MINTUS – Beiträge zur mathematisch-naturwissenschaftlichen Bildung

RESEARCH

Frederik Dilling · Felicitas Pielsticker · Ingo Witzke *Editors*

Learning Mathematics in the Context of 3D Printing

Proceedings of the International Symposium on 3D Printing in Mathematics Education

Springer Spektrum

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3D Printing in Mathematics Education—A Brief Introduction

Frederik Dilling, Felicitas Pielsticker and Ingo Witzke

1 Motivation

3D printing currently excites many users, perhaps because the technology is also used in many areas (manufacturing, medicine and health care, food, fashion, transportation, etc.). With the help of 3D printing, one's ideas can be turned into objects that can be held. A 3D printer creates solid, 3D objects. Many 3D printers also offer a certain ease of use. Although 3D printing technology may seem complicated at first glance, the application is surprisingly simple to learn. These reasons could all contribute to 3D printing technology, while still a very new medium, being successfully used in education. Technology and digital media play major roles in human social life, today as never before. This is also accompanied by some changes of varying scope. Especially for teaching and learning mathematics, 3D printing technology has been discovered. 3D

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printers can be used in various mathematical fields, not only in geometry, the obvious application, but also in learning environments for algebra, analysis, arithmetic, and probability. The diversity of uses for 3D printing in mathematics was particularly evident at the last International Symposium on 3D Printing in Mathematics Education, September 16–17, 2021.

This volume presents a synopsis of the symposium's findings. It reflects the discussions and exchanges that occurred and is, itself, part of the negotiation process. This volume is also intended to contribute to the current 3D printing discussion in the field of mathematical education.

This introduction outlines the subject matter. The reader will be briefly introduced to the functionality and implementation of 3D printing technology. Thus, a reader can use this introduction to learn about the basics of 3D printing technology, as an understanding of these is often assumed in the included articles. Therefore, this introduction will first deal with the technical basics of 3D printing in section 2. An educational theoretical framing will be offered in section 3, "3D Printing in Mathematics Education." In the fourth section, some central aspects of the individual contributions are discussed to provide a quick overview of their contents.

2 The Technical Basics of 3D Printing

3D printing technology includes not only the 3D printer as the hardware to produce objects. It also includes software for creating objects and controlling the printing process. If students are to use 3D printing technology, they should be able to handle both components. Learning how to use the technology should not only be part of computer science or technology lessons but also be explicitly discussed in mathematics lessons. Thus, technical competencies can be directly linked to the other content of mathematics lessons. This method ensures that the possibilities and limitations of the technology can be learned (Dilling & Witzke, 2019).

The following steps of the 3D printing development process can be distinguished (see Fig. 1). First, an object to be developed should be planned carefully—this can avert many problems in later stages. After planning, a 3D model is created with computer-aided design (CAD) software. Various design methods can be used and are implemented with different software types. The appropriate software is determined by both the object to be made and the level of instruction. The designed 3D model is exported as a standard triangulation language (STL) file and can then be prepared for printing with slicer software. When

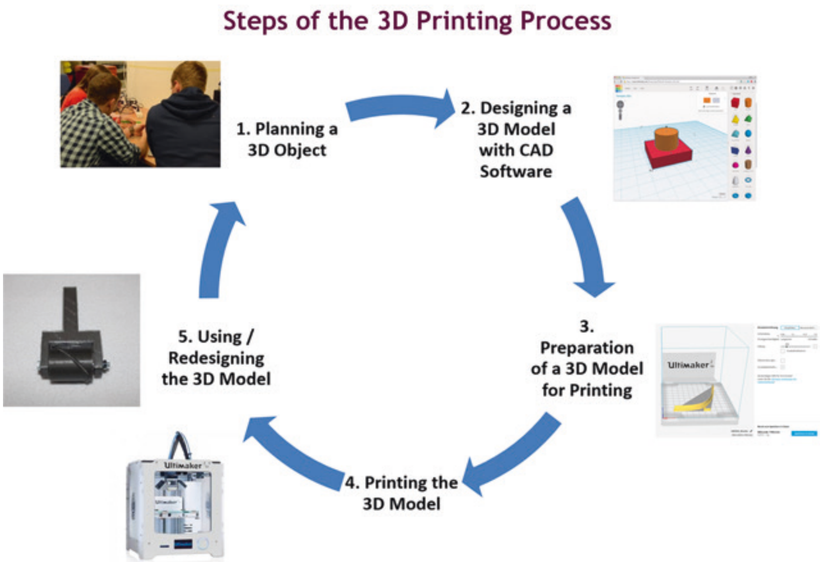

Fig. 1 Steps of the 3D printing development process

slicing a 3D object, various parameters, including the size and quality, are set for a specific printer. The data are then exported as a G-Code file, making them readable to the 3D printer. After the object is printed, the adhesive and support materials are removed and it is usable in the classroom or can be revised if the result is unsatisfactory.

2.1 Computer-Aided Design—Building 3D Models

Computer-aided design (CAD) software is a type of computer software that can be used to create virtual three-dimensional models. CAD software can be divided into types, although sharp distinctions are not always possible (Dilling & Witzke, 2019):

- *Direct modeling* allows the creation of objects by combining basic solids.
- *Parametric modeling* is based on two-dimensional sketches that are transformed into three-dimensional objects by extrusion.
- In *script-based modeling*, the objects are defined by coding or coding blocks.

- Some *mathematical tools*, like *GeoGebra*, have a 3D printing export option.
- *Sculpting* simulates modeling with materials like clay rather than scaled constructions.

The different types of CAD modeling have specific advantages and disadvantages and thus offer a variety of application perspectives for mathematics teaching. Programs should be selected based on the learning group's needs as well as the mathematical content. Simple objects can be created intuitively and quickly with direct modeling (e.g., dice; Pielsticker, 2019). However, if content that focuses on two-dimensional drawing is to be used, programs that offer parametric modeling or mathematical tools should be chosen (e.g., tessellation; Dilling & Witzke, 2020a). Script-based modeling can be used when algorithms are prioritized (e.g., a brick generator; Dilling & Vogler, 2022) or to code small applications for use in the classroom (e.g., "Graphendrucker"; Dilling & Struve, 2019). Sculpting is rarely relevant to mathematics education but can be used, for example, in the arts. Direct and parametric modeling will be briefly explained because they are the classical CAD modeling methods.

2.1.1 Direct Modeling

Direct modeling is a process in which 3D objects are created based on basic 3D solids. This procedure will be explained with the widely used program *Tinkercad*™ from the company Autodesk® as an example. It is a free, browser-based application. Thus, installation is unnecessary and it works on any operating system. Furthermore, a class account can be set up, so that teachers will have access to all of their students' work. Figure 2 shows a screenshot of the software.

The basic solids of which the object is to be composed can be dragged and dropped onto the workspace. A variety of simple geometric solids (cuboid, cylinder, sphere, cone, pyramid, etc.), as well as more complex solids (icosahedron, paraboloid, torus, etc.), are available. In addition, editable text blocks and numbers as well as shapes that are less relevant for mathematical purposes can be selected (a hand, glasses, etc.). Tinkercad™ also offers shape generators that produce objects that can be modified according to predefined aspects. For example, function graphs of functions with two variables can be inserted. To do this, the designer would select a relevant generator, enter a function equation, and adjust the size of the intervals with a slider. Shape generators can also be coded for Tinkercad™ with JavaScript and distributed online.

The selected solids can be moved on the working plane (x–y plane). By dragging an arrow, they can also be moved along the z-axis. Objects can be precisely located with the ruler function. This can be fixed at any point in the

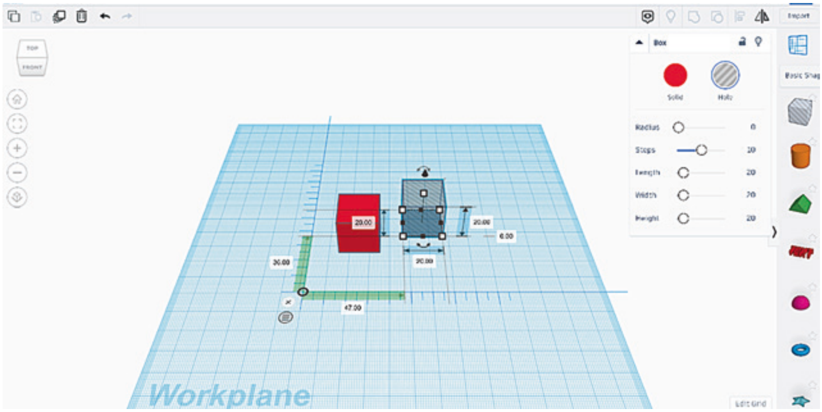

Fig. 2 Screenshot of Tinkercad™ by Autodesk®

plane and thus determine the zero point of a coordinate system. The distance of a solid from the zero point in the x-, y-, and z-dimensions is then indicated with numerical values and can be changed by adjusting these values. Additional working planes can be added on the surfaces of solids.

The shape and size of the selected basic solids can be changed by adjusting their parameters (length, width, height, and radius). This can be done by dragging the corners and edges of the solids, entering numerical values, or operating sliders. For curvilinear solids (cylinders, spheres, etc.), the resolution, i.e., the quality of the approximation using triangular surfaces, can also be manipulated with a slider.

Independent of shape and size, a solid can be set as "color" or "hole". This setting is decisive when solids are connected with the "group" function. If two "color" solids are grouped, the union of both solids is formed. A "hole" solid, in contrast, is cut out of a "color" solid when they are grouped; this is described in terms of set theory as a difference. The connection between objects can also be reset with the "ungroup" function, allowing the objects to be edited individually again.

The view can be changed during the construction process by pressing and holding the right mouse button while moving the mouse or by rotating a cube in the upper-left corner of the display. The mouse wheel can be used to zoom in and out.

The direct modeling method is rather intuitive because changes to the solid are immediately visible. This allows spontaneous changes and a more experimental approach. However, creating complex objects is time-consuming compared with other modeling methods.

2.1.2 Parametric Modeling

Parametric modeling is based on the creation of two-dimensional sketches in a chosen sketch plane. This description will use *Fusion360* software from the company Autodesk® as an example, as it is widely used in the 3D printing sector and can be used free of charge by educational institutions. A screenshot of the software is shown in Fig. 3.

At the beginning of the design process, a plane is selected in which to create the two-dimensional sketch. This can be any one of the xy, xz, or yz planes or a specially defined plane. The view is then automatically set perpendicular to the selected plane with the origin centered in the image, so the 2D sketch is displayed undistorted. However, the view can be changed during the design process by dragging a cube located in the upper-right corner. The mouse wheel can be used to zoom in and out. Pressing and holding the mouse wheel while moving the mouse moves the view within the sketch plane.

Two-dimensional figures (rectangles, circles, ellipses, etc.) or lines (lines, arcs, splines, offset curves, etc.) can be inserted into the sketch plane and

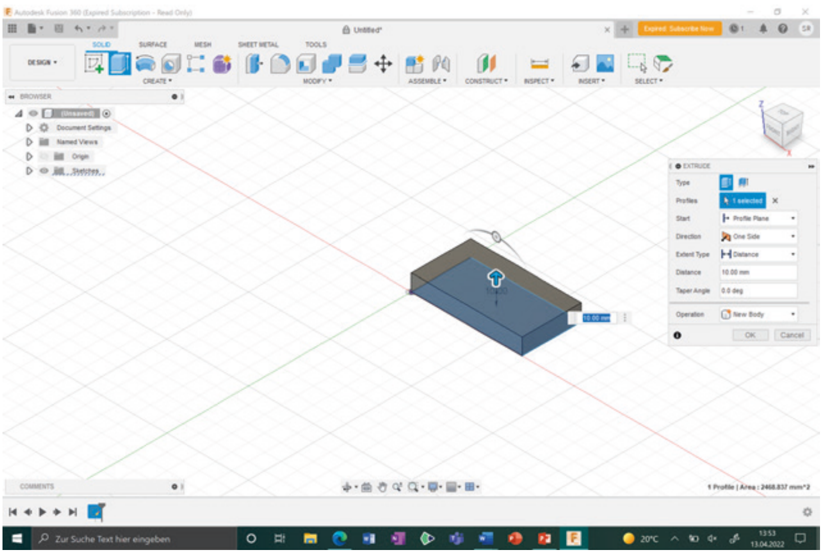

Fig. 3 Screenshot of Fusion360 by Autodesk®

positioned with the mouse. Some parametric programs (e.g., *Inventor*, also by Autodesk®) also permit the insertion of parametric curves, which can be useful in calculus education. By inserting relationships between lines or points (parallel, perpendicular, collinear, etc.) and measurements (distances, angles, etc.), the positions and other parameters of the elements can be defined.

A surface drawn in a 2D sketch can be transformed into a solid in a process called "extrusion." Different types of extrusion can be selected for this purpose. Linear extrusion ("extrusion") extends the object perpendicular to the sketch plane. Rotating extrusion ("revolve") creates a solid of revolution from the sketch when a rotation axis is selected. The designer can also choose to extrude along a path defined by a sketch ("sweeping") or converge to a defined point ("loft"). During extrusion, further adjustments can be made; for example, the distance, angle, etc. of the extrusion can be changed. Like the "color" and "hole" labels in Tinkercad™, extrusions can add a solid to the building space or cut a corresponding void from an existing solid.

On the surface of the created solid or in another defined plane, a new sketch can be created for further work. There are also various functions for directly modifying solids, such as functions to round or bevel edges. Certain parametric programs also allow the creation of 3D sketches, such as sketches with space curves or with simple bodies inserted directly.

The term "parametric modeling" originates from the parameters used to uniquely define the objects, which can be changed after modeling. The object does not have to be redrawn but rather adapts automatically, due to the inserted dependencies. This allows subsequent corrections to be implemented easily.

Simple and complex objects can be created in a few steps with a parametric modeling program. The range of functionality far exceeds that of a direct modeling program. However, it is also less intuitive, so some familiarization and a well-planned approach are particularly important. Compared with direct modeling, this demands more of the user's spatial imagination.

2.2 Slicing—Preparation for Printing

Slicing refers to transferring a 3D model created with CAD software into the control commands of the printer. The name comes from the layered structure of the 3D object in the printer and the specification of control commands for each layer—the software "slices" the object in thin parallel layers. The following explanation refers to the open-source software *Ultimaker Cura* (see screenshot in

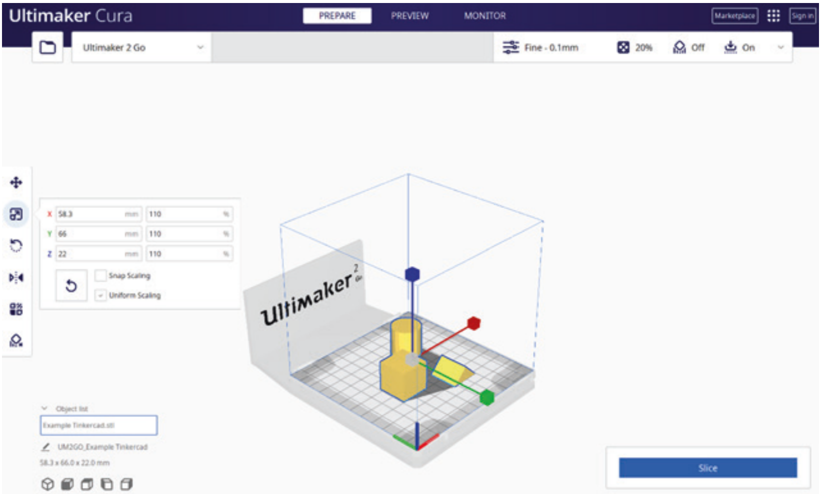

Fig. 4 Screenshot of Ultimaker Cura

Fig. 4), which can be used to control a variety of 3D printers. These instructions can be transferred almost completely to other slicer software.

The standard triangulation language (STL) format serves as the interface between the 3D CAD model and the slicer program. It approximates the model with triangular surfaces. Other export files (e.g., OBJ format) can also be read by most slicer programs.

In the slicer software, the constructed 3D object can be positioned in a virtual 3D printer. The software automatically adjusts the size of the build space when the user selects a 3D printer. The object can be moved parallel to the x-, y-, or z-axis, scaled along one of the axes, rotated around an axis, or mirrored on a plane. The settings can be adjusted either by dragging the object with the mouse or by entering numerical values. Several objects can be imported into the slicer software one after another and aligned in the virtual printer to be printed in a single pass.

Further settings can be defined before printing. The layer thickness and printing speed can be varied; these affect both the quality of the 3D-printed objects and the required printing time. The infill of the objects can be set between 0 and 100%. The 3D printer will fill the object with honeycomb structures accordingly. The desired wall thickness can also be set.

Depending on the object structure a support structure or printing plate adhesion may also be useful. The support structure ensures that overhanging parts of an object can be printed. Plate adhesion (called a "brim") prevents the object from detaching or deforming (warping) during the printing process. Plate adhesion, which is achieved by printing a thin layer around the object, is recommended for objects that are particularly tall and narrow, as well as particularly broad objects.

The slicer software outputs a G-Code file that contains the control instructions for the 3D printer. It consists of the coordinates of the points to be controlled in sequence and the amount of material to be extruded.

2.3 3D Printer—Manufacturing Objects

3D printing encompasses a range of special processes and is also known as additive manufacturing. In 3D printing, material is successively assembled into a three-dimensional object. This contrasts with subtractive manufacturing processes, in which parts that do not belong to an object are removed from the material (e.g., milling). Among hobbyists and in the educational sector, fused deposition modeling (FDM, also known as fused filament fabrication or FFF) is the most widespread 3D printing method.

In FDM, the material, called filament, is rolled up on a spool and pulled in gradually by the printer. A transport unit consisting of two opposing small wheels is connected to a stepper motor for this process. The transport unit presses the filament into the extruder, which consists of a heating block that maintains a fixed temperature to melt the filament as it passes through and a nozzle through which the liquefied filament is forced. The standard nozzle diameter is 0.4 mm, but smaller or larger nozzles can be installed. The liquefied filament is then applied to the print bed in layers. The nozzle moves up by a predetermined distance after each layer. Some printers' beds can be heated to increase adhesion and, thus, print quality. A schematic of the FDM process and a photo of a 3D printer are shown in Fig. 5.

A standard FDM printer has only one extruder, so it can print with only one color per print pass. Multicolor FDM printing or printing with different materials is possible with a second extruder, a mechanism for automatically exchanging the filament during printing, or by automatically coloring the filament after each layer is printed.

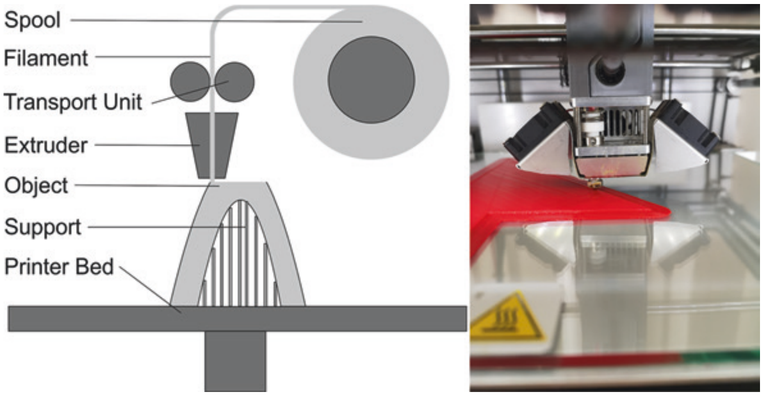

Fig. 5 Schematic representation of the FDM method and photo of a 3D printer

Various plastics and plastic mixtures are available as standard-sized 1.75 mm and 2.85 mm diameter filaments wound on spools for the FDM process. ABS (acrylonitrile butadiene styrene) is the material most commonly used in industry (e.g., to produce toys). It is comparatively robust, but hazardous vapors are produced during the printing process, so this material should only be used with closed printers or fume hoods. In addition, it requires a heated printing plate. PLA (polylactic acid) is a compound of bound lactic acid molecules. It is biodegradable and food-safe, which makes it very suitable for student work. It is slightly less robust than ABS but still produces sturdy 3D objects. Many different colors and hardness levels are available—even transparent and flexible objects can be printed. Thus, for health and environmental reasons, PLA is preferable to ABS for most classroom uses. In addition to PLA and ABS, various special filaments are available, including nylon, carbon, wax, and wood.

Fused deposition modeling is the most cost-effective 3D printing process. Its advantages include the comparatively short printing time, the variety of materials available, and the robust resulting products. However, compared with other 3D printing methods, only low resolutions are possible and the dimensional accuracy is lower.

In addition to 3D printers, there are 3D printing pens. Like an FDM printer, these can melt plastics and press them out through a nozzle. The same materials (usually at a 1.75 mm diameter) can be used for this purpose. Instead of computer-controlled movement, however, 3D pens are controlled by hand, allowing the user to make three-dimensional drawings. The objects created in this way are

Fig. 6 3D printing pen

less precise than 3D-printed objects but can be produced quickly and easily in the classroom. By handling 3D pens, students can also develop a feel for how a 3D printer works, making these pens particularly suitable for introductions to the new technology. Figure 6 shows a picture of a 3D pen.

3 3D Printing in Mathematics Education

3D printing is still a relatively new digital technology for education. The technology allows teachers and students to realize a variety of (individual) manipulatives and illustrative material for and within the mathematics classroom (see the examples in Fig. 7).

If 3D printing is used appropriately, the technology provides opportunities for *problem solving*, *discovery*, and *reasoning* in mathematics education. The concept of empirical mathematics education offers a basis for the use of 3D printing and 3D-printed manipulatives (cf. Burscheid & Struve, 2020; Witzke, 2009; Pielsticker, 2020; Dilling, 2022). The key idea is that mathematics education cannot and should not be organized as a formalistic science about abstract objects but rather in relation to real (empirical) objects. Consistently orienting mathematics education on empirical objects can be described as viable and authentic for learning theoretical and scientific historical findings and surpasses the artificial separation of mathematics and reality at school. Such an understanding of mathematics makes empirical objects (e.g., printed products such as cubes, edge

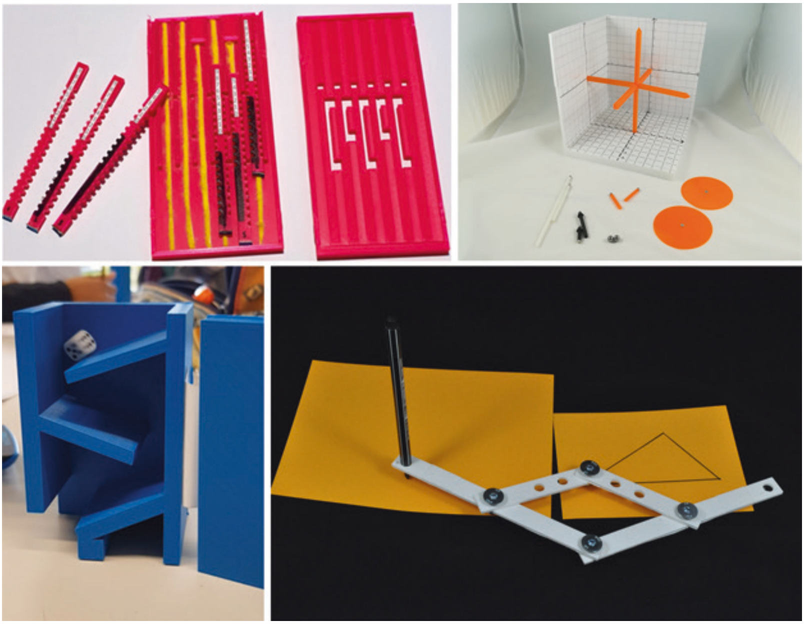

Fig. 7 Examples of 3D printed manipulatives (Clockwise from top left: A mechanical addiator or add/subtract calculator, a vector representation model, a pantograph, and a dice tower)

models, prisms, 3D graphs, etc.) essential to students' knowledge development using processes of discovery and reasoning. In this context, 3D printing technology makes the creation of such objects possible in several mathematical contexts that are suitable for experimentally discovering mathematical phenomena and can motivate and initiate logical-deductive derivations.

By dealing with different *representations* (Bruner, 1966), learners can (further) develop mathematical concepts in a constructivist sense. In this context, the learners' interactions with the representations are particularly important, i.e., performing certain actions, reading and interpreting pictures and graphic representations in a certain way, or reading and setting up algebraic symbols according to certain rules. 3D printing technology allows new representations to be added to mathematics lessons in selected situations—for example, in geometry, probability, algebra, or calculus—and thus presents new approaches for students to learn mathematical concepts.

3D printing technology can influence mathematics education in various ways. In particular, four ways to implement the technology in mathematical learning processes can be distinguished (Dilling & Witzke, 2020b; Witzke & Heitzer, 2019). First, the 3D printer can be used to *reproduce existing materials*. Many manipulatives and tools have already proven useful in the classroom independently of 3D printing technology and have been scientifically evaluated. Corresponding materials can be reconstructed with CAD software and printed with a 3D printer. Furthermore, exchange platforms for 3D models (e.g., Thingiverse, https://www.thingiverse.com/) can be searched for suitable materials, which can then be customized to meet individual needs.

Once the STL file for a desired object is available, it can be printed any number of times. This permits work in different social formats. For example, large demonstration versions of manipulatives can be produced for the teacher and smaller versions for students' individual work. Compared with handcrafted paper-based materials, 3D-printed objects are long-lived and can be made available to other teachers in a school collection or given to students for further work. The costs of 3D-printed materials are also significantly lower than the purchase price of most manipulatives that are commercially available (Dilling, 2019).

Alternatively, *teachers can develop manipulatives and tools* for use in mathematics classes. While teachers could previously develop three-dimensional manipulatives only with many restrictions and great effort, 3D printing technology makes their creation particularly simple and fast. The design possibilities are almost unlimited. Thus, materials can be adapted to the needs of a learning group or a particular student. Thus, the development of materials is not left solely to manipulatives publishers but can be done by anyone interested in the further development of teaching. This leads to a wide variety of manipulatives and tools, which reflects the heterogeneity of students. The teacher can involve students in the development process at any point. Thus, even printing objects in class or assembling printed individual objects can contribute to their knowledge development and prevent the objects from being perceived as black boxes.

Probably the greatest transformation of learning processes is achieved when *students develop suitable materials in mathematics lessons*. This can happen either during the introduction of a new topic or during practice phases when applying learned knowledge. The tasks should be sufficiently defined so that the work is done on the mathematical content, but open-ended enough for students to contribute their ideas and understandings when designing objects. This allows a variety of very different objects to emerge, which can lead to interesting discussions and negotiations among the students (Pielsticker, 2020). Additionally, the teacher can review individual learners' levels of knowledge and perceptions.

The structure of teaching is fundamentally changed by the use of 3D printing technology in the classroom. Instead of fixed sequential steps for all students towards a predefined learning goal, learners can work together on a project-oriented task over a long period. By going through the entire development process, from planning, to design with a CAD program and printing with a 3D printer, to the use or revision of the object, the focus is essentially on examining the mathematical topic at different levels. Identifying and solving problems and challenges during development also strengthens students' problem-solving and collaboration skills. There is a natural shift from teacher-centered to student-centered instruction, where students take responsibility for their own learning processes.

Finally, *3D printing technology is also an interesting artifact* whose study in the classroom can reveal applications of mathematical concepts. Many mathematical concepts are applied in 3D printing technology and can underpin mathematical investigations of the devices (see Dilling & Witzke, 2020b).

Among the various components of 3D printing technology, the distinct positioning of objects with coordinates plays a decisive role. The build space of the CAD software is a parallel projection of three-dimensional Euclidean space. In the STL file, an object is described using triangular surfaces by specifying their three vertices in coordinates and describing the surface normal as a vector. The slicer software, finally, represents the object in a virtual printer and again describes the position with coordinates. Finally, as a control command for the 3D printer, a G-Code file is generated that specifies the extruder's individual movements to create the object layer by layer by specifying x and y coordinates as well as the amount of material to use. The various forms of description represent an authentic application of coordinates.

Similarly, interesting connections can be found in the approximation methods used in 3D printing. For example, the printer approximates a line defined as a curve as a polygonal line. Students can perform procedures to determine a suitable polygon course themselves. Similarly, surfaces are filled with narrow strips, which can become a subject of instruction explaining the concept of integrals. Finally, solids are layered vertically, which has parallels to Cavalieri's principle.

4 This Book

Some overarching aspects are addressed in the individual contributions to this volume that can serve as contextual bridges between them. The first aspect is:

- The use of 3D printing in classrooms—chances and challenges

This aspect describes the use of 3D printing in regular classrooms from the students' and teachers' perspectives. Various opportunities and challenges in the use of 3D printing technology are highlighted. In this volume, the articles by Ng and Ya, Dilling and Vogler, and Knöppel and Pielsticker address this aspect in a variety of ways.

Another overarching aspect is:

- Interdisciplinarity and 3D printing (e.g., mathematics and chemistry, mathematics and architecture, STEAM, etc.)

3D printing is used in various fields. Thus, it is worthwhile to investigate other disciplines in connection to mathematics. The contributions by Tejera, Aguilar, and Lavicza; Dilling, Vogler, and Milicic; Bedewy, Haas, Lavicza, and Houghton; and Eva, Anđić, and Lavicza particularly address this aspect.

A higher-level aspect can also be seen:

- Rediscovery of a focus on "Stoffdidaktik" or subject/content matter instruction.

The focus of this aspect is the content and development of mathematical content for the classroom. This aspect appears in the contributions by Hrach, Päßler, and Dilling.

Another overriding aspect of this volume is:

- The power of "Anschauung" or visualizations.

This aspect is especially evident in Pielsticker and Stoffels' contribution.

A final overarching aspect involves the use of technology in mathematics education in general. 3D printing technology offers a practical example of technology in education. Its contribution to education and what can be learned from the educational benefits of technology will be explored.

- The aspect of technology in mathematics classrooms: What we learn by and from the use of 3D printing.

Wulff, Heinze, and Wilken's contribution, as well as Greenstein, Akuom, Pomponio, Fernández, Davidson, Jeannotte, and York's, address this aspect in particular.

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3D Transformations for Architectural Models as a Tool for Mathematical Learning

Shereen El Bedewy, Ben Haas, Zsolt Lavicza and Tony Houghton

1 Introduction

In this chapter, we aim to shed light on the use of technologies in mathematics education. We propose STEAM (Science, Technology, Engineering, Arts and Mathematics) practices that require mathematics knowledge encapsulation and technology integration. The background to this chapter is also described in El Bedewy et al., (2021) which focuses on mixed reality visualization for the proposed practices that connect architecture, culture and history to mathematics education. This paper aims for proposing the STEAM practices to mathematics teachers that fosters mathematics education by modelling architecture while connecting to culture and to history. The modelled architectures can be visualized using various technologies as 3D printing and Augmented reality (AR). These STEAM practices consist of unguided tasks that would encourage participants to follow the reasoning and problem-solving in order to be able to solve them (El Bedewy et al., 2022; Olsson & Granberg, 2019). These practices introduce architectural modelling using "dynamic geometry software" GeoGebra (https://www.geogebra.org/). Also, because architecture is connected to culture and

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history, these practices use this connection by allowing participants to collect and research some cultural and historical information about these architectures. As the terminologies that are being introduced in these practices are so broad, we need to define each one and how we will integrate them in these practices.

Architecture is a constructional object that exists in the real world, starting from our homes, schools, museums or monuments, they could fall under the category of ancient or modern architectures. But in these STEAM practices we encourage participants to choose interesting architectures that have cultural stories and historical facts to reflect their motivations behind these choices and to practice mathematics during the modelling of these practices.". Moreover, the modelling of these architectures should be achievable with GeoGebra. The Architecture should consist of basic forms, shapes, and relations that can easily be simulated with GeoGebra tools and functionalities. The first step is introducing the GeoGebra tools and functions encapsulated in the modelling process such as polygons, prism and pyramid extrusions, curves, transformations, the surface of revolutions and many others. The participants get familiar with these tools by modelling several foundational architectures in a step-by-step approach during the intervention. This stage acts as a preparatory step for the participants to allow them to get to know the GeoGebra tools before choosing their own architectures for modelling.

The next terminology is the cultural relation, which means the architectural cultural relation to the participants. This could be elaborated by specifying the motivations behind choosing a certain architecture by the participants. This could reflect a cultural connection to a certain place, country, or region (Hessam & Sotoue, 2016). We refer to the historical terminology which reflects the historical facts connected with the participant's chosen architecture; these facts can vary from historical stories or mathematical facts describing the architecture construction.

The last generic terminology is the technology or technologies adopted in such practices, namely GeoGebra 3D Graphics and Geometry applications that allow participants to perform the architectural modelling, the visualization technologies such as digital and physical augmented reality (AR), 3D printing, and others that we refer to in our practices roadmap (e.g. the 3D scanning, Origami, 4D Frames), but do not discuss in this chapter because they didn't take place in the presented intervention in Upper Austria. 4D Frames are manipulative and tangible tools that were first initiated in Korea after their inspiration by Korean Architecture by redeveloping geometric shapes and investigating them (Lavicza et al., 2018). The practices proposed in this chapter encourage participants to visualize architectural models in many forms digital and physical, allowing a 3D transformation process to take place. The last terminology is mathematical knowledge, and we refer to it as the mathematics know-how the participants have or will acquire during the architectural modelling, and this is considered the contribution of such practices to mathematics learning which will be discussed later in this chapter.

Figure 1 displays the project map showing the research's main modules or parts. This map shows how we try to provide the participants with freedom of choice in several categories that all combine to form the main building blocks of these practices. The first category defines the participant's age because this will affect the other categories' choices in that it suggests the student's mathematical knowledge. The second category is the architectural model choice that includes ancient, modern, based on mathematical insights, freedom of choice, or inventing their own architectural models. The third category is the environmental choice, this provides the participants with freedom of choice of the classroom, online, outdoor, or in museum practice. The last category is the technologies that show the two paradigms we are tackling: digital and physical. As the teachers choose from the four criteria then they define the building blocks or main components of such practices.

After specifying the four criteria as shown in Fig. 1, the teachers completed the lesson plan. Other aspects the teacher may need to add for the lesson preparation such as the duration, assessment strategies, and other aspects are left to the teacher's own teaching methods and cultures.

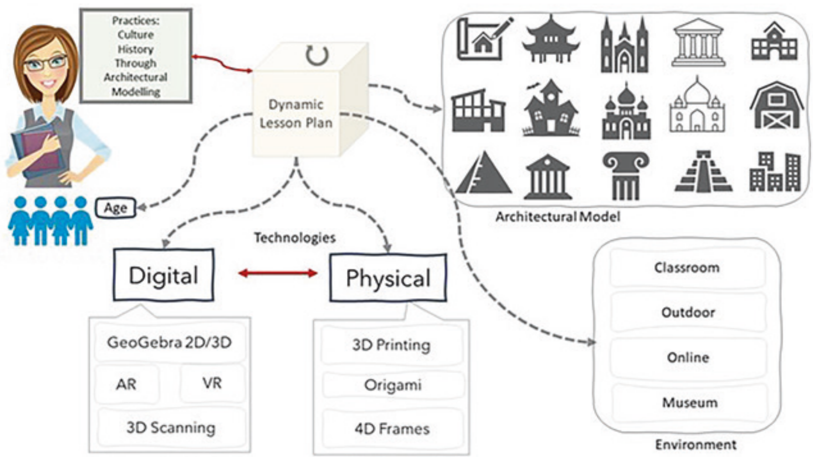


Fig. 1 The project map with the four criteria to implement the proposed STEAM practice

1.1 Dynamic Lesson Plan

As visualized in Fig. 1 the journey a participant would take to fulfil the four main practice criteria is mapped to a web interface that was developed using Unity gaming program to provide the participants with real-time response. This web interface we refer to as the Dynamic Lesson Plan (DLP). Figure 2 displays the DLP components, once the participant fulfils all the main practice criteria from this rotating cube, they press the button "Get the Link". This will provide the teachers with redirection links to a GeoGebra book that would open in a new tab, giving them examples, instructions and guiding them through each of the four criteria they chose from the DLP as seen in Fig. 3. The motivation behind designing the lesson planning as it is displayed in the DLP was to allow teachers to have multiple options to choose from the criteria referred to earlier and not to restrict them to a certain age, architecture, environment, or technology. Other aspects that are considered crucial for completing the lesson planning phase are the teaching methods, syllabus of study, curriculum structure and the assessment strategies, but these aspects could be culturally variant therefore these parts were left to the freedom of teachers to define outside the DLP tool. Therefore, we aim to provide teachers with DLP without any cultural dependencies during our interventions. Lesson planning is culturally related as mentioned by John (2007) that teachers during their lesson planning should "select and prepare resources, and plan for

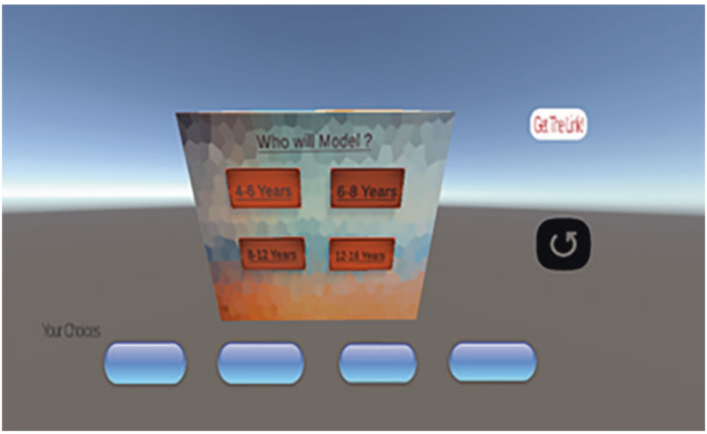


Fig. 2 DLP components placed on the rotating cube in the Unity scene

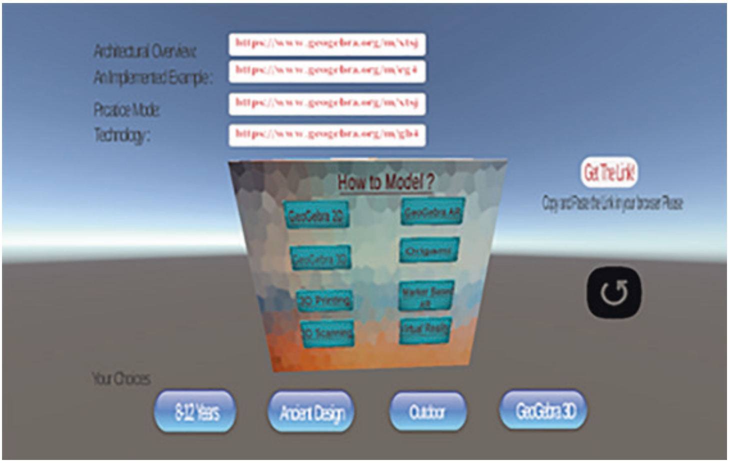


Fig. 3 The redirection links that the teacher can use to direct them to a GeoGebra book with more explanations on the chosen criteria

their safe and effective organization, taking account of pupils' interests and their language and cultural backgrounds." (John, 2007, p.484).

We are eager to promote creative teaching that as stated by Rinkevich (2011) should start in the phase of lesson preparation of the content to be taught and how it will be taught in a creative means.

By following the examples provided, the teachers would be able to understand the parts of the practices to be applied under their chosen preferences from each category. It assumes that the teachers have an idea about GeoGebra mathematical modelling to establish the basic foundational level in order to guide their students in these practices. Or as in the case of the intervention presented later in this chapter, the teacher and the students attended foundational workshops together in order to be able to apply these practices.

2 Literature Review

The literature discussed relates to each of the main modules in the proposed practices as technologies, STEAM practices, and the 3D transformation process of architectural models. The proposed research track falls under the STEAM umbrella. Marrero et al. (2014) states that STEM education aims for the growth of the main four disciplines of Science, Technology, Engineering, and Mathematics by connecting them in a single lesson, exercise, or practice, and to real-world challenges. STEM allows the growth in the skills of the students both for themselves and for the professional track because they encourage research analysis, problem resolutions, creativity, working with each other in a group in a collaborative way which will enhance communication skills (English, 2016). Other studies believe that the arts if integrated in STEM disciplines to STEAM would be of great benefit while utilizing the STEM learning goals the art integration could be of a great use (Boy, 2013; Connor et al., 2015; Watson & Watson, 2013). This also applies to our proposed STEAM practices that utilize the arts that exists in the architectural elements during modelling. Additionally, these views direct the research track to implement STEAM practices that may help in enhancing educational practices in general while encapsulating architectural modelling to foster mathematical competencies. The architectural modelling initiative in this study is based on the Egyptologists' work by Rossi (2004) to understand the ancient Egyptian mathematical systems, which was represented as texts of mathematics components written on Papyrus and leather such as the Rhind Mathematical Papyrus, the Moscow Mathematical Papyrus, the Kahun Papyri, and the Egyptian Mathematical Leather Roll and the relation to ancient Egyptian architecture. This connection between ancient architecture and mathematics makes the authors of this chapter believe that it may be a great opportunity for classroom explorations and applications. The cultural and historical initiative emerged from the idea of combining mathematics with other disciplines. The connection of architecture to mathematics is very clear also in ancient architecture which was represented in a couple of sources such as Peet (2017) who wrote in his article about ancient Egyptian architecture which was based on mathematical formulas and equations that we study and know today. Stating that sources as the Rhind, the Moscow papyri, and the Egyptian Mathematical Leather Roll, which contains a table of 26 decompositions of unit fractions, Egyptian mathematics became clear. As well, the Reisner Papyrus represents the mathematical practical applications in the field of construction and architecture.

Not only the mathematical connection to ancient architecture was represented and studied, but it also appeared in modern architectural designs as the new term emerged called Architectural Geometry, which is a combination of differential geometry, computational mathematics, and architectural design and engineering as said by Pottmann(2008).

The technology integration in these practices such as 3D printing benefits educational practices (Szulżyk-Cieplak et al., 2014). Szulżyk-Cieplak et al. (2014), believe that 3D printers are a good tool for supporting teachers because the 3D prints can help in the improved interpretation of certain courses. Moreover, they believe it is a tool that can help students' participation and engagement because they can physically explore their outcomes during the classes and act as a measuring tool for the students in detecting their achievements or limitations. They also believe that "3D printing allows the students to transfer their ideas into reality" (Szulżyk-Cieplak et al., 2014, p. 100). Leinonen et al. (2020) investigated the results of integrating digital fabrication into the educational system and observed its effects through ethnographic research. Their focus on digital fabrication is on the process of 3D modelling and printing. They observed that it helped their participants in allowing them to design, create while doing a practical task to produce a personalized output. From an emotional point of view, Leinonen et al. believe that 3D printing increased the feeling of encouragement, feeling engaged, and above all happiness with the design and production process. Leinonen et al.'s findings which address the 3D modelling and printing during the process of digital fabrication show that enrolling the participants in the whole process may be efficient and maybe of positive effects on the learning outcomes.

This research provides teachers with various technological options to visualize architectural modelling, for example, 3D printing is a very useful tool to represent the modelled architectures in a physically tangible form. Lieban et al. (2019) argue that 3D printers are now of affordable costs to be adopted in schools and for educational practices. They believe that 3D printers are helpful for supporting teachers to apply mathematics education by printing the students' work and investigating its mathematical basis. They conducted two interventions with teachers which resulted in further explorations in mathematics education. Furthermore, Lieban et al. (2019) believe that 3D printing may foster the modelling process because its application would enhance the student-centered learning approach. This may have a great effect on students' motivation when representing things, they created themselves. From these findings, we can capitalize on adding 3D printing as a tool to represent students' architectural models in a physically tangible way. Huleihil (2017) argues that 3D printing offers another perspective as it allows students to investigate the printed models which enhances the process of perceiving the designs mentally and allows them to think about the models' mathematical development. These findings encourage us to provide the participants with a tangible edge by printing the 3D architectural models. AR encapsulation in educational settings also proved its advantage by allowing participants to impose digital overlays on their real-world experiences while still not being totally immersive.

> "Augmented reality is a system that enhances the real world by superimposing computer-generated information on top of it." (Furht, 2006; Wellner et al., 1993)

In their work to investigate the effectiveness of using GeoGebra AR in geometry learning through some exercises, Series (2021) practiced with the students to apply geometrical shapes from real-life examples such as triangles in the upper part of houses. They believe, GeoGebra AR allowed the students to use their five senses in practicing these tasks, and their tangible, spatial, and hearing abilities proved to be of positive impact.

In their review Sırakaya and Alsancak Sırakaya (2020) investigated the effects of using AR in STEM education. Their overall results promote the use of AR in STEM education but for that to take place innovative practices should be available. These practices should allow "piloting prior to actual implementation, prolonging the implementation period" (Sırakaya & Alsancak Sırakaya, 2020, p.10) before being implemented in actual education STEM settings.

In order to ensure enhanced learning outcomes, the interest of educators and students in VR, AR, 3D printing, and modelling was studied by Trust et al. (2021). In their study, they used the TPACK (cf. Mishra & Koehler, 2006, 2008) model to see where teachers and educators are in their understanding of these technologies. They concluded that educators are interested in these technologies but lack knowledge on how to adopt them in the classroom. They suggest that novel designs and practices are needed to be utilized in the adoption of such technologies in the classroom. From this point, we hope that this research focus is one of these novel contributions and it helps in delivering use cases with various technologies to be adopted by teachers and educators. From these publications, it is believed that integrating the wide spectrum of technologies may aid teachers in supporting their student's understanding of the virtual representations and the physical representations of the architectural models.

The next section will walk the readers through the methodology, theories and present the Upper Austria intervention and the student outcomes in a 3D transformational representation.

3 Methodology

The research is following the Design-Based Research (DBR) approach. The DBR is an iterative process encapsulating multiple components with diverse types that help in tuning practice to meet what really works and to affect theory in a constructive way (Cobb et al., 2003). This is of a dynamic nature in the form of cycles that continuously contribute to each other, to the theory and to the final goal of the research. Brown (1992) described the DBR and the theory and practice relationship as two way direction that the theory advises the design and vice versa. Therefore, we will introduce the theoretical framework and how it contributes to the research.

