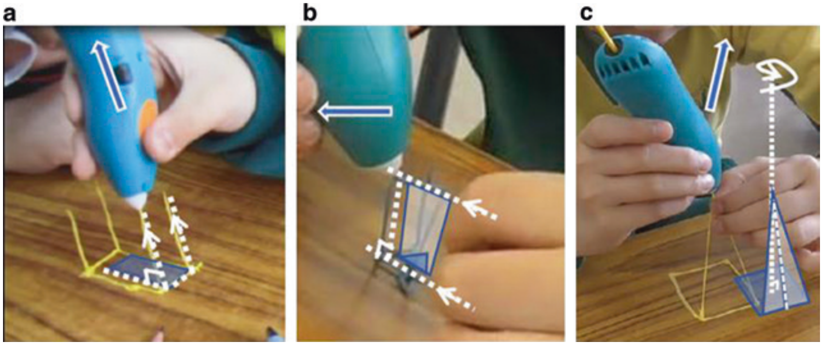

Fig. 11 Hand movements conveyed meanings, such as (a–b) the perpendicularity and parallelism between lines and/or planes and (c) reflectional and rotational symmetry (Ng & Ye, 2022)

on the "lines" one is drawing relative to the shape. As illustrated in Fig. 11, the process of constructing prisms requires students to observe parallel or perpendicular relationships between lines or lines and planes when moving their hands to shape the target object (Fig. 11a–b). In addition, when drawing a square-based pyramid, students also need to pay attention to the reflectional and rotational symmetry pertained in the 3D solids. In embodied cognitive terms, these particular gestures express the geometrical meanings behind the objects being constructed. The gestures generated by such embodied activity strengthens students' visualization of 3D solids, as they actively using and constructing geometric relationships and properties as opposed to simply viewing with a 2D representation of the solids.

6 Conclusion and Future Direction

We conclude this chapter by discussing the role of 3D pen as a form of 3D printing and presenting a four-fold characterization of making in mathematics education (Ng & Ferrara, 2020). Building on a previously developed notion of "learning as making", we were compelled to conceptualize the tool of 3D pen as a highly transformative and constructionist environment for supporting mathematical thinking and learning. As a result of our five years of research, we were able to identify some characteristics of these constructionist practices with 3D pens which we think has implications for tool-based in mathematics teaching and learning.

1. Making Involves Co-Constructing Meanings

Due to the characteristics of 3D pens (speed and hardness, etc.), students coordinate their hands and eyes when constructing 3D solids and have to consider the features of 3D pens to successfully construct mathematical objects. The materials, tasks, students' actions, and the final artifacts interact in the meaning-making process in a co-constructive way. As argued by the theory of inclusive materialism, if one of these elements were to be changed, the final artifact would change as well. Aligned with constructionism, students use 3D pen actively to construct a personally meaningful artefacts while constructing mental schema about the process, upon which the artefact can be shared with others for peer learning. Therefore, making is a process of co-constructing mathematical meaning.

2. Making Is Mathematizing

In the process of making, students' gestures, artifacts, mathematical concepts are often co-implicated. When a student constructs an artefact with 3D pens, mathematical meanings (such as segments, planes, bases, sides, triangles, etc.) are also generated during the making process. Importantly, these artefacts are not external representations of their "mind"; rather, based on the embodied cognition, the students' mathematical thinking and bodily movements are co-constituted while making. Hence, making is a process of creating something new externally (in a physical sense) and mathematically (in a cognitive sense), i.e., making is mathematizing, which deepens one's mathematical knowledge. In addition, working with 3D printing involves a process of mathematical experimentation. Students are not required to make prisms of a certain size and to follow specific construction methods. Instead, students are free to explore and develop their own construction processes. Therefore, students are actually thinking and doing mathematics, or mathematizing, while making.

3. Making Is Assembling with Technology

From the materialist perspective, students engage in gesture-diagram interaction that is unique to the tool of 3D pens and, at the same time, engage in considering the mathematical properties of the objects. In this process, a "making assemblage" focusing on the evolving relationship among human, material, and mathematics is formed. That is, "mathematics was not some abstract concepts to be conceived […] but it emerged as an assemblage with technology from the students' drawing and gesturing hands" (Ng & Ferrara, 2020, p. 941).