4 Theoretical Framework

The theoretical framework governs this research scope, levels and stages. We will tackle this intervention process from its initiation stage and then walk through the research stages to cover all the related theories that are introduced in this theoretical framework. At the end of this section, we will display how the theories are linked with each other, displaying their intersections and how they affect each other to inform the design and practice.

4.1 TDS

The first step is initiating the idea with the teacher who will practice this intervention with students. Therefore, the first stage requires lesson planning which is defining the building blocks of these practices, for example, from the DLP, specifying the student's age group, the architectural model type, environment of practice, and finally, the technology used. For these criteria to be gathered and fulfilled we developed the web interface of the DLP for the teachers as an aid for completing the lesson plan stage. From that perspective, we think the Theory of Didactical Situations (TDS) (Brousseau, 1997) is the most compatible when it comes to the lesson planning and the DLP because they are both of a dynamic nature and not specific to certain inputs in the lesson plan. TDS rather promotes the dynamic nature than the static and describes it as the interplay between the teacher, student and the environment which they refer to as the milieu. We are interested in the para-didactical situations which are the teachers' milieu that appears during the lesson planning when defining the milieu to the students. Our focus is this milieu, but from the teacher's perspective, i.e. the intentions they have and their hypothesis of such practices they create during the para-didactical situations. Moreover, the teacher's reflections and feedback from these practices at the end of the intervention affect their milieu outcomes off their students in the post didactical situations. Therefore, the TDS allowed us a dynamic nature in defining the variables of focus when it comes to the teacher's definition of the milieu to their students.

4.2 TPACK

Another aspect of the theoretical framework is the Technological Pedagogical Content Knowledge (TPACK) by (cf. Mishra & Koehler, 2006, 2008), where we try to focus on the teacher's TPACK Fig. 4.

The TPACK framework is an extension of Shulman's (1986) PCK framework that describes teacher knowledge necessary for effective integration of technology into teaching a discipline in a specific context. The central component of TPACK framework is an integrated TPACK domain, which is a knowledge that is

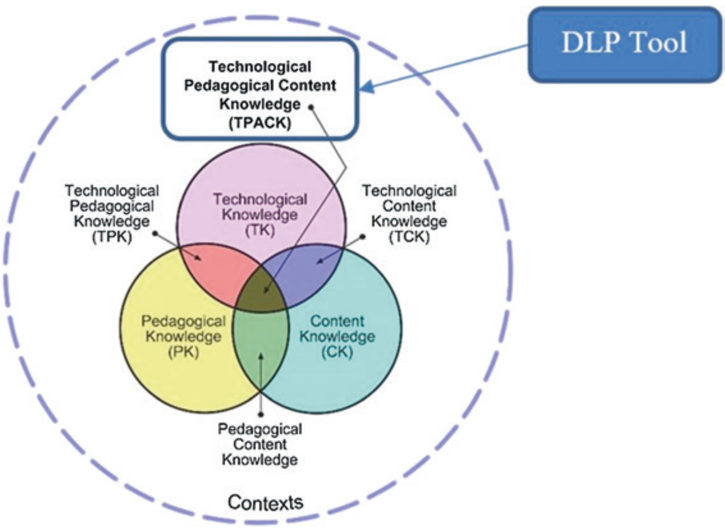


Fig. 4 Adapted from TPACK framework circles of knowledge (Mishra, 2019) to the teachers' lesson planning using DLP tool

formed through integration of technological, content, and pedagogical knowledge in a way that these inputs are no longer distinct. In this study, we focus on this integrated TPACK, specifically on how it can affect and influence the teacher's choices from the DLP tool during the lesson planning phase. So, we explore the teacher's choice on an architectural model rather than another and what are the motivations behind this choice, it could be for example that they want to focus on a certain mathematical concept that is well utilized in such an architectural model. In that case the teacher's content knowledge is influenced by their TPACK as the architectural model choice is one of the choices in the DLP. Another choice in the DLP that reflects teacher's content knowledge is the age of the students' criteria, were we observe how teachers select architectural models that suit their student's mathematical knowledge. The environment choice in the DLP is also a pedagogical aspect and the visualization technology choice in the DLP is a technological aspect where both are influenced by the teacher's TPACK. In the proposed practices the TPACK domain of teacher knowledge influences the lesson planning part with the aid of the DLP encapsulation as it is a web tool which is also technology based and in influencing the teacher's pedagogy in specifying the age of the students and the environment of practice. The TPACK domain influences the content by specifying the architectural models to be modelled and specifying a certain architectural category. While the technological choice is influenced by the TPACK domain which can appear in specifying the 3D transformational technologies whether physical or digital to be used in these practices. The technology adoption interferes in this major aspect of the 3D transformational choice and how they will allow teachers to decide on which technologies to model architecture with the help of physical and digital tools. All these criteria using the DLP tool contribute to the lesson planning by the teachers which is influenced by their TPACK. For example, if the teacher chose the intervention to take place online then what are the technologies to be used during the intervention (e.g., devices and tools), or if the intervention took place in the classroom would it be on a smartboard or tablets. Then comes the challenges of technology literacy if it takes place and how to overcome them by encouraging teachers to use the technologies in the most efficient way. So, this level of technology integration also affects the technology knowledge integrated into the final self-reflective teacher's TPACK. The TPACK domain enlightens us on how teachers perceive technologies when learning, using, instructing, and guiding their students.

4.3 AMOEBA

The Adaptive, Meaningful, Organic, Environmental-based architecture for Online course design (AMOEBA) by Gunawardena, Wilson, and Nolla (2003) is basically the cultural contribution to these practices. This cultural wheel of practices that make them applicable across cultures takes into consideration the cultural boundaries and differences that may affect these practices. The AMOEBA governs the language, the means of communication, and the technology affordances. This framework sheds light and points out the diversity between various cultures that participated in this research, to allow us to keep these diverse factors into consideration when designing such interventions, keeping into consideration the diverse architectural choices outputs from these interventions and what it tells us about cultural diversity. This framework is important in regulating the cultural variations in an online context because most of the interventions that took place in several countries were online during the time of the pandemic.

4.4 Program Development

The program development has four main components as stated by Demir (2011) which are inseparable; they depend on each other and dynamically affect each other. The program development was included in the theoretical framework to add a logical sequence to the steps of the practice that may take place. The first step in the program or practice development is determining the goals of the program, which justifies the need for such a program and the goals of the target group. Afterwards, the four components follow with the objectives, content, teaching-learning situations, and assessment phases. The first component is the objectives which state the required knowledge or skills that need to be covered in this program to the target group. The objectives have to be tailored to meet the target group's needs to enhance their shortages and add on their current knowledge while taking into consideration the cultural differences. The second component is setting the content which is what is to be taught to the target group within this program, the content provided should be suitable for the target group. The third component is the learning-teaching situation which is the actual implementation of the situations or interventions that fulfil the required objective with the added contents. The last component of the program development is the assessment which measures the target group's achievements in satisfying and meeting the objectives of the program. This cyclic model in Figure 5 is the integrational

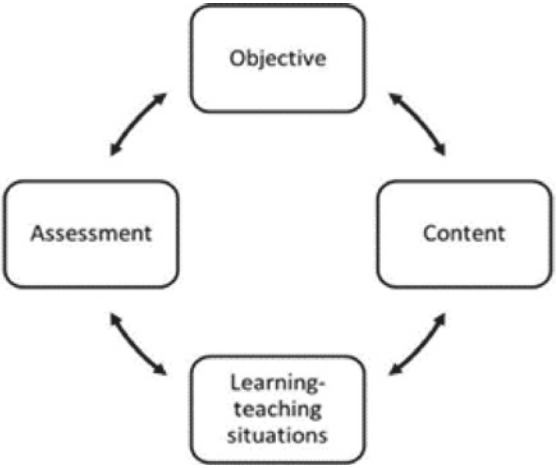


Fig. 5 Process of program development

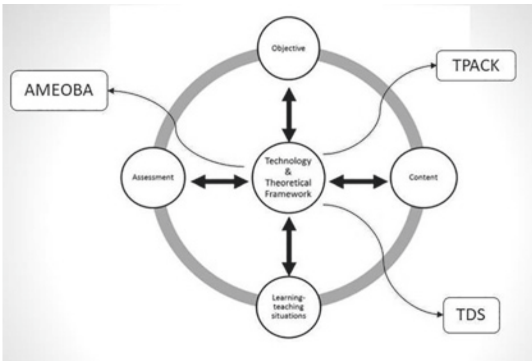


Fig. 6 The program development relation with the theories

program (Demir, 2011) that mainly aims at introducing programs to practice technology while keeping into consideration the theoretical impact.

Figure 6 displays the relationship between the cyclic program development stages and the proposed theories: The TDS is used in the lesson preparation stage as well as after the program takes place in the assessment part in the

Para didactical situations. The TPACK takes place during the four stages of the program development due to the technology interference in each of these stages: During the lesson preparation using the DLP, during the didactical situations using the 3D transformation, and representing architectures in many forms, also in the assessment part which is done through technological tools such as the GeoGebra classroom, PowerPoint presentations, or videos. The AMOEBA interferes in the program development in the four stages as it regulates the method of interaction, languages and technology affordance based on cultural preferences. The program development was a method to describe the nature of such practices we are proposing, how they can be divided into stages and how we visualize the theories' interference in each of these stages to complete the integration of these practices to foster mathematics education as shown in Fig. 6.

5 Upper Austria Intervention

One of the interventions took place in Upper Austria with a female mid-age teacher and 35 students aged 15 – 16 years in a high school in Linz, Upper Austria. The school was of a vocational type focusing on construction and technology goals. The intervention took 5 months. The methods we used for the data collection were four video-recorded interviews with the teacher and the final projects of 23 students. The students' final projects were pdf documents describing their motivations for the architectures they chose, historical facts, and the GeoGebra files they modelled along with a hybrid form of architectural visualizations.

5.1 Upper Austria Teacher Interviews

The semi-structured interviews were written beforehand, and the interview protocol was piloted before the intervention with three other teacher participants. The first interview that took place with the Austrian teacher was to initialize the project idea and explain to the teacher the research concepts and expectations. The second interview was to allow the teacher to explore the DLP and set the workshop criteria by choosing from the four modules in the DLP. In the second interview, the final project's structure and the assessment strategy for the students' work was also agreed on. The third interview took place after the workshop with the students, so it was mainly considered as a mid-cycle reflection, to answer the teacher's architectural modelling inquiries and to see the challenges that the teacher and the students met during the intervention. The last interview was to collect the students' work and the teacher's final reflections, opinions, and recommendations for future interventions.

5.2 Upper Austria Workshop

During the interview with the teacher, she defined the lesson plan through the DLP. The teacher chose an ancient architecture to be practiced with the students during the workshop. It was the Carnuntum located in Austria as seen in Fig. 8. The teacher chose this architecture because the students studied it earlier in their history classes. So, she believed this could be a good connection to cultural and historical perspectives. The workshop took the form of hybrid presence because it was during the pandemic time, so half of the students attended online and the other half from the classroom with the teacher. GeoGebra was new to the teacher,

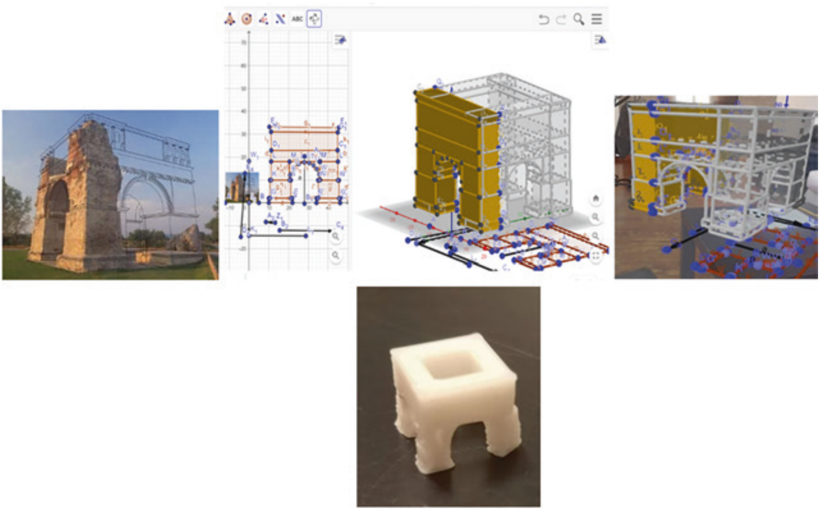


Fig. 7 The Carnuntum Austria, and its modelling in GeoGebra, GeoGebra AR and 3D printing visualization

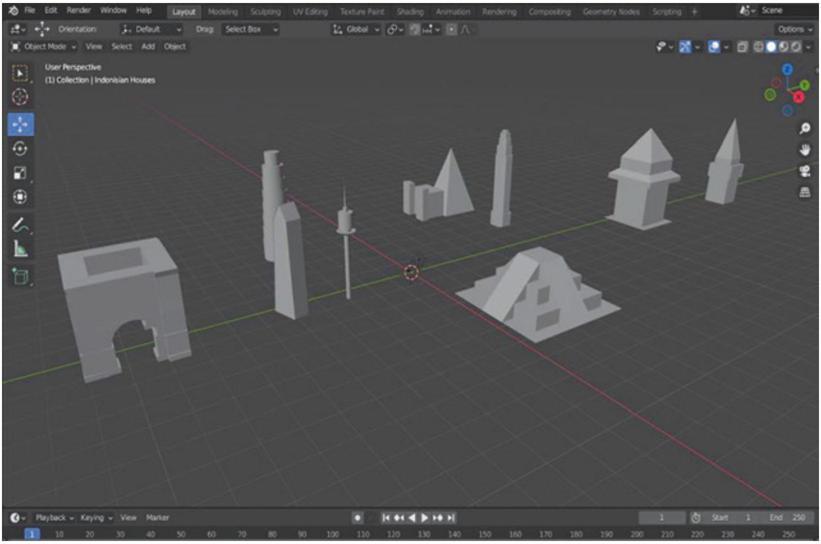


Fig. 8 The Student's work which is 3D printable displayed in Blender environment before printing

but most of the students were familiar with it and had used it before in other subjects. But the 3D modelling part was new for everyone during the workshop. We joined the online group and were projected for the other student group in the classroom. The teacher chose AR as the technology to be used for visualization during the workshop. During the workshop we try to analyze architectural models by replicating the main forms these architectures consist of, we don't model complex architectures, wall carvings or complex decorations using GeoGebra. This is to not overwhelm the students with complex architectures and to allow them to focus more on the main building components and analyze them mathematically to meet the practices' didactical goals. Figure 7 displays the Carnuntum modelling in GeoGebra, we walked the participants through a step-by-step approach displaying how the modelling took place. and giving them foundational knowledge of the GeoGebra tools and functionalities that would assist them in their final projects. The workshop lasted 90 minutes.

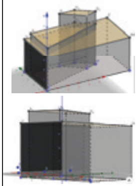
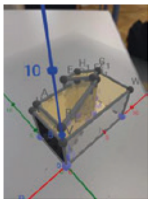
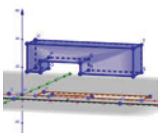
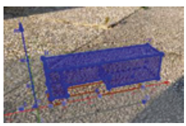
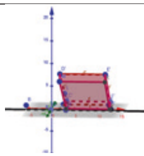
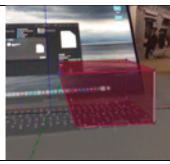
5.3 Students' Final Projects

After the workshop was over the teacher asked her students to use any architecture they would like to represent from any place around the world. They should mention their motivations and collect some historical facts about these architectures. The teacher recommended her students to use several technologies such as AR which we practiced during the workshop, 3D printing if it were available to the students, Origami, 4D frames, or even any type of manipulatives that can be encapsulated to visualize these architectures in many forms, digitally or physically.

5.3.1 Students' work in AR

The teacher thought the GeoGebra AR feature is helpful for the students because it doesn't require any extra development or any other program or application to be installed with GeoGebra. She believed GeoGebra made AR very accessible for everyone. Therefore, we showed the students the AR feature during the workshop by connecting our mobile phones and displaying the architectural models in AR

Table 1 The student's work displayed in GeoGebra Modelling and in GeoGebra AR, the last column describes the mathematical tools and functions

Architecture Name/Location	GeoGebra Modelling	GeoGebra AR	Mathematical Significance
Music Theater/ Musiktheater Linz, Austria 			• Irregular Polygons • Multiple Prism Extrusions to different altitudes
Lentos Kunstmuseum (Art Museum) Linz, Austria 			• Irregular Polygons • A Prism Extrusion • Rotation on a line
Ars Electronica Center, Linz Austria 			• Polygons • A Prism Extrusion • Vector Translation

form. Table 1 displays some of the students' work in AR and the mathematical tools and GeoGebra tools used in these architectural modelling.

Now that we have seen some samples from the student's work displayed in the digital technological form using AR, we will now explore the physical ones using 3D printing.

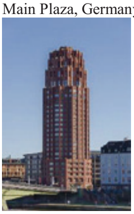
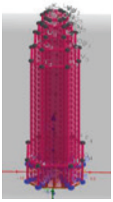
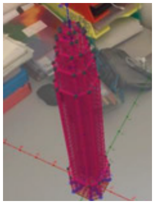
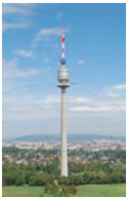
5.3.2 Students' work in 3D Software

As a transitional step before attempting to 3D print the architectural models, we have a quick view of the models using the Blender software. There are other 3D software that can help participants in exploring their models such as Mesh Mixer or Tinkercad and many others. We exported the GeoGebra files as an STL format and imported them into the Blender software. This step may be important if the participants won't do the 3D printing by themselves and would send it to a maker space for 3D printing. One of the learning strengths of the 3D printing process is that it allows participants to foster problem-solving while converting non-printable shapes to printable shapes. Examples are adding thickness to planes or overcoming hidden internal faces and many other aspects that need to be considered for the 3D printing process to successfully take place. This allows the participants to alter their designs to meet the 3D printing technology's affordances. Figure 8 shows our transitional step attempt to explore the student's GeoGebra models in the Blender 3D environment.

5.3.3 Students' work in 3D Print

The teacher was aware of 3D printing and its educational benefits, she also mentioned that she had access to 3D printers but could not use them at the time of the intervention due to the pandemic situation. Therefore, she recommended students print their models if they have access to any 3D Printers. Some students mentioned that they had access, but their parents didn't have time to print the models. Therefore, we selected some of the students' work to display the 3D printing concept in an educational context based on the students' work. Because not any 3D model can be printed, we selected the printable ones that had a reasonable thickness and consisted of solid polygons while keeping into consideration the materials used (Sculpteo, 2015). This will demonstrate that if 3D printers were to be placed in schools, students could model architectures and continue by visualizing them in a physical form through 3D printing. By elaborating the 3D transformation concept to the students by visualizing the same models but in various forms, reasoning, problem-solving and hands-on activities are fostered (Lieban et al., 2019). Table 2 displays some of the students' modelled architecture work

Table 2 The student's work displayed in GeoGebra Modelling, in GeoGebra AR, and the 3D prints, the last column describes the mathematical tools and functions

Architecture Name/Location	GeoGebra Modelling	GeoGebra AR	3D Printing	Mathematical Significance
Main Plaza, Germany 				1.Irregular Polygons 2. Multiple Prism Extrusions to different altitudes
Donau Tower, Austria 				1.Regular Polygons 2. Multiple Prism Extrusions to different altitudes 3.Pyramid Extrusion downwards 4.Vector Translations

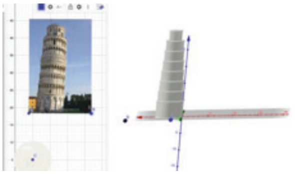
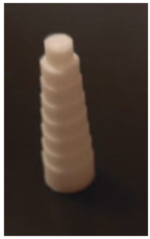
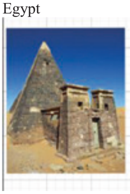
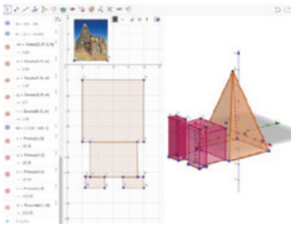
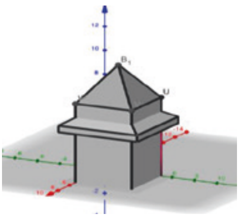
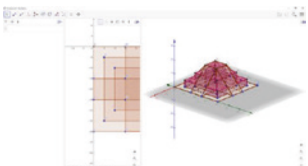
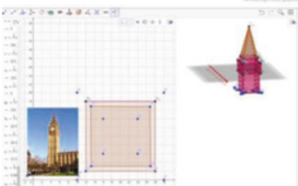
as well as if present their representation in GeoGebra AR and their mathematical significance by stating the GeoGebra tools and functions.

Table 2, show some students' outputs who used AR for their architectural representations. In Tables 2 and 3, show the students' architectural models which were good for promoting the 3D printing architectural visualization.

5.3.4 Reflections on the 3D Printing process

In this part, we reflect on the 3D printing process and discuss the multiple options during the printing. One should keep in mind the level of details of the models in order to adjust the printing quality because if low-quality printing is applied to models with details, the details are omitted, and the final 3D objects become distorted. This isn't the fault of the 3D printer, but the 3D models should be manifold and ready for printing. The definition of a non-manifold geometry from Sculpteo (2015) is the following: "A non-manifold geometry is a 3D shape that cannot be unfolded into a 2D surface with all its normal pointing the same direc-

Table 3 The student's work displayed in GeoGebra Modelling and the corresponding 3D prints, the last column describes the mathematical tools and functions

Architecture Name/Location	GeoGebra Modelling	3D Printing	Mathematical Significance
Ibiza Tower, Italy 			1.Regular Polygons 2. Multiple Prism Extrusions to different altitudes 3.Rotation on a Line
Nubian Pyramids, Egypt 			1.Regular Polygons 2. Multiple Prism Extrusions to different altitudes 3.Pyramid Extrusion
Graz Clock Tower, Austria 			1.Regular Polygons 2. Multiple Prism Extrusions to different altitudes 3.Pyramid Extrusion 4.Vector Translations
Azteken Pyramide America 			1.Regular Polygons 2. Multiple Prism Extrusions to different altitudes
Elizabeth Tower"- The Big Ben, England 			1.Regular Polygons 2. Multiple Prism and Pyramid Extrusions to different altitudes

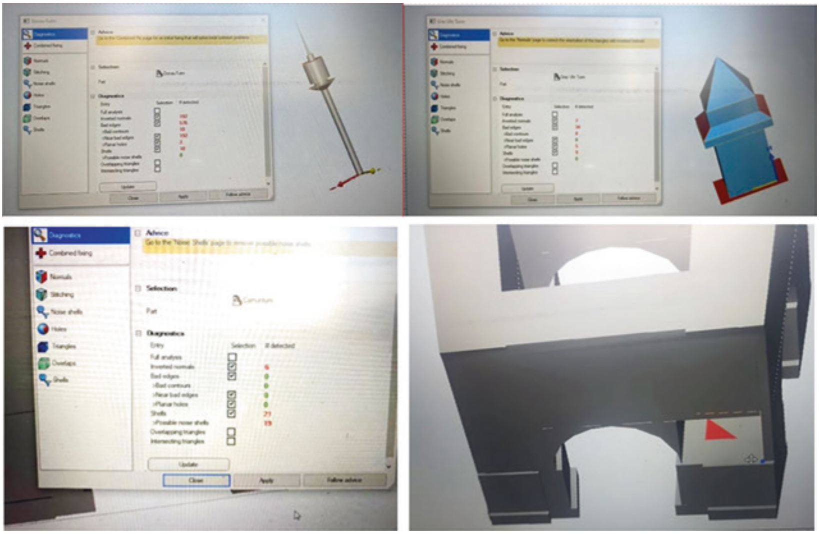


Fig. 9 This figure displays examples from the 3D models that were defective and had red parts highlighted as well as the printing software showing the number of errors that hinder these models from being printed.

tion." Normal is simply the perpendicular vector on a surface of an object. One of the common mistakes that can apply to 3D models and hinder it from printing, or it could be printed but with very poor quality is the non-manifold geometry feature in the 3D models. Examples of non-manifold geometries are geometries that include "Multiple connected geometries" which is three faces sharing the same edge in a 3D model, or many surfaces connected with a single vertex. Surfaces with no thickness could cause a problem while 3D printing, also 3D objects with hidden internal faces and finally objects with two adjacent faces having opposite normal (Sculpteo, 2015). All these issues lie under the category of a non-manifold geometry that can affect the 3D printing process. But thankfully some printing software is capable of detecting these errors in the models before printing and capable of fixing them automatically, for example, adding thickness to the faces with no thickness, inverting the surfaces with opposite normal, detaching objects that relate to a single vertex or an edge connecting three faces. So, all these issues are fixable but should be addressed for a smooth printing experience.

Figure 9, shows some of the 3D models and the error message that appears as well as highlighting the defective parts in red for the users to fix before print-

Table 4 The student's work displayed in Blender 3D environment, and two 3D printing attempts one low and one high with the explanation of the differences

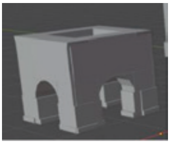
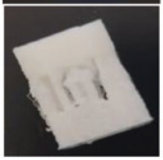
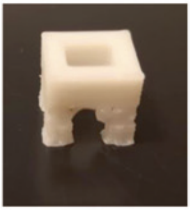
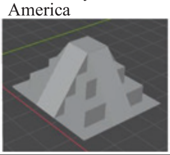
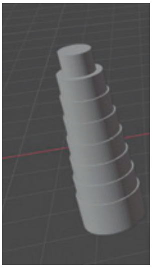
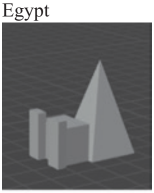
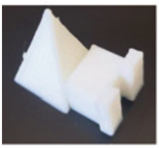
Blender View of the STL files before Printing	3D Printing with no fixes and Low-Quality Printing	3D Printing After Fixes and Higher Quality Printing	3D Printing Remarks
Carnuntum, Austria 	 		This model had a problem of inner faces and missing faces. It was detected using the printing software and automatically fixed.
Azteken Pyramide America 			The base of the pyramid wasn't printed because it didn't have any thickness. This was regardless of the printing quality.
Donau Tower, Austria 	 		In the first printing trial, two parts from the tower were printed and they were separate parts, the model had various inverted normal and wrong edges which prevented it from printing in the low-quality trial. We added a base for the model to stand on as a fixing step before high-quality printing.
Clock-Tower, Austria 			The printer added support around the lower part of the model and it had to be cut off to reflect the model's true shape.

Table 4 (continued)

Ibiza Tower, Italy 			The Ibiza tower details were not printed in the first trial, the software highlighted every second cylinder in the tower and therefore it resulted in an unstable, missing model. We allowed the auto-fix to fix the model printing and it printed the missing parts.
Main Plaza, Germany 			This model had many connected wrong edges internally, so we explored it in a transparent mode in the 3D printing software and we removed all the internal faces and connected edges. This caused the printer to add support which was attached to the missing parts of the model. When all these issues were fixed the model was printed correctly in a higher quality form.
BigBen, England 			This model had an extra inner face that connected two edges with a single vertex.
Nubian Pyramid, Egypt 			The only difference between the two models is that in the low printing option for the parts were all connected whereas, in the high quality one, the pyramid was separate from the whole shape to reflect how it was modelled in GeoGebra.

ing. Some of the errors show bad edges, inverted normal, planar holes. Once the model is fixed by using the software there will be no red parts in any of its components and this means it is 3D printing ready.

In order to visualize the 3D printing distortions that could happen if the models were not fixed and have manifold geometries in Table 4, we display the two-printing processes, before fixing the models and after fixing the models as well as the defects that caused the models to be non-manifold geometric models and how the software fixed these models. Table 4 shows the model in Blender and in 3D printing in high quality and in low quality showing the level of details that can disappear before and after fixing the 3D models.

In all the high-quality printing models the printer settings have been adjusted as the retraction distance and speed, the temperature, the movement speed, and many others that also affect the printing quality. Table 4 showed that there are various printing outputs that could be interesting to explain to participants of such practices. If the participants are aware of such issues, then they can learn to produce 3D models that are manifold and avoid the errors that affect the printing process. The next part will display the student's opinions in 3D printing.

5.4 Students' 3D Printing Reflections

We have explored the teacher's opinion on 3D printing during the interview. We are also curious to see the students' opinions on the 3D printing technology. Therefore, we distributed a survey to the students after they handed in their final

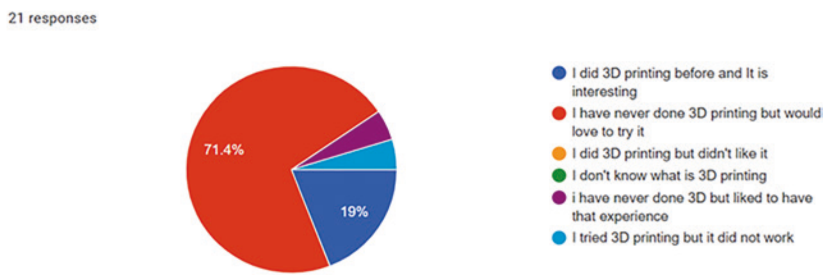


Fig. 10 21 Students' opinion on 3D Printing

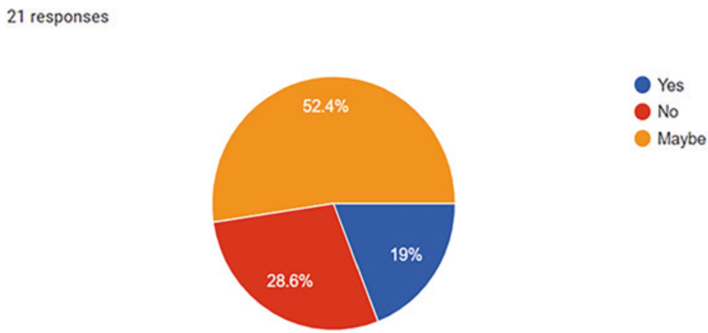


Fig. 11 21 Student's opinion on 3D Printing their own 3D architectural modelling during this intervention

projects to capture their reflection on the 3D printing concepts and we received 21 responses. Figure 10 displays the pie chart with percentages of the student's opinions on 3D printing.

In Fig. 10 the pie chart shows that the majority of the students didn't experience 3D printing but would love to try it out. Another question was introduced to the students to measure their preferences on printing their 3D architectural models they constructed using GeoGebra during this intervention or not. The responses to the question "Would you like to 3D print your architecture?" is displayed in Fig. 11.

In Fig. 11 the pie chart shows that more than half of the students who participated in this survey were not sure on printing their 3D architectural models, the other half of the students were split into two unequal groups, some were sure on printing their 3D prints and some students didn't show interest. These results could be due to many factors as the 3D printing availability to the students and that not all the students have access to 3D printers. These two pie charts displayed in Figs. 10 and 11 give us an overview of the student's interest in 3D printing, in future papers we will display their reactions and reflections after receiving the images of the 3D printed models of their own architectures and see if this would affect their opinions on the 3D printing technology.

5.5 Reflections on the Intervention's Data

The previous section showed the technology encapsulation in visualizing the architectural models using digital technologies like AR and physical technologies such as 3D printing. We will analyze these findings and connect them to our proposed theoretical framework. In Tables 1, 2 and 3, the students' work has been visualized in a sequence of steps. In Table 1, 2 and 3, the first column represents the architectural choice which when we reflect on our theoretical framework, the AMOEBA theory is very significant in the architectural choices and the cultural relationship. Architectures tell a lot about cultures, according to Hessam and Sotoue (2016). The teacher allowed the students to choose their architectural choices, which is significant from their cultural perspective. The teacher's cultural freedom resulted in vast diversity in architectural models from various places worldwide.

This intersects with TDS in how teachers prepared the lessons and the components she chose from the DLP in the para didactical situations, which are now visualized in the two tables of the student's work. The teacher's architectural choice is the freedom of choice and the technologies used to visualize such architectures as AR and 3D printing. The teacher also encouraged students to use technologies, and therefore, we find that some of the students adopted the AR technology as a visualization tool besides the GeoGebra modelling part. The teacher was proud of her students when she was handing in their work. This was obvious in her speech and how she described the students' work as she opened each one and described it and even mentioned how she attended the modelling process of such architectures in the classroom and saw them in the making phase. Where some students needed help, and she guided them through using GeoGebra tools. This may also be perceived as a teacher motivation because she lacked the GeoGebra knowledge before this intervention. However, during and after the intervention, we noticed a behavioural change as the teacher made a GeoGebra account, started to build things to guide her students and started to discuss with them and research some tools and functions which she also asked us about during the second interview as building irregular shapes, copying, or duplicating objects. She mentioned that she recommends Boolean operations in GeoGebra to take place to make the modelling process smoother. Another reflection from the teacher about the technologies was that she liked GeoGebra AR so much and thought it is an excellent opportunity for a hybrid visualization in a straightforward way that does not require any different technologies to be integrated with GeoGebra. Connecting to the TDS, these reflections can be perceived as the teacher's reflections and feedback on their own milieu in post didactical situations.

In Table 2, the third column shows the technologies adopted for visualization as the AR. This makes TPACK significant in how teachers chose the technology that worked for the intervention as GeoGebra and AR, not only for the intervention for the situation that faced them at that time, the pandemic. The teacher believed AR would be very convenient for this period. Moreover, the teacher developed the content to be used during the intervention of the Carnuntum, which is an architecture they studied in history class, so the teacher built the cultural and historical connection. Furthermore, the pedagogy used by the teacher is significant in her lesson planning before the intervention and how the teacher formulated the requirements of the final project that students have to state the motivations and facts about these architectures they choose. Moreover, the pedagogical knowledge that the teacher covered could be evident in the teaching methods the teacher followed, such as allowing the students to follow inquiry-based learning (IBL), to be able to solve the modelling tasks on their own while only providing guidance if she was asked to. The teacher provides her students with a space to research, explore and come up with ideas, representations, and motivations on their own. The teacher promoted this learning style of the unguided tasks approach in defining the pedagogy used (Goldston et al., 2013). As we fuse the three circles of knowledge, we can assume that the teacher developed her TPACK for these practices in encapsulating the technology for content achievement with a pedagogical meaning. This is by looking at the intervention as a whole.

In Tables 1, 2 and 3, the second column shows the GeoGebra models that the students created, which we analyzed to create the last column in the two tables, the mathematical significance. We determined this by analyzing the function and tools the students used to result in these architectural models using GeoGebra. The mathematical tools and functions vary from one architectural representation to another, and this is because each architecture has its own nature. Moreover, each architecture can be modelled differently, which is an essential aspect of the architectural modelling to highlight that fosters problem-solving strategies. Lieban and Lavicza (2019) stated that it allows participants to solve problems in various ways to achieve the architectural constructions' modelling. This last column in Tables 1, 2 and 3 shows how these interventions are connected to mathematical learning and allow students to encapsulate the mathematical knowledge they have in such architectural creations and practices in principle. As we analyzed Tables 1, 2 and 3, Table 5 combines the significance of the theo-

Table 5 Mapping the intervention output visualizations to the proposed theories based on columns in table 1, 2 & 3

Theories (Teacher Focus)	Architecture Name/Location	GeoGebra Modelling	GeoGebra AR		3D Printing	Mathematical Significance
TDS	TDS took place when constructing the lesson planning for these practices. And in Teacher's milieu as a reflection on these outcomes					
TPACK	TPACK in defining content as the Architectural model's category		TPACK influenced teachers by the technology adoption during lesson planning (DLP) and 3D transformation of the architectural models. And the pedagogy in defining the final project's structure.			Display the pedagogical approach used as IBL that gave diverse outputs and mathematical approaches
AMOEBA	AMOEBA in Defining Architectural Choices		AMOEBA was used in defining which technologies and means of communication can be adopted in these practices			

ries described in these tables columns to display the theories' encapsulation and effects on these practices in each of the intervention outputs.

The effect of these theories' representation was significant in meeting the optimal goal of these practices to promote mathematics education and in training teachers TPACK, to foster mathematics education while adopting technologies like 3D printing and AR. To combine architecture, culture, history with mathematics education.

5.6 Upper Austria Summary

The Upper Austria intervention was the first intervention to take part with students, all the other interventions which we will refer to in other publications took place with teachers only. Although our focus is on teachers in this research track, the students' outputs were very important to see, analyze, and understand the effects of the proposed practice on students. This specific intervention highlighted the effects of such practices on the teachers and how they got transmitted to students, their emotions, hypothesis making and reflections on the research idea. Also, we evaluated the practices applicability and the research idea on the students' abilities and mathematical knowledge. From these intervention results and students' final projects, we may assume that these proposed practices could be applied by students while keeping into consideration all the variables such as their ages, mathematical knowledge, technology awareness and readiness, and GeoGebra previous knowledge. Therefore, all these factors give us a possibility of integrating such practices with teachers and students in mathematics learning. The proposed theoretical framework was significant in such practices although the proposed theories intersected in various aspects such as during the lesson preparation and the technology encapsulation or during the cultural regulations and the lesson planning and technology adoption. But these intersections crystallize our thoughts in building up the practices and in moving back and forth between the research design as well as theory which is one of the natures of DBR which we try to adopt in this research.

6 Discussion and Future Steps

As discussed earlier, we propose STEAM practices that use architecture, culture, and history to promote mathematics education in a new learning form. In these practices, we provide participants with various visualization methods such as 3D printing and GeoGebra AR.

In order that teachers adopt these practices as integrational tasks or programs to their curriculum, one of the challenges is the lack of support from schools in providing 3D printers, in particular the lack of sufficient guidance through training from their schools and educational entities that they belong to (Trust et al., 2021). Trust et al. assume that if a teacher attends a workshop on 3D printing and would try to adopt this technology and map it to the TPACK as to the content and pedagogy, this would be still challenging without the school support and affording such technologies. This was one of the reasons that made us avail several physical and digital technologies to the teachers through the DLP to overcome technological equipment deficiencies in schools. Moreover, we also noticed that teachers should learn 3D printing as part of the introductory phase of these practices in order to overcome all the common possible mistakes during the printing and try to consider them in the design phase. In the methodology, the DBR, we believe these practices can be best introduced as an integrational aspect to the standard curriculum. It uses many disciplines, so it fuses many fields of study and technologies, making it broader than just a single scope in a certain curriculum. Therefore, we propose these practices to be part of the integrational programs that follow the DBR and seek continuous feedback and enhancements. The proposed practices are considered unguided tasks to leave room for the participants for exploration, creativity, reasoning, and problem-solving while ensuring the teachers' guidance during the whole process in order for the unguided tasks using GeoGebra to be well solved and achieved by the students (Olsson & Granberg, 2019).

7 Conclusions

The proposed STEAM practices introduce a new mixture of recipes in the mathematical field combining architecture, culture, and history. These practices allowed Upper Austrian students to model various architectures from different places while connecting to its historical facts and cultural motivations. In this chapter, the architectural models were visualized digitally by the students using GeoGebra

AR or physically by us as a prototype using 3D printing. These technologies that result in different visualization forms are crystallizing the concept of 3D transformations for the participants that can add value to the participant's reasoning, creativity, problem-solving, exploration to result in a more engaging learning environment. Lieban and Lavicza (2019) believe that both physical and digital representation of assets can significantly influence the mental efficiency of reasoning and thinking towards the models that are geometrically based.

To conclude, the output of a DBR cycle for this research that has been advised by (Plomp, 2013) is that it should get empirically tested, and there should be clear output evidence and prove its applicability for further cycles and further adoption in the educational field. Our proposed contribution is the teachers' reflections and students' outputs in applying such practices. This chapter has also highlighted mapping the practice applications to the proposed theoretical framework and how the theories affected the design and vice versa in the Upper Austria intervention. The theories fusion managed to help us bring these practices' scattered ideas and notions into one by integrating many disciplines of study into one practice that combines architecture, culture, history, to mathematics education. The intervention result concluded that these practices might be applicable, keeping in mind all the variables that can affect its implementation, such as the technology affordance, the teacher and students' readiness for technology awareness, and GeoGebra know-how. As discussed, all these variables are important and should be kept into consideration when approaching these practices. However, DLP opened a window of hope for teachers who can adopt these practices to suit their readiness and without any limitations from the various options they can choose from to apply these practices. This will be discussed in our coming publications when presenting data from other countries that reflect other cultures and how these practices took new forms and new designs to be applied in various places to meet diverse cultures.

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Increasing the Skills on Occupationally Relevant Digital Technologies Among Students in Southern Denmark and Northern Germany: 3D Printing as a Learning Context in Regular Mathematics Classes

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1 Introduction

Within this conceptual paper, the project DiASper (Digital Working World from School Perspective) will be presented. The project is a cooperation of German and Danish network partners and is lead managed by the IPN—Leibniz-Institute for Science and Mathematics Education in Kiel. The aim of DiASper is, among other things, to develop teaching concepts for mathematics lessons that take up 3D printing (as an example of a digital technology of the digital workplace) as a learning context. This means that students should learn mathematics within the frame of the 3D printing processes in their regular mathematics lessons.

The background to this project goal are the changing demands on future workers in the digital working world. Since students in general education schools are

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to be educated to become responsible citizens, among other things, preparation for the challenges of the digital world of work should be an integral part of teaching. This paper therefore provides an insight into the conceptual framework of the DiASper project, starting by identifying some of the requirements of the digital workplace. In particular, Additive Manufacturing/3D printing will be discussed as an example of a digital tool that has high potential for the current and future labor market (Sect. 2). The mathematics behind the processes of 3D printing will also be discussed in order to motivate the potential connection points to the regular content of mathematics lessons of German and Danish curricula, which will be examined in more detail in Sect. 4.

Likewise, a brief insight into the state of digitization in German schools and the use of digital media in mathematics education will be given in Sect. 3. Sect. 4 then introduces the DiASper project with its conceptual framework and the identified points of connection between the mathematics behind the technical processes of 3D printing and the content of regular mathematics lessons. For illustration purposes, an example of a teaching idea will be presented shortly. Within a short conclusion, the next steps of the project will finally be presented.

2 Digitalization of the Workplace

The increasing digitalization of the world has led to constant changes in society as well as the values and norms linked to it. As a result, everyday-life needs to be adapted accordingly. However, social areas are not the only ones transformed by the digitalization: During the last decade, the working world has undergone a digital transformation (Kirchner, 2015; BMAS, 2016; Kaiser et al., 2021). In a research report the German federal ministry of labor and social affairs (*Bundesministerium für Arbeit und Soziales* - BMAS) evaluated the digitalization of the workplace (BMAS, 2016). The report states, that the use of information and communications technology (ICT) in job-related contexts has increased (BMAS, 2016). However, ICT is not the only form of digital technology that has experienced a growth in the working world. Indicators of digitalization can be found in different sectors: Digital technologies are used in production processes, but also in creative domains as well as service and administration (Calvino et al., 2018). As a consequence, the BMAS speaks of an advancing change from analogue and/or mechanical processes to automatized processes. This transition influences not only processes of the working world (e.g. production processes), but first and foremost the required qualification profile of employees and future workers (Grundke et al., 2017; BMAS, 2020). This means that (future) employees need the knowledge to work with different machinery and technologies than before. This knowledge includes both technical knowledge and the ability to work with the given digital technology. According to research conducted by BMAS in 2020, some employees assume that the increase of digital technologies in job-related contexts demands the acquisition of new competences. After an analysis of competence profiles related to workplaces influenced by the digitalization, the BMAS states that competences that are required in the digitalized workplace cannot be seen as replacements of competences needed previously. They can, however, rather be characterized as a further development of formerly required skills (BMAS, 2020).

2.1 Additive Manufacturing as an Example of a Digital Technology

3D printing is an important new digital technology, which is increasingly used in various manufacturing sectors in industry and craft. There are various 3D printing processes that are collected under the umbrella term '3D printing'. This makes it difficult to clearly distinguish processes that cannot be assigned to this umbrella term. For this reason, one could refer to the following working definition: *3D printing is a process, in which a material is either first liquefied by a melting procedure or taken in its liquid or pulverized form. The liquefied mass is then either applied in layers from a movable, usually heated nozzle onto a base or the material, which has already solidified again, or solidified in layers with the help of sintering or other solidifying processes. The process is iterated so that an object which has previously been modeled is created layer by layer* (based on Fastermann, 2014; Junk, 2017). Due to this additive production method 3D printing is also called Additive Manufacturing. Based on the working definition, processes associated with 3D printing can be distinguished from traditional subtractive (e.g. milling, machining) or formative (e.g. casting) processes.

The advantages of the 3D printing process are the possibilities for innovation, since objects that are to be printed can expeditiously be adapted and individualized in the preferred modeling software and the possibility to produce intricate or especially small objects. It is also worth mentionable, that the printing process described in the working definition enables the use of 3D printing by non-professionals who do not have extensive technical or craft knowledge. This potential has also increased in recent years as alternatives to complex modeling software that are usable by laymen or printer-specific slicer programs can be accessed.

In 2018 and 2019, 314 Danish production companies were asked about their use of Additive Manufacturing. At that time, according to the survey, 25% of the asked companies were using 3D printing processes. Nevertheless, the report draws attention to the high potential of the technology, although it can be stated that 3D printing played a rather subordinate role in 2018 alongside robot technologies and the associated automation of production processes (Blichfeldt et al., 2020).

In February and March 2021, the German Association for Information Technology, Telecommunications and New Media (*Bundesverband Informationswissenschaft, Telekommunikation und neue Medien*—BITKOM) asked several industry branches in Germany about their views on 3D printing. The survey showed that 44% of the questioned companies were already using 3D printing in work processes and another 42% were planning and discussing the manufacturing process. Consequently, it is already possible to speak of a broad use of 3D printing technology in the German industry, which could be further expanded in the future, thus showing the potential and significance of the technology. The survey also provides a reasoning for this: The questioned companies state that the use of the 3D printing technology increases the possibility for the flexible design of work processes, for bridging supply bottlenecks (for example during the Covid19 pandemic) and for the individualization of products (BITKOM, 2021). In addition, the 3D printing technology is also expanding the possibilities in non-manufacturing and non-manual areas with processes such as bio-printing or Contour-Crafting.

2.2 The Mathematics Behind 3D Printing

To be able to analyze the mathematics behind the 3D printing, the entire process of 3D printing is divided into four interdependent technical processes: Modeling, tessellation/triangulation of the surface, slicing and printing (see for example Espera et al., 2019). In the following, these processes are described in more detail with regard to their mathematical content.

2.2.1 Modeling

The first step of 3D printing is the modeling phase, in which the object to be printed is constructed digitally. There are several modeling methods (cf. Dilling & Witzke, 2019): A real object can be scanned with the help of a 3D scanner. The scanner then provides the user with a digital model of the real object. Furthermore, an object can be modeled in a CAD (*Computer Aided Design*) software.

Some CAD software work on the basis of solid geometric figures which are composed (direct modeling). This means that the model is built together with the help of the Boolean Operators AND, OR and NOT. These operators are applied to solid figures in order to examine the basic operations Unions, Intersections and Complements. Other CAD software operate so that a two-dimensional area is modeled and eventually either extruded to a prism or a pyramid or rotated around a fixed axis into a body of rotation (parametric modeling). Another way of creating a digital model is via code: An object is built digitally with the help of variables, loops and logical operators (script-based modeling).

When the digital model is finished, one has to think about the optimal printing position. As overhanging parts for instance can only be printed with support structures, the optimal positioning of the digital object during the modeling process can save material.

2.2.2 Tessellation or Triangulation

Within the tessellation or triangulation process, the surface of the modeled object is described by a net of primitive geometric shapes (usually by squares or triangles, hence the term triangulation) and thereby positioned in a virtual printing space. Thereby, the modeled object is converted into a “language” that can be understood by computers. The most common format is the STL (Standard Triangulation Language)-format. However, different tessellation formats exist, each with a different focus. Within the STL-format, the description of the modeled object can be divided into different blocks with four parameters each (specification of the coordinates of three vertices as well as the surface normal of the triangular surface) (Dilling & Witzke, 2020).

Rounded surfaces are described by a finer net of triangles and consequently with a higher number of triangles. This is done to increase the accuracy of the representation of the modeled object. The approximation of a rounded surface is based on Leibniz’s idea that every curve can be approximated by a polygonal chain (Guicciardini, 1999).

2.2.3 Slicing

The slicing-phase is the final preparation phase for the printing process. To prepare the triangulated file for printing, the model is sliced into individual layers. Control commands such as the temperature of the print nozzle, a description of the travel path and the amount of material to be applied along the path traveled by the print nozzle are added. Beside control commands, support structures can be added, if necessary. If surfaces are rounded, the resolution of the print can be refined through adjustment of the layer thickness of the printing process.

2.2.4 Printing

The sliced model is eventually printed in layers. Within this process, the material is added in voxels, three-dimensional data points, which are the equivalent to pixels in a two-dimensional image. Depending on the selected printing process, the individual layers can be glued, welded or sintered to previous layers.

The previous Sect. 2 focused in the digitalization of the workplace and accompanying changes (e.g. in the competence profile of (future) employees). The following sections focus on the training and education of future employees. An important component of the preparation for the digital workplace is general education, which students attend over a period of nine to 13 years. Preparing students for the digitalized working world can be seen as one of the central missions of general school education.

3 Digitalization of German Schools

General education schools play a significant part in preparing children and young people for life in today's and tomorrow's society—including the preparation for the digitized workplace. Already in 2019, the Digital Education Charta[1] was launched in Germany as an alliance of representatives from politics, business, science and society. The Charta defines digital education as an integral part of general education, which is why digital education should also be part of the curriculum in general education schools, according to the alliance.

This is reflected both in the strategy for education in the digital world (*Strategie zur Bildung in der digitalen Welt*) defined by the Standing Conference of the Ministers for Education and Cultural Affairs of the Länder in the Federal Republic of Germany (*Kultusministerkonferenz* - KMK) and in the handout Digital Education—The Key to a Changing World (*Digitale Bildung—Der Schlüssel zu einer Welt im Wandel*) published by the Federal Ministry for Economic Affairs and Energy (*Bundesministerium für Wirtschaft und Energie* - BMWI) in 2016. Both publications assign general education schools a high status and consequently a great responsibility in the training of future skilled workers and their preparation for the digital work environment. While the BMWI's handout addresses the need for digital education in all phases of life and education, the KMK's digital strategy shows the extent to which digital education could be implemented in general

[1] See further https://charta-digitale-bildung.de/ (Gesellschaft für Informatik e.V.).

education schools. The KMK's digital education goals focus on active and independent participation and self-regulation in digital contexts. These digital educational goals are to be implemented in every subject at general education schools. The link between education and workplace is also highlighted in literature: The embedding of technology in educational settings can be seen as essential for learning processes (e.g. to empower students to self-regulate their learning processes), for supporting lifelong learning, for accessing knowledge and for motivating students (see for example Starke-Meyerring & Wilson, 2008; Willinsky, 2009; Selwyn, 2013).

Before the use of digital technologies in school can be discussed, the digital equipment of schools in Germany needs to be considered. This has already been done in comparison to other countries in the *International Computer and Information Literacy Study* (ICILS) (Fraillon et al., 2020). According to this study, the student-computer-ratio adds up to 9.7:1, which means that there is one computer, laptop or tablet-computer for about ten students. These computers are in Germany often stored in computer labs, which means that students often have no direct access to these computers. But in order to be able to work with most of the existing digital technologies, a computer is needed. It is worth mentionable that the Covid19-pandemic has benefitted the digitalization of German schools, as teachers and pupils needed to work together and communicate digitally as no actual meetings were allowed (Mußmann et al., 2021).

Beside the digital equipment, the existence of an IT contact person that helps with technical issues is important when working with digital technologies at schools. ICILS recorded that most schools in Germany rely on external support if at all (Fraillon et al., 2020). This could result in slower support and therefore a less amount of teachers working with the given digital technologies. The importance of accessibility is therefore highlighted in the ICILS report. Furthermore, there is also a call for pedagogical support (Fraillon et al., 2020). This refers to support given in order to include digital technologies in individual school lessons.

If schools were equipped in an optimal way both with digital equipment and IT contact persons, teachers still would need to be educated in how to use and how to incorporate the given digital technologies (Härtig et al., n. d.). Therefore, there is a need for further education for teachers. There is a range of possibilities that could help teachers with the use of digital equipment in their individual lessons. If digital technologies were to be incorporated early in teacher education, future teachers would already have basic knowledge when arriving at schools. Moreover, support structures both at schools and from the government would be needed to support teachers on their individual competence level.

If all the aforementioned conditions are given, teachers still need to be willing to incorporate digital technologies in their regular lessons. In order to develop teaching ideas and concepts that are easily accessible and usable in teaching, there should be a focus on existing curricular and mathematical content (Lipowsky, 2009). As a result, content that is already taught, could be taught in a different context or with a different focus. Still, the issue of added value remains that is also important when constructing teaching ideas that potentially could replace existing teaching ideas.

3.1 Digitalization in Mathematics Education

In order for mathematics education to be able to fulfill its general education mission in today's digitized world, digital goals must be placed in the focus of teaching in addition to, among others, logic, problem-solving, and modeling skills. According to Winter (1995), in order to have a general educational effect, mathematics instruction at lower and upper secondary education should enable the following three basic experiences: Firstly, mathematics education should address phenomena of our environment which concern or should concern us all. Secondly, mathematics education should enable students to get to know and understand mathematical objects and facts as a deductively ordered world of its own kind. And lastly, mathematics education should enable students to acquire problem-solving skills that go beyond mathematics in the confrontation with tasks.

As digital technologies are increasingly found in all areas of life, an examination of them and their modes of operation is inevitable. Therefore, the digital educational goal of active and self-determined participation in the digital world (KMK, 2017) is supported in particular by Winter's first basic experience in that learners should become familiar with and understand the basic processes of digital technologies through mathematics education. Likewise, learners should acquire a higher level of self-regulation competence (i.e. gradually assuming responsibility for their own learning) through the use of digital technologies in the classroom (KMK, 2017). In particular, this educational goal can be achieved by using digital technologies to enhance the design of lessons by incorporating digital learning environments and learning opportunities. By moving toward process- and outcome-based teaching and learning, learners can be supported in increasingly taking charge of planning and designing their learning goals and learning paths that are individually adapted to their learning levels (KMK, 2017). Consequently, this educational goal can be considered on the basis of Winter's second and third basic experiences. For mathematics education it can be deduced that especially modeling and problem-solving processes that have a relation to digital technologies can be useful to let learners rethink their own ways of thinking and to learn and use heuristic skills.

4 3D Printing as a Learning Context in Regular Mathematics Teaching

Within the research project that is connected to the DiAsper project (funded by the EU Interreg-program), 3D printing will be considered as a learning context. This approach can be distinguished from existing approaches in research: The two main foci in international research are to use 3D printing to visualize (complex) mathematics (see for example Sun & Li, 2017; Dilling & Witzke, 2019; Dilling et al., 2021) or to address the modeling aspect of the digital technology (see for example Aguilar, 2020; Lavicza et al., 2020; Bedewy et al., 2021). Within both approaches 3D printing or parts of 3D printing are used for descriptive mathematics teaching, in which 3D printed objects are used either for lesson preparation or as a visualization opportunity. Consequently, 3D printing can in both research approaches be seen as an effective vehicle to facilitate the students' access to the mathematical content of the individual lessons. Other research studies focus, for example, on the incorporation of CAD software for teaching the concept of volume of solids (see for example Ng, 2017) or art-related mathematical projects (see for example Vanscoder, 2014; Menano et al., 2019).

In the DiASper project we want to incorporate 3D printing as a *learning context* in mathematics education. Thus, we chose a different focus, as the primary role of 3D printing in this educational context is not specifically to facilitate the learning of mathematical content. Instead, we want to develop teaching ideas for mathematics lessons for Grades 5–13 in which students can learn the curricular mathematics content, on the one hand, and get an idea about the 3D printing processes as part of their digital education and preparation for the digital workplace, on the other hand. The teaching concepts that are created within the project use the four technical processes of 3D printing (modeling, triangulation, slicing, and printing process) to generate learning opportunities for mathematics learning in the context of 3D printing. These learning opportunities are intended to promote students‘ digital competences in addition to mathematical competences (see for example MBWK SH, 2018) through exposure to a digital technology. Hence, an integration of 3D printing into regular mathematics lessons is pursued. Research questions to be pursued in the DiASper project include how these teaching concepts should be designed (design principles) so that they not only have didactic added value for regular mathematics teaching in terms of promoting mathematical and digital skills, but also give students an insight into the digital workplace.

4.1 Conceptual Framework

A main focus of this research project is the implementation of the 3D printing process in mathematical school lessons based on the given curricula (educational plans, subject requirements, and schools' internal curricula). This approach and procedure were chosen in order to potentially increase acceptance among teachers, since it must be kept in mind that changes in new content are adopted with greater willingness by teachers if they have a focus on already existing curricular content and can thus be directly integrated into the already existing subject lessons (Lipowsky, 2009). Therefore, we chose to rely on the curriculum of the federal state of Schleswig–Holstein and Denmark to connect existing mathematical content with the four technical processes of 3D printing.

Another goal of the project is to contribute to vocational orientation and preparation with the help of the teaching concepts and by connecting the industry and schools. Therefore, the connection process starts with the general frameworks of the industry and schools. These include for instance competences that are expected from future employees or the current curriculum. The general frameworks of industry and schools result in a list of expectations and needs that have to be addressed or fulfilled by teaching ideas with the intention of their connection. Consequently, the given expectations and needs influence the development of teaching ideas. After developing a teaching idea on the basis of the given curricula, it is evaluated with the help of teachers and industry contacts[2] before it is tested in schools. Within these evaluation cycles, the teaching ideas are first presented to teachers. They are asked to comment on the following questions through written feedback:

- *Is the concept written in a comprehensible way? (Can I, as a teacher who does not have extensive knowledge of 3D printing, understand the concept? Should aspects be emphasized more clearly or explained better?)*
- *Are the tasks appropriate for my students in terms of the level of challenge or is greater differentiation needed?*

[2] The project's industry contacts include contacts from various industries, including medical technology, manufacturing technology and prototype production. Most of the contacts have a focus on the vocational educational perspective, as they are active in the training of future employees.

- *When I imagine that I, as a teacher, use the exercise blocks given in the teaching concept in my mathematics lessons: Where might I need further help or support? How can this help/support look like?*
- *Is the connection to 3D printing actually palpable?*

While the teachers' evaluation cycle focuses on the feasibility and acceptability of the teaching concept, the evaluation cycle in the industry is dedicated to the authenticity of the integration of 3D printing in the teaching concepts. Within the second evaluation cycle, industry contacts are also asked to comment on a set of questions through written feedback:

- *Are the competences addressed in the teaching concept relevant for (future) employees in digital workplaces that use 3D printing and other digital production processes?*
- *Is there a connection between the actual authentic use of the 3D printing technology in the digital workplace and the use of 3D printing in the teaching concepts?*

This concluding evaluation in the process has the purpose of combining theoretical and practical aspects, i.e. the revision of the researchers' theoretical input through practical input of teachers and industry contacts. The creation and evaluation of the teaching ideas is based on the design-based research approach and follows the known evaluation cycles of expert feedback, where the experts both stem from schools and industry, and revision (Plomp & Nieveen, 2013).

4.2 3D Printing as a Learning Context in Mathematics Education

As the teaching concepts are constructed on the basis of the given curricula, this chapter shows possible points of connection to today's lower and upper secondary mathematics education in Denmark and Germany. The connection points are divided according to the four technical processes of 3D printing (modeling, triangulation, slicing, printing process).

4.2.1 Modeling

Of the various modeling options, only modeling in CAD-software, in which Boolean operators are used to create digital objects, is considered in this research approach. The basis of modeling competence is the understanding of a real situation or if modeling is based on inner-mathematical condition, the understanding of an inner-mathematical situation. This given situation has to be simplified or idealized in a subsequent mental process so that it can be translated into mathematics (Stillmann, 2019). As mathematical modeling[3] is one of the six general subject-related skills in mathematics curriculum of the federal state of Schleswig–Holstein (MBWK SH, 2014) as well as the Danish curriculum (BUM, 2019), fostering this competence is one of the central elements in mathematics education.

In the following list elements of the curricula of Denmark and the federal state of Schleswig–Holstein are listed that can be taught using 3D printing as a learning context:

- *Drawing a construction of an object that is to modeled on a piece of paper can serve as preparatory measures that can make the modeling process easier.*
- *To master the transition from a two-dimensional sketch or an idea to a three-dimensional model, students need the ability for spatial visualization. This competence is especially necessary when modeling in the cartographic coordinate system (e.g. when using GeoGebra).*
- *In order to model a closed solid, students first and foremost need knowledge about solids (e.g. polyhedron on the basis of the definition of the simplex), as basic solids (e.g. cuboids, cubes, pyramids or prisms) can be used in some CAD software in order to construct the final object.*
- *As already mentioned, Boolean operators are important tools in some CAD programs. With the help of the Boolean operators AND, OR and NOT applied to solids, complex objects can be constructed. The first Boolean operator (union) can already be thematized in lower secondary education as the composition of bodies with disjoint interiors. The second Boolean operator (difference) does not play a role in the curricula of Denmark and the federal state of Schleswig–Holstein. The third Boolean operator (intersection) is only addressed in the curricula of the lower secondary education in both Denmark and Schleswig–Holstein within the educational area of functional relationships (i.e. special intersections of two or more functions), but not in geometry. On the other hand, the geometric consideration and mathematical discussion of intersections of curves and planes are part of analytic geometry in upper secondary education in both parts of the DiASper project.*

[3] For further research on mathematical modeling with the help of digital tools, see for example Hankeln (2018) on mathematical modeling with a dynamic geometry software.

- *The last mathematical elements that can be thematized within the learning context 3D printing are (point) reflection, rotation and translation. After the object is modeled, it makes sense to think about the optimal printing position of the modeled object. The search for the optimal printing position includes considerations of the object's alignment. As overhanging parts for instance can only be printed with support structures, it could be advisable to mirror or to rotate the modeled object. As a consequence, less structural support is needed during the printing process.*

4.2.2 Triangulation

During the triangulation phase the surface of the object is covered with simple geometric shapes, usually triangles or squares. The resulting code can vary depending on the chosen format. The less extensive OFF (Object-file-format)-code can be used to consider the definition of simplexes on the basis of their vertices, as the vertices of the modeled object, each connected by an edge, are specified one by one within this format. Thus, the OFF-code uses only the coordinates of the vertices of the modeled object and can therefore be used for a simple mathematical representation of polyhedron. As a result, only polyhedron can be described with this format, as rounded surfaces cannot be grasped by the code.

Another possibility is the STL (Standard Triangulation Language)-format: Within this format, each surface of the subject is divided into triangles which are the basis for each block of the code. Every code-block of the STL-format is made up of one coordinate set for each vertex as well as the facet normal of the triangle given by the three vertices. Thus, the entirety of the code can be understood as the object resulting from the intersection of two planes each. In the upper secondary education, students learn about parametrical and coordinate descriptions (e.g. parameter form, three-point form, coordinate form or normal form) of lines and planes in the three-dimensional space. Through the previously named consideration of the STL-format as an intersection, the known plane representations can be extended by a form relevant for the digital working world. Moreover, the new form of representing planes and intersections of planes consists of known elements (points that lay on a plane as well as the surface normal).

When looking at the triangulation of rounded surface neither the consideration of the OFF-code as a mathematical representation of polyhedron nor the consideration of the STL-format as an intersection are sufficient. By addressing the triangulation of rounded surface, the students can look at limits and limit approximation processes. The finer the net of triangles describing the rounded surface, the better the coverage of it and the higher the final printing resolution. The possibility of approximating rounded surfaces by a net of triangles is based on

Leibniz's infinitesimal calculus theory (i.e. that every curve is composed of infinitely many infinitesimally small rectangular pieces).

4.2.3 Slicing

Within the slicing process, control commands are added. Especially the travel path of the extruder head can be content in mathematics education, as the mathematically optimized print-way can be described with the help of trigonometric functions during lower secondary education. This can be done, if the mathematically optimized travel path is considered as a continuous curve which is limited to the area of the layer that is to be printed. In the case of a uniform travel path in a rectangular form, the print-way can be represented by a trigonometric function. Additionally, if the parameters are adjusted, the actual travel path can be approximated. Within the upper secondary education, functional descriptions of the travel path of non-rectangular layers can be addressed.

As the G-code also consists of information regarding the material, the constant distance of the extruder head to the previous material layer can be considered in upper secondary mathematics education in order to be able to record settings related to the movement of the print nozzle. These geometric descriptions, such as the distance of a point (e.g. the extruder head) to a plane (e.g. the last material layer), are part of the analytic geometry curricula both in Denmark and Schleswig–Holstein.

4.2.4 Printing Process

The printing process is more of a technical than of a mathematical nature. However, it can be used as an approach for two different mathematical issues:

- Via the printing process, an enactive (i.e. action-based) approach in the sense of Bruner (1966) to Cavalieri's principle can be pursued through the layer-by-layer construction of an object in the process. Different objects which have the same area in each layer can therefore be discussed. However, the principle would then have to be formalized mathematically in further steps.
- In the preparation of teaching students about volumes, printing the same object with different diameters of the extruder head opens up the possibility of thematizing at least the upper sum, since the filling of a surface by 3D printing becomes more accurate the smaller the diameter of the nozzle. However, it is not possible to thematize the lower sum. Consequently, this would have to be substituted accordingly with the help of other enactive processes.

4.2.5 Digital Competences in Mathematics Education

In addition to the mathematical content that can be taught in the learning context of 3D printing and the mathematical competences to be fostered in the emerging teaching concepts, the implementation of 3D printing also offers opportunities to expand the students' digital competences: By working with a 3D printer, the students are confronted with a digital tool whose potential in the digital working world is currently being expanded. Likewise, the students come into contact with software (e.g. CAD and slicer software). These digital contact points can be used to promote the competences of students to use digital tools according to their needs and to use technical and/or digital tools to plan and carry out development and production processes.

4.3 Example of a Teaching Idea

A first teaching idea has been developed whose content are composite figures which is part of the Grade 6 curriculum in Germany and Denmark (MBWK SH, 2014; BUM, 2019). In both countries you often find that three-dimensional objects are taught with the help of axonometric depictions. CAD software such as TinkerCAD[4] also make use of this graphical procedure, but within this program students can look at an object from each side (Eryilmaz & Deniz, 2021). The composition of shapes can also be done easily through a 'drag-and-drop' function. This can be seen as an alternative to putting actual geometrically shaped objects together by hand in geometry lessons. Before teaching about composite figures, students should already have sufficient knowledge about solid geometric figures, such as prisms, cubes, cuboids, cylinders, pyramids and cones.

Every teaching idea consists of different exercise blocks with one of the four technical processes of 3D printing as the learning context. Within this example the technical process of modeling serves as the overall learning context, whereas the concept of humanoid robots is taken as the direct context of the tasks. Within this exercise block students are asked to model humanoid robots and to perform mathematical calculations on given models or those modeled by them. A first task is the construction of a digital version of a humanoid robot based on a given sketch (see Fig. 1). In this task students have to master the transition from a two-dimensional sketch to a three-dimensional modeled object. To help students with

[4] See further https://www.tinkercad.com.

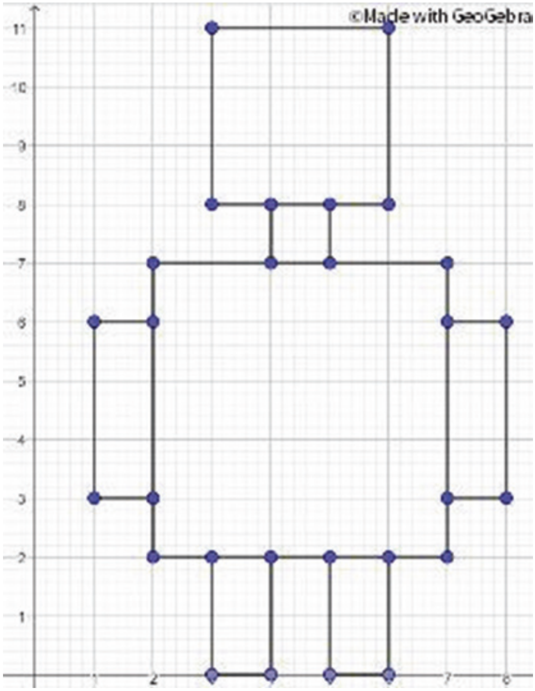

Fig. 1 Given sketch of a humanoid robot

this transition more information can be given. The two-dimensional sketch that is given Through the use of a two-dimensional coordinate system, the height and the width of the humanoid robot are given, but further measurements can be an example for helpful structures in the task. The information can be given as part of the task: *Aydan has made a sketch for her robotics project. The sketch shows the front of her version of a humanoid robot. Help her with constructing a digital object on the basis of her draft. Every part of the robot should have a depth of 1 cm.*

This exercise can be used to foster both mathematical and digital competences as defined in the Danish and German mathematics curriculum (MBWK SH, 2014; BUM, 2019). As students are asked to model something on the basis of a given sketch, they are going through a modeling process,[5] in which the given

[5] Competence "Mathematical modeling": K3 - Mathematisch modellieren in the German mathematics curriculum, K2 - Modellering in the Danish mathematics curriculum.

information is taken and built into a new object which has to be constructed by the individual student. Furthermore, the sketch is a mathematical representation that has to be used in order to fulfill the task.[6] Students have to take the given information (height and width of the individual elements of the robot as well as the total height) from the sketch and transfer it into a three-dimensional space. To construct a polyhedron (here: cuboids or cubes), the further information of the depth can be used, if necessary. Then, based on the sketch, the students have to assemble the resulting cuboids and cubes to resemble the sketch of the robot. Hereby, students learn to assemble individual basic geometric bodies into a composite figure. Within the task, students are asked to construct a digital object, which means that they have to work with technical elements of mathematics.[7] Finally, through the use of a digital CAD software (e.g. TinkerCAD), students encounter a digital tool to develop and produce a digital model. They have to be able to use the elements of this digital tool and utilize it in a way that fits the task.

As already mentioned, the task can be adapted to adequately fit the students' individual current state of learning. The above mentioned task could be modified so that it is not based on a given sketch. Thus, the students would be able to work independently in a creative way. To ensure students' progress during the different modeling phase (i.e. planning, constructing a two-dimensional sketch, constructing a three-dimensional model), one can work with different requirements (e.g. requirements about mandatory body-parts) and support structures (i.e. scaffolding).

The above mentioned task can also be used to target the guiding principle of measuring (cf. MBWK SH, 2014; BUM, 2019). By adding extra information like the height of the actual built or printed robot and a question targeting the surface area, the modeling of an object can be combined with actual calculation processes*: Aydan has built her humanoid robot out of cardboard. It has a height of 1.10 m just like Aydan's younger brother. She wants to glue silver paper on its surface so that it looks like a real robot made of metal. How can she calculate, how much paper she will need? Explain and calculate afterwards.* As students are

[6] Competence "Using representations": K4 - Mathematische Darstellungen verwenden in the German mathematics curriculum, K4 - Repræsentation og symbolbehandling in the Danish curriculum.

[7] Competence "Using symbolic, formal and technical elements of mathematics": K5 - Mit symbolischen, formalen und technischen Elementen der Mathematik umgehen in the German mathematics curriculum, K6 - Hjælpemidler in the Danish curriculum.

confronted with a problem of how to calculate the surface area, this task would foster their problem-solving skills as well as their mathematical modeling competence and their ability to use mathematical representations (MBWK SH, 2014; BUM, 2019).

5 Conclusion

The societal changes connected to the digitalization of the world have affected the private and especially the working world in recent decades and pose new challenges for general school education. Students need to be prepared for elements of the digitalized workplace in general education schools, as they play an integral role in preparing students for the future working world. Therefore, elements of the digitalized workplace should already be integrated into general education subject lessons. Among the innovative digital technologies, 3D printing in particular has an important role in today's digitalized world of work and the potential to remain relevant in the workplace of the future. Additive Manufacturing can be found in production as well as in planning and development processes, so that knowledge of and around 3D printing technology can be beneficial for future employees.

Previous research on 3D printing in mathematics education developed approaches how to use this technology as a vehicle for facilitating learning mathematics (cf. Sect. 4). Within the present research project, 3D printing will be implemented as a learning context by using the technical processes of 3D printing to address mathematical content in schools in order to enhance students' mathematical competences. In addition to the mathematical competences to be promoted, the teaching concepts also focus on digital competences through the integration of digital tools such as 3D printers and software (for example modeling or slicing software).

The example given in Sect. 4.3 is taken from the first developed teaching idea in the context of the DiASper project on composite figures. This teaching idea has already gone through the first evaluation cycle and is further coordinated with teachers at different schools in Denmark and Schleswig–Holstein. Initial feedback showed among other results a desire for more in-depth introductions to the various software. The teachers also indicated that a focus should be placed on the differentiability and adaptability of the teaching idea in order to accommodate heterogeneous learning groups. Further feedback from teachers will be used to derive design principles for instructional approaches in mathematics teaching that

- *integrate aspects of the digital world of work into mathematics lessons in a way that is appropriate for the learner,*
- *focus on and promote students' mathematical and digital competences, and*
- *give students an insight into the digital world of work and associated skills profiles, thereby contributing to career selection and preparation.*

In order to finally test these design principles to determine whether teaching ideas, based on the formulated principles, add value to mathematics teaching in schools in Denmark and Schleswig–Holstein, a pre-post testing will be conducted. The teaching concepts will initially be tested only at partner schools of the DiASper project in Schleswig–Holstein (Northern Germany) and the regions Syddanmark and Sjælland (both Southern Denmark). These schools have been equipped with 3D printers in cooperation with the project. The authors are well aware that not all schools in these regions have access to a 3D printer. However, since the teaching concepts use 3D printing as a learning context for regular mathematics lessons instead of a learning content, access to a 3D printer can be deemed as not necessarily crucial, even though it can certainly be seen as beneficial.

Finally, the resulting teaching ideas and materials will be made available via an *open educational resources* (OER) platform to teachers who did not participate in the project or who do not work at a school within the project region. In the long term, all concepts will be made available to teachers via an OER platform that makes it possible to develop them further.

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Vignettes of Research on the Promise of Mathematical Making in Teacher Preparation

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1 Introduction

Prospective elementary teachers (PMTs) have been characterized as coming to teacher preparation with limited conceptions of mathematics (AMTE, 2013) and a model of mathematics teaching that is oriented more toward the transmission of rules and procedures (Ball, 1990; Ma, 1999; Thompson, 1984) than to the cultivation of conceptual understanding. Consequently, teacher preparation must offer

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opportunities that challenge this model of mathematics teaching and learning, and provide gateways to meaningful interactions and deepened understanding of both content and pedagogy. Connecting with a body of research that conceives of Making in education as the creative practice of designing, building, and innovating with analog and digital tools and materials (Halverson & Sheridan, 2014), we present one such opportunity that we centered in a novel Making-oriented experience within mathematics teacher preparation. That experience tasks prospective teachers of elementary mathematics (PMTs) with digitally designing, 3D printing, and sharing an original manipulative with a child to support and promote their mathematical understanding. In seeking to determine what this experience might offer PMTs as they prepare for the work of mathematics teaching, our work has pursued a number of theoretical directions and methodological approaches to address research questions at the intersections of teacher identity, teacher knowledge, pedagogy, and design.

Schad and Jones's (2020) review of the research on the Maker movement in K12 education finds that students' learning through Making dominates that literature, with foci that include the improvement of STEM learning outcomes, increasing student motivation and interest in STEM, and increasing equity by broadening notions of what counts as Making in STEM education. The extent to which Schad and Jones's review mentions research on what *teachers* learn through Making is through studies of how they learn to design and run makerspaces, and how they learn to integrate maker-centered learning strategies (Clapp et al., 2016) into their own curriculum. Thus, there is almost no research on supporting teacher learning through Making. We situate our work within that gap in the research.

1.1 Chapter Structure

In this chapter, we share vignettes of several research projects that address the research question that broadly frames this work: *What are the potential benefits of a Making experience within mathematics teacher preparation?* These vignettes provide snapshots of our work; within footnotes at the headings for each of these vignettes, they also direct the reader to where they can read more about it. As the Making experience we designed is central to each of these vignettes, we begin by describing it in detail in order to enable a grounding for their theoretical and practical rationales, which are presented afterwards. As the chapter outline in Fig. 1 illustrates, the overarching research question frames the project, and the pilot study becomes the launching point on the research trajectory of our other

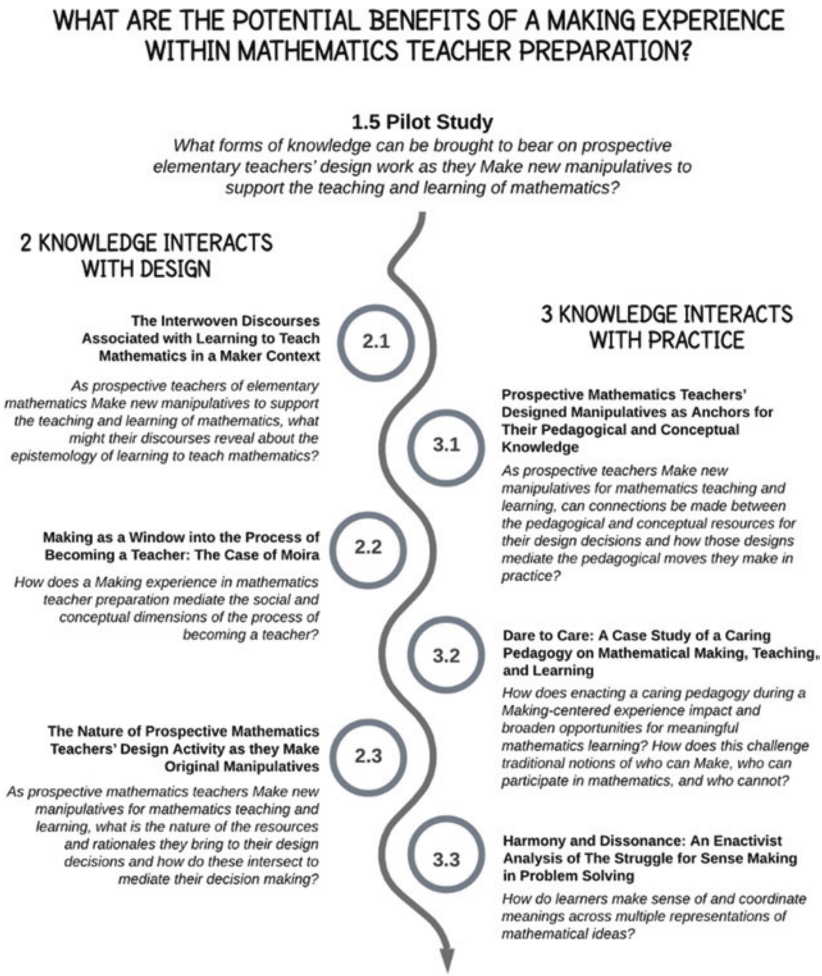

Fig. 1 The outline for this chapter

projects. Following the presentation of the pilot study, we organize the presentation of our other projects into two sections. The first of these appears in Sect. 2, where we present those that occurred within the design environment of PMTs' making, where "knowledge interacts with design." The second appears in Sect. 3, where we present those that occurred outside of it, where "knowledge interacts with practice." Each vignette follows a similar structure: Introduction, Theoretical Framing & Methods, Findings, and Implications. Fig. 1 provides the names of each of these vignettes, the research question(s) they address, and the sections in which they appear.

Lastly, we wish to share that the trajectory of this research has hardly been linear. Only some of these vignettes flow "naturally" from others (e.g., 2.1 and 2.2; 2.3 and 3.1), and this has implications for how the reader may wish to engage with this chapter. Although the initial objective of this project was to explore what prospective teachers of elementary mathematics learn—in relation to knowledge and about themselves—as they make new manipulatives, that trajectory generated new limbs as the qualitative inquiry "allow[ed] for things to emerge on their own terms" (Thorp, 2005, p. 117). The wavy spine of the timeline in Fig. 1 is meant to depict the path of our research as it was "laid down while walking" (Varela, 1987). It follows, then, that any path the reader wishes to take as they explore these vignettes would be a fruitful one.

1.2 Making in Mathematics Teacher Preparation

This work connects with a body of literature that frames *teachers as designers* (e.g., Brown, 2009; Maher, 1987) of learning experiences and of the material resources that mediate them. We conceive of design quite broadly to include the "intentional activity of transforming ideas and knowledge" (Carvalho et al., 2019, p. 79) into "tangible, meaningful artifacts" (Koehler & Mishra, 2005, p. 135). Our purpose in doing so is to introduce a pedagogically genuine, open-ended, and iterative design experience into mathematics teacher preparation that is centered on the Making of an original physical manipulative for mathematics teaching and learning. We hypothesized that the experience would afford unique pathways of diversified engagement that could promote an epistemic shift toward inquiry-oriented creative and participatory practices that support teaching and learning mathematics with joy and understanding. Accordingly, we view this Making experience from a constructionist perspective (Harel & Papert, 1991), which argues that meaningful learning happens through the designing and sharing of digital or physical artifacts "that learners care about and have some degree of agency over" (Schad & Jones, 2020, p. 2). Indeed, when teachers take agency over the design of their own curriculum materials, they assume ownership over them and the learning environments they generate and come to see themselves as agents of curricular and pedagogical reform (Leander & Osborne, 2008; Priestley et al., 2012).

1.3 Curriculum Context

Data collection for this research took place over three implementations of the Making experience between the spring of 2017 and the spring of 2020. The study occurred within a specialized mathematics content course for prospective mathematics teachers (PMTs) at a mid-sized public university in the northeastern United States. Although the university is a Hispanic Serving Institution (HSI), the majority of the students in the three classes were not Hispanic. Over 90 % of all three classes of participants identify as female. These demographics are typical of the prospective elementary teacher population.

Situated in an instructional context in which the teacher educators of the courses practiced an inquiry-oriented pedagogy grounded in a constructivist theory of learning, the course engaged students in a Making experience defined by the following task: "The purpose of this project is for you to 3D design and print an original physical tool (or 'manipulative') that can be used in teaching a mathematical idea, along with corresponding tasks to be completed by a learner using the tool." To realize their project, the PMTs learned to use the web-based Tinkercad (Autodesk, Inc., 2020; see Fig. 2, left) digital modeling platform. The design of their manipulative and a corresponding set of problem-solving tasks aimed to reflect a) PMTs' knowledge of what it means to do mathematics and how we learn with physical tools, b) their knowledge of elementary-level mathematics content, and c) their perspective on pedagogy and curriculum in mathematics education. In addition to the design of the tool, four written project components comprised the data corpus: 1) a "Math Autobiography" that calls on students to reflect on their experiences as a student of mathematics and consider how those experiences might inform their future work as mathematics teachers; 2) an "Idea Assignment" that describes PMTs' initial thoughts about a manipulative they want to create; 3) a "Project Rationale," which is an account of how their design

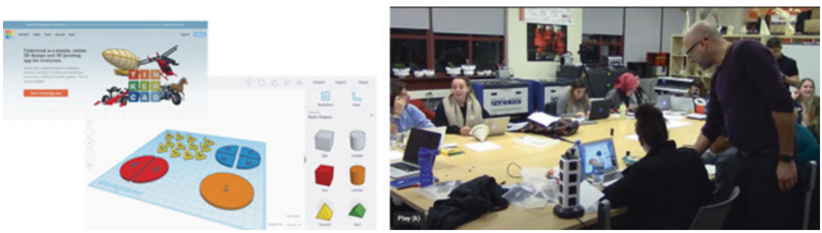

Fig. 2 The Tinkercad design environment (left) and the design setting (right)

reflects an understanding of what it means to know and learn mathematics; and 4) a “Final Paper/Reflection” that presents findings from a “Getting to Know You” interview and problem-solving interviews conducted by the PMTs with their tool and an elementary-age target student.

Although the particular approaches varied somewhat across the three implementations of the Making experience, in each instance, PMTs worked on their designs during in-class design sessions during three or four of the thirteen weekly class meetings. All design sessions—as well as all other class meetings—were held in a large design lab (Fig. 2, right), which we chose deliberately because we imagined that the PMTs’ design activity would be more inspired in an environment intentionally configured with affordances to support it and that can accommodate the kind of immersive, collaborative social space that nourishes it. Digital fabrication technologies, including about forty 3D printers and two laser cutters, lined the perimeter of the space, as did bookcases of evocative 3D-printed objects designed by students in other courses.

1.4 Practical and Theoretical Rationales

As we have already alluded to in passing, the design of the Making experience is grounded in the learning theories of constructivism and constructionism. These theories recognize that knowledge is actively constructed by a learner, with constructionism adding the dimension that the knowledge should be constructed during the process of making a shareable object (Harel & Papert, 1991). Then we took a *Learning by Design* approach (Koehler & Mishra, 2005) to leveraging and potentially advancing this knowledge. Learning by design engages PMTs in the active inquiry, research, and *design*—or the purposeful imagining, planning, and intending—that precedes and interacts with Making. In doing so, it honors the proposition that it is productive to develop teacher knowledge within a context that recognizes the interactions and connections among these constituent domains of knowledge. The approach has methodological advantages, as well, since it opens a window into the interplay between a PMT’s iteratively evolving artifact and the application of teacher knowledge domains in the artifact’s development. Indeed, Pratt and Noss’s case study (2010) offers a proof of concept that a learning by design approach provides a venue for characterizing the interplay among a participant’s knowledge domains as they are invoked during the design process.

1.5 The Pilot Study[1]

The pilot study for this project was an exploratory one. It took place in the spring of 2017 and its intention was to broadly discern the value of engaging PMTs in Making and design practices that were made possible by increased access to human-centered design practices and digital fabrication technologies, and that we hypothesized would inform their pedagogical and conceptual thinking. Our analysis of the data had its genesis in a narrower focus, which is expressed in the following research question: *What forms of knowledge can be brought to bear on prospective elementary teachers' design work as they Make new manipulatives to support the teaching and learning of mathematics?*

1.5.1 A Teacher Knowledge Analysis of Pedagogical Content Knowledge in Interaction with Design Activity

We took a grounded theory approach (Corbin & Strauss, 2008) to the analysis of PMTs' written project artifacts using the teacher knowledge literature (e.g., Ball et al., 2008; Koehler & Mishra, 2009; Shulman, 1986) to establish base codes followed by iterative analyses of design cases to generate new ones. We found from that analysis that students used a variety of forms of knowledge in the course of their design activity, including knowledge of mathematics, specialized mathematical knowledge, knowledge of standards and curriculum, knowledge of research on student learning, and knowledge of how students learn with tools as informed by a constructivist perspective. PMTs wrote about how manipulatives aim to embed mathematical knowledge, how ideas may be abstracted to construct ideas through students' manipulations of their tools, and how learners can use their manipulatives as tools to learn through problem solving rather than memorization. They also spoke to how affective concerns and personal experiences informed their design ideas. A complete articulation of these results that provides evidence of each identified form of knowledge can be found in (Greenstein & Seventko, 2017). We only wish to mention that the surprising range of knowledge that PMTs brought to their design activity convinced us of the promise of a Making-oriented experience within mathematics teacher preparation. Consequently, we were convinced of the promise of further exploring the potential benefits of that experience.

[1] Greenstein et al. (2019); Greenstein and Olmanson (2018); Greenstein and Seventko (2017)

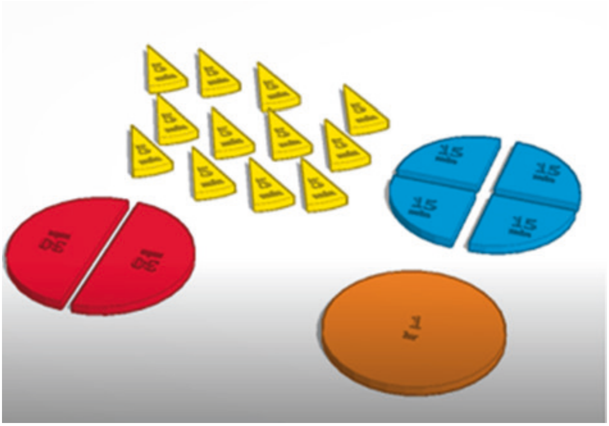

Fig. 3 Casey and Mia's Minute Minis

As a next step, we took a revelatory case study approach (Yin, 2009) to uncover deeper insights that could help us better understand what we deemed to be an underexplored phenomenon. Specifically, we sought to discern a relationship in the emergent interactions between PMTs' pedagogical content knowledge (PCK) and their design activity. Working individually and asynchronously, two of the authors took their own grounded theory approach (Corbin & Strauss, 2008) to open coding an initial pass of the data using base codes derived from the literature on teacher knowledge. And in an iterative fashion, following each independent review of the data, they came together for a collaborative consultation where they shared and refined their codes and interpretations (Patton, 2002). At the culmination of that analysis, they had each selected the same two working groups for case studies: "Avery" (working alone) and "Casey and Mia" (working together). The researchers shared that they were each drawn to the ways these PMTs made authentic connections between mathematics and the tools they were designing. Here we share some of the findings of the case of Casey and Mia's "Minute Minis" project (see Fig. 3). In (Greenstein et al., 2019), we also present with the case of Avery's "Even Number Tool."

1.5.2 Findings

Casey and Mia were inspired to design a manipulative that could help children reason about the abstract concept of time. In their design rationale, they hypothesized about breaking through the ordinarily obscure nature of time to make it more accessible to learners: “The main goal of this project is to give a concrete representation of the relationship between hours and minutes. Using manipulatives is especially important when exploring new concepts, and [since] time is a very abstract concept, it is especially pertinent that students have something concrete to work with.” This idea came from Casey, whose thinking was informed by coincidental work as a student teacher:

> Currently, most of the 2nd graders in my class can tell time to the nearest half hour, yet I am unsure of how they know how to do this. Is it just because they know that when the minute hand is pointing at the 6, I say _:30 and when it’s pointing at the 12, I say _:00? Or do they have a… deeper understanding of time and how a clock works?

These considerations reflect how Casey’s PCK (wondering about students’ conceptions of time) informed her design. Over the course of the project, these questions developed into other strands of knowledge that she and Mia used to investigate these issues as the two were driven by a desire to transform potentially limited conceptions of time from memorized models into deeper mathematical meanings. Drawing upon other aspects of PCK and of mathematical and curricular knowledge, Casey and Mia took an existing design of fraction circles and used concepts from geometry to amend it for their objectives. They wrote:

> [We] will be using the same concept of fraction circles, yet instead of labeling them with a fraction, they will be labeled with minutes. For instance, a whole circle will be labeled ‘1 hour,’ while two half circles will be labeled ‘30 minutes.’ [We] will also have [fraction] circles for 15 and 5 minutes.

One of Casey and Mia’s key design decisions focused on being able to “visually illustrate the concept of minutes as fractions of an hour.” The circular shape was important to them in ensuring “that students would be able to use the Minute Minis directly on the face of a clock. This would aid [them] in exploring the relationship between where the minute hand is pointing and the number of minutes past the hour.” They deemed this design affordance essential in supporting student inquiry of the fractional ideas embedded in the tool so that the child could assemble the fractional pieces to compute time.

In summary, Casey and Mia shared reflections that leveraged their knowledge of fractions and area to mediate a bridge between abstract and concrete representations of time. By supplementing the traditional focus of instruction about time with a concrete representation that facilitates conceptual connections between a clock face and its underlying area properties, they drew on this knowledge to articulate the mathematical richness underlying their manipulative and its possible uses by a child.

1.5.3 Implications

As PMTs assumed the multi-faceted role of teachers as designers of instruction within a space of technological possibilities, they created powerful and innovative tools, and their work demonstrated a rich and mature repertoire of knowledge domains that we are not typically afforded opportunities to see (AMTE, 2013). The identification and advancement of this knowledge suggested to us the promise of a Making experience within mathematics teacher preparation and convinced us of the value of discerning what other benefits the experience might offer. Accordingly, our pilot work became the launching point on that trajectory of research. In the next two sections, we present selected vignettes of research on this trajectory.

2 Knowledge Interacts with Design

With the theoretical and practical rationales for the Making experience now laid out, and with evidence from pilot work that speaks to the potential benefits of that experience, we now present vignettes of other research we have conducted on this trajectory. In this section, we share findings of studies that occurred within the design environment of PMTs' Making. Then, in Sect. 3, we present those that occurred outside of it and within approximations of practice. We made this distinction in our research as we sought to explore the potential for transfer of PMTs' learning from teacher preparation and into their practice. We did so, because that connection too often proves rather difficult to sustain.