4. Making Is Inventing

In the cross-section lesson, students came up with creative ideas when using 3D pens to explore the cross-sections of 3D objects. After completing the construction of the first cross-section with a 3D pen, students gained an initial visualization of the cross-section. Specifically, they knew they could trace the outline of the anticipated cross-sections with a 3D pen and could detach them from the solids to inquire the shape of the cross-section (e.g., a rectangle in the case of a perpendicular cut to the base of a cylinder, a circle in the case of a horizontal cut to base of a cylinder). Afterwards, a student can directly create a new cross-section by considering the position of the cross-section based on past experience of constructing the cross-section. In particular, one student adjusted the height of a pyramid's cross-section to get a trapezoid of a different size (Fig. 12). Another student used his gestures to describe the process of generating the cross-section of a cylinder, which is also an invention of an action (Fig. 13). In both cases, the students came up with new ways of thinking about cross-sections, which are generally difficult for students to visualize. These cases show that making is a process that enables exploring, engaging with, and inventing mathematics.

Overall, our research has been one of the first attempts to examine the role and use of 3D pen as a form of 3D printing in mathematics education. Our empirical investigations point to the potential changes in thinking, learning, and doing that may result from the use of 3D pens, which enable mathematics to be performed

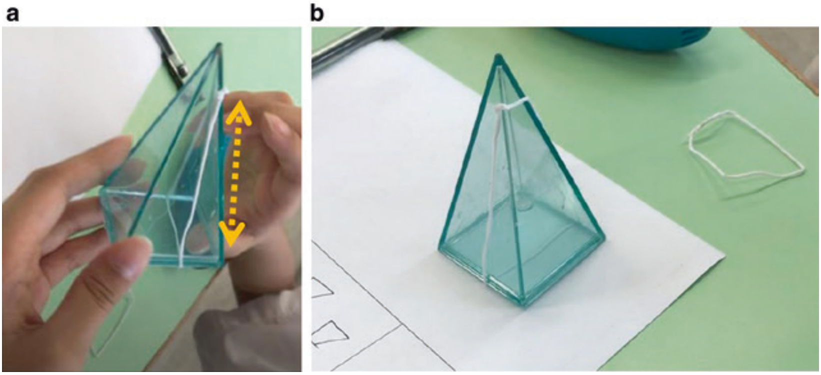

Fig. 12 (a) A student's gestures for highlighting the height of his artifact; (b) two artifacts in the shape of trapezoids were constructed at the end of the task (Ng & Ferrara, 2020, p. 941)

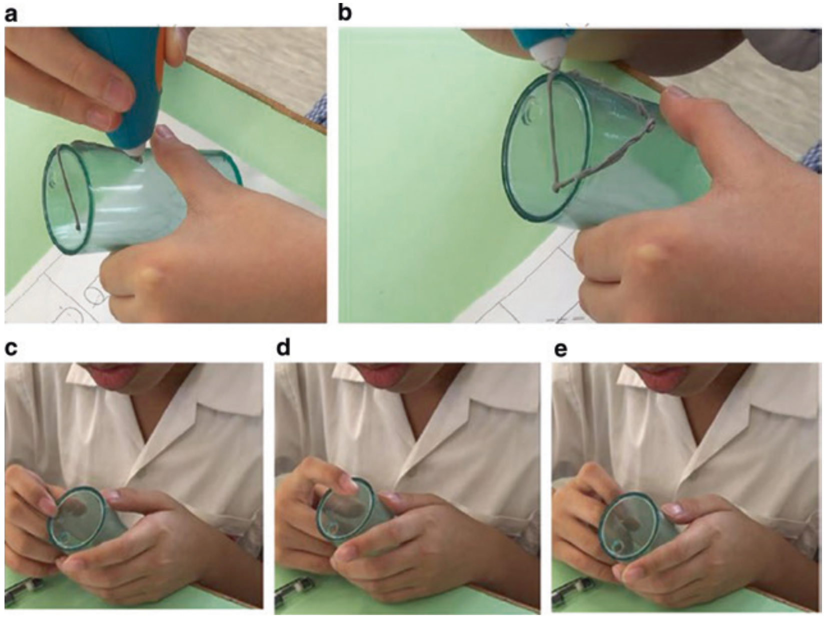

Fig. 13 (a–b) Constructing the outline of a cross-section with an oblique cut to the cylinder; (c–e) a student deciding to move his index finger along the surface of a cylinder to visualize a cross-section with an oblique cut to the cylinder (Ng & Ferrara, 2020, p. 940).