2.1 The Interwoven Discourses Associated with Learning to Teach Mathematics in a Maker Context[2]

Recent conceptualizations of teacher knowledge build on previous characterizations of distinctive knowledge domains in order to promote a wider focus on

[2] Greenstein et al. (2020)

their integration (Scheiner et al., 2019). In this phase of our project, we adopted this perspective by viewing *teachers as learners* and foregrounding their identities (Sfard & Prusak, 2005) in order to recognize what affective, interpersonal, and social matters can bring to this conversation. That is, by honoring the interrelationship between the learning of mathematics and the learners themselves, the promise of this approach is suggested by the proposition that teachers' "invention[s] of 'objects-to-think-with'… [offer] the possibility for personal identification" (Papert, 1980, p. 11).

2.1.1 A Discourse Analysis of Identity in Interaction with Mathematical and Pedagogical Design Activity

We adopted a commognitive perspective on learning (Sfard, 2007, 2008), which is one that encompasses both interpersonal communication and individual cognition. Our objective in doing so was to explore the premise that learning to teach mathematics can be seen as changes in discursive activities that include narratives about mathematics and identity (Heyd-Metzuyanim & Sfard, 2012; Sfard & Prusak, 2005). The following question guided the research: *As prospective teachers of elementary mathematics Make new manipulatives to support the teaching and learning of mathematics, what might their discourses reveal about the epistemology of learning to teach mathematics?*

We addressed the question through a revelatory case study (Yin, 2009) of a prospective elementary teacher named "Moira" and by framing mathematics learning as the interplay between discourses about mathematical objects *(mathematizing)*, participants of the discourse *(identifying)*, teaching and learning *(pedagogy)*, and design activity *(designing)*. This framework provided us with a lens through which to study how the process of making a manipulative can provoke the four discourse activities and make visible the intertwined nature of a teacher's learning. We chose Moira because her initial design was a tool intended to simulate the "keep-change-flip" algorithm for fraction division. However, when the course's teacher educator pushed back on the idea because it did not meet the project's expectations for a tool that would support a students' conceptual learning, she tried to make sense of the algorithm but could not, and eventually she abandoned the idea altogether. We sought to understand this change through the lenses of the four discourses.

2.1.2 Findings

In this section, we present just two central results. These came from a follow-up interview we conducted with Moira in order to understand her rationale for the change in her design idea. The first concerns our analysis of this change through the discourses of Mathematizing *[M]*, Pedagogy *[P]*, Designing *[D]*, and Identifying *[I]*, and is as follows: "I wanted to make something that could be interpreted in many different ways *[M/P/D]* … that wasn't something that I was just forcing them to, like, all right, you have to use it this way." As she considered her initial "keep-change-flip" tool, she explained how she realized that, "flipping the fraction upside down in my initial tool… it was just not useful *[M/P/D]* … Then I came up with this [fraction comparison tool]" *[M/D/I]*.

These reflections revealed how Moira's decision to abandon her fraction division design was not just about mathematizing, it was also about identifying. As a teacher, it was important to her that her students have the opportunity to develop their own ways of thinking about fractions with a tool that can be used in a variety of ways. Moira acknowledged that the pedagogy promoted by the instructor in the classroom was also part of her decision to change her design: "Well, [the change of design] was because we were talking and you [the teacher educator] said, 'you're just teaching them how to—you're just giving them a way to solve the problem.' And I realized, you're right … It wasn't helping them learn how to do a problem" *[M/P/D/I]*. By switching to a design for comparing fractions, Moira found that she could participate in the discourse endorsed in the course and honor the teacher she wanted to be.

A second result emerged from our awareness that her current tool (see Fig. 5 below, left) *could* be used to make sense of fraction division and a question about whether Moira realized this capability in her tool. Our query to her about this possibility using the problem, ½ ÷ 2, prompted Moira's in-the-moment reflections such as this one: "½ divided by 2. ½, this divides it into two equal parts, and I know this equals fourths, so this is ¼" *[M/I]*. Next, as she was investigating 1 ÷ 1/3, Moira took the 1 and 1/3 ring, guessed that the answer was 3, and said, "I know I can do it, and I'm seeing it, but I don't know how to describe it" *[M/I]*. Moira was using her tool to make sense of this problem when we prompted her to explain whole-number division (e.g., 6 ÷ 3). As she reasoned through whole-number examples *[M]*, she exclaimed, "Oh! So, so, if I am dividing 1 by ⅓, there are three thirds in 1, so it's 3! Yes! You can do division with these… Wow! Fractions make so much sense now" *[M/I]*.

Moira's shift from a partitive conception of division to a measurement one gave her sought-after language to describe her tool's utility in her understanding of fraction division. That mathematical discovery was intertwined with an expression that revealed how emotionally invested she was in this realization. As she used her tool to think through fraction ideas *[M]*, Moira came to recognize its potential not only for her own learning, but also for teaching fraction division in a way that aligned with her identity as a teacher *[P/I]*. Her expressive body language and energy substantiated her enthusiasm for the discovery.

2.1.3 Implications

As in a woven tapestry, learning to teach mathematics weaves together four threads—or discourses—that are unique to a PMT's discursive experiences and particular to a learning community where an inquiry pedagogy is promoted. In this sense, to characterize Moira's learning to teach mathematics as a complex structure of discursive activities interwoven in dialectical unity is to illuminate the brilliance of a tapestry threaded by what she wants to teach (mathematizing), how she wants to teach it (pedagogy), decisions about what resources to make available (designing), and the kind of teacher she wants to be (identifying). Collectively, these threads contribute to an apparently honest depiction of the "organic whole" (Scheiner et al., 2019, p. 165) that is learning to teach mathematics.

This finding of the intertwined nature of the four discursive activities establishes that identity is as central to learning to teach mathematics as is the learning of mathematics, pedagogy, and design. And its implications speak to the potential of interdisciplinary experiences like the design experience as venues for the meaningful learning of learning to teach mathematics within teacher preparation coursework.

2.2 Making as a Window into the Process of Becoming a Teacher: The Case of Moira[3]

The process of becoming a mathematics teacher entails the development of an integrated knowledge base including both content and pedagogy. Much research has been done to better understand the forms that this knowledge base might take and how its development in PMTs might be supported. What is less clear is an image of the experiences these PMTs find formative as they progress through their teacher preparation coursework. Thus, in this phase of our project, we sought to illuminate the processes of *a teacher becoming* as they are mediated by a variety of social and conceptual resources within teacher preparation. The following question framed the inquiry: *How does a Making experience in*

[3] Greenstein et al. (forthcoming).

mathematics teacher preparation mediate the social and conceptual dimensions of the process of becoming a teacher?

2.2.1 Framing Making as Mediated Learning

We addressed this question through a revelatory case study (Yin, 2009) in order to better understand the processes at play in design activities that were hypothesized to underpin moments on a trajectory of becoming a teacher. Again we chose Moira, and we did so because, more than any other student, she *designed aloud*. Her words, gestures, and other embodied actions gave unprecedented access to elaborate and interwoven discourses of mathematics, pedagogy, identity, and design in Moira's design conversation (Schön, 1992). We were able to document her trajectory of becoming via an analysis of those discourses.

We brought situated and sociocultural theories of learning to bear on our attention to practice and to the social and conceptual artifacts that mediate knowledge and identity formation through that practice. In particular, we grounded this work in Engeström's (1987) cultural-historical activity theory (CHAT) and Holland et al.'s (1998) concept of figured worlds. The CHAT perspective offered a lens with which to situate Moira's activity in relation to the design context in which it arose. At the same time, a figured worlds perspective captured the mediating role of social, material, and conceptual artifacts on learning as identity formation. Together, the two perspectives proved useful in our analysis of the data to reveal and craft narratives of salient moments in Moira's becoming a teacher.

2.2.2 Findings

At the conclusion of our analysis, we chose four "moments of becoming" from the collection of moments that had been coded. CHAT and figured worlds analyses of the data then continued as we crafted narratives of each of the four moments and as we looked across them subsequent to the initial constructions of narratives. Moments 1 through 3 lead up to Moment 4, which occurs in the context of the follow-up interview whose findings we shared in the previous vignette. The three additional moments leading up to it were chosen because they were beginning to help us understand elements of that fourth moment more deeply. Although we made connections across all four moments, Moment 2, which details the change in Moira's initial design idea, is strongly linked to her problem-solving activity, which appears in Moment 4. Moment 1, which helps us understand Moira's passion for creativity, is strongly linked to Moment 3, where we find Moira leveraging both her mathematical and creative knowledge at a salient point in her design process. These four "moments of becoming" a teacher are presented next in chronological order.

2.2.2.1 Moment 1: *"I Should not Be Allowed in This Class. I'm Having Too Much Fun."*

This moment explains how an assemblage of design-related practices offered a space of authoring (Holland et al., 1998) for Moira's identity. Iterative enactments of agency involving conceptual, material, and social artifacts mediated her identity development as she "reconceptualize[d] what and who" (Vågan, 2011, p. 49) she was from one lived moment to the next. In addition, within that space of authoring, Moira was afforded a space of "improvisational play… [the] predominant form of agency" (Chang, 2014, p. 33)—the "medium of mastery, indeed of creation, of ourselves as human actors" (Holland et al., 1998, p. 236). Moira's improvisational play appeared to have been fueled by creative tendencies that she described as being embraced in elementary mathematics but later suppressed through the message that "math and science will probably come difficult" to students with "creative minds." In contrast, these creative capacities were not only invited into the design session, they were "demand[ed]" in response to the "possibilities of a design situation" (Schön, 1992, p. 4). Furthermore, as she was sought out by classmates for help with their designs, she was positioned by them as an expert. The community-subject interactions that mediated the rules by which participants related to each other spoke to the situated nature of activity in the figured world that emerged for Moira as she was once again positioned as a "top student," just as she was in elementary school.

2.2.2.2 Moment 2: *"Ugh, You're Right. We can't Flip the First One. I'm Gonna Work on This."*

As we analyzed the evolution of Moira's design within this Moment, we came to realize the saliency of several artifacts that mediated her design decisions. A lived history of all-too-common experiences in the figured worlds of traditional mathematics classrooms informed her interactions with fraction concepts and the concept images (Tall & Vinner, 1981) that were co-determined in them. In addition, pedagogical commitments and her knowledge of content and students (Ball et al., 2008) interacted with design activity that was further shaped by broader flow of social interactions distributed across the design environment. Taken together, we were reminded of Roth's (2012) assertion that we are only able to understand the actions of a subject on the object of their activity when we consider all the relations that mediate every aspect of the activity. Thus, this situated network of individual and collective activities that mediated the distinctive evolution of Moira's manipulative offered concrete markers of the formation of her identity as she emerged as a more central participant in an educational design community.

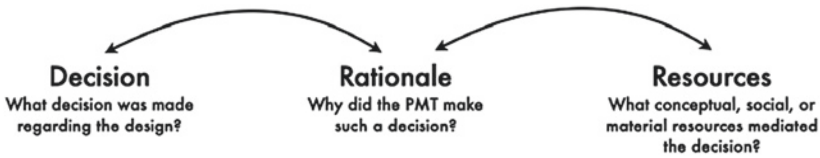

Fig. 4 The 3 elements of a design decision

2.2.2.3 Moment 3: *"Do I Know What a Torus is? No. Am I Using It? Yep!"*

In this third moment we centered our analysis on Moira's mathematics and how it mediated her design activity. From a mathematical perspective, equal partitions of a unit whole express the part-whole relationship that is fundamental to fraction knowledge. From a design perspective, the notches that form these partitions embed that part-whole relationship into Moira's manipulative (see Fig. 4, left). In addition, aesthetic considerations regarding the form of those notches mediated Moira's design as she decided to cut them using a torus, a mathematical object she had hardly been familiar with but whose affordances she discovered in action. In this regard, we found evidence of the "double arrow" (Engeström, 1987) of mediating interactions between mathematics and design.

2.2.2.4 Moment 4: *"I just Figured Out so Much Math!"*

In this moment of becoming, as Moira realizes that she's used her tool to *learn* fraction division, she's also realized her initial objective, the one she had abandoned and the one she had now achieved, which was to design a tool to *teach* fraction division. And yet again, affect and cognition arose as manifestations of the same accomplishment (Roth, 2012). In an act that is a manifestation of community relations (Engeström, 1987) that had the researchers participating alongside Moira in her quest to make sense of fraction division, everyone celebrated along with her. "Wow," she exclaimed. "Fractions make so much sense now. That blew my mind… I've learned so much. This is a great day!"

2.2.3 Implications

The use of CHAT and figured worlds perspectives made apparent the formative power of the Making experience by affording Moira opportunities to make choices that leveraged her mathematical, pedagogical, and design knowledge and moved her along a trajectory of a teacher becoming. This finding contributes to the research on teacher learning and identity formation in teacher preparation.

In addition, by further demonstrating the value of accounting for identity formation in research on mathematics teacher education, this finding also generates new opportunities to move the field forward in relation to research into the potential value of constructionist, STEAM-integrated curricular experiences in teacher preparation.

2.3 The Nature of Prospective Mathematics Teachers' Design Activity as They Make Original Manipulatives[4]

Positioning teachers as designers of their own curricular resources invites opportunities for their explorations of innovation at the intersection of content, pedagogy, and design. And given that there is almost no research on supporting teacher learning through Making outside of our own project, this gap in the research misses the opportunity to explore the design decisions teachers make through their design activity. This vignette addresses this gap as it posed the following question: *As prospective mathematics teachers Make new manipulatives for mathematics teaching and learning, what is the nature of the resources and rationales they bring to their design decisions and how do these intersect to mediate their decision making?*

We addressed this question through an exploratory case study approach (Yin, 2009) to understand PMTs' design activity by considering the three elements of each of their design decisions (Fig. 4).

2.3.1 Theoretical Framing

Grounding this work in a Learning by Design approach (Koehler & Mishra, 2005) enabled us to characterize the interplay between a designer's knowledge, experiences, intentions, and other resources as they are invoked during the iterative design of the shareable object. In addition, Schön's (1992) notion of "knowing in action" (p. 2) enabled us to characterize and organize the resources that mediated PMTs' design decisions. That notion provides that a "designer sees what is 'there'…, draws in relation to it, and sees what he/she has drawn, thereby informing further designing" (p. 5).

[4] Akuom and Greenstein (2021a)

2.3.2 Findings

Here, we present the cases of "Moira" and "Anyango," as their written work expressed the greatest number of design decisions from among the thirty-four projects we analyzed. In the subsections that follow, we share and contrast several findings from our analysis of their design activity that convey the diversity of design decisions, rationales, and mediating resources that they entailed.

2.3.2.1 Student-Centered Design

Both Moira and Anyango chose the child they had worked with in problem-solving interviews earlier in the course as their intended user of a tool. Moira explained that the student she tutors was having trouble with fractions, so she decided to create a tool to help him. Anyango shared that the student she was working with said she enjoys fractions, so she wanted to nurture that interest. Although both Moira and Anyango decided to create a fraction tool, they provided different rationales for that decision. Moira hoped to help her child make better sense of fractions; Anyango hoped to extend her child's current thinking about them.

2.3.2.2 The Nature of the Tools

Moira's design (Fig. 5, left) is a tool for fraction comparison. It consists of "a series of rings that rest on a cylinder… The notches help divide the rings equally

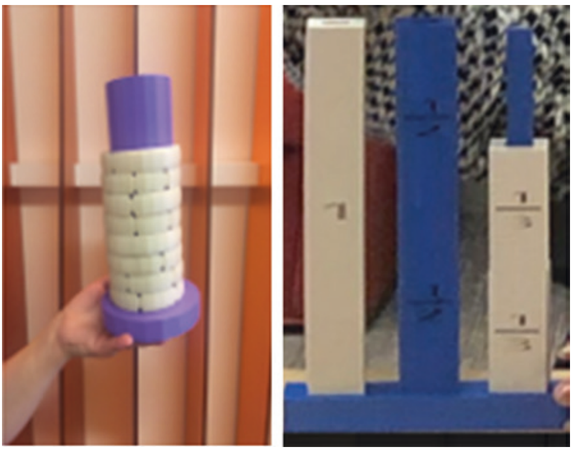

Fig. 5 Moira's (left) and Anyango's (right) fraction tools

up into pieces to represent parts of a whole. Each ring represents a different number of parts, like sixths and eighths." Anyango also designed a tool for fraction comparison (Fig. 5, right), yet her design is markedly different from Moira's. Anyango described her tool as "a 3D version of fraction strips. Each strip was made to be a rectangular/square piece that slides into individual pegs…[the] blocks stack vertically… to indicate height as value and amount."

While the mathematics of fractions and the knowledge of technology mediated both their design decisions, fraction concepts are embedded differently in their designs. For Moira, these are represented as arc lengths of the partitions of a continuous ring; for Anyango, these are represented as discrete fractions of the height of a referent whole. Relatedly, Moira's imagined utilization scheme involves aligning notches so that, for example, "the rings are able to be compared, showing how many fifths are in one half." Central to Anyango's scheme is that "all the fractions [can be] mounted on one platform… so that the student could begin to grasp how all the smaller parts can equate and compare to the whole."

2.3.2.3 The Role of Aesthetics

To Anyango, "The colors didn't matter much." Giving each fraction block its own color would have been "aesthetically pleasing, but it did not affect how the manipulative worked." Moira made the same design decision, but with a different rationale mediated by different conceptual resources. She explained that all her "rings have the same color," because if each ring had a unique color, it might "take away reasoning from children. If a student believes that a yellow ring represents 1/6ths, they will immediately reach for yellow the second that they hear sixths." By giving the rings the same color and leaving them "unmarked," Moira ensured that children will construct their own meanings in relation to each of the rings, thereby giving her tool the promise that it can "be used in multiple ways." Thus, epistemological knowledge mediated a decision that seems to reflect Moira's commitment to an inquiry pedagogy that affords multiple means of engagement.

2.3.3 Implications

As our research continues to discern the intellectual influences of the Making experience on PMTs' pedagogical and curricular thinking, this research adds more nuance to findings from prior research that revealed the breadth of teacher knowledge that they bring to their designs. It does so by revealing the particular power of design activity—namely the diversity of design decisions that PMTs must make in order to meet the expectations of the experience—to elicit, articulate, and advance that knowledge. Thus, this finding speaks to the generative power of an open-ended and iterative design experience in terms of the agency prospective teachers enact throughout their design activity and the wealth of knowledge and experiences that mediate it.

3 Knowledge Interacts with Practice

In the previous section, we shared vignettes of research on activities that took place within the design environment of PMTs' making. In order to explore the potential for transfer of PMTs' learning from the design setting and into their practice, in this next section we share vignettes of research that occurred within approximations of practice. We begin with a vignette that extends the one just presented in Sect. 2.3, which identified the knowledge resources PMTs brought to bear on some of their design decisions.

3.1 Prospective Mathematics Teachers' Designed Manipulatives as Anchors for Their Pedagogical and Conceptual Knowledge[5]

This next vignette presents research that sought to discern whether connections could be made between the pedagogical/conceptual knowledge that PMTs construct in teacher preparation and how that knowledge is enacted in their teaching. We wondered whether their designs could possibly mediate—or be some sort of anchor for—their pedagogical visions. Specifically, we asked the question: *As prospective teachers Make new manipulatives for mathematics teaching and learning, can connections be made between the pedagogical and conceptual resources for their design decisions and how those designs mediate the pedagogical moves they make in practice?*

3.1.1 Research Design

To answer this question, we took a sociocultural perspective and grounded this work in the notion of mediated activity, derived from Vygotsky (1978) and advanced as instrumented activity by Verillon and Rabardel (1995). We use the term *embedding* to connote an intentional (Malafouris, 2013) design element that

[5] Akuom and Greenstein (2021b)

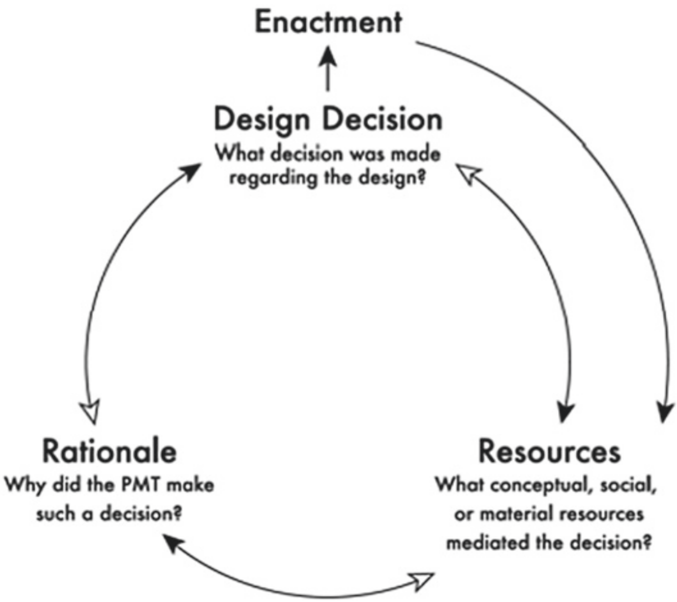

Fig. 6 Conceptual resources inform rationales for design decisions and may also be evoked in enactment. Open arrows acknowledge that feedback is reciprocally informing

embeds a PMT's pedagogical and/or conceptual knowledge in their tool. PMTs do so, so that in practice, the tool can afford (Gibson, 1977) particular utilization schemes (Verillon & Rabardel, 1995) that the PMTs hypothesized would enable the child to abstract (Piaget, 1970) percepts that are the constitutive elements of concepts. We then took an exploratory case study approach (Yin, 2009) using grounded theory (Corbin & Strauss, 2008) to discern instances in PMTs' teaching when the use of their artifact implicated the pedagogical and/or conceptual knowledge underlying their design rationales. The locus of these particular research efforts among the broader research project is depicted as the arrow from "Design Decision" to "Enactment" in Fig. 6. Note that central to the figure are the elements of a design decision that appear in Fig. 4.

3.1.2 Findings

Here we present two excerpts from "Roda" and "Anyango's" tool-based problem-solving interviews that demonstrate how their embeddings of pedagogical and/or

Fig. 7 (**a**) Roda's decimal snake and (**b**) Anyango's fraction tool

conceptual knowledge in their designs served as an anchor for that knowledge in practice. Their manipulatives are shown in Fig. 7.

3.1.2.1 Reasoning About the Unit Whole

Roda's tool is a "Decimal Snake" (see Fig. 7a) that she designed in order to teach a child about decimals and decimal comparison. The tool consists of ten pieces, each equally partitioned into ten parts. Thus, the decimal snake can be used to represent any value between 0.01 and 1 to two decimal places. These design features are Roda's embeddings of the concepts of the whole and its decimal parts. During the interview, when Roda asked her target child to use the snake to compare 5.5 and 5.47, the child manipulated the tool and used it to demonstrate his response: "5.47 is 5 and 47 hundredths, because it's 3 hundredths away from 5 and 5 tenths." While Roda's design intention was for the child to *only* compare the decimal parts, after 60 s of struggle, the child located 5.5 at (what we would identify as) 0.55 (if the entire snake represented 1), and 5.47 at 0.47. We inferred from his solution that he had *unintentionally* designated each piece of the snake as 1 (as opposed to 0.1) and each partition of a piece as 0.1 (as opposed to 0.01). Thus, he changed his designation of the entire snake from 1 to 10, and consequently, each piece of the snake now represented 1. In such case, 5.5 would be presented as the 5th partition of the 5th piece.

Determined to help the child identify and resolve the confusion, Roda asked the child to "Show me one tenth" on the snake. He pointed to one of the tenth pieces. "Two tenths?" he pointed to the second (tenth) piece. Next, Roda asked, "Where is 5 and 5 tenths?" Roda's questioning perturbed the child's thinking and provoked disequilibrium. As a result, the child declared, "Oh, wait! This [entire snake] is one whole! 5 and 5 tenths, you can't even make it out of the snake!" Roda then leveraged the affordance that each piece of the snake could represent a tenth of a whole to help him resolve his confusion about the representational capacities of the tool.

> *Roda:* You need how many snakes to make 5.5?
> *Child:* You need 5– No, 6 snakes!
> *Roda:* How can we compare [5.5 and 5.47] using 1 snake? Is that possible?
> *Child:* We can pretend that each piece is one snake.

In this excerpt, Roda leveraged her *unintentional* pedagogical embedding of a conceptually resourced design decision that allows for flexibility in naming the unit whole in relation to the snake and its pieces, revealing the student's thinking and posing purposeful questions to advance his mathematical reasoning.

3.1.2.2 Noticing in Action

While Anyango's tool (see Fig. 7b) is similar to Roda's in purpose, Anyango explained that she designed her tool "to help the student visualize and deepen their understanding as they explore fraction relationships." Her decision to design a tool that affords a vertical stacking of fraction pieces rather than in horizontal arrays allows the child to have "all the fractions mounted on one platform with the 1 (whole) always being visible, so that the student could begin to grasp how all the smaller parts can equate and compare to the whole." Anyango also engraved the fraction name of each piece on one of its lateral faces. In practice, she posed the task, *Jack and his two friends each had the same size pizzas for lunch. Jack ate 5/8 of his pizza. Judy ate 2/3 of her pizza. And Sam ate 3/6 of his pizza. Who ate the most pizza? Who ate the least?* In response, the child stacked five one-eighth pieces, two one-third pieces, and three one-sixth pieces, each on their own pedestal with their labels facing her (Fig. 7b, right). Anyango's pedagogical intention was for the child to compare "heights as amount" and identify the tallest as the one "who ate the most,"' and vice versa. In contrast, when asked, "So, if we just look at this, who ate the most?" the child attended exclusively to the symbolic representations engraved on each piece. This led her to decide that, "It's Jack" (represented by the 5/8 piece). She justified her answer by saying that "5 out of 8 is the biggest of all of them… 2 out of 3 is smaller and 3 out of 6 is… kind of small." Then, when Anyango asked the child what made her think it is smaller, the child explained that, "The top is two and the bottom is three." We infer from this response that the child was basing her comparisons on interpretations of fractions not as parts of a whole but as two separate whole numbers. That's why, for the child, 5/8 is greater than 2/3.

We interpret Anyango's next move as a noticing one (Sherin et al., 2011) that leveraged her pedagogical knowledge about the efficacy of attending to, interpreting, and responding to students' thinking:

> *Anyango:* If I turn this [pedestal] around [Fig. 7b, left, such that the child's gaze can no longer be restricted to the fraction labels on the pieces], who has the most?
> *Child:* This one <points to the stack of two one-third pieces, which corresponds to Judy's share>.
> *Anyango:* Who has the least?
> *Child:* This one <points to the stack of three sixth-pieces, which corresponds to Sam's share>.

In this excerpt, Anyango's "flipping" move leveraged an unintentional design affordance that we suggest served as an *anchor* for her pedagogical knowing in action mediated by that affordance. Reinterpreting Schön's (1992) concept of "knowing-in-action" as a noticing-in-action, we suggest that in this instance, Anyango saw what was there, made a move in relation to it, and saw what that move accomplishes, thereby informing her next steps. Thus, by returning the tool to its initial, label-facing orientation so that the child could connect the physical representation of the amount to the symbolic one, the child was supported in determining a correct response to the question, "Who ate the most?"

3.1.3 Implications

In an attempt to solve the perennial problem that teachers tend to face considerable challenges in transferring their theoretical knowledge into practice, this work explored teacher learning at the interface between theory and practice by discerning whether connections could be made between the pedagogical and conceptual knowledge that PMTs construct in teacher preparation and how that knowledge is enacted in their teaching. As PMTs used the manipulative they designed in a problem-solving setting, we analyzed instances when their manipulative served as a mediating anchor for the pedagogical and conceptual knowledge they acquired in teacher preparation and subsequently embedded in their designs. In relation to practice, our identification of these instances of anchoring phenomena suggests that the Making experience yielded material epistemic scaffolding (in physical manipulative form) that supported PMTs and their commitments to the models of knowing and learning they construct in teacher preparation. And in relation to theory, findings from this study and the prior one that it builds upon suggest the analytic value of our design, rationale, resource, and practice (DRR-P) framework for revealing the promise of the Making experience.

3.2 Dare to Care: A Case Study of a Caring Pedagogy on Mathematical Making, Teaching, and Learning[6]

This case study investigates the interaction between a *caring pedagogy* and Making, and how the two informed each other in our project. Because the subject of mathematics (Gutiérrez, 2017; Stinson, 2004) and the Maker culture (Barton et al., 2017) can be interpreted as exclusionary to so many students, we wondered about the possibilities that *caring pedagogies* could bring in broadening opportunities for learning and learners in these spaces. The three central participants in this section include the teacher educator (TE) and the PMT, "David," each of whom brought caring pedagogies to the project and viewed themselves as interlopers to the Making culture. And then there's "Vincent," a kindergarten student who is on the autism spectrum and whose energetic ways of learning are not typically embraced in traditional mathematical classrooms. By focusing on *caring-centered relationships*, we illustrate how *together*, the participants redefined values associated with Making, traditional mathematics, and what can get celebrated as learning. The following research questions to guide the inquiry: *How does enacting a caring pedagogy during a Making-centered experience impact and broaden opportunities for meaningful mathematics learning? How does this challenge traditional notions of who can Make, who can participate in mathematics, and who cannot?*

3.2.1 Theoretical Framing and Methods

The fact that Making and caring can elicit both cognitive and affective concerns suggests a need for a framework that accounts for these dual traits. Hackenberg (2010) terms a *mathematical caring relation* (MCR) as one that honors both the mathematical and affective parts of learning. She recognizes a teacher's sensitivity to a student's learning needs and their ability to participate in the activity at hand as central to supporting meaningful MCRs. Hackenberg describes how *cognitive decentering* can help a teacher to navigate an MCR by decentering "from his or her own perspectives… to help students realize and expand their ideas and worlds" (p. 239). In our project, we honor and utilize the mathematically open-ended nature in designing and Making a manipulative; the sometimes, uneasy navigation through emergent mathematical "unknowns"; the child's unique

[6] Fernández et al. (2021)

experiences and needs; and the tensions that are negotiated by carers (Noddings, 2012) in balancing these considerations.

In exploring the larger question of how the PMTs see themselves as mathematics teachers, we were drawn to *caring relationships* that developed between project participants and utilized the methodological stance of *purposeful sampling* (Creswell, 2007). We opened our analysis to participants' verbal utterances and intonations, body language, actions, and mutual positionings (Simmt, 2000) as revealing defining moments in MCRs. The possibility of intersecting caring theories with Making and the novelty of our data suggested a grounded theory approach (Glaser & Strauss, 1967) to analyzing and cross-referencing our sources.

3.2.2 Findings

Initially, David created a quick and easy answer to the task of Making the project manipulative (by way of designing an already-existing manipulative with a fellow classmate). However, he was invited to reconsider this approach by his TE, who noticed the special and warm interactions between himself and Vincent that David had recorded in a "Getting to Know You" session. Overcoming trepidation of her own, the TE invited David to design a manipulative that responded to these interactions, and made clear that she would support David in this initiative when he realized it would require more time, thought, and *care*. We recognized this as the TE accepting responsibility for supporting David in caring for Vincent and in navigating the discomfort and tensions (Noddings, 2012) that accompany this pedagogical decision. David, in turn, opened to accepting responsibility for Vincent's care by sharing and utilizing Vincent's knowledge and his love of diverse shapes. David attempted to understand Vincent's strengths with shapes, and after a few sessions with Vincent, opted to design triangular, square, and hexagonal prisms with holes and corresponding inserts intended to create a one-to-one matching task (for example, *Which of these shapes fit together?*).

During a design session, David noticed that multiple printed inserts did not fit into their intended holes. The TE took advantage of this moment of struggle to support David through his technological anxieties, and recommended including the extra "mis-shapes" in the matching task (for example, *Which of the multiple hexagonal inserts can fit into the hexagonal hole?*). David reflected on this as being a "teachable moment," such that his "mis-shapes" could become usable for Vincent's learning. In another teachable moment, Vincent showed David how every shape and insert need not match to fill the holes (e.g., Vincent drops hexagonal inserts into the square hole). These uninhibited moments of insight suggested a transition in Vincent's attention from a shape's sides to whether or not

Fig. 8 Vincent sees similarities in different-shaped holes

it has a hole—a driving force in understanding the concept of topological equivalence. These explorations culminated when Vincent aligned the hexagonal and square prisms with unlike holes to peer through them, and in response, David arranged the pieces between himself and Vincent to form a telescope (see Fig. 8)! Together, they locked eyes and exchanged laughter and words of affirmation in an MCR where David decentered from the intended activity to *literally see his child's point of view* (Hackenberg, 2005).

3.2.3 Implications

The increasing pressures and responsibilities faced by teachers and teacher educators can make enacting caring pedagogies seem especially daunting. Our project's focus on Making something *for* and *with* a specific student enabled both a TE and PMT to leverage their caring-centered pedagogies, and speaks to the inclusivity that caring brings to learning. Vincent, a member of the students with disabilities (SWD) community, approached and demonstrated learning with animated physical enthusiasm. The TE and David enacted caring-centered pedagogies that embraced Vincent's inclination to learn with his body, explore open-ended mathematical ideas *together*, and recognize that design "mis-shapes" could become viable learning tools for Vincent.

By inviting David to substitute a more open-ended investigation for his initial, "easy" project solution, the TE set in motion a ripple effect that challenged traditional notions of mathematics learning in which uncomfortable discoveries such as David's "mistakes" are dismissed as divergent from intended tasks (Lampert, 1990). Instead, David embraced those mistakes as an important part of his learning and celebrated Vincent's mathematical discoveries. In doing so, he defied the exclusionary notion that SWDs should not be expected to participate in problem solving, and welcomed the unexpected (but worthwhile) mathematical interpretations that open-ended investigations can bring. By providing a platform "to demonstrate care for individual students and for the subject matter itself" (Bartell, 2011, p. 54), this case demonstrates how Making and designing can create a novel opportunity to embrace mathematical struggle, surprise, and discovery for all types of learners.

3.3 Harmony and Dissonance: An Enactivist Analysis of the Struggle for Sense Making in Problem Solving[7]

The teaching of mathematics requires that teachers give "explicit attention to the development of mathematical connections among ideas, facts, and procedures" (Hiebert & Grouws, 2007, p. 391). Indeed, the National Council of Teachers of Mathematics (NCTM, 2000) emphasizes the value of representing mathematical ideas in a variety of ways, and that these representations are fundamental to how we understand and apply mathematics. While much research has been done regarding the ways in which teachers can support students' engagement with multiple representations, what is less well understood is the process by which multiple representations of a concept can be leveraged and connected to contribute to learners' meanings of the referent concepts for those representations. Thus, in this phase of our research we aimed to address that gap as we posed the following question: *How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?* We did so through a revelatory case study (Yin, 2009) of the problem solving of two learners, "Dolly" (a researcher-participant) and "Lyle," as they aimed to make sense of fraction division by coordinating meanings across two artifacts, one being a "Fraction Orange" physical manipulative that Dolly designed and the other being a written expression of the standard algorithm (see Fig. 9, left and right, respectively).

3.3.1 An Enactivist and Semiotic Analysis of Emergent Problem-Solving Activity Involving Multiple Representations of Fraction Division

This study is grounded in the enactivist theory of cognition (Maturana & Varela, 1987), which recognizes that the learning environment (e.g., a task, setting)

[7] Greenstein et al. (2021)

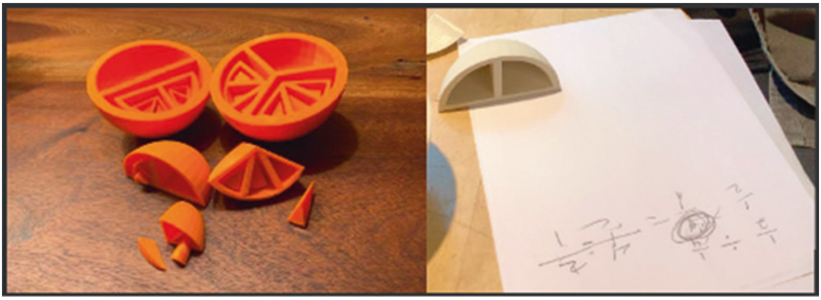

Fig. 9 The Fraction Orange and the algorithm

and the solver(s) (e.g., a student, teacher) are structurally coupled and determined through a dynamic, emergent, contingent, and "ongoing loop" (Proulx, 2013, p. 319) of interactions of problem solving. Verillon and Rabardel (1995) add the dimension that sense making is inextricably linked to the material and symbolic tools that mediate its learning. Thus, we considered what an enactivist analysis might reveal about the processes at play in mathematical meaning making as it develops through the complex interplay of signs and meanings (Maffia & Maracci, 2019) associated with learners' engagement with multiple representations. Maffia and Maracci's (2019) concept of semiotic interference is thus used in tandem with the enactivist analysis to analyze the dynamic, emergent, and contingent (Proulx, 2013) interactions that Dolly and Lyle have with the Fraction Orange and the algorithm.

3.3.2 Findings

While the 13-minute problem solving interview video offers many opportunities worth sharing, here we present just two central moments in order to demonstrate what our theoretical lenses revealed. Note that all fraction pieces of the Orange (Fig. 9, left hemisphere) are named in our analysis just as Dolly and Lyle name them: the hemisphere of the Orange is the whole, and that whole is partitioned into halves, fourths, eighths, and sixteenths.

3.3.2.1 Embarking on a Path of Problem Solving

The problem-solving interview opened with Dolly posing the problem, ½ ÷ ¼, on paper alongside her fraction orange. Lyle chose the pen and paper (over the Orange), performed the flip-and-multiply algorithm (½ ÷ ¼ = ½ × 4/1 = 4/2 = 2/1), and declared his answer to be 2. We interpreted

Lyle's application of the standard algorithm as a structurally determined action informed by a lived history of structural coupling with traditional school mathematics. To Lyle, this execution of an algorithm and the answer it yielded was deemed "good enough" to "survive" in school. It constituted what Lyle needed to do to achieve harmony with his mathematics learning environment.

Next, Dolly directed Lyle's attention to the Orange and asked, "Can you show me with this?" From there, the pair set off to navigate a complex interplay of signs literally at hand. Our semiotic analysis enabled us to identify moments of semiotic interference (Maffia & Maracci, 2019) that they experienced as they pursued a non-linear path of problem-solving activity punctuated by moments of what we referred to as either *harmony*, a pleasing fit, or *dissonance*, a displeasing conflict or lack of fit. Next, we present one of those moments.

3.3.2.2 A Crowning Achievement

Here we present what appeared to be a crowning achievement for Dolly and Lyle in their search for harmony in meanings for fraction division mediated by the two artifacts. By enchaining signs (Presmeg, 2006; Bartolini Bussi & Mariotti, 2008) across pieces of the Orange and elements of the algorithm—specifically by translating interpretations of parts of the Orange to interpretations of quantities in the algorithm (i.e., ½ and ¼)—they made sense of those quantities. Then, they engaged in similar sense making in order to find interpretations for the ½ and ¼ in the posed problem, ½ ÷ ¼.

> *Dolly: <referring to ½ ÷ ¼>* We wanna take a half of one and divide it by a quarter of one, right?
> *Lyle:* Yes.
> *Dolly:* Take a half of one and divide—oh, that's what it is!
> *Lyle:* It's 2.
> *Dolly:* We wanna take this <points to the half piece of the orange> and see how many of those <*now pointing to quarter piece*> fit in there <*points to the half piece again. Then, with confidence:*> And that's why our answer is 2.
> *Lyle:* Yes.
> *Dolly:* There's still two halves in a whole, 'cuz this<*the expression, ½ ÷ ¼*>is in regards to a whole.<*rephrasing*>This is in regards to 1. So a half of 1 divided by a quarter of 1 is 2, because 2 quarters fit into 1 half. Or<*returning to the expression,*>4 quarters fit into 2 halves.
>
> *Lyle:* Yeah.

In this excerpt, we observed the meaning Dolly makes of the expression, ½ ÷ ¼, by enchaining interpretations of ½ and ¼ in light of the measurement meaning of

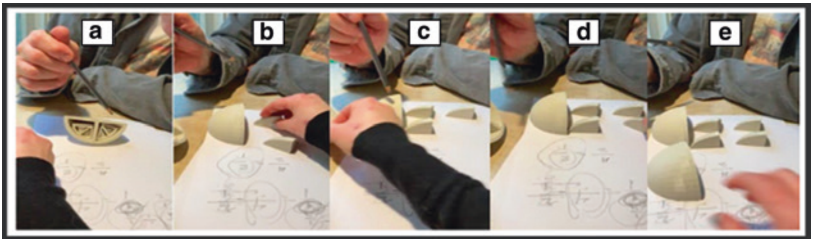

Fig. 10a–10e Lyle re-enacts Dolly's understanding of "4/2 = 2/1."

division she and Lyle enacted earlier, as well as the meanings they enacted for ½ and ¼ in the algorithm. Next, Lyle re-enacted the interpretation for himself.

> Lyle: So this is half of a whole and this is a quarter of a whole. <*Next, he turns his attention to the orange (Fig.* 10*a) and points to the half piece resting on the paper.*> Half of a whole. <*Next, he takes his pencil and points to each quarter piece*:> Quarter of a whole <*Then, pointing to the two quarter pieces:*> is 2.
> Dolly: <*pointing to the 2 quarter pieces*> Yeah, 'cause there's two quarters of a whole.
> Lyle: Yeah, that makes sense.
> Dolly: 'Cause there's two of these <*She pulls out the quarter pieces and sets them next to the half piece (Fig.* 10*b).*> for every one of these <*she says as she touches the half piece*>.
> Lyle: <*with a sigh, perhaps of relief*> Yes.
> Dolly: Or there's four of these. <*She takes the quarter pieces out of the other half piece.*>
> Lyle: <*points to the half piece and extends Dolly's thinking (Fig.* 10*c)*>: For two of those.
> Dolly: <*revoicing Lyle*> For two of those. <*As she speaks, she aligns all of the quarter pieces as well as the second half piece on the page (Fig.* 10*d and* 10*e).*>

As if to establish his own meanings for fraction division and its coherence in representations across artifacts as Dolly had just done, Lyle used the pencil to re-enact a physical bridge between the elements of the problem (½ ÷ ¼) and the pieces of the Orange. He uttered "half of a whole" as he pointed to the ½ on paper, and "quarter of a whole" as he pointed to the ¼. Then he repeated these phrases on the other side of the bridge: "half of a whole" as he pointed to the half piece, and "quarter of a whole" as he pointed to the quarter piece. We interpreted this activity as a matching of his interpretation of half of a whole and quarter of a whole in the symbolic representations (½ and ¼, respectively) to the representations he identified in the orange (the half piece and the quarter piece, respectively). These embodied epistemic actions seem to reify the harmony that had finally emerged from recursive interactions that culminated in an enchaining of signs signifying the sense he and Dolly had made. This reification can be viewed as a newly coupled structure of fraction division for Dolly and Lyle, one that offers a stark contrast to the structurally determined response to fraction division that they enacted at the outset of their activity.

3.3.3 Implications

In analyzing the iterative cycles of harmony and dissonance experienced by Dolly and Lyle, enactivist and semiotic analyses enabled us to see the apparent structural coupling they had with traditional school mathematics and that constituted their felt experiences throughout their drive for fit. In light of this finding, we offer recommendations for pedagogical and material resources in mathematics classrooms that enable, support, and honor this sort of loosely structured problem-solving activity. As Proulx (2013) reminds us, students' paths of problem solving emerge in interactions with the environment and are contingent on their particular mathematical structures and interactions. "Average" paths and tools presumed viable for sense making simply cannot be determined *a priori*. Rather, resources should be provided that are responsive to students' creative and agentive efforts at sense making.

4 Conclusion

The Making experience at the center of this body of work had prospective teachers of elementary mathematics innovating at the intersection of mathematics, pedagogy, and design. That experience provided them with an opportunity to consider the interplay between the iterative design of an evolving artifact and the application of teacher knowledge domains in the artifact's development. In addressing the broadest question, *What are the potential benefits of a Making experience within mathematics teacher preparation?,* our research has revealed a number of positive outcomes. These findings and their implications for teacher learning have been shared at the conclusion of each of the vignettes we presented above. We provide only a summary overview of them here.

Over and over, our findings demonstrate the formative value of immersing prospective teachers in a communal design environment of collective social making and tasking them with a pedagogically genuine design experience centered on the Making of an original physical manipulative for mathematics teaching and learning. In particular, these findings show that the experience informed the pedagogical, mathematical, and design thinking of prospective teachers, while also demonstrating that identity formation (e.g., as a mathematics teacher) is just as central to their learning to teach mathematics as those three forms of thinking. That revelation in turn allowed us to determine that the experience supported prospective teachers' movement along a trajectory of participation that we called a "teacher becoming," with the potential cultivation of a caring pedagogy and of knowledge about the formative power of embodied activity in sense making as just two aspects of that becoming. Lastly, we shared evidence of the experience's potential impact beyond teacher preparation in that it yielded epistemic scaffolds in material form that can support the connection between teacher preparation and teachers' practice.

All in all, we propose that these findings contribute to the bodies of research on both teacher learning and identify formation in teacher preparation. They also generate new opportunities for research that moves the field forward regarding the potential value of constructionist, STEAM-integrated curricular experiences in teacher preparation. Future research could more closely explore the design of these environments in teacher preparation, the teacher educator's role in designing and facilitating these experiences, and the subsequent in-service instruction of teachers who participated in these experiences during teacher preparation.

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Plane Tessellation

Rudolf Hrach

1 Introduction

Tessellation is a connection between mathematics and arts. Tessellation means to fill a wall or a floor with a periodic pattern. In art, there are many famous drawings and wall paintings of plane tessellations. In mathematics, the structure and symmetry of such patterns is studied. For example, by starting with a simple triangle one can produce further triangles by translating, rotating and reflecting the original triangle and in this way fill the whole plane.

New technologies now offer the possibility of 3D printing such tessellations. This serves as a great motivation for students to study mathematics. Applications at school can be found in the course of this article.

As a first example and motivation, we print copies of normal stones for paving ways and places in the ratio of 1 to 2 (Fig. 1).

With these "stones" one can start a first tessellation (Fig. 2).

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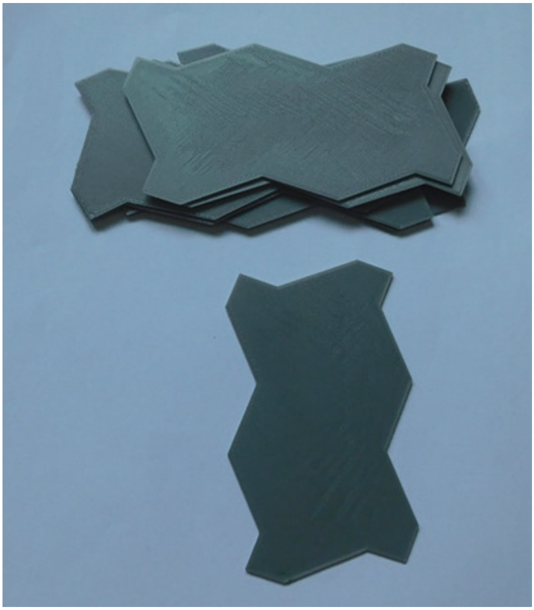


Fig. 1 3D printed stones

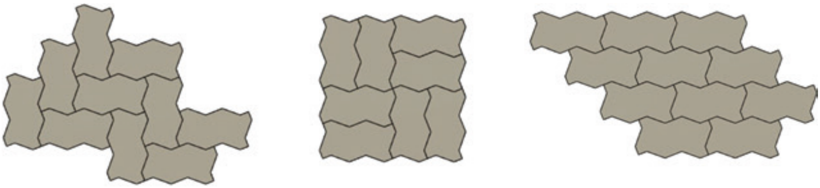


Fig. 2 Three possibilities

2 Platonic tessellation

A tessellation is called platonic if the whole plane can be filled with congruent regular polygons. There are only three congruent regular polygons that can be used to create platonic tessellations: triangles, squares and hexagons, all equilateral.

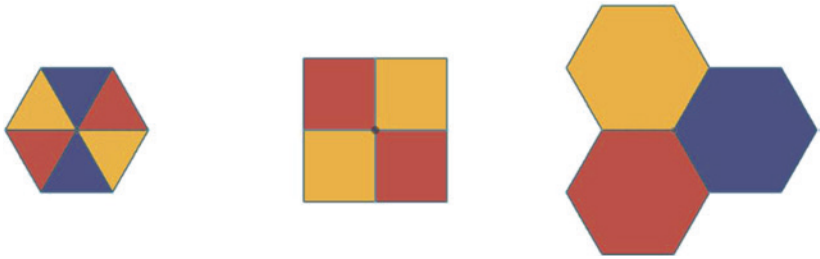


Fig. 3 Platonic tessellation

That is because the sum of the angles at each vertex must add up to 360° (Fig. 3)

n of Polygon	3	4	5	6	7	8	9	10	11	12
Angle at vertex	60°	90°	108°	120°	128.57°	135°	140°	144°	147.27°	150°

Formula for the respective angle at the vertex: …
angle at vertex $= 180° - \frac{360°}{n}$
Therefore, only in the cases 3, 4 and 6 platonic tessellation is possible (Fig. 3).

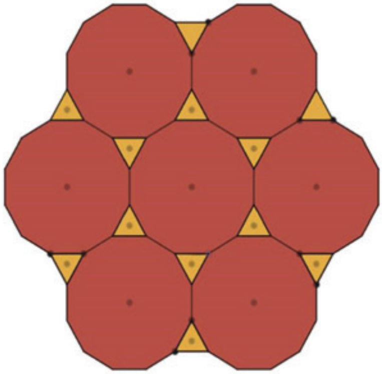
3 Archimedian tessellation

A tessellation is called archimedian (in short AT) if the whole plane can be filled with different sorts of regular polygons. Again, the sum of the angles at each vertex must be 360°. It is a combinatorical problem to find all possibilities. There are eight possibilities for arranging the polygons.

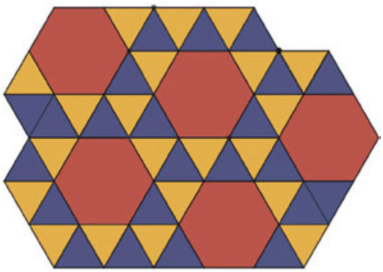
We introduce a notation for them:

At each vertex certain n-gons meet. You can now specify them by writing the order n of the n-gons in the positive sense. Thus, AT(4 6 12) stands for the archimedian tessellation where at every vertex one 4-, one 6- and one 12-polygon meet in this sequence.

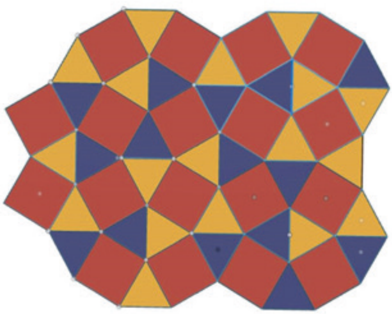
A list of the 8 archimedian tessellations is provided as follows:



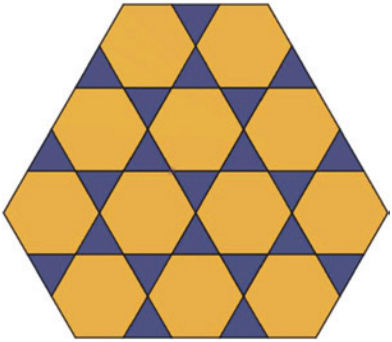
AT(3 12 3 12)



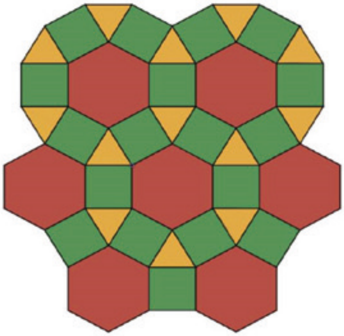
AT(3 3 3 3 6)



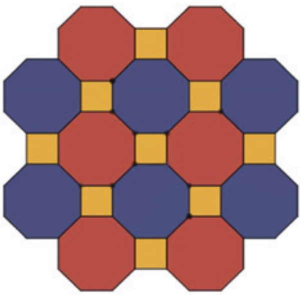
AT(3 4 3 3 4)



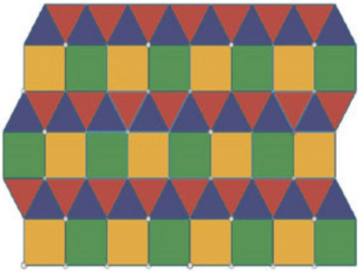
AT(3 6 3 6)



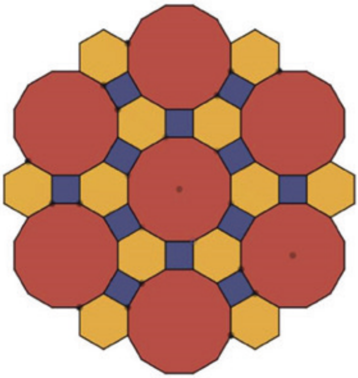
AT(6 4 3 4)



AT(8 8 4)



AT(3 3 3 4 4)



AT(12 6 4)

4 3D printing of tessellations

To print a coloured tessellation as shown in chap. 2 and 3 we can use two methods.

a) 3D print the polygons you need in different colours (filaments) and glue it on plexiglass. It fits very well together if you take a special plastic modelers' glue. But it is difficult to get it in an accurate way.
b) You can 3D print layers in different colours by pressing the pause button, changing filament and resuming the 3D print. Slicer programs (we take the Prusa slicer) contain the possibility of programming these pauses automatically.

Example

Looking from the top at Fig. 4 you want to see the three objects in three colours. This can be done in the Prusa slicer (Fig. 6). The bar in the middle you can use for determing the height of the filament change. This effects that, during the print process, the printer pauses at these heights and one can change the filament. ◀

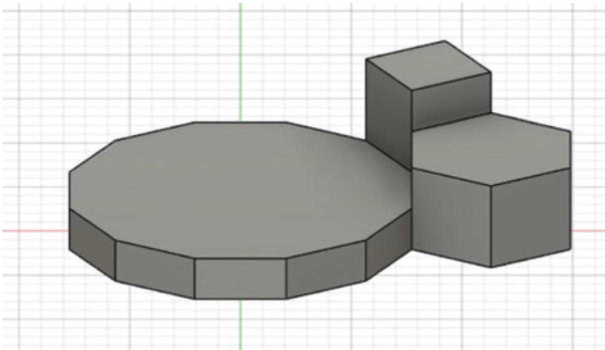


Fig. 4 Extrusion in FUSION 360 with different heights

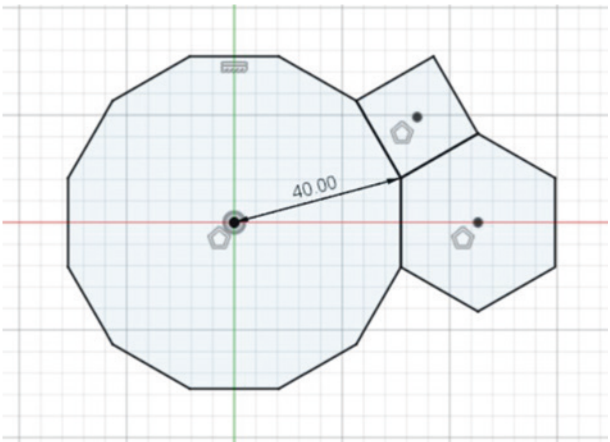


Fig. 5 Sketch in FUSION 360

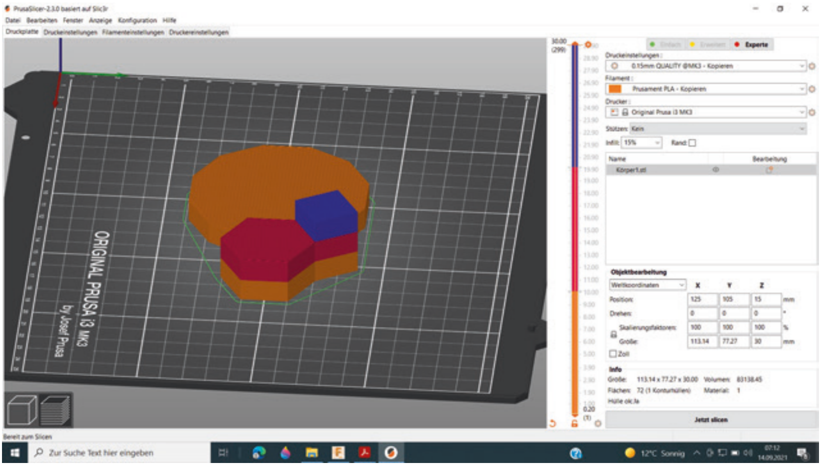


Fig. 6 Prusa slicer

5 Application of Archimedian tessellations at school

Polygons are an important subject in two-dimensional geometry and to hold them in one's hand is a great motivation for studying.

Therefore, regard the basic polygons (edge 2 cm, 1 mm thick) in Fig. 7. The students get a "lot" of them. Now they try to build the shapes of the platonic and archimedian tessellations that we have shown in the AT list above or even try to find other constellations.

6 Laves Lattices

The archimedian tessellations can serve as a source for further tessellations.

The duals of the platonic and archimedian tessellation are called Laves Lattices (in short LL) and they are also tessellations. All in all, we receive eleven Laves Lattices. The three platonic duals are self-dual.

The dual of an archimedian tessellation is constructed as follows:

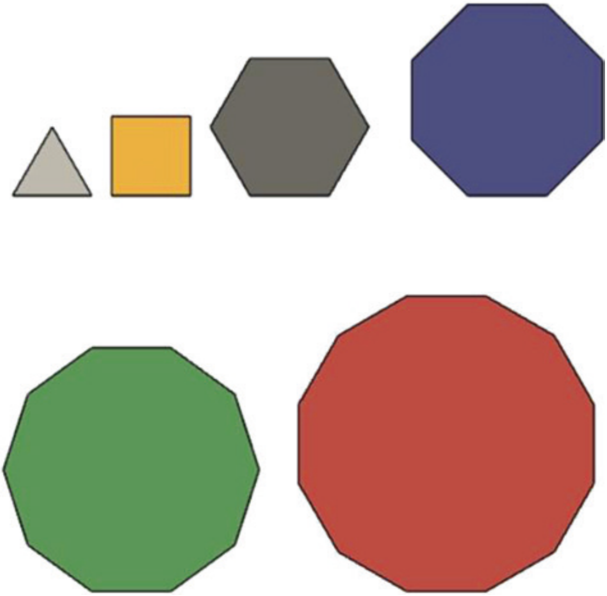


Fig. 7 Regular polygons as a basis for tessellations

Example

Description of constructing the dual:

Take AT(33336) and any vertex A (Fig. 8). All the vertices are congruent. Construct the perpendicular bisectors of the edges which lead to A. The intersections of the perpendicular bisectors form the pentagon. Notice that the polygons are non-regular. We give the LL the same number as the corresponding AT

It is evident that this process leads in the case of the equilateral 3gon, 4gon, 6gon to themselves. They are self-dual (Fig. 9, 10).

For further explanations we examine and compare AT(3 6 3 6) and LL(3 6 3 6)

Comparison of the Archimedian tessellation and the Lave Lattice with the numbers 3636 (Tab. 1)

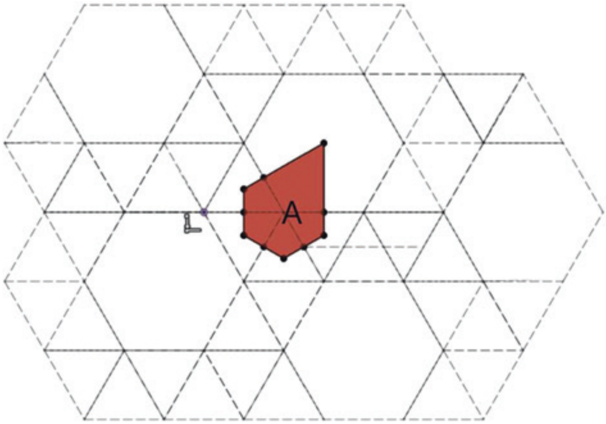


Fig. 8 Construction of Laves Lattice 1

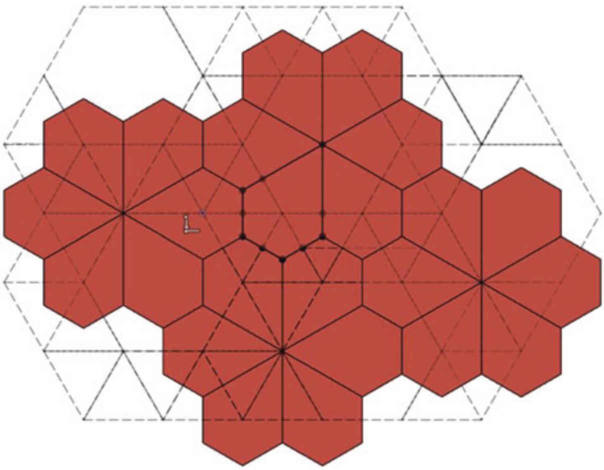


Fig. 9 Construction of Laves Lattice 2

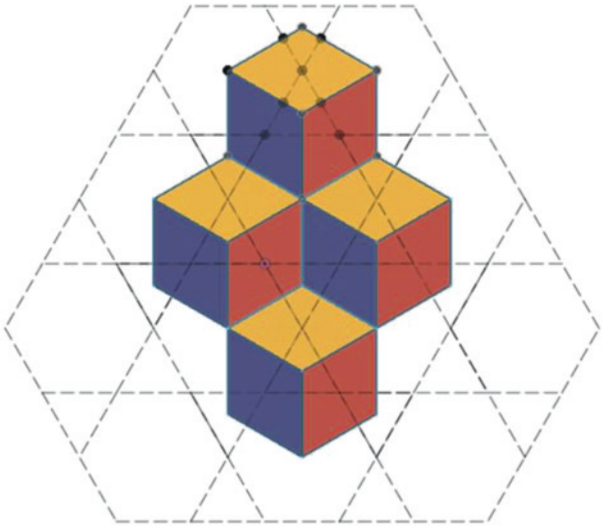
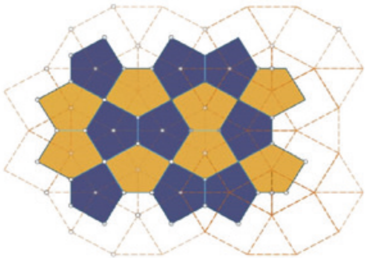


Fig. 10 AT(3 6 3 6) and LL(3 6 3 6)

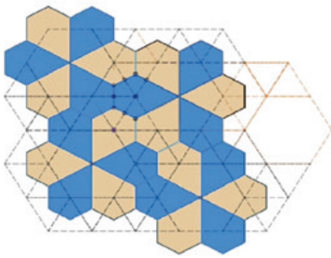
Table 1 Comparison of an AT(3 6 3 6) and the corresponding LL(3 6 3 6)

Archimedian tessellation	Laves Lattice
AT(3 6 3 6)	LL(3 6 3 6)
Congruent vertices	Congruent faces
each vertex is surrounded by 3gon 6gon 3gon 6gon	each face has 4 vertices: V1 V2 V3 V4 V1 is surrounded by 3 faces, V2 by 6, V3 by 3, V4 by 6

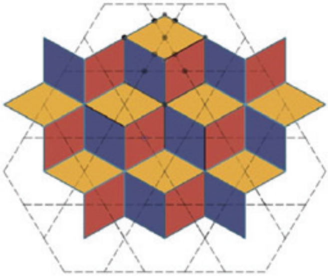
The list of the 8 Laves Lattices with their corresponding Ats



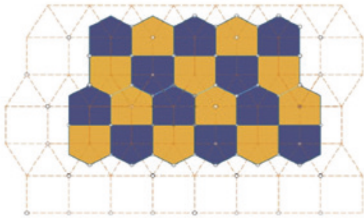
LL(3 3 3 4 4)



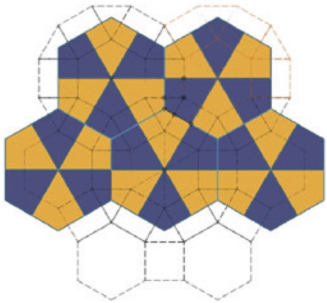
LL(3 3 3 3 6)



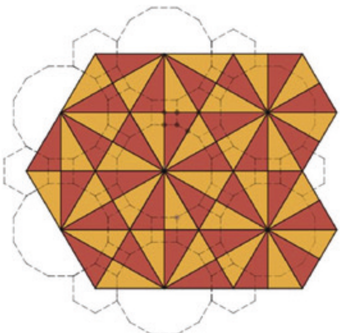
LL(3 6 3 6)



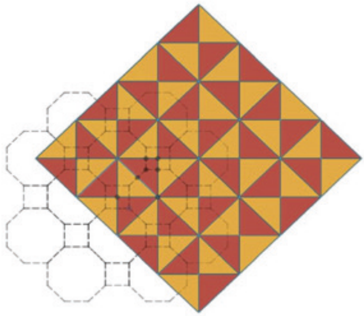
LL(3 3 3 4 4)



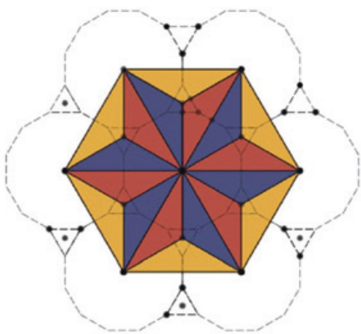
LL(4 3 4 6)



LL(4 12 6)



LL(4 8 4 8)



LL(3 12 12)

◀

7 Application of Laves Lattices at school

As done with the Ats one can 3D print some copies of each non-regular face and build the LL. Apart of the mathematical beauty of Laves Lattices there is another aspect that can be treated at school:

It is impossible to cover the plane with regular pentagons. But 3 Laves Lattices contain non-regular pentagons. It is a delightful task for students to calculate these pentagons by given sidelength a of the corresponding archimedian tessellation. With the 3D printed pentagon-tiles one can fill the plane.

8 Quadrilateral (4gons)

The question arises if 3gons or 4gons tessellate the plane. One can print arbitrary 3gons/4gons and let the students try to tessellate. That this is possible, one can show as follows: (with the reflection of a 3gon on one side one gets a parallelogramm, thus a 4gon)

The sum of the angles in any quadrilateral is 360°. Therefore, a tessellation is possible with any quadrilateral. The vertex star around point B forms a tile. With this tile, by composition of 2 translation vectors, you can cover the whole plane.

Construction of the tile:

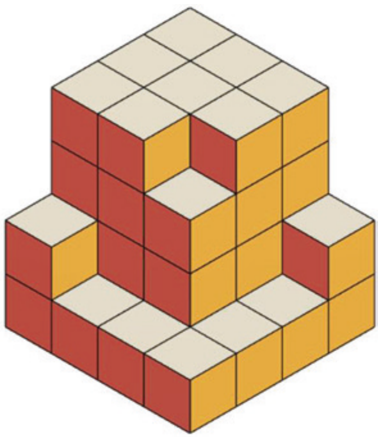
Rotation of red ABCD around the midpoint M_a with 180° results in the blue one. The rotation of the blue one around midpoint M_d with 180° leads to the green one. Rotation of the green one around midpoint M_c with 180° results in the yellow one. Thus, at vertex B all angles meet. (Fig. 12)

9 Art and Mathematics

Tessellations inspired among others two famous artists: Vasarely and Escher.

9.1 Vasarely

Victor Vasarely (1906–1997) was a hungarian-french painter and sculptor who worked amongst others with mathematical objects.



A cube model (LL(3 6 3 6)) and a sculpture of Vasarely (wikipedia.org/wiki/Victor_Vasarely

Vasarely worked with 2D tiles that produce the impression of 3D cubes. This can easily done by students themselves. With many copies of the three congruent rhombi (see Fig. 11, side 2 cm, thick 1 mm, angle 120°/60°) exciting 2D projected cube models can be built (Fig. 13).

9.2 Escher

M.C. Escher (1898–1972) was a dutch graphic artist. His work was inspired by mathematical objects. In Fig. 14 you can see a hexagonal tessellation with lizards. The mathematics which is involved in the background is explained in the following.

We want to create a 3D printed picture similar to Eschers famous Lizards.

Therefore, (Fig. 15) in a hexagon, construct three mathematical edge figures F1, F2, F3 and rotate them around the centers Z1, Z2 and Z3 with 120°. Then, extrude the sketch. Rotate the whole figure around Z with 120° and –120° and

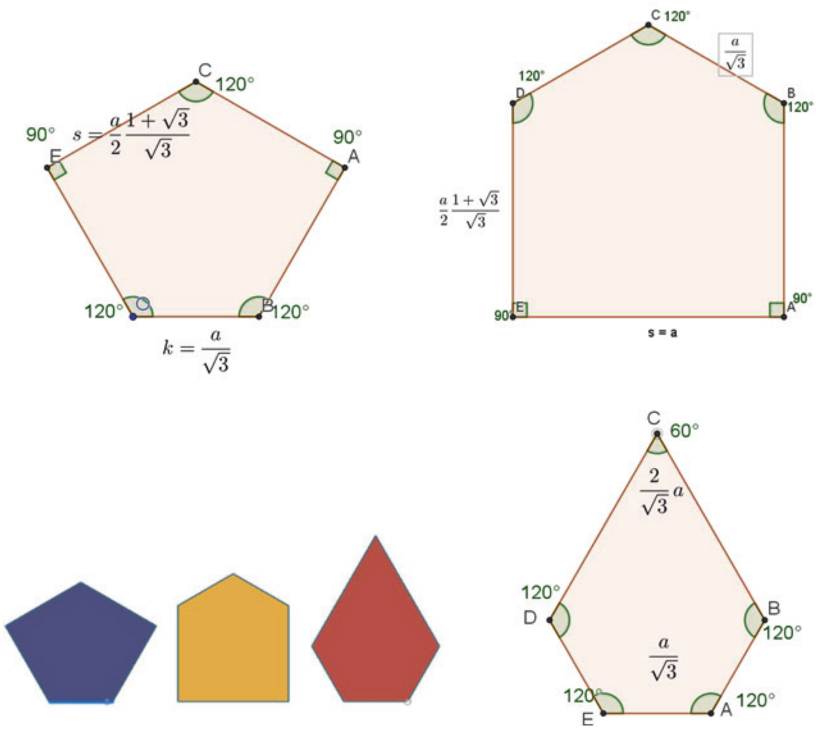


Fig. 11 3 pentagons that tessellate the plane

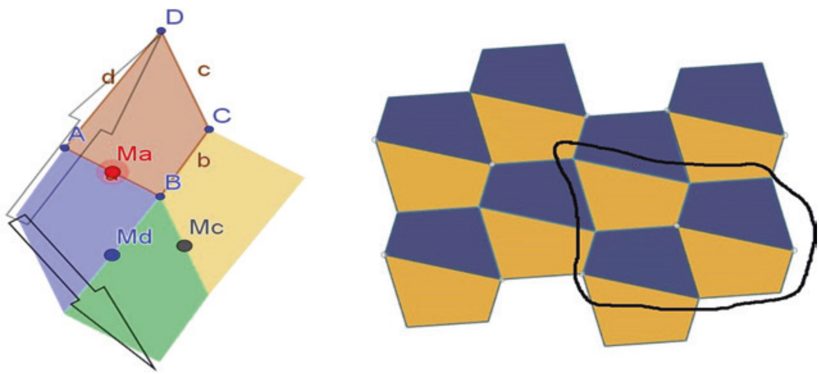


Fig. 12 a tessellation with 4gons

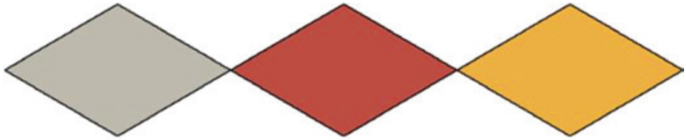


Fig. 13 rhombi 120° / 60°

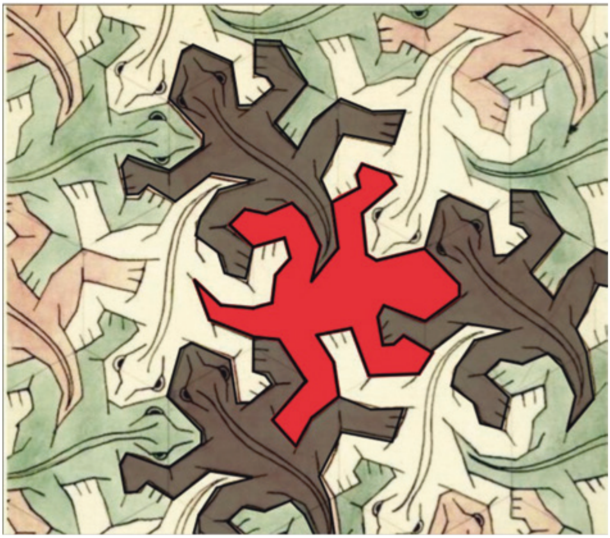


Fig. 14 Escher Lizards (www.geogebra.org/m/zxuaxtk8)

translate it in the direction of the shown vectors. By further translations you get the result.

Students can apply similar methods to create analogues shapes.

The construction in Fig. 15 can also be created with the shape of a lizard. Many coloured 3D printings of the lizard give a great puzzle (Fig. 16).

If you print with thin layers, the object is translucent (Fig. 17).

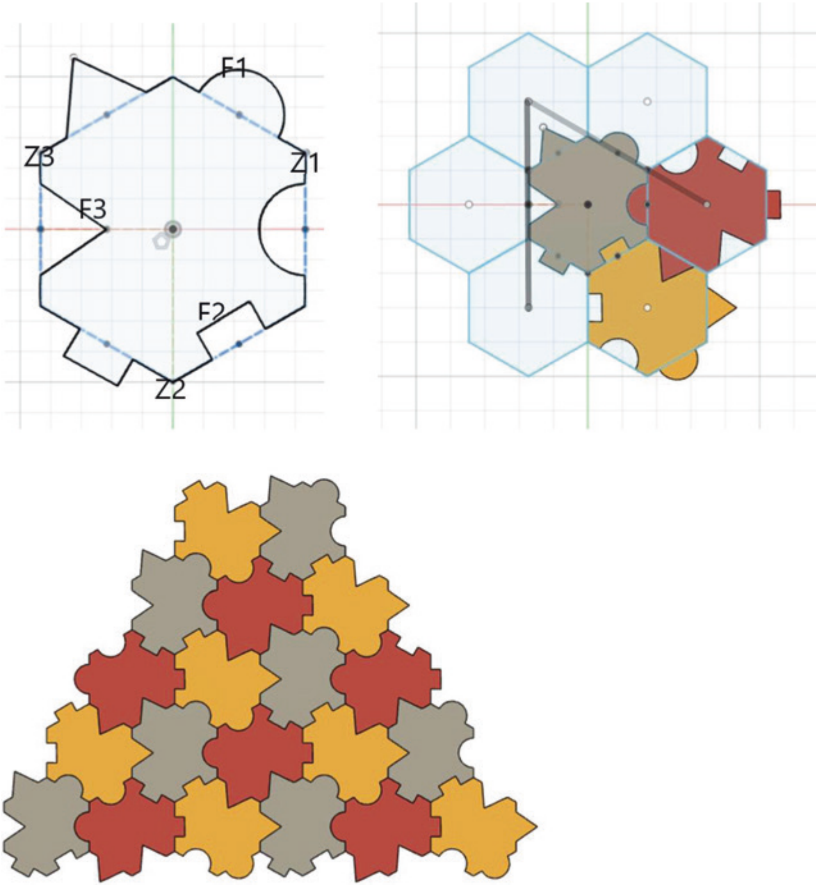


Fig. 15 A simple example for a Escher tessellation, constructed in FUSION 360

9.3 The Alhambra

The **Alhambra** is an islamic palace located in Granada (Spain). It contains many plane tessellations done with mathematical precision and is therefore aesthetically pleasing (see www.ams.org/notices/200606/comm-grunbaum.pdf) (Fig. 18)

We give a short instruction of modelling such patterns. (Fig. 19)

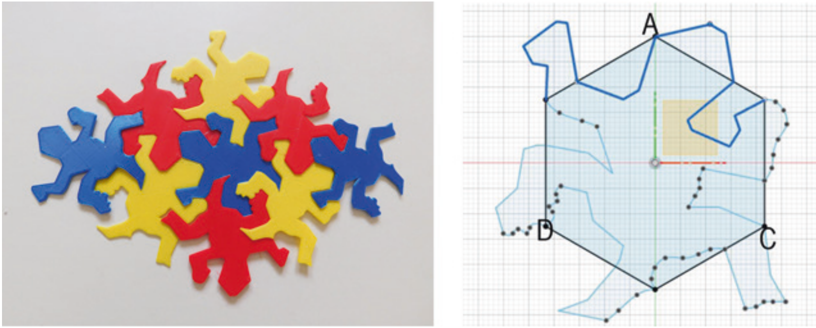


Fig. 16 a puzzle with lizards, constructed in FUSION 360

Fig. 17 Escher: black and yellow horsemen 3D printed with thin layers

Draw in a square, divided by a diagonal in two parts, an arbitrary pattern and reflect it on the diagonal. Rotate it fourfold. Afterwards translate it vertically and horizontally in the way that you get the pattern as big as you want. By the

Fig. 18 Alhambra wall paintings, 3D printed

method explained in chap. 4 extrude in different heights and 3D print it with different filaments.

10 Conclusion

Tessellation can be very useful in teaching mathematics as it has been indicated in the examples used in this article. Pupils can create their own patterns with the help of their knowledge of geometry. Besides, they can take their art work home as a 3D print.

The geometric methods used here are rotations of 60°, 90°, 120° and 180°, translations, reflections and glide reflections.

For constructing the shapes we used FUSION 360. It is free for students and not hard to learn.

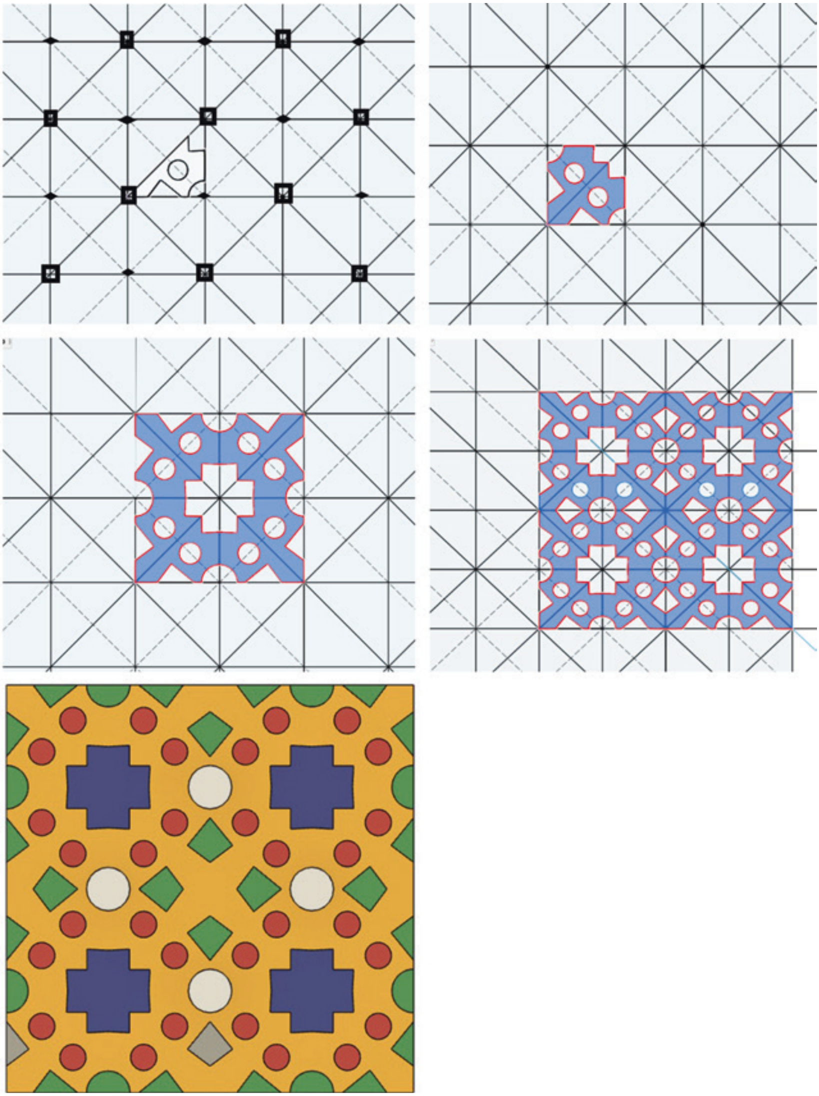


Fig. 19 5 steps in constructin a wall paper pattern (FUSION 360)

The Platonic Solids as Edge-Models

Rudolf Hrach

1 Introduction

The five Platonic solids are attractive subjects in space geometry since Euklid's The Elements. They are built mainly as face-models. In this article they are built as edge-models. Each of the five edge-models consists of 2 sorts of material: the edges are wooden kebab skewers bought in the supermarket, the vertices to hold the wooden edges are 3D printed objects. In Sect. 1 we show and compare shortly face- and edge-model. In Sect. 2 we describe how the vertices are constructed. This method of construction may help interested teachers to print their own vertices (better called vertex-connectors for they connect the edges) even for other polyhedrons than the platonics. In Sect. 3 we show applications of the vertex connectors. We build attractive models (the so-called Kraul model similar to the Kepler planet model and a great soccer ball) and we show how edge models can visually help to study the inner region of a platonic solid.

2 The Platonic Solids

The Platonic solids are regular polyhedrons and consist of the tetra-, hexa-, octa-, dodeca- and the icosa-hedron. They can be built in a compact (face-model) and in an open (edge-model) form (see Fig. 1). The compact models are constructed in

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https://doi.org/10.1007/978-3-658-38867-6_6

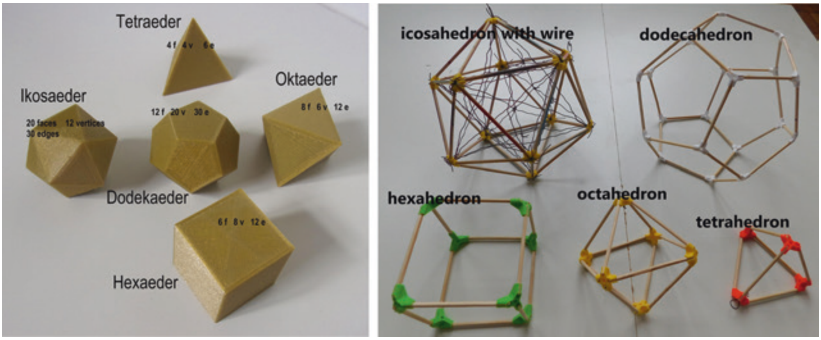

Fig. 1 face models (left) and edge models (right)

FUSION 360 and are practical for studying regular polygons. For completeness, the numbers of edges e, faces f and vertices v are added. Concerning the edge models, they have coloured vertices (these is the subject of this article) and in the icosahedron you can see wire to show which other figures are contained in the inner region. This method with the wire inside is a possibility to teach the platonics in a deeper way.

3 The Vertex-connector

As already mentioned above, this chapter is for teachers who are interested in the construction of vertex connectors and not only in the application of them. We will take the tetrahedron as example.

The tetrahedron has 4 rotation axis. Each runs through a vertex to the midpoint of the tetrahedron, that is the midpoint of the circumsphere. So the vertex connector we want to construct has also a rotation symmetry. If we know the angle α between the axis and an edge we can construct one edgeholder and get the two others by rotating with 120°. This is very helpful in the CAD construction. The calculation of α is shown in Fig. 2, the CAD construction with the software BLENDER in Fig. 3.

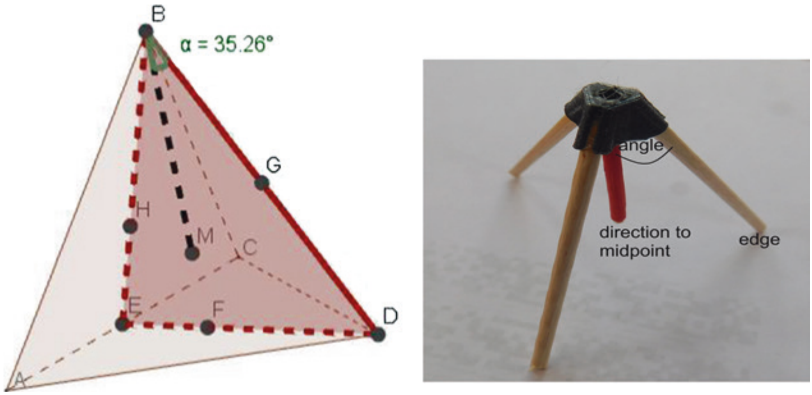

Fig. 2 Calculation of the angle (left) and the angle in the model (right)

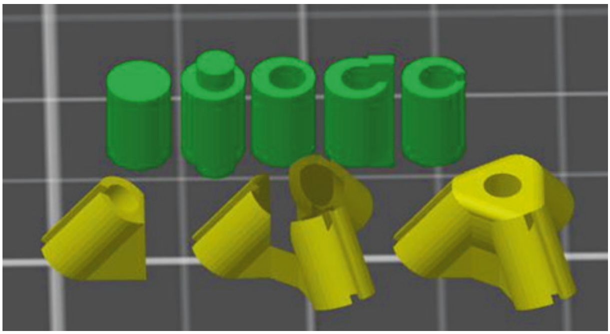

Fig. 3 The CAD-construction of the vertex connector

3.1 Determination of α

Regard the red isosceles triangle BED with midpoint M (Fig. 2). BD is the edge a, BM the radius R of the circumsphere of the tetrahedron and so we get:

The angles of the other Platonic solids are calculated in a similar way and we get the following data:

Platonic solid	Dodeca	Hexa	Tetra	Octa	Icosa
α	69,09°	54,74°	35,26°	45°	58,28°
Number of vertices	20	8	4	6	12

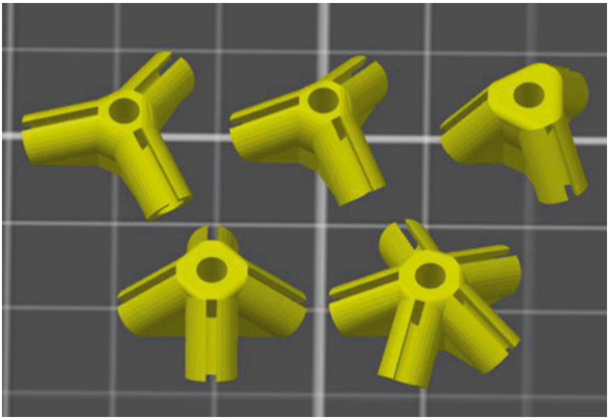

Fig. 4 The 5 vertex connectors

3.2 Construction of the Vertex-connector

We construct the vertex-connectors with the CAD software BLENDER. In Fig. 3 you see the development of the construction. In the green line one of the edge-holder is constructed. The left model in the yellow line is crucial: here the holder is tilt with $\alpha = 35.26°$ and stabilized. Afterwards two times duplicated and arranged with 120° and connected in a last step.

The hole in the middle of the vertex-connector can be used to install wire or threads in the inner region of the Platonic solid.

In a similar way you get all vertex-connectors (Fig. 4).

4 Application of the Vertex Connectors

4.1 Derivation of Icosahedron Formulas

As a possibility for an application the edge models may help students to determine as many as possible formulas that can be seen in the WIKIPEDIA picture. With an edge model in their hands it is much easier. The blue icosahedron in the WIKIPEDIA picture is exactly the model we have built. As example we take the icosahedron. In addition we formulate questions for teachers to explore the icosahedron (Fig. 5).

$V = \frac{5}{12} \cdot a^3 \cdot (3 + \sqrt{5}) \approx 2{,}182 \cdot a^3$
$A_O = 5 \cdot a^2 \cdot \sqrt{3} \approx 8{,}660 \cdot a^2$
$r_u = \frac{a}{4} \cdot \sqrt{10 + 2 \cdot \sqrt{5}} \approx 0{,}951 \cdot a$ [3]
$r_k = \frac{a}{4} \cdot (1 + \sqrt{5}) \approx 0{,}809 \cdot a$ [4]
$r_i = \frac{a}{12} \cdot \sqrt{3} \cdot (3 + \sqrt{5}) \approx 0{,}756 \cdot a$ [5]
$\frac{V}{V_{UK}} = \frac{1}{2 \cdot \pi} \cdot \sqrt{10 + 2 \cdot \sqrt{5}} \approx 0{,}605$
$\alpha = 60^\circ$
$\beta = \arccos\left(-\frac{\sqrt{5}}{3}\right) \approx 138^\circ\ 11'\ 23''$
$\gamma = \arccos\left(-\sqrt{\frac{1}{6} \cdot (3 - \sqrt{5})}\right) \approx 110^\circ\ 54'\ 18''$
$\delta = \arccos\left(\frac{1}{4} \cdot (1 - \sqrt{5})\right) = 108^\circ$
$\Omega = \arccos\left(-\frac{95}{243} \cdot \sqrt{5}\right) \approx 2{,}635\ \mathrm{sr}$
$\Psi = \sqrt[3]{\frac{(7 + 3\sqrt{5})\,\pi}{30\sqrt{3}}} \approx 0{,}939$

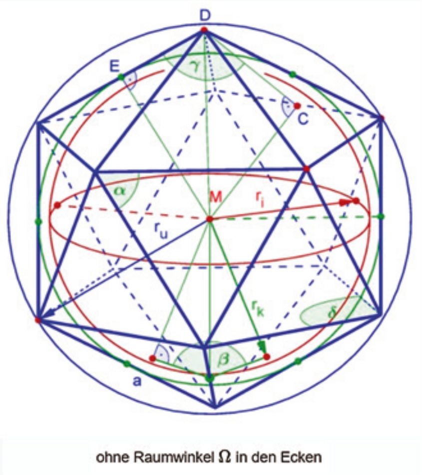

Fig. 5 The icosahedron in WIKIPEDIA (https://de.wikipedia.org/wiki/Ikosaeder)

i) How many faces, edges, and vertices has the icosahedron? Verify Euler's formula f + v = e + 2!
Solution: (icosa means 20), 20 faces, 12 vertices; 30 edges, faces + vertices = edges + 2

ii) What about the red, green, and blue circle?
Solution: they symbolize spheres: red insphere touches the faces; green midsphere touches the edges; blue circumsphere touches the vertices

iii) How many diagonals are there?
Solution: 6 from each vertex: 1 long and 5 smaller → divided by 2 results: 36

iv) Do you discover "golden" rectangles (the sides fulfill the golden ratio: short/long $= \frac{\sqrt{5}-1}{2}$) in the icosahedron? How many? (See Fig. 6)
Solution: With the edge-model it is very easy to construct the golden rectangles inside the icosahedron with threads 30:2 = 15

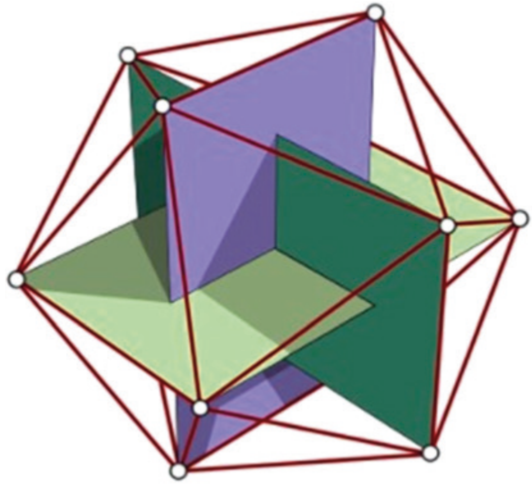

Fig. 6 The "golden" rectangles

4.2 The Kraul Model

Above we have stretched wire in the icosahedron and we watched other figures (like stars) in the inner region. The question arises to put the five Platonic solids altogether in one model. This is possible for you can change the size of edges (cut the skewers smaller) and therefore you can build big and small platonics. But in what sequence and how attach them?