in the third dimension and helping to make certain mathematical concepts tangible through touch and moving one's hands. Our future direction includes conceptualizing constructionist learning from the perspective of realistic mathematics education (Van den Heuvel-Panhuizen & Drijvers, 2020), which views doing mathematics as a human activity and connected to reality. As argued by the framework of realistic mathematics education, much opportunities should be given to children to reinvent mathematics through hands-on and informal experiences, which is in line with our developed conception of 'learning as making' to some extent. Therefore, we encourage future research to consider how students make connections and reinvent meanings of 3D-printed (mathematical) object, such as a 3D printed triangle, in the real-world contexts, and how their matematical discourse *evolve* before and after their interactions with the 3D printed models. This line of research should contribute toward providing a basis for further research into 3D printing and mathematical cognition from the perspective of realistic mathematics education.

Funding Acknowledgement
This book chapter is fully supported by Research Grants Council, Early Career Scheme (RGC Ref No. 24615919).

References

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247–286.

Châtelet, G. (2000). *Figuring space: Philosophy, mathematics, and physics*. Kluwer.

Cook, S. W., Mitchell, Z., & Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition, 106,* 1047–1058.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.

Ehrlich, S. B., Levine, S. C., & Goldin-Meadow, S. (2006). The importance of gesture in children's spatial reasoning. *Developmental Psychology, 42*(6), 1259–1268.

Hegedus, S., & Tall, D. (2016). Foundations for the future: The potential of multimodal technologies for learning mathematics. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 543–562). Routledge.

Jackiw, N., & Sinclair, N. (2009). Sounds and pictures: dynamism and dualism in dynamic geometry. *ZDM: The International Journal of Mathematics Education, 41*(4), 413–426.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372–415.

Ng, O., & Ferrara, F. (2020). Towards a materialist vision of 'Learning as Making': The case of 3D printing pens in school mathematics. *International Journal of Science and Mathematics Education, 18*(5), 925–944. https://doi.org/10.1007/s10763-019-10000-9.

Ng, O., & Sinclair, N. (2018). Drawing in space: Doing mathematics with 3D pens. In *Uses of technology in primary and secondary mathematics education* (pp. 301–313). Springer. https://doi.org/10.1007/978-3-319-76575-4_16.

Ng, O., & Tsang, W. K. (2021). Constructionist learning in school mathematics: Implications for education in the fourth industrial revolution. *ECNU Review of Education*. https://doi.org/10.1177/2096531120978414.

Ng, O., & Ye, H. (2022). Mathematics learning as embodied making: Primary students' investigation of three-dimensional geometry with handheld 3D printing technology. *Asia Pacific Education Review*. https://doi.org/10.1007/s12564-022-09755-8.

Ng, O., Shi, L., & Ting, F. (2020). Exploring differences in primary students' geometry learning outcomes in two technology-enhanced environments: Dynamic geometry and

3D printing. *International Journal of STEM Education, 7*(1). https://doi.org/10.1186/s40594-020-00244-1.

Ng, O., Sinclair, N., & Davis, B. (2018). Drawing off the page: How new 3D technologies provide insight into cognitive and pedagogical assumptions about mathematics. *The Mathematics Enthusiast, 15*(3), 563–578. https://scholarworks.umt.edu/tme/vol15/iss3/14.

Novack, M. A., Congdon, E. L., Hemani-Lopez, N., & Goldin-Meadow, S. (2014). From action to abstraction: Using the hands to learn math. *Psychological Science, 25*(4), 903–910.

Núñez, R. E., Edwards, L. D., & Filipe Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*(1), 45–65.

Papert, S., & Harel, I. (1991). Situating constructionism. In S. Papert & I. Harel (Eds.), *Constructionism* (pp. 1–11). Ablex.

Penner-Wilger, M. (2013). Symbolic and non-symbolic distance effects in number comparison and ordinality tasks. *Canadian Journal of Experimental Psychology, 67*(4), 281–282.

Tall, D. (2003). Using technology to support an embodied approach to learning concepts in mathematics. In L. M. Carvalho & L. C. Guimarães (Eds.), *Prooceedings of the First Coloquio de Historia e Tecnologia no Ensino de Matemática at Universidade do Estado do Rio De Janiero*. https://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2003a-rio-plenary.pdf.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 713–717). Springer.

Weisberg, S. M., & Newcombe, N. S. (2017). Embodied cognition and STEM learning: Overview of a topical collection in CR: PI. *Cognitive Research: Principles and Implications, 2*(1), 1–6.