Walter Kraul (1926–2016) was a Waldorf-Teacher at the Rudolf-Steiner school in Munic who was fascinated by polyhedrons and he produced many paper models. Amongst other things he created a nested model of the 5 Platonic solids with paper.

This gives us the idea to solve the question of the sequence of our edge-models. It is possible to nest the Platonic solids in a certain way. This reminds of the wellknown Kepler-model.

But the Kraul-model differs from the Kepler-model. In the Kraul-model all vertices of inner solids are connected with vertices or edges of outer solids while in the Kepler model the platonics are carried by hard spheres.

The Kraul-model presents all 5 platonics in one model and shows by this the relationship of the platonics.

The sequence in the Kraul-model is:

dodeca- hexa- tetra- octa- icosa- hedron

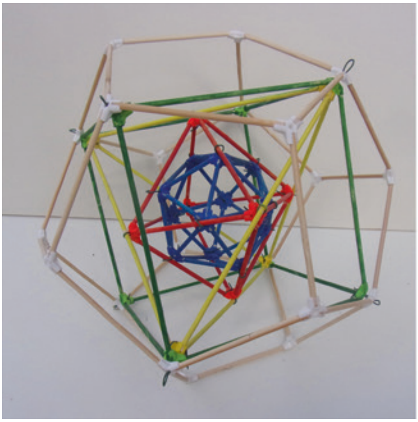
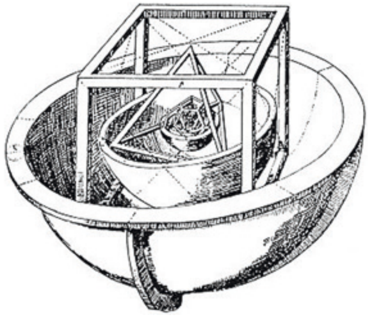

Fig. 7 Mini Kraul model (left) and the famous Kepler model (right)(wikipedia.org/wiki/Mysterium_Cosmographicum)

The first three are connected at their vertices, the last three at their edges (octa in the middle of an tetra-edge, icosa in the golden section of an octa-edge) (Fig. 7).

If you construct the vertex-connectors big enough to carry wooden edges of diameter 10 mm you can build a greater Kraul model exerting a fascination for nearly all people (Fig. 8).

4.3 Other Models with Vertex-connectors

There are many other polyhedrons with edges and vertices. After studying the angles at the vertices it is possible to construct the vertices of these polyhedrons by the method shown in 2.

With the method described in Sect. 1 we constructed the 13 archimedian and their duals, the 13 catalanian polyhedrons. The duality can be shown if you put two duals together. Figure 9 shows as an example the Truncated Cube (red) and its dual the Triakis Octahedron (blue). The connection between the red vertices and the blue faces (and vice versa) is obvious. Red and blue edges are orthogonal.

As a high motivation and for many people very astonishing is the fact that a normal soccerball is connected with the platonics. The soccerball is namely a truncated icosahedron. In Fig. 10 you see a huge model. In order to examine this model, two questions with answers as suggestions for teachers are added (Fig. 11).

Fig. 8 Large Kraul model (in front the book of Walter Kraul)

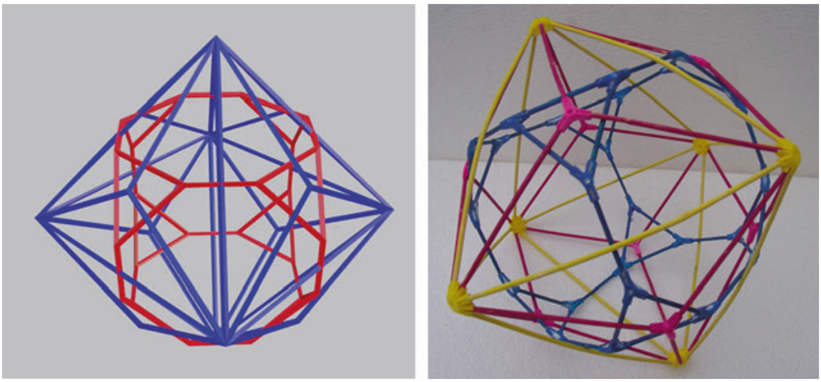

Fig. 9 Truncated cube and triakis Octahedron in BLENDER software and in reality

Fig. 10 two balls (wikipedia.org/wiki/Truncated_icosahedron#/media/File:Comparison_of_truncated_icosahedron_and_soccer_ball.png)

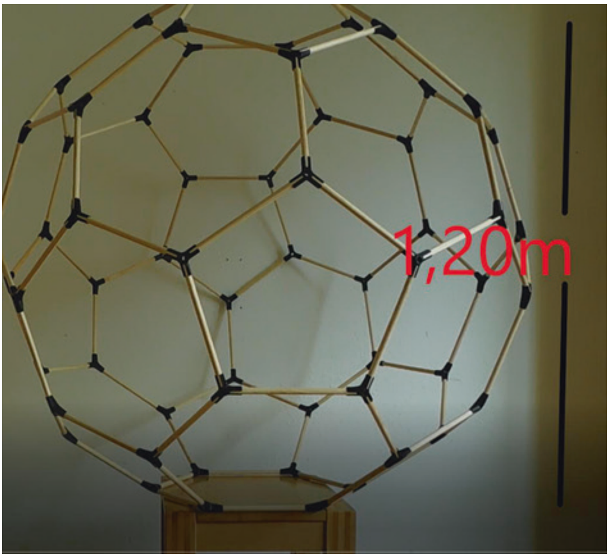

Fig. 11 Football is great!

For the following questions remember the icosahedron (f = 20; e = 30; v = 12)!

a) Describe the faces and count them!
Solution:
each truncation at a icosahedron vertex (v = 12) generates a pentagon, thus 12 pentagons; each triangle (f = 20) becomes a hexagon, thus 20 hexagons, each pentagon is surrounded by 5 hexagons.
b) How many vertices and edges?
Solution:
each truncation at an icosahedron vertex generates 5 vertices, thus 5 * 12 = 60 vertices; 12 new faces + 20 icosahedron triangles = 32 faces; each edge becomes 3 edges, thus 30 * 3 = 90 edges.

4.4 Application for Geometry at School

Suggestion:
Each group of students gets a set of vertex-connectors. They cut the skewers by themselves. By constructing the Platonic solids they learn the shape of polygons and polyhedrons. They now have a look in the inner region of a polyhedron. They can count diagonals, calculate their lengths and measure them. With the light of a lamp (mobile light) they can cast shadows of the Platonic solids. With help of the holes in the vertex-connectors they can stretch threads and by this way observe new figures in a Platonic solid.

4.5 Conclusion

We showed a method of construction vertex connectors. With this method one is able to build nearly any common polyhedron. Once a school owns a set of vertex-connectors this gives some advantages: the connectors and the skewers are easily to store, the students themselves build the corresponding polyhedrons (a step that cannot be underestimated), last not least the free inner region of the polyhedrons can be measured. The great models can be placed somewhere at school as beautiful monuments of mathematics and in addition as living memory of the process of learning.

Doing Mathematics with 3D Pens: Five Years of Research on 3D Printing Integration in Mathematics Classrooms

Oi-Lam Ng and Huiyan Ye

1 Motivation

Our work has been inspired by Seymour Papert, who put forward a new conception of learning known as *constructionism* or "learning-by-making" (Papert & Harel, 1991, p. 1), which holds that knowledge is constructed during human contact with external materials. In turn, while hands-on making, students simultaneously construct their knowledge as well as build meaningful products in the physical world. In addition, the artefacts are shareable in the community to facilitate peer learning. Among his contributions, Papert has shown that technology-based constructions can change students' ways of thinking about mathematics. While we were interested in educational programming languages such as *Logo*, which enables students to express their ideas in a computational manner, we also wanted to explore new forms of multimodal technologies, such as touchscreens, 3D printing, and the like, which provide a more direct way of interacting with and expressing mathematical ideas (Hegedus & Tall, 2016). These hands-on technologies have created more opportunities for learners to interact with technologies or products, thereby facilitating learners' knowledge construction processes in the mathematics classroom.

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Grounded in the theory of constructionism, we believe that 3D printing is a powerful technology for "learning-by-making." 3D printing can extend 2D products to the 3D environment; moreover, unlike traditional manipulatives preselected by teachers, 3D printing can provide students with hands-on opportunities to generate 3D models. Such learning experiences correspond to the "learning-by-making" approach advocated by constructionism. Furthermore, considering the unique characteristics of hands-on and "embodied making" (Ng & Ye, 2022), in our research, we used 3D pens as a form of 3D printing in mathematics education to explore the potential transformations that 3D printing can induce in mathematics teaching and learning (Ng & Ferrara, 2020; Ng & Sinclair, 2018; Ng & Tsang, 2021; Ng et al., 2018). In this chapter, we review the first authors' five years of research on the use of 3D pens in mathematics education. First, we describe the practice of mathematical diagramming and discuss the potential possibilities of 3D diagramming in engendering students' mathematical thinking as a way to introduce three affordances of diagramming with a 3D pen. Then, we illustrate the theoretical background that underpins our research and describe some lesson designs with 3D pen. Finally, we present both quantitative and qualitative results to discuss the role of 3D pens in mathematics learning and suggest future research direction.

2 Affordances of Diagramming with a 3D Pen

2.1 Support Visualization of 3D Geometrical Objects

The first unique feature of using a 3D pen is the ability to draw in 3D, which overcomes the limitations of paper and pencil and improves the visualization of 3D geometrical objects. For example, one way to draw a cube (Fig. 1), is firstly to draw four straight "segments" on a surface to form a square, then four vertical "segments" that join the four vertices of the square, and four more "segments" in the air, while drawing an identical square parallel to the base. Note that the constructionist practice mentioned here parallels the process of drawing, for example, a square in Papert's Turtle Geometry in the sense that a sequence of actions is taken to construct a figure, which is also the very process that defines the figure itself. The difference between the two constructionist practices lies in the tools used, which facilitated different (programmable and non-programmable) artefacts, where 3D pens afforded a mode of making merely by moving one's hands. We note that in the process of drawing such a 3D object, one can visualize vertices, segments, and planes and observe how these 0D, 1D, and 2D objects

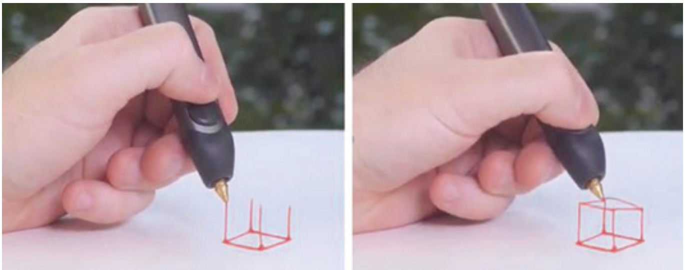

Fig. 1 Drawing a cube with 3D pen

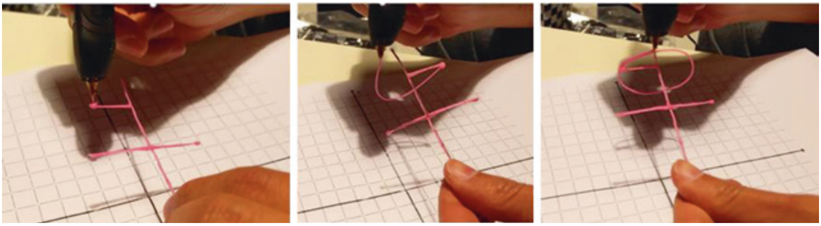

Fig. 2 Thinking while drawing with a 3D pen

compose the 3D object. The hand movements involved in the process of drawing are a significant learning component, one that is not present in screen-based media and different from operating with premade objects, as the movements themselves (moving one's hands to make a square) imply geometrical meanings.

2.2 New Modes of Thinking

Besides drawing in 3D, the diagramming process with a 3D pen can uniquely stimulate new modes of thinking. This is because the process of 3D drawing simulates the very process of gesturing; as the hand moves with the 3D pen, a 3D model is generated. The unique nature of interacting with the 3D model with one's hands affords some interesting movement that could not be possible on the flat surface of a paper-and-pencil environment. For example, as shown in Fig. 2, the drawing of a circle can involve the experience of spinning an axis of rotation with one's left hand while the 3D material continues to be added to a point at a distance from the axis at which the right hand remains still. Such material interactions stimulate new modes of thinking and facilitate meaningful bodily experiences, i.e., physically picking up and rotating an axis (for 360 degrees) to locate the trajectory of a point at a fixed distance from the axis, resulting in a circular trajectory with the help of two hands that are equidistance from each other.

2.3 The Dual Nature of Diagramming and Manipulating

Finally, 3D pens provide additional tactile experience of drawing 2D figures by affording students the ability to touch and feel 2D models drawn. Diagrams that would have been drawn using paper and pencil, such as a triangle, can be recreated and become physical objects that can be held, moved, and turned when drawn by 3D pens (Fig. 3). This enables one to interact with 2D figures in ways that one could not when using the traditional paper-and-pencil medium, as they can be physically transformed or manipulated during the meaning-making process. For example, our most recent research suggests that students conceive triangles as flat figures that could not be, in any way, manipulated in the third dimension. In particular, they no longer recognize it as a triangle if it was drawn by a 3D pen and made "standing up" rather than "lying flat on the table". This indicates that students' meaning making are constrained by the paper-and-pencil medium, where the triangle will stay dormant on paper once drawn. As such, students often lack the tangible experience of manipulating with geometric shapes after drawing them. In response, 3D drawings have a dual nature: they are both a *diagram* and a physical *manipulative*.

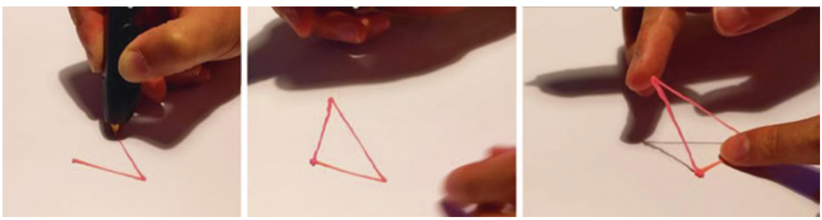

Fig. 3 Drawing a triangle with a 3D pen

3 Theoretical Background

Our research interest is framed from the perspective of embodied cognition, which focuses on exploring some salient aspects or less easily observable features that occur in the teaching and learning of mathematics until recent years. These aspects include language, the body, gestures, non-verbal communication, which reflect the movement from static, individualistic learning to a broader level and more dynamic features of learning that considers multimodality and sensorimotor experiences, as well as the role of the body in mathematical cognition. Embodied cognition is a study in cognitive science that frames a deep understanding of what human ideas are, and how they are organized in vast (mostly unconscious) conceptual systems grounded in physical, lived reality (Núñez et al., 1999). Further, Núñez et al. (1999, p. 50) defines embodiment as follows:

> Embodiment is not simply about an individual's conscious experience of some bodily aspects of being or acting in the world. Embodiment does not necessarily involve conscious awareness of its influence. Nor does embodiment refer to the physical manipulation of tangible objects, or to virtual manipulation of graphical images and objects […] an embodied perspective does not constitute a prescription for teaching in a 'concrete' way.

Therefore, an embodied cognitive approach to learning is not simply learning in a physical way but to draw on the individual's bodily experience and conceptual system in sense-making. In addition to theoretical considerations around embodied cognition, our work is also informed by research evidence that our hands contribute significantly to cognitive processes: a) there is a connection between spatial reasoning and gestures (Ehrlich et al., 2006)—for example, gestures accompany spatial information in speech; b) there is evidence of finger perception (Penner-Wilger, 2013), in the sense that college students' finger perception predicts calculation scores and that finger perception in Grade 1 is a better predictor of mathematics achievements in Grade 2 than test scores; and c) there is evidence that gestures contribute to effective communication (Alibali & Nathan, 2012). Therefore, gestures are highly important in thinking and learning. At the same time, mathematical cognition is deeply rooted in embodied interactions with the environment and materials (e.g., tools). As Nemirovsky et al. (2013) proposed by the term *mathematical instruments*, 3D pens serve as "material and semiotic device[s] together with a set of embodied practices, enabling the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) as acknowledged within the culture of mathematics" (p. 376). Overall, as reviewed in Ng et al. (2020), approaches to learning that take on the perspective of embodied cognition predict that sensorimotor experiences, including visual perceptions and bodily actions, strengthen students' sensemaking processes, especially their visualization capacity and spatial reasoning in the science, technology, engineering, and mathematics (STEM) disciplines (Weisberg & Newcombe, 2017). These empirical results are further extended by research showing that the transition from action to abstraction in mathematics and science learning can be supported via gestures (Novack et al., 2014). In relation to embodied learning, the use of 3D pens can facilitate hand movements that support gestural forms of thinking about mathematical concepts.

We are also interested in de Freitas and Sinclair's (2014) theoretical approach of *inclusive materialism*, which sheds light on the ontologies of the body and mathematics. This framework aims to redefine the boundaries of the body and claims, based on the interactionist perspective, that materials are constantly interacting with one another and with the human body rather than being inert. This approach reflects the relatively more social aspects of meaning making compared to the conceptualist tradition, which "ultimately demotes activity to simulation rather than full-body Making" (Ng & Ferrara, 2020, p. 928). This perspective offers a re-conceptualisation of classroom learning as assemblage of mathematical knowledge, teacher, students and material surrounding. Since materials do not have confined properties of their own, boundaries of materials, mathematics and the human body are re-defined, and learning is redistributed across the situation amongst the players and their material surrounding.

Given that the use of 3D pens constitutes a material interaction and the diagrammatic nature 3D drawing, the work of Châtelet (2000) is important, as it considers diagramming (the making of diagrams) and gesturing to be inseparable processes as well as creative embodied acts that constitute new relationships between mathematics and the material activity. In other words,

> [G]estures and diagrams are sources of mathematical meaning, which presuppose each other. They are never complete and share similar mobility and potentiality: gestures give rise to the possibility of diagramming, while diagrams give rise to new possibilities for gesturing. (Ng & Ferrara, 2020, p. 926)

As mentioned in the previous section, the use of 3D pens offers new gestural forms of thinking (Ng & Sinclair, 2018; Ng et al., 2018). This means that 3D diagramming is not only iconic representations of mental operations, they also affect the individuation of mathematical meaning. As de Freitas and Sinclair

(2014) state: "Does mathematics really just stand there, silently waiting for the breakthrough insight or shift in attention? Or might it somehow be much more implicated in the moving hands and the configuration of [materials]?" (p. 30). Due to the close resemblance of 3D diagramming and gesturing, we find it useful to adopt a materialist perspective when exploring the unique prospect of using 3D pens in mathematical activities, as embodied diagramming/gesturing during material creation which engenders new possibilities for encounters with mathematical concepts. Consequently, the hands-on production of artifacts with 3D pens is not only a form of making but also a kind of assemblage or emerging intra-action among the learner(s), concept, and tool.

4 Lesson Designs with 3D Pens

In view of the aforementioned theoretical framings and the three affordances of 3D diagramming, our research team has engaged in research contextualized in the mathematics classroom to improve our understanding of 3D pens' impact on learning mathematics. In this section, we will describe some examples of lessons designed for elementary mathematics topics (e.g., geometry) and secondary mathematics topics (e.g., functions and calculus).

4.1 Example 1: Primary School Geometry (Ng & Ferrara, 2020)

In Ng and Ferrara (2020), we developed a lesson design for teaching and learning properties of prisms, pyramids, and cross-sections of 3D solids. Using an inquiry-based and student-centered approach, in the lesson on the properties (faces, vertices, and edges) of prisms and pyramids, students used 3D pens to draw different prisms and pyramids and to investigate the target properties (Fig. 4). The students worked in pairs and exhibited high engagement with the inquiry-based learning activities (Fig. 5). Among the results, we found that the students construct triangular prism with different strategies which also indicated they have visualized a triangular prism differently. For instance, many students started off with drawing a triangular base and then constructed three vertical pillars in the air, followed by moving the 3D pen from the top of one vertex to another in a triangular path to form another triangular base at the top. In another construction, a student began by drawing a rectangular base, then created two facing triangles perpendicular to the rectangular base on both sides and finally drew a line connecting the two

Fig. 4 Students used 3D pens to draw different prisms and pyramids

Fig. 5 Students worked in pairs on 3D pen activities

triangles in the air. From the perspective of embodied cognition, the combination of students' linguistic expressions about 1D ("lines") and 2D shapes ("triangles", "rectangle", etc.) together with their gestures (e.g., perpendicularly moving, rotating, making four right-angled turns, etc.) suggest that "3D making did not only yield a product that was physical and sharable, but it was also a material process of thinking mathematically that outline differences in how to make a triangular prism" (Ng & Ferrara, 2020, p. 936).

In another lesson, students drew cross-sections using different cutting methods for different kinds of prisms and pyramids when learning the cross-sections of 3D solids. The 3D pen provides an alternative "cutting" method, which allows students to visualize a cross-section of the 3D solids. Students can hold a 3D pen and draw the outlines of the cross-sections of prisms and pyramids on the physical models themselves. When they pull their drawing out of the physical models, they could see the corresponding geometrical shape (e.g., circle, rectangle, trapezoid), which is the very shape of the cross-section (Fig. 6). From the perspective of inclusive materialism, when students were making an oblique or perpendicular

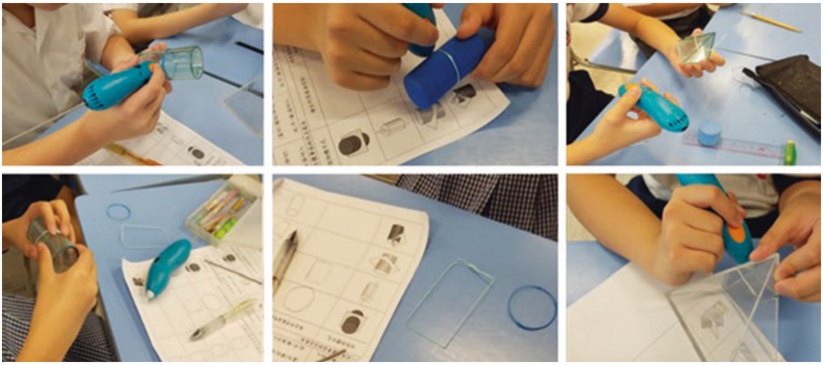

Fig. 6 Students draw cross-sections on different models of prisms and pyramids (Ng & Ferrara, 2020, pp. 932, 939)

cut to the cylinder, the notion of time, distance, angle and turns were made manifest in students' learning assemblage because of the human-material interaction of using a 3D pen. In so doing, the mathematical meaning of cross-sections, such as ellipse, rectangle, squares and trapezoids, were co-implicated due to the movement of the hand holding the 3D pen. This shows that 3D drawing provides opportunities for students to explore and create new mathematical meanings.

4.2 Example 2: Functions and Calculus (Ng & Sinclair, 2018)

In this subsection, we provide another example of 3D pens being integrated in the teaching and learning of functions and calculus. There were several aspects of 3D drawing that were significant to the lesson design. First, the "ink" of a 3D pen is extruded continuously, which makes 3D drawing into a process of continuous construction. Such continuous construction facilitates the understanding of functions as processes and objects. As we know, when students draw functions with paper and pencil, it is simply a drawing process that does not create an artifact. However, when they draw with 3D pens, students can produce physical objects. They can pick up a 3D drawing object (the graph of the function) and manipulate (translate and reflect on) it to touch and feel the function. Our observations show that, linguistically, student always say nouns "slope", "point" and "tangent line" as a singular which is different from the plural in the textbook. This indicates

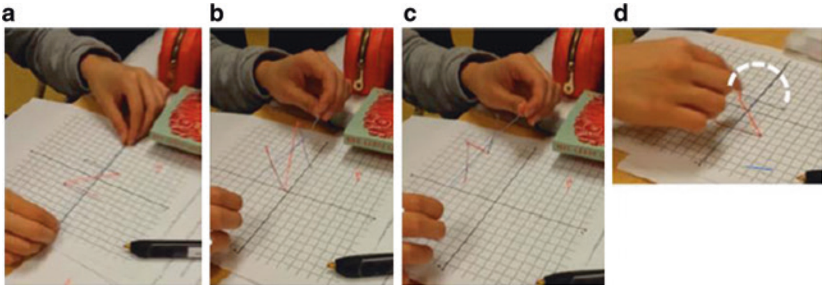

Fig. 7 (a–c) Picking up the drawn graph and rotating the axis to visualize the solid formed; (d) gesturing a semi-circle above the diagram (Ng & Sinclair, 2018, p. 307)

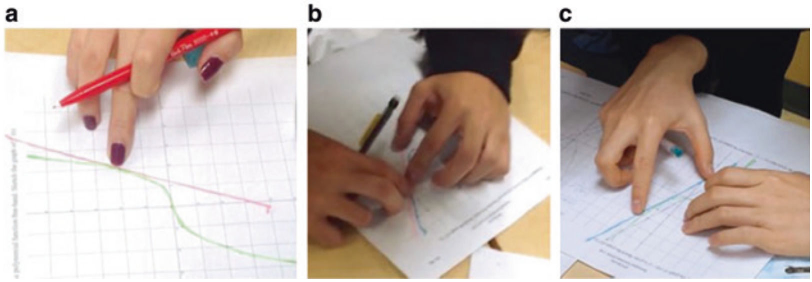

Fig. 8 (a) Physically feeling tangent to a curve with one or (b) two fingers. (c) Gesture-diagram interaction facilitated by using two fingers (or two points) to anchor the line tangent to the function, i.e., making a gesture expressing the slope of tangent on the diagram (Ng & Sinclair, 2018, p. 309)

that students considered the tangent line as a continuously moving object along the function graph rather than a discrete change of "slopes". When asked by the teacher to draw "solids of revolution", students leveraged the affordance of 3D drawing (as both a diagram and a manipulative) to draw a curve along with the coordinate axes and then spinning the x-axis (Fig. 7), thereby improving their visualization of how the solid is formed.

In the lesson on derivative functions, we designed a task in such a way that drawing with a 3D pen would offer new gestural forms of thinking. More specifically, when learning about slope of lines tangent to a curve, students used their fingers to push drawn 3D "lines" to be tangent to a curve, that is, to embody the tangent line (Fig. 8). The study showed that 3D drawing enables creating a physical instantiation of mathematical ideas, which can make students physically "feel" the idea of local linearity and the point of tangency via the sense of touch. As shown in Fig. 8, this was evident in students' gesture-diagram interaction which included students using their fingers to push the tangent line toward the curve and try to re-orient the tangent line to make it locally linear or "parallel" to the curve (Fig. 8a–c). Thus, 3D drawing enables what Tall (2003) referred to as the embodied mode of thinking about tangents as the "changing slope of the graph" and the idea of local linearity (p. 10). To restate, we can see that in 3D drawing environments, students could draw mathematical objects and manipulate these objects to construct meaning.

5 Empirical Studies with 3D Pens

Over the past five years, the research team led by the first author has conducted quantitative and qualitative studies on the role of 3D pens in mathematics learning. Adopting different research methods has allowed us to better investigate the potential benefits of using 3D pens in mathematics education; some of these benefits are illustrated in this chapter.

5.1 Study 1: Ng et al. (2020)

We adopted a quasi-experimental research design to investigate the differences in geometry learning outcomes between a dynamic geometry environment (DGE) group (which used DGE technology for instruction) and a 3D-pen group (which used 3D pens for instruction) (Fig. 9). The participating students were in the sixth grade from two primary schools. The DGE group contained 65 students from School A, while the 3D-pen group contained 101 students from School B. The learning topic was "measure, shape, and space" and was meant to help students explore 3D shapes and understand the relationship between the number of sides of the base, the edges, and the vertices of prisms and pyramids.

The teaching intervention took place over two 70-min lessons, and the two groups shared an almost identical lesson procedure except for the technology used in the class. Every student pair in the 3D-pen group used one 3D pen to draw different prisms and pyramids, while in the DGE group, students used a pre-made DGE to explore the properties of prisms and pyramids.

To compare the differences between the DGE and the 3D-pen group, different tests were used to assess students' learning outcomes, including pre-tests (T0),

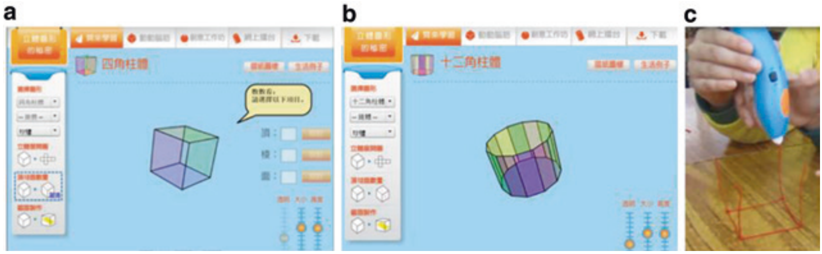

Fig. 9 (a–b) A computer applet that performs virtual transformations of various 3D figures; (c) constructing a physical artifact (i.e., a cube) with a 3D pen (Ng et al., 2020, p. 3)

post-tests (T1), and delayed post-tests (T2). The three tests used the same types of questions (Table 1) but had different assessment purposes. Pre-tests took place before the teaching experiment started, with the aim of assessing students' prior knowledge. Post-tests took place immediately after the interventions to assess students' learning outcomes, while the delayed post-tests, meant to assess students' knowledge retention, took place five months after the interventions. Students did not study the target concepts and relevant topics during this five-month period.

Using quantitative analysis, we obtained some findings on the effects of these two teaching interventions. First, the post-test indicated that after both technology-enhanced interventions, all students received a higher score in all (sub-) categories. Moreover, a higher improvement rate was obtained from the DGE group compared to the 3D-pen group regarding "vertices," "edges," "advanced," and "advanced questions on vertices (AV)." What's more, the delayed post-test showed that, compared to the DGE intervention, the 3D-pen intervention had a greater retention effect, particularly in relation to "vertices," "advanced," and

Table 1 A summary of the types of questions used in student assessments (Ng et al., 2020, p. 6)

Category	Sub-category	Number of items	Sample question	Total items
Simple: Determining the number of faces, vertices, or edges in a given prism or pyramid	Faces (SF)	5	How many faces does an 8-gonal pyramid have?	15
	Vertices (SV)	5	How many vertices does an 11-gonal prism have?	
	Edges (SE)	5	How many edges does a 9-gonal pyramid have?	
Advanced: Working backward given the number of faces, vertices, or edges in a prism or pyramid	Faces (AF)	4	A solid has 14 faces. This solid could be a _____-gonal prism or_____-gonal pyramid.	12
	Vertices (AV)	4	A solid has 9 vertices. This solid could be a _____-gonal prism or_____-gonal pyramid.	
	Edges (AE)	4	A solid has 18 edges. This solid could be a _____-gonal prism or_____-gonal pyramid	
				27

"AV" questions. Finally, we found the unexpected result that the 3D-pen group's T2 scores were consistently higher than the T1 scores, and that five months after the teaching interventions, the T2 scores did not differ significantly in all the sub-categories between the DGE and the 3D-pen group. This implies that over the long term, there was no significant difference in students' geometric thinking levels between the two groups.

According to the results of Study 1, it can be concluded that embodied interactions with 3D pens have a positive and sustained effect on geometry learning. From the persepctive of embodied cognition, the results can be explained by the consideration of the pen-hand movement as "concrete gestures that preserve the embodied nature of the interaction as found in physical manipulation" (Ng et al., 2020, p. 11). Corroborating with previous studies on gestures and memory, the results are consistent in showing the effects of gesture in promoting long-lasting learning (e.g., Cook et al., 2008). Moreover, the 3D Pen environment could make a much stronger "connection between mathematical and pedagogic dynamisms" (Jackiw & Sinclair, 2009, p. 418) due to the direct, hands-on interaction of mathematical representations as opposed to other tool-mediated devices such as using a 2D mouse to navigate a 3D scene. This study also raises the following question: *What role can 3D pens play in students' mathematical meaning construction?* There seems to be abundant room for future research regarding the benefit of learning mathematics using 3D pens; consequently, we adopted qualitative methods to answer the aforementioned question which we elaborate in Study 2.

5.2 Study 2: Ng and Ye 2022

In Study 2, we further explored how students' linguistic expressions and gestures produced while using 3D pens to construct 3D solids supported the students' thinking and mathematical-meaning construction when studying the properties of prisms and pyramids.

(1) Constructions with 3D pens Supported Students' Composition of 3D Solids by Their 0D, 1D, and 2D Parts, as well as Improved Students' Visualization of the Relationships of these Parts in Embodied Ways.
Regarding the study of 3D solids using 3D pens, we asked a student who had completed a rectangular prism construction using a 3D pen how she might construct a pentagonal prism. When asked to count the number of "straight lines" that were required to complete the drawing, she responded "13" and was encouraged to anticipate the process of drawing a pentagonal prism using her finger

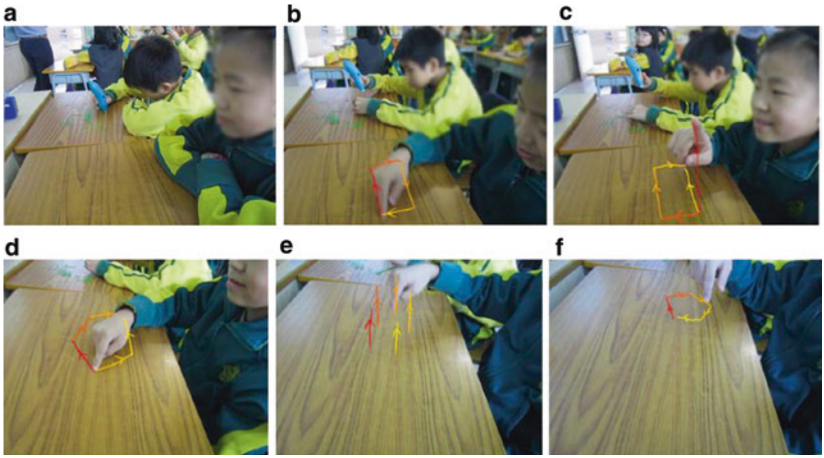

Fig. 10 (a–f) Snapshots of a series of gestures by a student (Ng & Ye, 2022)

(Fig. 10). At this point, she began to describe the drawing process while using her fingers on the table and in the air, imitating the method and process of drawing with the 3D pen previously. Moreover, the student used mathematical terms, such as base and sides, to describe her construction method and process, which indicates that the student thought of 3D solids using both 2D and 1D perspectives. Therefore, based on the linguistic expressions and gestures of students who use 3D pens to construct 3D solids, we can conclude that 3D pens can help students understand and reconstruct 3D solids from a low-dimensional level (0D, 1D, and 2D) and that such a visual construction process helps students understand the relationships between vertices (0D), edges (1D), and faces (2D).

(2) Constructions with 3D Printing Pens Gave Rise to Gestures Conducive to Learning the Properties of 3D Solids.

At the same time, drawing with 3D pens gave rise to gestures with geometrical meanings. When a student constructed 3D solids with a 3D pen, the hand holding the 3D pen moved constantly according to the shape of the constructed solid. Let us say that we are making cubes: students first need to draw a square with four congruent sides. The process of moving one's hand when drawing the square base reveals a property of squares: all sides must be of equal length, and they have to be perpendicular to one another. Or, when a student draws pyramids and prims with parallel or vertical lines and planes, the hand movement changes depending

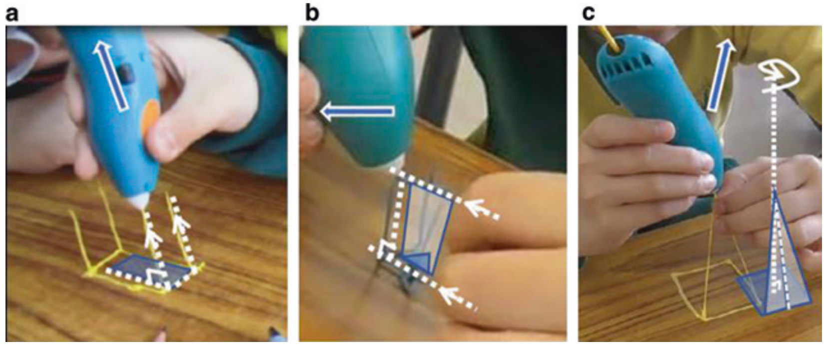

Fig. 11 Hand movements conveyed meanings, such as (a–b) the perpendicularity and parallelism between lines and/or planes and (c) reflectional and rotational symmetry (Ng & Ye, 2022)

on the "lines" one is drawing relative to the shape. As illustrated in Fig. 11, the process of constructing prisms requires students to observe parallel or perpendicular relationships between lines or lines and planes when moving their hands to shape the target object (Fig. 11a–b). In addition, when drawing a square-based pyramid, students also need to pay attention to the reflectional and rotational symmetry pertained in the 3D solids. In embodied cognitive terms, these particular gestures express the geometrical meanings behind the objects being constructed. The gestures generated by such embodied activity strengthens students' visualization of 3D solids, as they actively using and constructing geometric relationships and properties as opposed to simply viewing with a 2D representation of the solids.

6 Conclusion and Future Direction

We conclude this chapter by discussing the role of 3D pen as a form of 3D printing and presenting a four-fold characterization of making in mathematics education (Ng & Ferrara, 2020). Building on a previously developed notion of "learning as making", we were compelled to conceptualize the tool of 3D pen as a highly transformative and constructionist environment for supporting mathematical thinking and learning. As a result of our five years of research, we were able to identify some characteristics of these constructionist practices with 3D pens which we think has implications for tool-based in mathematics teaching and learning.

1. Making Involves Co-Constructing Meanings

Due to the characteristics of 3D pens (speed and hardness, etc.), students coordinate their hands and eyes when constructing 3D solids and have to consider the features of 3D pens to successfully construct mathematical objects. The materials, tasks, students' actions, and the final artifacts interact in the meaning-making process in a co-constructive way. As argued by the theory of inclusive materialism, if one of these elements were to be changed, the final artifact would change as well. Aligned with constructionism, students use 3D pen actively to construct a personally meaningful artefacts while constructing mental schema about the process, upon which the artefact can be shared with others for peer learning. Therefore, making is a process of co-constructing mathematical meaning.

2. Making Is Mathematizing

In the process of making, students' gestures, artifacts, mathematical concepts are often co-implicated. When a student constructs an artefact with 3D pens, mathematical meanings (such as segments, planes, bases, sides, triangles, etc.) are also generated during the making process. Importantly, these artefacts are not external representations of their "mind"; rather, based on the embodied cognition, the students' mathematical thinking and bodily movements are co-constituted while making. Hence, making is a process of creating something new externally (in a physical sense) and mathematically (in a cognitive sense), i.e., making is mathematizing, which deepens one's mathematical knowledge. In addition, working with 3D printing involves a process of mathematical experimentation. Students are not required to make prisms of a certain size and to follow specific construction methods. Instead, students are free to explore and develop their own construction processes. Therefore, students are actually thinking and doing mathematics, or mathematizing, while making.

3. Making Is Assembling with Technology

From the materialist perspective, students engage in gesture-diagram interaction that is unique to the tool of 3D pens and, at the same time, engage in considering the mathematical properties of the objects. In this process, a "making assemblage" focusing on the evolving relationship among human, material, and mathematics is formed. That is, "mathematics was not some abstract concepts to be conceived […] but it emerged as an assemblage with technology from the students' drawing and gesturing hands" (Ng & Ferrara, 2020, p. 941).

4. Making Is Inventing

In the cross-section lesson, students came up with creative ideas when using 3D pens to explore the cross-sections of 3D objects. After completing the construction of the first cross-section with a 3D pen, students gained an initial visualization of the cross-section. Specifically, they knew they could trace the outline of the anticipated cross-sections with a 3D pen and could detach them from the solids to inquire the shape of the cross-section (e.g., a rectangle in the case of a perpendicular cut to the base of a cylinder, a circle in the case of a horizontal cut to base of a cylinder). Afterwards, a student can directly create a new cross-section by considering the position of the cross-section based on past experience of constructing the cross-section. In particular, one student adjusted the height of a pyramid's cross-section to get a trapezoid of a different size (Fig. 12). Another student used his gestures to describe the process of generating the cross-section of a cylinder, which is also an invention of an action (Fig. 13). In both cases, the students came up with new ways of thinking about cross-sections, which are generally difficult for students to visualize. These cases show that making is a process that enables exploring, engaging with, and inventing mathematics.

Overall, our research has been one of the first attempts to examine the role and use of 3D pen as a form of 3D printing in mathematics education. Our empirical investigations point to the potential changes in thinking, learning, and doing that may result from the use of 3D pens, which enable mathematics to be performed

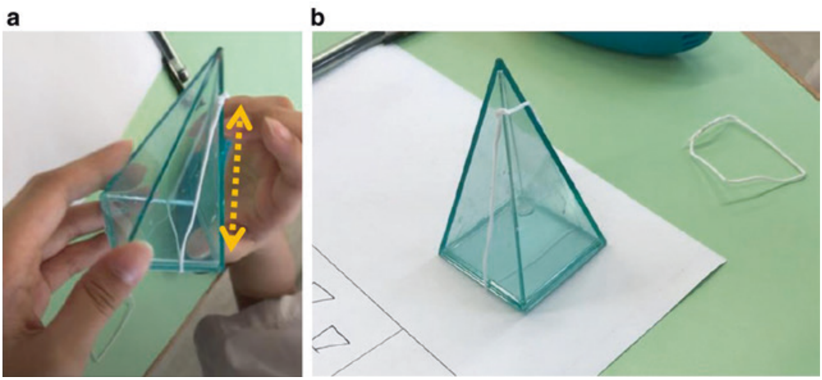

Fig. 12 (a) A student's gestures for highlighting the height of his artifact; (b) two artifacts in the shape of trapezoids were constructed at the end of the task (Ng & Ferrara, 2020, p. 941)

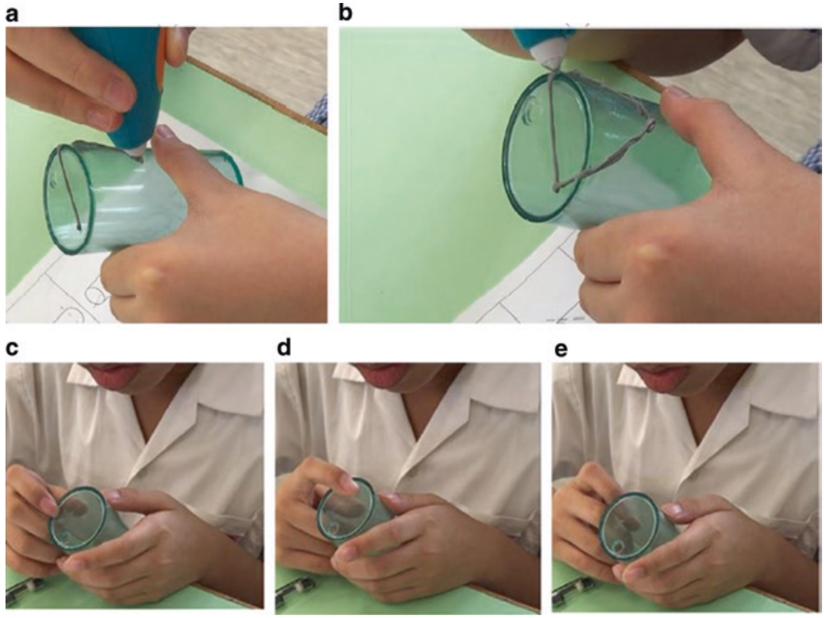

Fig. 13 (a–b) Constructing the outline of a cross-section with an oblique cut to the cylinder; (c–e) a student deciding to move his index finger along the surface of a cylinder to visualize a cross-section with an oblique cut to the cylinder (Ng & Ferrara, 2020, p. 940).

in the third dimension and helping to make certain mathematical concepts tangible through touch and moving one's hands. Our future direction includes conceptualizing constructionist learning from the perspective of realistic mathematics education (Van den Heuvel-Panhuizen & Drijvers, 2020), which views doing mathematics as a human activity and connected to reality. As argued by the framework of realistic mathematics education, much opportunities should be given to children to reinvent mathematics through hands-on and informal experiences, which is in line with our developed conception of 'learning as making' to some extent. Therefore, we encourage future research to consider how students make connections and reinvent meanings of 3D-printed (mathematical) object, such as a 3D printed triangle, in the real-world contexts, and how their matematical discourse *evolve* before and after their interactions with the 3D printed models. This line of research should contribute toward providing a basis for further research into 3D printing and mathematical cognition from the perspective of realistic mathematics education.

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Possibilities for STEAM Teachers Using 3D Modelling and 3D Printing

Eva Ulbrich, Branko Andjic and Zsolt Lavicza

1 Introduction

Many educators struggle with visuospatial challenges and this can be of importance when using new technologies that base on or require 3D visualisations. Having visuospatial skills and understanding connections between 2 and 3D are not only important in mathematics lessons but also in the future careers of students. They can be trained by using technologies like AR (augmented reality) and 3D modelling and printing (3DMP), however, such technologies are not widely used in schools. This hints towards a need for assistance, new course structures, or special learning materials for teachers to know how to use them in their lessons. Our research goal is finding hints towards what Science, Technology, Engineering, Arts, and Mathematics (STEAM) teachers need to be able to learn about and use such technologies in their lessons.

In an AR and 3D modelling workshop at an Estonian university for teacher education, mathematical concepts were explored by creating virtual mathematical mazes large enough to walk through them. One of the participants was surprised that he was not stopped by one of the virtual walls when he went backwards but

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passed right through. The participant was reminded that the walls were purely virtual so he could enter and leave the maze at free will. Some participants in the audience were surprised about the misunderstanding but in fact, this had not been the first time participants were surprised they were able to pass virtual walls. This incident shows that not all teachers always fully understand connections between virtual and physical representations pointing towards visuospatial challenges.

Visuospatial skills are needed for 3DMP which is expected to become valuable for students in the next decades. Trends predict that more and more consumer goods will be produced locally and additive manufacturing reaches many aspects of our lives from producing small items to large objects. The ING DIBA Analysis, 2017 predicts that this technology will reduce global trading by about 40% in the upcoming future and 50% of produced goods could be 3D printed within the next 20 years (ING DIBA Economic and Financial Analysis Global Economics 2017). This analysis also claims that the third most used application of this technology is 3D printing in and for teaching and education. Creating objects using 3DMP requires skills from STEAM fields as materials, forms, machines, and software are involved in the 3D modelling and the production of the model. These skills can also be found implicitly and explicitly in eight core competencies defined by the European Union in 2018 that should be fostered in education. The focus was on basic, entrepreneurial, and digital skills as well as languages aiming to enable everyone to participate actively in society (European Commission, 2018). Within these core competencies, science, technology, engineering and mathematics (STEM) related skills receive particular emphasis to lead people into careers in STEM fields. STEM and STEAM related skills are also considered valuable in the US. They can be part of 3D modelling and 3D printing as is described in the 2013 NMC Horizon Report: K-12 Edition. It predicts that this emerging technology will be used more regularly in schools and for learning practices (Johnson et al., 2013). In addition, the 2020 EDUCAUSE Horizon Report: Teaching and Learning Edition names technologies connected to 3D modelling such as 3D printing and AR as technologies with high significance in education (Brown et al., 2020). These technologies might also support spatial reasoning helping with visuospatial challenges.

We want to help teachers to become sensitive to their own and their student's visuospatial challenges and be able to create materials they find useful for their STEAM lessons. In this work, we reflect which attributes teacher training in the form of workshops and or course structures have that can help them learn and teach 3DMP using Bloom's Taxonomy and the Technological Pedagogical Content Knowledge model (TPACK).

1.1 The Role of 3D Modelling and 3D Printing in STEAM Education

The discrepancy between understanding the two dimensional representations of a three dimensional object might root in misunderstandings between virtual and physical connections. It can be challenging to cope with the representation of 3D objects in virtual or semi virtual worlds where parts of a problem has virtual and other parts have physical attributes. One example of connecting virtual and physical worlds can be using emerging technologies like AR or 3D printing. Creating objects by identifying a physical world problem, mathematizing it and modelling a virtual solution that can then be produced in the physical world requires an understanding of three dimensionality, virtual and physical possibilities and skills from STEAM subjects. 3D printing also can help to connect problem solving strategies taught in schools to students' daily lives for additional motivational boost and they help in preparing students for future challenges.

School activities that integrate the student's environment out of the classroom into teaching at the classroom aim to develop skills and strategies in students which can help them solve problems in the real world later in their lives. Such exercises can encourage and motivate students to explore skills in STEAM. By understanding the purpose of these skills in interaction with their personal life, students are encouraged to explore these STEAM skills. Such project based learning that involves 3D printing can help students to train their content knowledge, spatial abilities, and STEAM related skills. It can train modelling, critical thinking, problem- solving, students creativity and also their technological literacy starting from a very young age (Choi & Kim, 2018; Leinonen et al., 2020). Teachers of a variety of grades reported in a study that their students developed those skills and improved their self-directed learning (Trust & Maloy, 2017). Ford & Minshall (2019) identified six fields on how 3D printing is used in education. Namely, to teach students or educators how to use a 3D printer, as a supporting technology during lessons, to create artefacts such as manipulatives or models, to create assistive technologies for teaching, or to support outreach activities. These settings from STEAM education might play a significant role in our future and should be broadly adopted in schools. As described, 3DMP has the potential to help develop and train STEAM skills in students and help teachers in their lessons and teaching process. In addition to skill development for students' later careers, this technology is not only valuable to understand but also supports the development of skills which can be used for educational purposes. Using 3DMP, three dimensional thinking, problem solving competencies and more skills can be fostered by STEAM lessons such as a visuospatial understanding and working with both virtual and physical attributes as stated above. 3DMP can therefore open new possibilities to teaching these skills and transport concepts from, for example, mathematics (Lieban et al., 2018).

Another important use of this technology can contribute to especially mathematics teaching by visualising virtual concepts to students. Visualisations of mathematics are an important part of the teaching process in mathematics education. Concepts in mathematics are often virtual as in they might not even be experienceable in real lives such as one- and two-dimensionality and sometimes hard to grasp for students. Following constructivist ideas, students will create their own personal mathematical concepts fitting their mathematical understanding. Certain problems and misunderstandings during this process can be avoided and additional insights gained by students if they are supported by visualisations or manipulatives (Lieban et al., 2018). However, creating visualisations in 3D such as manipulatives are always models and educators have to be careful not to lead to other misconceptions using visualisations of concepts. One way of creating tangible visualisations as for example manipulatives is using 3DMP (Knill & Slavkovsky, 2013; Vanscoder, 2014).

However, 3D printing technology is rarely found in schools. Teachers do not always possess the intrinsic motivation to find resources and the courage to master the basics on their own. A study that investigated the familiarity of teachers in the US with new technologies, only about half of them were familiar with VR (48%), less knew about AR (40%) and even less teachers knew about 3D printing (24%) which shows that teachers could benefit very much from support to learn about this technology (Trust et al., 2021).

Apart from the costs of machines, the slow adaptation of such technologies in schools suggests that teachers need support. This work investigates which benefits STEAM teachers can expect if they learn about and use 3D modelling and 3D printing and looks at which attributes courses and workshops have that helps them learn about and use this technology in their lessons.

1.2 Obstacles for Teachers to Use 3D Modelling and Printing

In the literature, there is some research on how students perceive and use technology, but little about what teachers need to learn. Even though the Covid19 pandemic forced many in-service teachers to adapt to new technologies, some still struggle. Germany, for example, invested 5 billion euro for secondary schools to help teachers with buying new technologies but until 2020, many districts did not spend the money due to bureaucracy and a need for useful pedagogical usage concepts (Kerres, 2020). This points towards that only acquiring new technologies does not suffice to provide a school with a new teaching tool.

A study involving 1071 in-service teachers pointed to a high impact of personal factors when it comes to digital competence and that contextual factors do play a role but the personal factors have more weight. (Lucas et al., 2021). This suggests that personal preferences of teachers have a great influence and they might tend to use what they know from their private lives. 3D printing is a technology that is not broadly used throughout the population yet so it appears to be more complex than it might be. 3D models, however, are part of most computer games and while younger teachers might have experience, their students are very likely to have a deeper knowledge about them. Many computer games allow the creation of houses, objects or adaptations to the landscape by using blocks, so called voxels which come in a variety of sizes and colours, or other techniques like deforming or even deleting voxels. Teachers might not have experience with this and they can hesitate to engage in modelling tasks with their students. They can be intimidated or embarrassed if they have the feeling that their students are better at something than them. Teachers should feel secure and confident in their abilities to use a specific technology (Zammit, 1992) (Fig. 1).

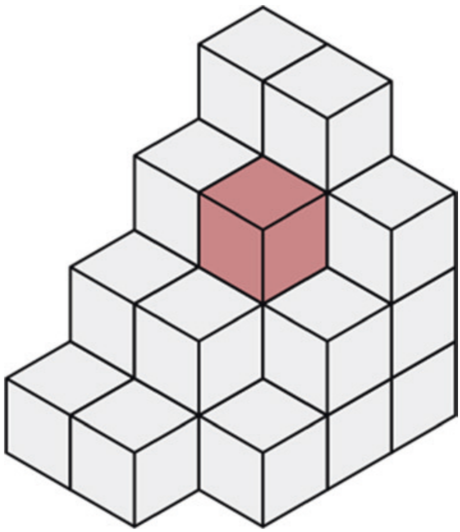

Fig. 1 Voxels with a voxel of a specific colour. (Copyright Vossman; M. W. Toews 2006)

Another important factor is that teachers should understand which options they have using a technology such as 3D printing and how to use the technology's potential during class for their subject. To do that, they need to know how 3D printing machines work as well as how to get or create 3D printable models. In addition, using technology can require extra effort. Used in lessons, 3D printing contains characteristics of an interactive lecture which requires interacting skills and also of a workshop which needs a good structure and preparation for resources (Menano et al., 2019). Moreover, 3D printing is highly dependent on the quality of the 3D model which is created in a software. This virtual model has to be designed carefully to prevent difficulties during 3D printing. Even the positioning of the 3D model on the printer bed changes the outcome. In the beginning, all of these "tricks" have to be thought of, practised, and might be challenging.

Some requirements have to be met so that teachers are able to use 3DMP. Availability and usability are named amongst the most important prerequisites needed (Holzmann et al., 2018; Knill, 2007). Provenzo and Cuban (1986) described teachers' decisions regarding technology integration being based on their perceived usefulness. Lam (2000) observed teachers who were supposed to use computers yet only did so when they were convinced of their beneficial aspects for students. This strongly indicates that teachers need the information about what benefit their students have when using this technology.

Experiences from implementing 3D printing show that addressing technological as well as motivational/knowledge topics can help teachers when analysing their approaches in lessons using Technological Pedagogical Content Knowledge (TPACK) (Song, 2018). We therefore developed and analysed workshops for 3DMP to work on a course concept suitable for mathematics and other STEAM teachers to motivate them overcoming the aforementioned challenges.

2 Approach

Design-based research can be described as a number of cycles where an idea of a design is tested, evaluated and revised. Each cycle consists of these three parts of the preliminary design phase, teaching experiment and analysis of the impact of the design to find interesting effects or possible hints for improvements that can help develop teaching principles. This method is suitable to display explicit decisions that otherwise would have been implicit and can help to evolve a researcher's work especially well if the theoretical framework is not yet fixed (Bakker & van Eerde, 2015). For the analysis step, we decided to use Bloom's Taxonomy to identify strengths and weaknesses of what teachers might learn with our approach and TPACK to learn whether the information we gave the teachers would help them in their lessons with their students.

The first cycle design was based on workshops and meetings with a handicrafts teacher. Technical or textile crafting is part of the curriculum in secondary high school as seen on Fig. 2. It involves practising STEAM skills like Physics or Mathematics combined with Art. The workshops and meetings lasted two afternoons, were located in a makerspace and provided the possibility of testing technologies. This was the start of collecting experiences about what kind of support teachers require when using 3DMP. 3D printing was interesting for this teacher as he aimed at teaching his class in CNC milling but had little budget and wanted to use the freedom benefits of 3D printing. The teacher asked for help with getting an overview about the technology, selecting a machine, help with choosing modelling software, and finding resources he could start with. The workshops and an analysis with him about what helped him understand the technology and possibilities of 3D printing were audio-recorded and discussed with the teacher.

This gave us a first impression about what a person needs as an introduction about learning the technology, what requirements a suitable machine should fulfil, which software was appropriate for his students and what resources he needed to get started to develop a lesson. In short, we gained an overview about technical criteria, the needs of teachers and students and which attributes teaching examples should have as he invited me to observe his first classes with the students. Observing what challenged and motivated him like having a clear order of which software is needed in which step and which requirements he has for a 3D printer helped us create a first outline of a workshop for STEAM teachers.

The experience gained was used in workshops in Montenegro that were conducted in ten schools. Schools were able to apply to be part of the project and participating schools were selected by criteria like inclusivity and structural aspects. For example, schools that focus on deaf students or are in remote areas

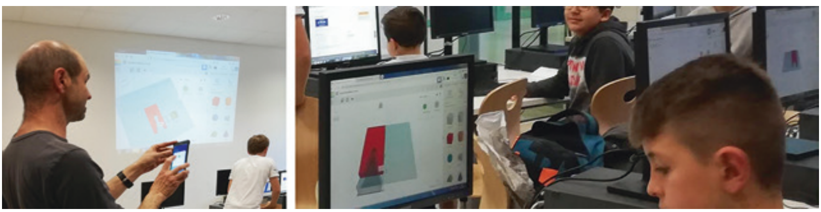

Fig. 2 A crafting lesson using 3D modelling, keyholes were created. (Picture source: Eva Ulbrich)

were ranked higher. Selected participants received the possibility for a workshop in 3DMP for interested teachers and were provided with a 3D printer each. We first prepared a workshop structure based on didactical experiences from accompanying STEAM teachers and the insights from the crafting teacher. We then created GeoGebra resources and searched for examples that we could use and present. The main idea was to teach teachers to find or design an object, to create a 3D model of it, and finally to 3D print it for their lessons in the best possible way.

We introduced the benefits of 3D printing to about 200 teachers who were novices in 3DMP from various subjects namely English, Chemistry, Physics, Informatics, Mathematics, and Arts, giving them the opportunity to try out 3D printing. Our specific aim was to trigger discussion and collaboration between them. Each workshop followed a specific structure and lasted between three to six hours depending on time constraints and teachers' motivations. Our workshop team consisted of two biology teachers specialising in teaching disabled students, three mathematics teachers using 3D printing, and two 3D printing experts. One of the biology teachers also had the role of a translator.

In general, the workshop was divided into two parts. The first part had a presentational and inspirational character where we presented ideas on how 3D printing could be used in real-life situations. Games created using 3DMP which could be used to teach mathematics are shown on Fig. 3. The second part focused on gaining experience in 3DMP. The technology was new to all of the teachers and none had seen a 3D printer in action before. To show the teachers how a 3D printer operates, we used a Fused Filament Fabrication printer and created

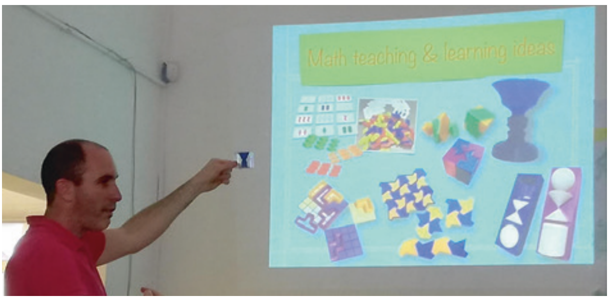

Fig. 3 3D printed objects in maths developed and explained by Diego Lieban. (Picture source: Eva Ulbrich)

Fig. 4 Teachers (a) observing a 3D printer, (b) getting information about 3D modelling while others learn about 3D printing, (c) using 3D modelling software in small groups

a cookie cutter resembling an image found within the school or in the shape of Montenegro to demonstrate personalisation possibilities.

To avoid wrong expectations, we explained how long the process takes, noted that their prints could sometimes fail, which can happen to the experts as well, and how they might get better printing results.

Teachers were divided into groups of two or three members as shown on Fig. 4(c). Each member of the group was given a certain responsibility, such as operating the computer, taking notes, or making sure to understand how the printing process works, as shown in Fig. 4a and b. The teachers responsible for using the printer were given extra instructions related to setting up and using a 3D printer. These instructions were documented in an online GeoGebra book which is available at https://www.geogebra.org/m/hdqxz2vj.

We observed reactions at the workshops refining the workshops for the following schools in a design based manner. We discovered that teachers engaged more if we used art from the schools instead of generic 3D models – something they could connect with emotionally. Motivationally, we had the best engaging results if we created cookie cutters with the outline from pictures drawn by students or teachers (Ulbrich et al., 2020).

As a further step, we aimed to encourage the teachers to keep evaluating what they learned and start to be creative in a competition. We asked them to create items they would find useful for teaching. We received various results from 3D printed cells to mathematical objects as seen on Fig. 5. Data was collected by audiotaping the discussions, collecting questionnaires before and after the workshops and open interviews of 37 teachers from those schools participating in the competition a year later.

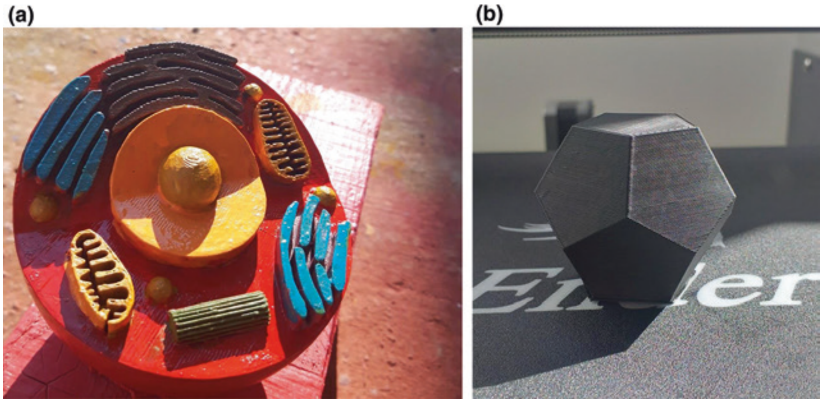

Fig. 5 A cell which was colored (a), a pentagonal dodecahedron (b). (Source: teachers from Montenegro, Biology and Mathematics)

3 Reflections on the Workshops

In reflection, we concluded that the model of Bloom's Taxonomy can be a good basis to explain the improved success of the later, more developed workshops. By addressing each level bottom up we wanted to give participants a structure they could repeat with their students. So we analysed what we did and checked whether and if so, where it can be connected to Bloom's Taxonomy to find a basis for future structures.

Bloom's Taxonomy was developed in 1956 aiming at creating a classification of cognitive skills and was named after one of the editors of the volume (Adams, 2015; Bloom, 1956). This ranking of levels of skills from little to high cognitive processing was chosen to identify which improvement steps could be the reason for more successful workshops. The steps range from simply remembering information to being able to understand and even apply the information to being able to be critical and analyse results until reaching a level of being able to create new applications of a gained skill. The taxonomy was used combined with 3D printing in medicine education in the past aiming at supporting higher levels of cognitive skills and training spatial recognition at the same time (Cai, et al., 2019). After looking at which order of steps in the end appeared useful, we found that we could connect the activities to the levels of Bloom's Taxonomy in the order shown in Fig. 6.

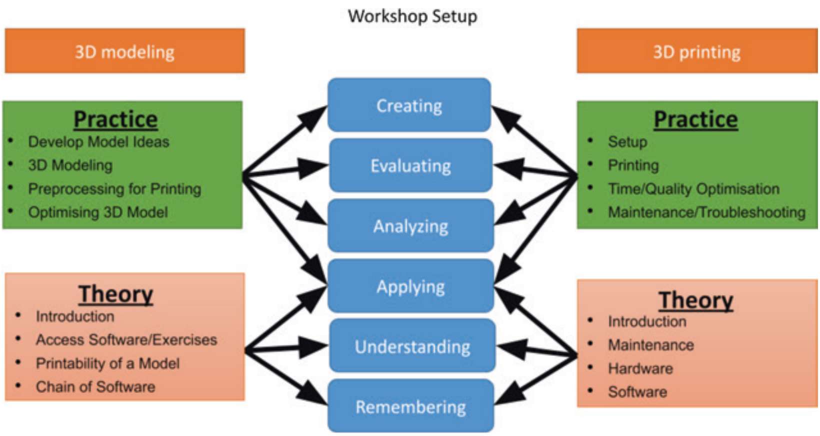

Fig. 6 Theoretical/practical parts of the workshops/lessons connected to Bloom's Taxonomy, visible in the centre. (Picture by Andjic and Ulbrich)

In the first workshop part we presented why 3DMP could be useful and how it works in theory to help them remember, understand 3DMP, and develop a first idea of applications as seen on Fig. 6 in light-orange. Teachers were shown possible benefits by examples from mathematics lessons, for example puzzle cubes demonstrating the division of a cube. We showed that 3D printing is an iterative process of modelling and printing. For the hardware theory part, we gave an introduction about the 3D printer machine, explained the parts and gave practical tips for using and maintaining the printer. Teachers were invited to observe the printer up close as seen on Fig. 4(a).

In the second part after the break, teachers were given the practical opportunity to practise 3DMP themselves. Previously chosen 3D printer colleagues were given an introduction on how to handle the machine while the other teachers were introduced to the process of 3D modelling, as can be seen on Fig. 4(b). We wanted them to practically experience 3D modelling, think about modelling implications on the printing process as can be seen on Fig. 4(c), and have them produce their model.

Not all teachers' subjects were mathematics but they considered themselves as STEAM teachers mostly. Some of the teachers were language teachers which helped translate instructions related to 3D modelling or 3D printing. The models we presented in the introduction and at online resources they could get predesigned models from were chosen by their attributes of whether they could be used to support children with special needs, in that specific school or in the subjects of the teachers present in the workshops. Examples were aids to hold a pen for spastic children, a prediodic table with heights hinting at the specific weight of the atom or musical instruments in a school for blind musicians. Our goal was to show them possibilities and to encourage them to search for and alter existing models fitting their needs. We in particular talked about didactical implications of models related to the examples from mathematics that were presented as seen on image 3 for example.

At first, we showed them free online sources for where to get 3D models for downloading. Next, we asked them to experiment with free online 3D modelling tools that are simple and which they could use also with their students. This aimed at analysing and applying what they learned before and implementing it afterwards. Little TPACK related ideas were used in these workshops as they were solely aimed at helping teachers to get acquainted with the technology, the TK side. As the teachers were foremost but not only STEAM teachers, we did not feel comfortable giving them advice on how to use the technology. Moreover, we only had a few hours to introduce the teachers to the technology and wanted to focus on one angle of TPACK in more depth rather than skimming all topics only briefly by showing examples from tools and teaching aids for children with special needs and using 3DMP in particular for mathematics. However, the mentioned collected data and in particular the interviews one year after the workshops pointed towards an importance of TPACK oriented ideas for teacher training. This gave us the idea for a TPACK oriented course for pre-service teachers.

4 Possible Options and Opportunities

We collected pre- and post questionnaires from all participating teachers, audio recorded the sessions and collected our own observations to analyse and improve the workshops. In addition, we interviewed 37 teachers a year after the workshops that were still using 3DMP for their lectures. Twenty open questions were asked in the interview encouraging the teachers to point out strengths and weaknesses of 3DMP and their experiences with using the technology in their lessons. We created a code and clustered the answers from the transcripts in relation to defined categories. This pointed us into the direction of possible benefits teachers can expect and their needs in lessons. The collected data was translated from Montenegrin to English and we clustered information according to broad topics like impact on students or statements regarding technologies, then created subtopics if a certain topic was mentioned several times such as language improvement or collaboration.

A first analysis of the feedback and the experiences showed us that there were interesting effects on teachers and they observed valuable developments in students. In addition, we got feedback on attributes about how teacher's lectures are impacted by using 3D printing.

For students, teachers observed that students were able to correct misconceptions they developed by challenges in the translation from 2D representations and virtual concepts to 3D representations and physical objects as seen on Fig. 7. One category of comments we created was a misconception category and we found comments in the transcripts where students had the opportunity to correct an idea. One example was that students realised that a cell in a human body was not a disc with strange looking things attached but it is actually a sphere that is cut in half in most books showing the inner parts of a cell. The students had no possibility to examine a real cell due to the small size of human cells and had only seen cells in books and gained a wrong impression. Using models and images is a common method for representation in Biology and giving more context to a representation seems valuable especially in this subject. An outcome from the interviews analysis was that teachers reported occasions where students were able to correct some misconceptions that teachers were not aware of initially.

Another important issue was that students, according to their teachers, started to understand relations between concepts better. By creating a 3D model and then being able to rotate it in their hands, they had direct feedback on their skill level by using the technology. Teachers reported that by for example lacking the skill to calculate something or understand the English information around it, they trained critical thinking by drawing their own conclusions and connected a physical instance of an object to virtual concepts they previously learned. In addition, they trained their self-evaluation by the immediate feedback of their skills and they exercised working in a physical and digital environment.

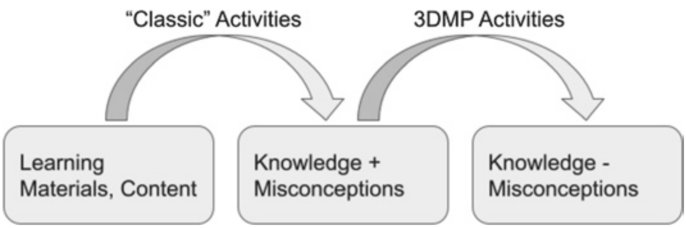

Fig. 7 3DMP activities can correct previous misconceptions due to a 2D representation. (Picture by Andjic and Ulbrich)

In addition, teachers mentioned in the interviews that their students realised that STEAM lessons are of importance for their future lives. We collected comments containing this in a subsection of the section with benefits for students. Comments contained remarks such as using this technology always overlaps with engineering and mathematics which would be more visible for students. The teachers reported that their students gained skills that are useful for a variety of subjects and that they understood the importance of transdisciplinarity. It even had an impact on their career choices as 18 of the interviewed teachers reported. They said that students mentioned they were much more inclined to choose careers connected to STEAM topics. The other teachers did not report on career choices.

For teachers, using 3DMP had a motivational aspect and also helped them develop their STEAM related skills. They trained applied mathematics and were able to work on open ended tasks. 3D modelling was especially considered a valuable exercise and they perceived 3DMP as a useful teaching tool for their professional development.

They reported that modelling objects had a positive influence on their abilities to teach STEM lectures and teachers also vastly improved their English language skills as most of 3D printing information material is not available in Montenegrin.

Another benefit that teachers reported is that teachers were forced to work in collaboration together with other teachers and also with experts outside of school. For example, maintenance of machines had to be done by reaching out to companies outside of school while English teachers helped with translations and Informatics teachers helped with software.

5 A First Developed and Tested Course Concept

These workshops and interviews provided further experience and insight into teachers' needs. After analysing the workshops and the feedback from the 37 interviews, we based a course for a semester on the experiences and findings. We developed three lessons as modules for a part of a course in geometry for 60 pre-service mathematics teachers, complementing the course's calculation part. An existing geometry course that incorporated exercises with GeoGebra was accompanied by the modules that were held in distance learning using Zoom. Most of the time, the group was split into groups of 30 students where one was exercising geometry and was supervised by professors and the other group attended the zoom lesson about 3DMP. As we learned that modelling itself already was a huge benefit and presence learning was not possible, our approach was to focus on 3D modelling and then print out the objects for the pre-service teachers before and after the last module.

Our idea is to support teacher's future needs by using the experience and analysis of Bloom's taxonomy and estimating their needs by orienting the course at TPACK.

5.1 Course Evaluated by Bloom's Taxonomy

We divided up the content into a theoretical overview about 3DMP showing benefits for students and connections to mathematics to fulfil Bloom's taxonomy regarding the understanding part and followed this part due to many email questions with an hour of Q and A in an additional lesson for more remembering and understanding.

In the second part, we introduced 3D printing theoretically describing useful models for the pre-service teachers' later students and what a 3D model should fulfil to be producible and which implications modelling might have on the outcome connecting to their understanding. In addition, we invited them to experiment with GeoGebra themselves, turning to practical 3D modelling to connect to applying and creating in Bloom's taxonomy. All pre-service teachers had to come up with an idea for their own projects and we asked them to give feedback to each other, connecting to analysing and evaluating.

In the third module, much was repeated that was presented in previous lessons, connected to remembering and understanding. After that, 3D printing was described theoretically in depth as seen on Fig. 8, as practical 3D printing was not possible due to pandemic restrictions. The project's models were discussed and their printability analysed, again connecting to analysing and evaluating from Bloom's taxonomy. It was also discussed how the pre-service teachers thought they would use them in future lessons and whether they could be improved.

5.2 Course Looked at from a TPACK View

As described, the course started with a lesson module giving a theoretical overview about the technology, how it can be beneficial for students and where it reflects mathematics topics. The course participants were given a quick overview of 3D printing technologies for orientation about TK and then were presented how this technology might be beneficial for their future lectures. Examples from

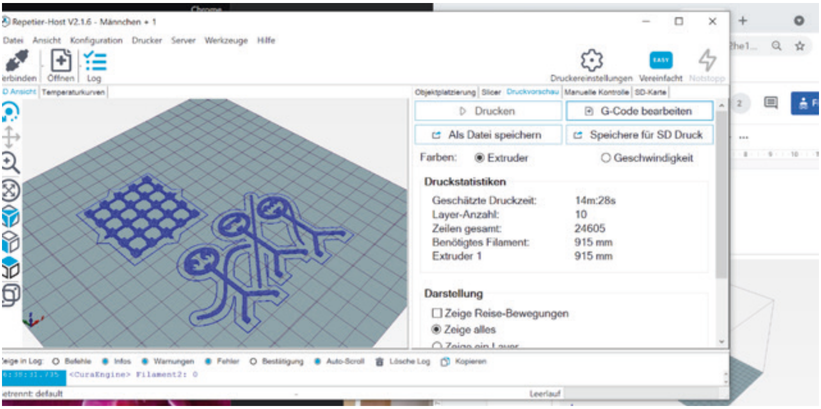

Fig. 8 A pre-service teacher showing her idea of dancing men resembling functions and a grid where the feet can be inserted, shown in the software Repetier-Host. (Screenshot by Ulbrich)

mathematics were presented that were created to demonstrate certain mathematical concepts such as calculating volumes or divisions of objects by cutting them into pieces of the same volume. Also TPK was touched when course participants were given the information that using this technology could be distracting due to fumes or noise and which motivational aspects 3D printed games, created by students, could have.

This part of the course aimed at technological and content knowledge (TCK) as seen on Fig. 9 on the right side which describes the knowledge about which content can be transported by a certain technology. We thought that showing them how knowledge about this technology can impact their students' experiences in lessons and examples for skill development would help them understand the benefits of 3DMP.

Assuming that the students learned the geometry content in one part of the course, we aimed to connect it to technology by showing how the content can be used in 3D modelling. As an example, vectors and triangles play a big role in the file format STL for 3D models. We showed them how triangles are used in architecture and art, presenting a 3D modelling project from a colleague where GeoGebra was used to model and later 3D print a model of a temple (El Bedewy et al., 2021). The pre-service-teachers were told that they should look at examples like 3D printing exercises on Thingiverse containing lesson plans and create their own project idea of connecting mathematics to 3DMP until the next lesson, as described.

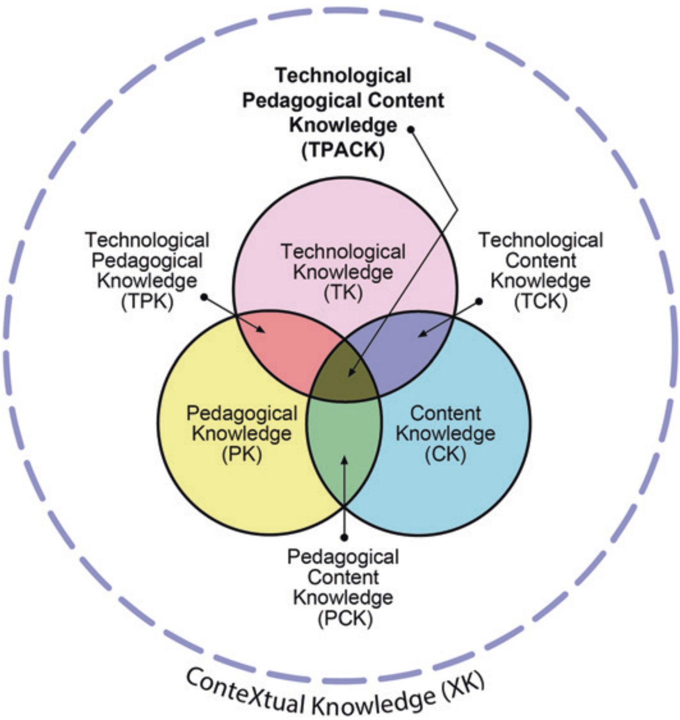

Fig. 9 The TPACK model, image © 2012 by tpack.org

The next course module aimed at repeating the steps of the theoretical 3D modelling part of the workshop. It started with a summary of concepts of the previous lesson and focused on an introduction to 3D modelling software, as described. GeoGebra and TinkerCAD were chosen as most mathematics teachers are well acquainted with GeoGebra and can use TinkerCAD due to its simplicity. We discussed questions related to teacher's projects, strengthening the TCK part of the lesson and starting to work on the technological and pedagogical connections (TPK) between 3D modelling, what a useful 3D model in a project for their later students could be, and TK as in which skills are needed for a later use of 3DMP. They were also asked to create their models for the next lesson as seen on Fig. 10 and apply their gained knowledge. The pre-service-teachers had to give feedback to two of their colleagues which was also documented in the GeoGebra books.

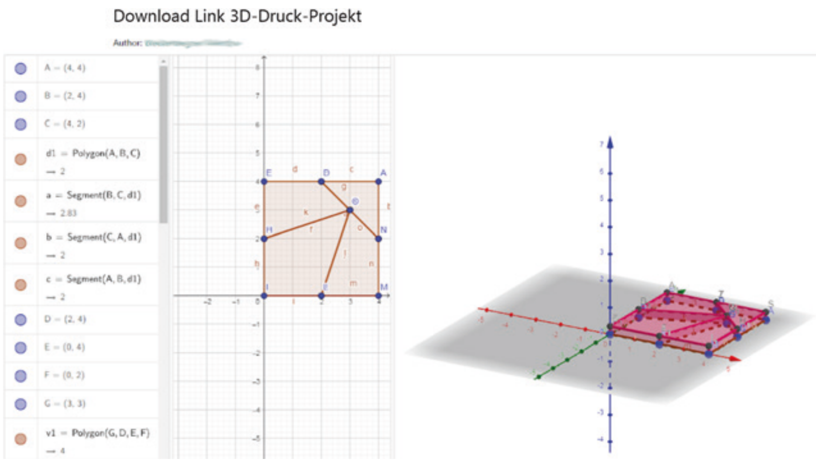

Fig. 10 A project created by a pre-service mathematics teacher using GeoGebra

The third part repeated the previously learned concepts before focusing on how printing works and their models' producibility, printing them out for the pre-service-teachers. We discussed how they would use it in their lessons, reflecting on their pedagogical and content knowledge (PCK). Course participants had to give small presentations on what the aim of their project is and what they want to reach with it or use it, how they want to realise their idea and which problems they think they might encounter technological wise. The feedback from the two other colleagues was presented and also discussed. As two other pre-service teachers were assigned to each project for feedback, every course participant had to analyse two other projects and got two opinions on their work plus later additional opinions from the course participants and the course instructors.

This part also reflected parts of the practical part of the workshops. Our emphasis was that the developed projects should be usable in pre-service teachers lessons and they should reflect on what their aim is, what they believe students can learn from and to have an idea whether and how they could transport their gained knowledge to their students.

All pre-service-teachers created projects, gave feedback and finished the course with handing in a printable model as well as their project development documentation in the GeoGebra book. The models were then produced by the authors as seen on Fig. 11 until the end of the geometry course and the pre-service-teachers had the possibility to pick up their produced model individually

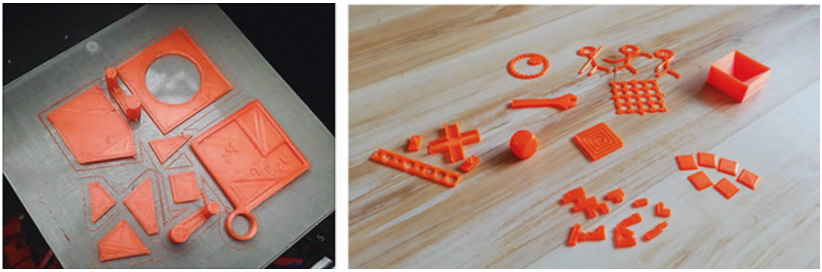

Fig. 11 3D printed projects like for example Pythagoras proofs and the function men

afterwards. When collecting their project's 3D printed outcomes at our office, many pre-service teachers said that they were happy and proud of their project and told us that this course motivated them to learn more about the technology and its possibilities. Presumably, only those students came to pick up their 3D printed projects that were motivated to see the outcome of their modelling process. To clarify the motivational aspects, interviews with roughly 10 students from the course are planned where about half the participants volunteered to do interviews and half of them were asked whether they agreed to interviews.

6 Discussion

We used the GeoGebra books, pre-service teachers' reflections on their colleagues' projects and non-mandatory short open questionnaires they filled out before, during and after the course to collect more information about their needs for each level of Bloom's Taxonomy. Their reflections on the modelling process were sorted into aspects of TPACK for a better understanding of which part of the course needs to be strengthened.

Our impression by clustering the answers from the questionnaires and student's reflections in their GeoGebra books was that some pre-service teachers had trouble imagining how this technology can be used for their lessons until they started to work on their own projects. Especially, giving feedback to their colleagues and looking at their peers' work and explanations seemed to spark their imagination as many projects were then altered and changed. About one third of the projects were further developed and got a new focus while only five of the projects stayed entirely the same. Pre-service teachers seemed to have little problems understanding the usefulness and limitations of the technology from a pedagogical perspective once they had started to get inspiration and in communication with their peers suggesting that this is a suitable method to give them useful information for a TPK coverage.

Emails with questions during the modelling phase suggest that the practical TK part of the course might need more emphasis. Some of the course participants felt unsure about how to use a 3D printer and which problems they would encounter as in this course they had no possibility to experience it but even more, they were unsure about how to do the 3D modelling. As they all had questions related to their projects, questions that concerned many projects were discussed in the next course while more specific questions were explained in small intermediate Zoom meetings with students. A major concern was repairing models as, for example, GeoGebra sometimes produces models with so-called manifold errors where some edges are not connected properly. These practical issues arose after participants had started with their 3D modelling process and might derive from the distance learning situation.

As pre-service-teachers were asked to document their processes, many reported concerns about their technological skills. They each supported their pedagogical and content wise ideas about their projects though, pointing at a sufficient coverage of the teachers feeling secure with issues related to PCK and TPK and a need for more on the TCK part. Some course participants gave feedback that they plan to test their projects with future exercise classes and then would like to share their experiences in an interview. There was no test about theoretical knowledge on the topic and students passed the course by producing a 3D printable model that can be produced in roughly 10 min printing time which was tested by actually printing the object. This seemed to puzzle some students as they were asking about a written exam in the last lesson. This will be taken into consideration in the future as it seems that the course participants might need motivation to read through their notes again.

7 Outlook

A first course concept to teach 3D printing to pre-service teachers, based on a series of workshops in Montenegro, was developed and tested in an online teaching environment. Previous experiences showed a benefit of 3D modelling even without additional 3D printing so this was emphasised more than practical 3D printing due to the distance learning situation. However, many technical questions arose implying that something was missing in the communication to the course participants giving them insecurity. Either the practical part should be done avoiding a distance learning situation or the 3D printing part should be more emphasised, giving participants the opportunity to self-test their models.

In the future, a mathematical modelling-like approach might help improving a self-assessment situation so the projects might be further developed. In addition, if the pandemic situation allows us to work in presence, this might also help answer questions directly.

In conclusion a next course, only focusing on 3DMP, will be conducted with the learnings from this course. It will be a non-mandatory course and we aim to give participants the possibility to practice their gained knowledge and ask questions during exercises instead of in between lessons per mail. Participants from this and the previous course will be invited to open interviews investigating whether this course can lead to even more opportunities during teaching. The course will aim at having the pre-service teachers feel more secure with the technology and give them more opportunity to support one another, will test first steps into investigating whether such courses can also support mathematical modelling skills in pre-service teachers and ask them to create projects they could test with their students in class. In case they try to test their projects, they will be invited for interviews such as those conducted with the Montenegrin teachers using the same 20 questions for comparison. In addition, an expert course is planned that will ask the course participants not only to create a 3D modelling project but an entire lesson plan including a proper assessment for their future students.

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“I Cannot Simply Insert Any Number There. That Does not Work” — A Case Study on the Insertion Aspect of Variables

Jenny Knöppel and Felicitas Pielsticker

1 Introduction and Motivation

It is widely agreed today to emphasize content aspects in learning algebraic language, so that algebra does not appear as a meaningless game with letters and other (Freudenthal, 1983; Malle, 1993; Vollrath & Weigand, 2007; Hefendehl-Hebeker & Rezat, 2015).

Students are not expected to form an accomplished automation of transformations, which ultimately can be performed faster and more reliably by computers, but rather to develop an understanding of the content of the algebraic language and to understand its meaning and use (Hefendehl-Hebeker & Rezat, 2015). To ensure this, the introduction of the Common Core State Standards Initiative for High School Algebra states:

> “An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability

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> to express the computation in general terms, abstracting from specific instances" (Common Core, 2021).

Especially the detachment from specific (numerical) examples mentioned in the last sentence will become interesting for the analysis of our case study with the students Harry and Luke (names changed).

Malle (1993), like some others, points out that many students have difficulties making the connection between formulas and underlying factual situations. Especially with the help of the 3D printing technology, authentic concrete situations can be designed that focus precisely on these connections. Students have the opportunity to assign meaning to variables and terms in a concrete 3D printing context. This is also in line with the claim that the "object aspect" (Malle, 1993) should be preferred and emphasized at the beginning of an approach to algebra. In particular, this can be used to advocate an empirical and real-world approach to algebra for students. The case study of the students Harry and Luke described in this paper takes this empirical approach to algebra very seriously. In terms of an empirical-oriented approach (Pielsticker, 2020), e.g. to the concepts of variables and terms, the students in our case study deal with 3D printed tiles. As we will elaborate in the theoretical part of the paper, an empirical-oriented mathematics class, is understood as a teaching concept in which the teacher intendedly makes educational decisions to work with empirical objects as *the* mathematical objects of math classes. In mathematics classes, the empirical objects (e.g. 3D printed tiles) are not intended to illustrate mathematical concepts that are abstract in nature, but they are rather the subject of the lesson (Pielsticker, 2020). In order to address this conceptual idea of an empirical-oriented approach to algebra, we will pursue the following research questions in this paper:

Q1: How do students develop mathematical knowledge of the concept of variables in geometrical empirical contexts with 3D printing?

and

Q2: How can students be supported in their mathematical concept formation process with a specially designed learning environment based on the usage of 3D printed objects?

To address these research questions, a case study with an exploratory setting is conducted. Thus, we describe a setting with 3D printed objects that are created by the students.

The students' 3D printed tiles are used in an in-depth learning environment about the Pythagorean theorem, namely the theorem about the so-called "Lunes of Hippocrates". In order to be able to describe and analyze the setting as well as the case itself, the article is structured in such a way that we would first like to discuss our theoretical background. We then present our methodological approach. The next section is about presenting the results of the case study of the students Harry and Luke. Finally, we discuss our findings in light of our research questions.

2 Theoretical Background

2.1 Empirical-Oriented Mathematics Classes with Self-Made Empirical Objects

Material-bound and content-visual work (as well as content-visual proofs, according to Wittmann & Müller, 1988) have increasingly become the focus of mathematics teaching in recent years, as they "contribute to a deepened and interconnected knowledge and […] to a secure handling of non-formal symbolic representations" (Kempen, 2019, p. 16, translated by the authors). Winter (1983) also calls for leaving room for demonstrative-empirical working processes. In the approach of empirical theories (Witzke, 2009; Burscheid & Struve, 2018, 2020; Schlicht, 2016; Schiffer, 2019; Stoffels, 2020), it is assumed that students acquire empirical (student-)theories about the means of observation – referred to as empirical objects in the sense of the approach – in a classroom that is guided by manipulatives and based on illustrations. Students bind their knowledge to the objects they use. This ontological binding to reality is justified in educational theory and developmental psychology by the task and goals of general education schools (Hefendehl-Hebeker, 2016).

In this context, an empirical-oriented mathematics class can be a possible answer to the question of how mathematics education can be designed against the background of a development of mathematical knowledge in the context of ontological ties to objects of empiricism. Students' knowledge about real phenomena can be described with the help of theories, usually empirical theories (Burscheid & Struve, 2018, 2020; Witzke, 2009). We do not claim that children can explicitly formulate these theories, but rather that the children's behavior becomes describable in these theories (Burscheid & Struve, 2020). In this context, an empirical theory is understood as a theory that describes and explains phenomena of reality (Sneed, 1971; Stegmüller, 1986; Balzer, 1982). These theories can have different sizes or ranges, so that everyday theories of children are also included (Struve, 1990). From the perspective of empirical theories or a related mathematics education that explicitly takes this concept into account, we speak of so-called empirical-oriented mathematics classes (Pielsticker, 2020). As we also stated in the introduction of our article, this is a concept of teaching mathematics in which the decision is made to work in conception with empirical objects as *the* mathematical objects of instruction.

The empirical objects of the case study described in this article are 3D printed tiles created by the students themselves. The students under consideration use them to demonstrate and develop their (mathematical) knowledge.

If we look at the historical development of algebra, it is interesting that Euler repeatedly emphasizes a reference to real size ranges in his work "vollständige Anleitung zur Algebra [complete instruction in algebra]". He states that "Mathematics is nothing but a science of quantities searching for means to measure those" (Euler, 1771, p. 4, translated by the authors). Witzke (2009) also describes that mathematical objects, which Euler considers to be represented in his analytical claims, are no blank variables as in formalistic theories, but rather quantities. As Euler (1771) considers certain arithmetic rules for numbers in his educational book, which are then also applied to letters. "We also make use of the same sign + plus, to connect several numbers together; for example, 7 + 5 + 9 signifies that to the number 7 we must add 5, and also 9, which make 21. The reader will therefore understand what is meant by 8 + 5 + 13 + 11 + 1 + 3 + 10, viz. the sum of all these numbers, which is 51 (Elements of Algebra, 1972, p. 3).

"All this is evident; and we have only to mention, that in Algebra, in order to generalise numbers, we represent them by letters, as a, b, c, d […]. Thus, the expression a + b, signifies the sum of two numbers, which we express by a and b, and these numbers may be either very great, or very small. In the same manner, f + m + b + x, signifies the sum of the numbers represented by these four letters" (Elements of Algebra, 1972, p. 3).

This indicates that letters represent numbers in Euler's textbook. According to Euler, the syntactic rules for the handling of numbers also define the handling of letters. Altogether, Euler's objects – as well as numbers and letters – originate from a real subject area in which they have been introduced.

In the following section, we would like to discuss how we approach technical terms in the context of empirical theories in order to describe student knowledge as well as their concept formation processes in the field of Algebra.

2.2 The Concept of Empirical Theories

The basis of the analysis presented in this article is a constructivist view on learning processes, the idea that students construct their mathematical knowledge by themselves in interaction with their environment. This is supported by the work of Hefendehl-Hebeker (2016), who states that the concepts and contents of school mathematics have their phenomenological sources predominantly in our surrounding reality. Today it is generally acknowledged that students constitute their own mathematical knowledge in action and in processes of negotiation (Krummheuer, 1984). In this sense, learning is understood as an active process, dependent on individual experiential areas, as constructing theories for an adequate cognition of certain phenomena (Burscheid & Struve, 2020). Looking at approaches from cognitive psychology (e.g. "Theory theory" by Alison Gopnik), in particular the idea that "children develop abstract, coherent, systems of entities and rules, particularly causal entities and rules, […] they develop theories" (Gopnik, 2003, p. 5), it seems to be quite reasonable to describe the behavior of students in the sense of theories. Students' knowledge can be described with the help of theories, usually empirical theories.

In the structuralist presentation of an empirical theory – especially an empirical student theory – the following terms (technical terms) in Table 1 are central.

What happens in school is the development of practical mathematics (Tall, 2013, 2020) and the formulation of theoretical definitions and deductions based on natural perception and reflection. With the help of the 3D printing technology, it is possible to produce individual visual materials and physical objects (empirical objects - in accordance with our theoretical framework) in mathematics classes. This technology makes the development as well as the production of empirical objects (manipulatives) in a variety of mathematical contexts quite easy and thus may facilitate empirical (or in Tall's terminology, "embodied",) approaches to mathematical contents.

Thereby, we assume that students strongly tie their knowledge to the (empirical) context of its development (Bauersfeld, 1988). Research results show that a transfer of this knowledge, which is tied to the contexts of its constructional process ("domain specific knowledge"), presents a big challenge for mathematical learning in the sense of empirical theories (Burscheid & Struve, 2020). Thus, in order to describe the mathematical knowledge of the concept of variables that students develop in empirical contexts, we use the approach of empirical theories established by Burscheid and Struve (2020).

Before we go into more detail about our case study, we would like to outline our methodological approaches.

Table 1 Technical terms to describe empirical (student) theories in empirical contexts[1]

Technical terms	Description
Intended applications	Each empirical theory aims to describe and explain certain phenomena of reality. These phenomena are called intended applications. The intended applications of an empirical theory (cf. corresponding technical term) are characterized exemplarily, i.e. they are defined trough the indication of paradigmatic examples
Empirical objects	In this study, empirical objects are understood as items and objects of reality that are immediately accessible to students, especially in a tactile or visual way
Non-theoretical term	T-non-theoretical terms and empirical concepts are terms of a theory T, which have been defined before the theory T was established (Burscheid & Struve, 2020). These include the "empirical concepts", i.e. terms that have empirical objects as objects of reference such as the term "cube" in the mathematics classroom, where dice are produced with the 3D printing technology. For all theories T it is true that concepts describing objects of reality, or in other words having objects of reference in reality, are T-non-theoretical
Theoretical term	T-theoretical terms are terms whose meaning is only established within the theory T (Burscheid & Struve, 2020). In particular, the meaning of such a concept cannot be determined in a theory T', which is independent of the theory T. The concept in question can exclusively be defined by T. An example for a T-theoretical term is the concept of force in relation to the Newtonian mechanics (Burscheid & Struve, 2020)

3 Methodology

3.1 Data Collection and the Learning Environment

Thematically, the teaching unit under consideration (9th grade, secondary school in NRW) is based on the theorem about the "Lunes of Hippocrates", which is attributed to the works of Hippocrates of Chios (second half of the fifth century BC) (Volkert, 2004). According to Volkert (2004), this theorem can be formulated as follows: "If one describes the triangle as a rectilinear lune, one can briefly state: The rectilinear lune is equal in area to the two curvilinear lunes together" (p. 3, translated by the authors). The theorem serves to apply the Pythagorean theorem as well as the formula for the area of the circle and the semicircle, respec-

[1] A more detailed list of technical terms and their description can be found in Pielsticker, 2020, p. 38ff. (German).

tively (Volkert, 2004). The learning environment of our case study was designed for four lessons à 60 min within two weeks. Two further lessons as well as the time in between the lessons were used for the completion of the students' models as well as the 3D printed objects.

The idea of this approach was to provide time and space for exploratory phases. According to Witzke and Heitzer (2019), the 3D printing technology can be used in and for the classroom. This includes the independent development of 3D objects, which creates occasions for mathematical negotiation processes (construction and argumentation) (Witzke & Heitzer, 2019). The Pythagorean theorem and the extended Pythagorean theorem for semicircles had been previously introduced in class. A symbolic justification of the extended Pythagorean theorem was noted on the blackboard and discussed with the students. After a short

Task 2 (in pairs)

Construct the triangle with the "Hippocratic Moons" in SketchUp. Use your notes on Worksheet 1 for help.

TIP: Press **AltGr and "+"** to add more "sides to the arc of the circle" while you are constructing the **first semicircle**. How often can you press these keys? What happens to the circle? Try it out!

Task 3 (in pairs)

a) When you have finished the construct, save
- the whole construct (the triangle and the moons)
- ONLY the moons
- ONLY the triangle

as STL-files.

TIP: Delete the figure(s) you do not need, save the file and undo your steps, so that you end up having all the figures on the screen again.

b) Prepare the whole construct – the triangle and the moons – for printing. Set the infill to 100%.

Fig. 1 Assignment to create the 3D printed tiles for the theorem about the "Lunes of Hippocrates"

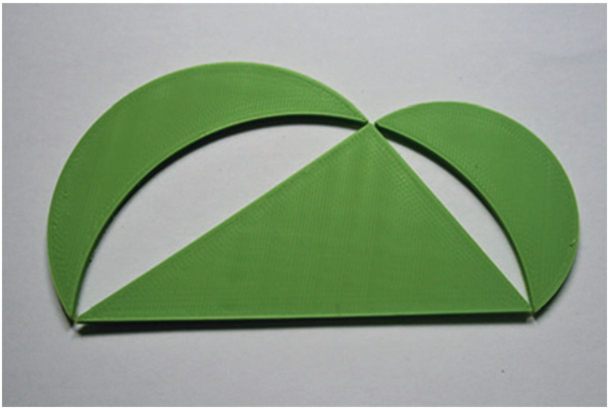

Fig. 2 3D printed objects of a group of students

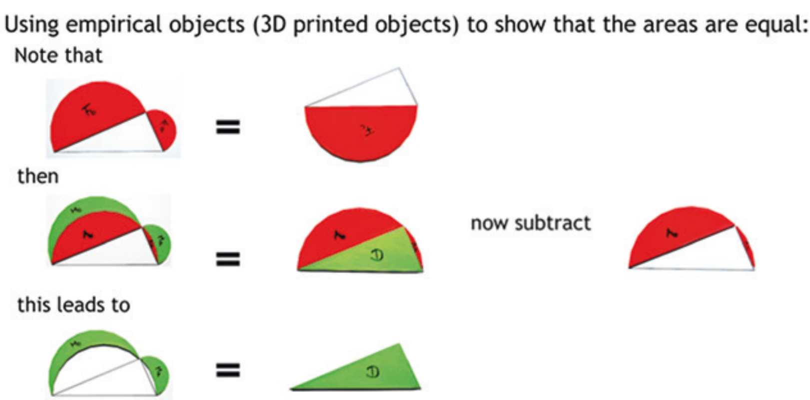

Fig. 3 Possible operative justification with the help of 3D printed tiles

introduction, the first phase of the learning environment for the justification of the theorem about the "Lunes of Hippocrates" began. In this phase, the students were asked to design their own 3D objects with the program SketchUp©,[2] which were printed out by the authors afterwards (Fig. 1).

[2] For another study using the Program SketchUp© in a learning environment cf. Dilling and Witzke, 2020

The students thus created a figure for the theorem about the Lunes of Hippocrates (cf. Fig. 2).

The 3D printed tiles were then used for further work. Luke and Harry had already attracted particular attention in class because they had finished the construction process of the 3D printed tiles particularly quickly.

One challenge for the two students was to find an operative justification for the set, as presented in Fig. 3. Therefore, in terms of an informative case study for our research question, the two students were selected for consideration in this paper.

3.2 Context of the Case Study – Interviews with the Two Students Luke and Harry

Three groups of students were videotaped during the teaching phases described above. This selection was done by the teacher. After the lessons, qualitative guided interviews were conducted with three groups of students (Mayring, 2002; Strübing, 2018). For this paper, the group consisting of Harry and Luke was chosen because they stood out for their good use of 3D printing technology on the one hand and their challenges in justifying the theorem on the other. For this purpose, the selection criterion “obvious relevance for the research question” according to Krummheuer (1992) was used. The interview situations were audio and video recorded and transcribed afterwards. During the interview, the student groups had access to their work results (Fig. 4).

We transcribed this video material for our analysis according to the rules of Meyer (2010). In addition to the transcripts of the classroom situations and student interviews, written statements of the students are also part of our investigation and descriptive analysis. The case of Harry and Luke was chosen for presentation because it can contribute to our questions to a particular extent.

Methodologically, we thus chose to use a case study approach (Stake, 1995). To describe students’ knowledge development processes in empirical contexts, we use the concept of empirical theories.

3.3 Case Study Research – Luke and Harry

In the spirit of a multiple-instrumental case study (Stake, 1995), the focus was on our two research questions (Sect. 1). With regard to generating informativeness or

Fig. 4 Interview situation with the students Harry and Luke

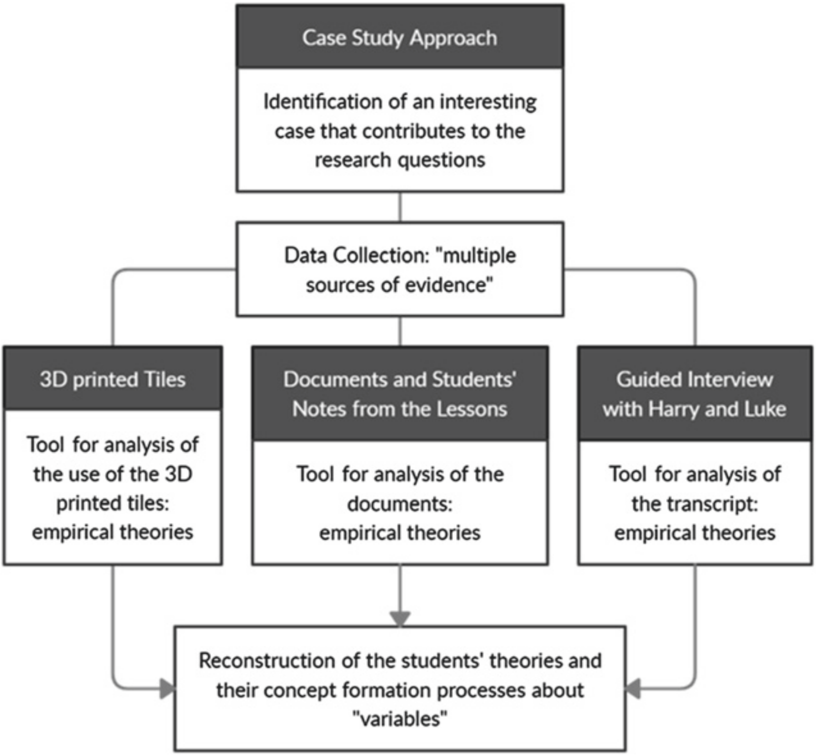

Fig. 5 Summary of the Methodology

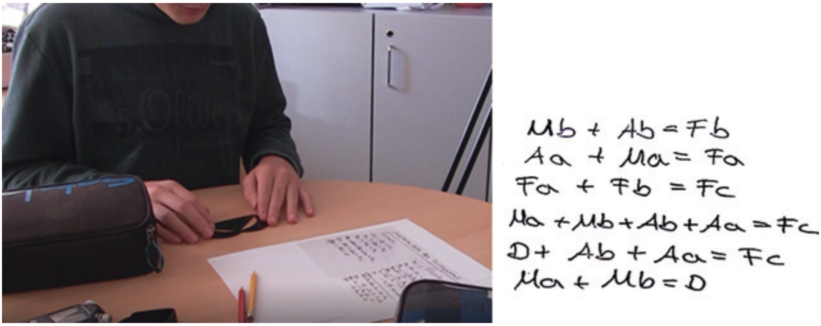

Fig. 6 Scene from the interview situation with Harry and Luke

a deeper understanding of the case key scenes have been selected: "Case studies are undertaken to make the case understandable" (Stake, 1995, p. 85).

In addition, the following can be stated for a case study: "For a case study, this is when a "how" or "why" question is being asked about a contemporary set of events, over which the investigator has little or no control" (Yin, 2003, p. 9).

As it is usual for a multiple-instrument case study, our case focuses on answering the research questions. An important contribution to the contextualization of our case study is provided by the interview, the documents from the two lessons, and, for example, the 3D printed tiles that the students created.

As presented in Fig. 5, a summary of the methodology, the empirical theories approach is used as a tool for analysis. As described in Sect. 2.2, empirical theories provide a very clear analytical descriptive framework that explicates the aspect of knowledge development when dealing with empirical contexts (in our example 3D printed objects – the students' self-made tiles).

4 Luke and Harry

4.1 Description

In our case study, we consider the students Harry and Luke (both 15 years old). We will focus on a transcript from the interview situation with both students in the following key scene. As shown in Fig. 4, the two students had access to all the materials from class during the interview. Both, the 3D printed objects as well as a slide with the students' justification of the theorem in the form of symbols and equations, are available on the table (cf. Fig. 6). In the following transcript excerpt, the interviewer asks a question about the equality of the areas of the triangle and the two lunes. At the same time, the interviewer refers to the equations of the two students (cf. Fig. 6) and asks the following question: "this means I could also, name it differently" (Table 2, 17:48).

Harry uses the term "variable" in his response to the interviewer's question. In addition, the student indicates – with a reference to the slide (cf. Fig. 6) – that it makes no difference to him how the lunes are named: "simply say lune c̲, that and lune, y or so" (Table 2, 17:50). As empirical objects we can describe the 3D printed "lunes" for Harry at this point (Fig. 6). Harry uses the names of the empirical object "lune" to assign meaning to the term "variable". In doing so, he picks up on the interviewer's impulse to "name it differently" (Table 2, 17:48). Here, the notion of variable can be seen as a potentially theoretical term (cf. Table 1) that takes on meaning for Harry in an empirical (student) theory. Similar to what Burscheid and Struve (2018) note for the variables *m* (mass) and *f* (force) in physics, the concept of variable takes on meaning for Harry in this key scene. At the same time, Harry does not care which variable is at stake: "say lune c" (Table 2, 17:50). Variables acquire meaning in Harry's empirical (student) theory – they become "observable" and communicable in his theory. For Harry, variables are therefore detectable in the successful application via the lunes. In this case, the terms "lune" or "triangle" can be seen as non-theoretical terms since Harry finds a reference object for them (in the form of the 3D printed tiles). At this point, as an intended application, we can describe the 3D printed tiles (lunes and triangle) that can be placed on the surface of the table, moved around and be held in one's hand. It is also worth mentioning that Harry holds the lunes in his hands the entire time, while he is speaking (cf. Table 2). In his remarks and argumentation, he "holds on to it", so to speak.

Table 2 Interview situation with Harry and Luke – "they are variables"

17:48	I	(*points to the slide in Fig. 6*) this means I could also, name it differently
17:50	Harry	*(holds one of the lunes in his hands, points to the slide)* these are only variables. […] *(points to the slide)* yes. you could also, simply say lune c̲, that and lune, y or so., they are **variables**. […]

Through Harry's impulses, Luke now adds to the conversation: "you could, insert, numbers, here" (Table 3, 18:00). For Harry, this interjection by Luke is not understandable and he immediately asks, "why numbers" (Table 3, 18:04). How-

Table 3 Interview situation with Harry and Luke – "insert, numbers"

18:00	Luke	*(points to the slide)* you could, insert, numbers, here., and ehm –
18:04	Harry	why **numbers**‘
18:05	Luke	well because it, is different in size
18:08	Harry	yes but, which numbers do you want to insert there. […] *(points to one of his lunes)* this is, this is 4 or what

ever, he also makes a suggestion (presumably ironic because of his expression: "or what" Table 3, 18:08) after another comment from Luke with a reference to one of the lunes and asks, "this is 4 or what" (Table 3, 18:08).

Luke would like to insert numbers as described in Table 3. The fact that Harry talks about the "lunes" in the sentences before, which can be named differently with variables, triggers him to use a number for a variable (he points to the slide, Fig. 6). The empirical objects Luke refers to in this scene are the 3D printed lunes and the triangle, which are "different in size" (Table 3, 18:05). It seems that Luke talks about size ranges in the key scenes. This is indicated in particular by his argument "well because it, is different in size" (Table 3, 18:05). So, for Luke, the variables as names of the 3D printed objects stand for size ranges. Thus, sizes can also be used. This is not surprising given the fact that they are geometric figures within a geometric context (Lunes of Hippocrates, cf. Fig. 2). A measurement and a handling of sizes is crucial here. In this case, the intended application in Luke's (student) theory is dealing with quantities (sizes) in terms of the lunes and the triangle (empirical objects). Harry cannot follow Luke's argument at this point with his (student) theory. Therefore, he asks: "which numbers do you want to insert there" (Table 3, 18:08) and points to the lunes. Harry does not think of size ranges and therefore suggests the number "4 or what". (Table 3, 18:08). For Harry, it is about variables as names for the lunes.

In the following sequence, Luke talks about a "side a" (Table 4, 18:11). As an exemplary reference, he uses the 3D printed triangle (cf. Table 4). He points

Table 4 Interview situation with Harry and Luke – "side a"

18:11	Luke	no. *(shows it on his triangle)* lo – for example at side a, is for example –
18:14	Harry	*(points to the slide)* where does it say side a ‘
18:16	Luke	I don't know., *(points to the slide)* here
18:17	Harry	yes well that does not **work**

Table 5 Interview situation with Harry and Luke – "to calculate"

		[…]
18:25	Luke	yes but you **can** insert numbers there., how else are we supposed to calculate it

to the triangle side of his empirical object and wants to give an example for the length of "side a".

Harry responds to the suggestion by asking "where does it say side a" (Table 4, 18:14). Thereby, he refers to the slide (Fig. 6), where in his opinion there are no side lengths. Eventually, Harry claims: "yes well that does not work" (Table 4, 18:17).

Harry and Luke talk past each other against the backdrop of their (student) theories in this sequence. For Harry, we can describe that in this sequence, he thinks about variables that are names of empirical objects (the 3D printed lunes or the triangle). Luke, on the other hand, interprets the variables as side lengths when dealing with size ranges ("for example at side a" Table 4, 18:11).

Luke still sticks to his idea. As it becomes clear in the following Table 5, in his opinion "numbers" can be inserted. In addition, for him it is a matter of "calculating something". And in order "to calculate something" he needs concrete numbers at this point. He does not see a conflict in the context of his (student) theory and the handling of sizes, as sizes can be measured and then "numbers" can be used to represent them.

For Luke and his (student) theory about size ranges, it is all about doing the calculations. For him, it is not about leaving a variable a as a designation for a lune on the slide (Fig. 6) and continuing to work with it. He thinks in the context of concrete size values, and these can be determined – measured. For him it is then of course also possible to determine the value of a side length and to continue calculating with it.

Based on their different (student) theories, and Luke's continuous request to insert concrete numbers into the variables which is revealed in his statement: "yes but somehow there **must** be a way to insert numbers otherwise –" (18:40), the students continue their discussion about the meaning of the variables on their slide (Fig. 6). Eventually, Harry concludes that the variables can stand for the areas of the respective geometric figures (cf. Table 6).

This short excerpt from the interview indicates that Harry, who thinks in the context of variables as names of empirical objects (the 3D printed lunes or the triangle), changes his context in order to be able to communicate with Luke, who

Table 6 Process of negotiation about the meaning of the "variables"

18:42	Harry	then perhaps we might have to write down different **formulae**
18:43	Luke	mm
18:44	Harry	yes. …
18:49	I	why would you need to write down **different** numbers there'
18:50	Harry	here it says, moon b –, I cannot simply insert any number there. that does not **work**. this is an **area** and you cannot, simply represent it by a number. *(shows it on one of the moons)* this is an area you would, somehow have to calculate the area, of such a **moon**, *(places the moon on the table)* I just don't know how it works, with regard to the triangle, I would, *(points to the slide)* here I could write down the area of the **triangle**. because I know how to calculate that one

thinks in the context of concrete size values. As Harry tries to understand which numbers Luke wants to insert into the variables, he extends his knowledge about the empirical objects – towards the idea of calculating the surface areas of the corresponding objects (18:50). So, Harry changes the context and talks about the calculation of surface areas and thus also about size ranges to communicate with Luke. But in this case, for him, concrete values (Luke's numbers) are only used when calculations are to be made: "you would, somehow have to calculate the area" (18:50). Thus, surface areas are to be calculated.

But he separates this context from his previous context "variables as names of empirical objects" by stating: "here it says, moon b –, I cannot simply insert any number there. that does not work. this is an area and you cannot, simply represent it by a number" (18:50). For him, concrete numbers are only useful for the calculation of areas.

While Harry claims that he "[knows] how to calculate [the area of a triangle]" (18:50), he still needs to figure out "how it works" (18:50) for the lunes. Based on his idea, the students then discuss how they could determine the area of a lune, by determining the area of semicircles. As the students have not dealt with the calculation of the area of a circle in class yet, they talk about potential working steps of calculating the area of a circle and dividing the respective area. The discussion ends with the fact that they do not know which area they would need to subtract in order to determine the area of a lune.

With regard to Harry's concept formation process, we can observe that the process of negotiation about the meaning of variables supports Harry in changing his context in this learning environment, in order to be able to communicate with Luke and to find something – areas, in this case – that they can calculate.

4.2 Results and Discussion

In this section, we will specifically address our research questions. The results described in Sect. 4.1 will be used to answer our research questions. First, we will address our first research question.

4.2.1 How Do Students Develop Mathematical Knowledge of the Concept of Variables in Geometrical Empirical Contexts with 3D Printing?

If we consider the term "variable" to be a potentially theoretical concept (Burscheid & Struve, 2020) it is not observable and, in simple terms, it has no reference object. In our case study, the term takes on meaning for the two students in the empirical context of the learning environment. Using the empirical objects, the two students in our case study negotiate their knowledge of the term "variable". In the context of their (student) theories, the theoretical concept becomes communicable for them. In doing so, we were able to describe that the two students were, in a sense, talking past each other. For Harry, we can describe that it is about variables that are names of empirical objects (the 3D printed lunes or the triangle). Luke, on the other hand, seems to interpret the variables as side lengths when dealing with size ranges ("for example at side a" Table 4, 18:11).

This case study can be used to illustrate that a theoretical concept can become an epistemological obstacle (Sierpinska, 1992) in mathematics education and in this way, it can generate conflicts. In the context of their (student) theories, Luke and Harry seem to talk past each other on the theoretical notion of variable - for which there is no reference object. Luke talks about size ranges and the associated side lengths (e.g. of a side "a") for which concrete values can be measured, determined and used. Harry, on the other hand, speaks of variables as designations for empirical objects – such as the lunes or the triangle. In the context of their respective (student) theory, both can communicate about the term "variable", but in different contexts and thus they communicate past each other, to some extent.

However, based on their different concepts about "variables" in their discussion, Harry changed his context in order to be able to communicate with Luke and his theory. He concludes that the variables represent surface areas of the lunes and the triangle and when it comes to calculating the surface areas, the use of concrete numbers is crucial. Thus, the students develop and discuss ideas of how they can determine the area of a lune, by determining the area of respective semicircles, for instance.

When empirical contexts play a role in mathematics education, such as in this case study (using the 3D printing technology), the vagueness of the empirical objects should be taken into account. In this way, empirical objects also promote a negotiation of knowledge. This negotiation of knowledge and the associated knowledge development processes should be allowed and encouraged in the classroom. In most cases, more time is needed for this in mathematics classes.

We also want to address our second research question.

4.2.2 How Can Students Be Supported in Their Mathematical Concept Formation Process with a Specially Designed Learning Environment Based on the Usage of 3D Printed Objects?

In the sense of our theoretical background, empirical theories and teaching mathematics consistently in an object-oriented and practical way leads Luke and Harry into a situation where they develop hypotheses, test them, and transfer them to other fields of application. With the usage of the 3D printed objects, the development of concept formation as a mathematical activity is promoted. Harry and Luke engage in concept formation processes about "variables".

With the help of the 3D printed objects, the two students demonstrate and share their knowledge with each other, always seeking to refer to the empirical objects. They also challenge each other repeatedly through the given empirical context. Therefore, with the 3D printed manipulatives we were able to offer new experiences to Luke and Harry and encourage them to construct more sophisticated ideas that are also appropriate for and applicable to further contexts. As a case study on digital technology, this study shows in detail that a meaningful use of the 3D printing technology may support these kinds of learning processes. It opens up a range of possibilities for an individualized use of empirical objects in mathematics classes. These 3D printed objects provide a whole set of opportunities for Luke and Harry to engage in concept formation and definition processes themselves – which our article can only give a very small account of. There is of course potential for further interesting research issues here: For example, it would be interesting to see whether and how generalized knowledge (about variables) can be developed when working on very specific (individual) empirical objects produced by the 3D printer. Certainly, it would have been additionally interesting to look at the data of the case study through the lens of the variable aspects according to Malle (1993), as in our case study, the student Luke, in particular, wants to insert concrete "numbers" into the variables. For him, it could be investigated whether the focus is primarily on the insertion aspect and for Harry, on the other hand, on the object aspect (Malle, 1993).

4.3 Conclusion and Outlook

To conclude, the excerpt from a lesson based on an empirical-oriented approach to teaching mathematics has indicated how the use of the 3D printing technology can support students' concept formation processes with regard to Algebra – in this case the theoretical concept of variables.

The unit on the Lunes of Hippocrates seems to provide a broad empirical environment to initiate discussions about areas and their relations to each other, even if the students have not dealt with the calculation of areas of circles in class. Furthermore, by creating 3D printed objects, such as tiles that represent the surface areas of lunes and a corresponding rectangular triangle and describing their relations with formulae – the notion of variables can be addressed and discussed. As shown in this case study, the empirical context promotes the negotiation of knowledge – based on the vagueness of the empirical objects for instance. With regard to these findings, further learning environments could be developed and analyzed in further studies.

Besides, the connection to the empirical objects – the ontological binding as well as the constant reference towards them – can be used as an indicator for further research and the development of further learning environments, in which students can create their own empirical objects. In this case study, the two students demonstrate and share their knowledge with each other, by using and referring to the empirical objects – which thus provide a basis for discussion as well as the exchange and negotiation of knowledge. Therefore, 3D printed manipulatives offer new experiences to students and encourage them to construct more sophisticated ideas that are also appropriate for and applicable to further contexts.

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Coding in the Context of 3D Printing

Frederik Dilling, Gregor Milicic and Amelie Vogler

1 Introduction

Algorithms are fundamental concepts in mathematics and computer science. Coding based on mathematical tasks can directly link the interaction with algorithms to problem- and process-oriented approaches and thus contribute to basic computer science and mathematics education. A particularly easy access offers block-based coding, in which commands are joined by a puzzle-like structure, so that no programming language syntax must be learned any more by the students. In this paper, block-based coding of 3D objects for 3D printing is discussed on the basis of mathematical problems.

First, there is a theoretical introduction to problem solving and algorithms in the context of coding. This is followed by a brief demonstration of block-based programming using the tool BlocksCAD. The core of the paper lies in the presentation of three mathematical example tasks and the explication of the theoretical considerations. The first task is about cube buildings in geometry. This example is used to explicate the connection between problem solving and coding. The second

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task deals with the coding of a building block generator. Using a transcript excerpt of a student working on this task, the development of algorithmic and spatial thinking is addressed. The third task deals with an algorithm for a spiral staircase. This task is used to explain ways to make coding versatile and also differentiating. The last section of the paper concludes the presented approach and gives an outlook. Ideas from this paper are taken from Dilling et al. (2022).

2 Coding and Algorithms—A Learning Opportunity for Problem Solving Skills in Mathematics and Computer Science Education

2.1 Problem Solving and Algorithms in Mathematics and Computer Science Education

In mathematics education, problem solving is understood as a mathematical activity that can be defined as the transition from an initial state to a final state, where the problem solving individual is not aware of any direct procedure to the transition at the beginning (Dörner, 1979; Newell & Simon, 1972; Schoenfeld, 1985). It is not the complexity of a task that matters but whether or not direct solving procedures are known (Smith, 1991). Therefore, the definition of problem solving is relative to the individual problem solver himself:

> "The problem solver does not have easy access to a procedure for solving a problem – a state of affairs that would make the task an exercise rather than a problem – but does have an adequate background with which to make progress on it […]." (Schoenfeld, 1985, p. 11)

This statement of Schoenfeld indicates the problem solver's background consisting of one's available knowledge and the (heuristic) strategies that are decisive for a successful problem solving. Hence, problem solving processes require the linking of one's knowledge and strategies. Schoenfeld (1985, p. 14) states:

> "Whether one wishes to explain problem-solving performance, or to teach it, the issues are more complex. One must deal with (1) whatever mathematical information problem solvers understand or misunderstand, and might bring to bear on a problem; (2) techniques they have (or lack) for making progress when things look bleak; (3) the way they use, or fail to use, the information at their disposal; and (4) their mathematical world view, which determines the ways that the knowledge in the first three categories is used."

Schoenfeld describes four categories that are relevant in the problem solving process: 1. The resources understood as knowledge for solving the problem. 2. The heuristics one must activate. 3. The control one must keep the whole process long. And finally, one's belief system (4.) that determines the ways the problem solver uses the available knowledge.

A well-known model for describing problem solving steps in mathematics education is the model of Pólya (1949) that is transferred from Greefrath (2018) into a problem-solving cycle (see Fig. 1).

As a starting point of the problem solving process serves a task embedded in an authentic problem context which the problem solver himself must understand. The next step includes the development of a plan followed by the conduction of the plan which leads to a solution. The last step is the evaluation or revision of the whole process, so the problem solver reflects his way to assess the achieved solution.

In computer science education in comparison, Müller and Weichert (2013) describe problem solving as follows:

> "The goal of a problem solution is, starting form a problem, to find a system that represents this solution. This procedure can be broken down into three steps. The starting point is a given problem for which a computer solution is sought. Subsequently, an algorithm is created by means of abstraction from the problem. In the next step,

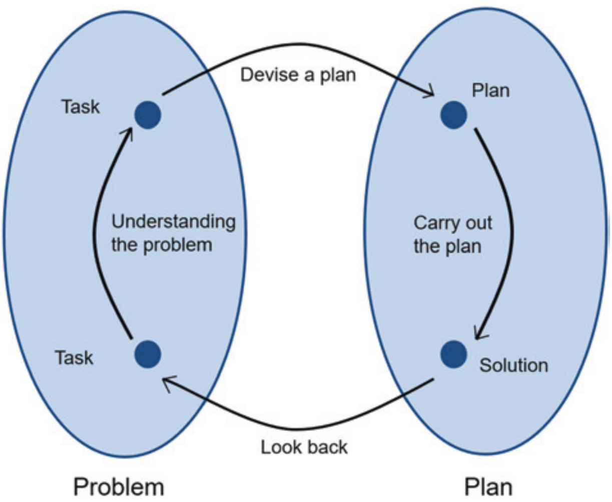


Fig. 1 Problem solving model following Pólya (1949) according to Greefrath (2018)

the algorithm is transferred into a program, which is then executed by a computer. The program represents the solution of the problem." (p. 16, authors' translation)

This defines an algorithm, implemented in a program, as the result of a computational problem solving process. Consequently, it seems, that developing of algorithms—instructions that transform input variables one-to-one into output variables in a finite number of steps—for specific problems is a core activity in computer science education. Algorithms can be seen not only as a fundamental idea in computer science (see e.g. Schubert & Schwill, 2011) but also in mathematics (see e.g. Ziegenbalg, 2015). Mathematical algorithms that are used in school are, for example, written calculation algorithms, the sieve of Eratosthenes for finding prime numbers or the Gauß algorithm for solving systems of linear equations. Müller and Weichert (2013) developed a problem solving model in computer science which is presented in Fig. 2.

Like the problem solving process in mathematics, the computer science problem solving process can be described as a sequence of certain typical steps (cf. Dilling & Vogler, 2022a). The problem must be formulated or identified in the first step. This is followed by the analysis of the problem in the second step. Here, a crucial question is whether there are one, no or several solutions to the problem. In addition, an abstraction takes place in this step by introducing variables for certain information. Possibly, the problem is also revised or decomposed into subproblems. The third step is designing an algorithm for possible subproblems

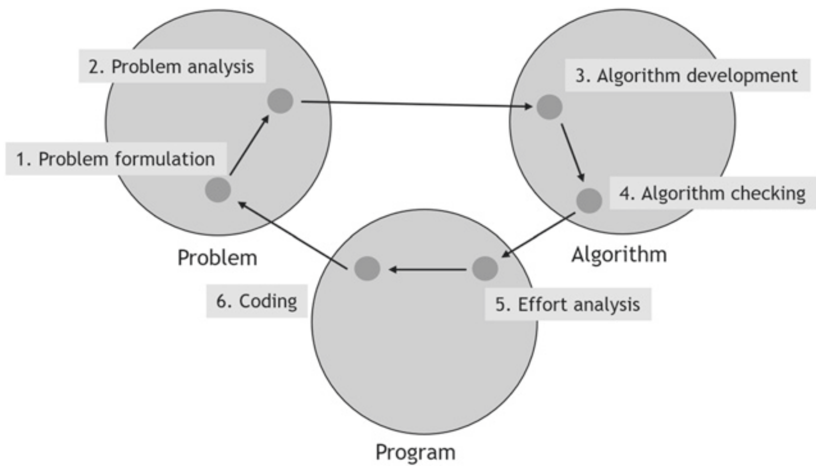


Fig. 2 Problem solving model according to Müller and Weichert (2013)

as well as the overall problem. In the fourth step, the correctness of the algorithm is checked (Does the algorithm terminate? Does it always deliver the correct result?, etc.) and in the fifth step, the implementation effort is analyzed. Only in the sixth step the actual coding is carried out, i.e., the algorithm is formulated in a selected programming language. Depending on whether the result is satisfactory, algorithm and code are improved in further problem solving cycles. The computer science problem solving processes in class do not always have to pass the whole cycle. For example, the students can develop an algorithm only, but do not implement this in a program, or an algorithm is given, which is then to be implemented in a given coding environment.

2.2 Problem Solving at the Intersection of Mathematics and Computer Science Education

Authentic problems are rarely bound to strict boundaries of subjects like mathematics or computer science (Dilling, 2020). For this reason, it seems comprehensible that appropriate coding contexts can encourage students' mathematical competencies as well as problem solving strategies and knowledge about algorithms. In Sect. 4, such possible appropriate coding contexts are represented and analyzed in detail. Beckmann (2003) emphasizes the opportunity to learn mathematical content like the influence of a variable on an equitation or methods like problem solving:

> "If coding […] is used to solve mathematical problems, it is to be expected that this is accompanied by learning mathematics. Because coding always requires an intensive examination of the topic, in this case mathematics. Beyond that, however, coding can also be used specifically to learn and deepen certain mathematical content or methods." (p. 16, authors' translation)

Already at preschool age, students can be introduced to coding contexts and thus also develop computational thinking, for example by playing with simple robots (e.g. Bee Bots) or minicomputers (e.g. Calliope). According to Wing (2006, p. 33),

> "Computational Thinking is a fundamental skill for everyone, not just for computer scientists. To reading, writing and arithmetic, we should add computational thinking to every child's analytical ability."

In North-Rhein-Westphalia, according to the current resolutions of the conference of education ministers (2016), coding is already part of the curriculum from grade

1 onwards. And investigations of Förster (2011) demonstrate that coding can help students with difficulties they have in mathematical problem solving because it provides a common language which allows them to gain their own experiences and makes it easier to talk about (self-developed) programs. For example, they can explain the program's structure, the development, and the relationships to other (student's) programs.

In the following section, it will be explained how coding can be related to 3D-printing contexts. Based on that, section 4 contains examples of possible coding tasks for mathematics classes from elementary school to upper high school.

3 Coding in the Context of 3D Printing

Several CAD-programs offer the opportunity to design a 3D model through block-based coding which is the basic idea in the tasks examined in more detail in section 4. In the following, block-based coding is explained regarding the web-based CAD-software BlocksCAD[1] which is suitable for middle and high school students. In elementary school, the block coding function of Tinkercad[2] can be recommended. The software interface is divided into two areas (see Fig. 3). On the left side, there is the area to create a script of the program with blocks which can be taken from the different categories on the left edge. On the right side, there can be seen a representation of the designed 3D model on a working plane (xy-plane). If you click the button "Render" below the representation area the 3D model is rendered due to the script on the left. Another important function in the program is that you can switch between the script in blocks and in code like a Java-Script. The button "Generate STL" generates a stl-file which is needed for the slicer-software to print the object with a 3D printer.

For the analysis of the example tasks, some of the block categories (see list on the left of the interface) are considered in more detail (cf. Dilling et al., 2022). First, there are 3D shapes (dark green), such as sphere, cube, and cylinder. The imprecise designation of the 3D shapes in BlocksCAD can be used as an occasion for exactification of the geometric bodies on the part of the teacher. By appropriately varying the variables, a cuboid can be created using the cube block as in the code in Fig. 3. Transformations in the plane and in space (dark blue), such as

[1] https://www.blockscad3d.com/editor/?lang=en (last access: 21.12.2021).
[2] https://www.tinkercad.com/ (last access: 21.12.2021).

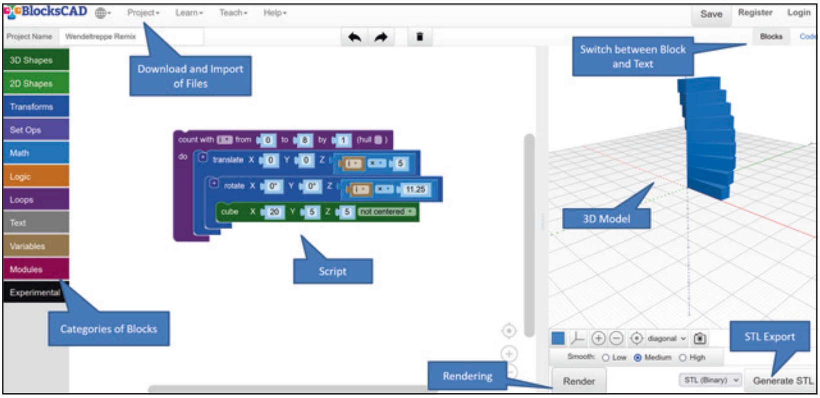


Fig. 3 Interface of BlocksCAD

moving, rotating, mirroring, and scaling, but also extruding plane figures, represent important commands in addition to the set operators: union, difference, intersection, and hull (light purple). The category of loops (dark purple) contains the loop command typical for computer science. The category of mathematics (light blue) has numerous mathematical parameters and operations, such as the basic arithmetic operations, root extraction or also the generation of a random number. The last category to be mentioned is named variables (brown) and contains the setting and changing of variables. The external puzzle-like form of the blocks specifies how they can be put together and therefore prevents errors on the level of syntax. Further logical errors can be seen by the operator through the preview of the created model on the working plane. Thus, students are encouraged to proceed in a more exploratory manner.

4 The Analysis of Three Example Tasks

4.1 Example 1: The Cube Building

Develop a code which creates a cube building matching the given construction plan. Proceed as follows:

a) *Plan your approach by first considering which blocks you need for your code.*

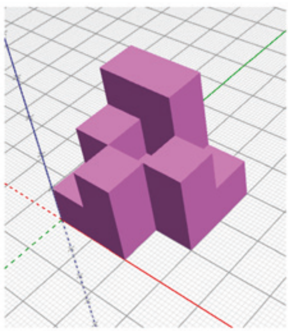
b) *Develop a code and test it by rendering. Improve your code if necessary.*

c) *Evaluate your code: Could you create a cube building to any construction plan with 4 × 4 fields by entering different values in your code? Improve your code if necessary.*
d) *Explain: How do you need to change your code if your construction plan contains more or less rows/columns?*

This first example is related to a very common content in elementary school. To develop spatial ability and further skills in spatial geometry, creating a building from cubes is a very popular activity from first grade onwards. An example task in this field could be to create a cube building with a given construction plan using wooden manipulatives. Based on experiences gathered in such tasks, students could get the task mentioned above to create a cube building with a given construction plan in the CAD-software BlocksCAD. A possible given construction plan and solution in BlocksCAD can be seen in Figs. 4a and 4b.

Subtask (a) aims at identifying relevant blocks like the cube block and translate block. Then to answer task (b), one could develop the code shown in Fig. 4b. Task (c) addresses the evaluation of the code, one should consider if one's code is appropriate to create a cube building to any construction plan with 4 × 4 fields. To solve this, the code can be enriched by variables for each field of the construction plan, as realized in the code in Fig. 5.

As an explanation (task (d)), one could describe, on the one hand, that if the construction plan with 4 × 4 fields contains different numbers, you have to type in these numbers related to the different variables in the code. Each of the variable determines one field of the plan, e.g., the variable "2;1" defines the field which is in row 2 and column 1. Thus, according to this sort of solution, one has to add more variables and related translate-blocks if the construction plan would have



1	2	3
2	1	3
0	2	1

Fig. 4 **a** Solution Example Task 1 (3D model and construction plan) **b** Solution of Example Task 1 (Block-code with comments)

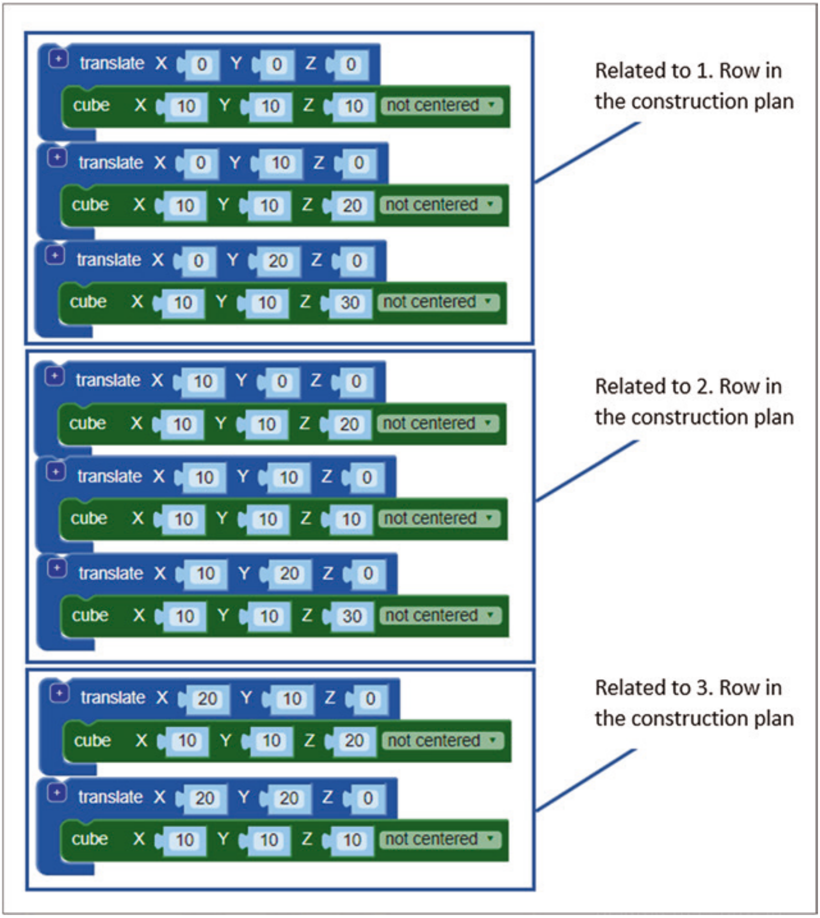


Fig. 4 (continued)

more rows and/or columns, and one could erase some variables if the construction plan would have less rows and/or columns.

In the following, the solution and the possible development process of this solution is theoretically assigned to the model of Müller and Weichert (2013), introduced in Sect. 2.1. Regarding the theoretical background (Pólya, 1949; Schoenfeld, 1985; Müller & Weichert, 2013), it is explained how the processing of the task can encourage problem solving procedures.

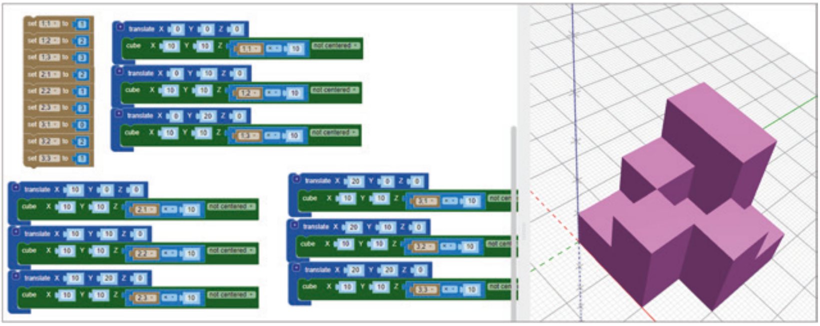


Fig. 5 Revised Solution of Example Task I (with variables)

First, the problem can be formulated or identified as follows: *A program is to be created which creates a cube building according to a given construction plan*. In the second step, this problem is analyzed. Here, the problem solver can think about the blocks that are needed for the code and how the algorithm can look like. A crucial aspect that can be identified at this step is that there are several solutions. Two possible algorithms are presented below:

1. Algorithm Without Using Variables:
    1. *Create a cube with the dimensions x: 10, y: 10, and z: 10 multiplied with the height of the cube building (in numbers of cubes) on the desired field of the construction plan.*
    2. *Move the cube of (1) to the desired field on the construction plan by translating it with the following coordinates: x: 10 multiplied with the row number of the selected field minus 1 and y: 10 multiplied with the desired column number of the selected field minus 1 and z: 0.*
    3. *Reply step 1 and 2 for each filled field on the construction plan.*
2. Algorithm With Using Variables:
    1. *Set a variable for a field of the construction plan which defines the height of the cube building (in number of cubes) on the desired field of the construction plan.*
    2. *Create a cube with the dimensions x: 10, y: 10, and z: 10 multiplied with variable of step (1).*
    3. *Move the cube of step (2) to the selected field on the construction plan by translating it with the following coordinates: x: 10 multiplied with the desired row number of the selected field minus 1 and y: 10 multiplied with the desired column number of the selected field minus 1 and z: 0.*

4. *Reply step 1 and 2 for each filled field on the construction plan.*
5. *Set the created variables to the desired number of cubes that are to be stacked on the field defined by the variable.*

Another possible solution could be to create and move each cube of the cube building individually. However, this would necessarily lengthen the code. Thus, it can be said that the proposed algorithms are already more sophisticated. Using the software BlocksCAD, the third step (Algorithm development) to the sixth step (Coding) were passed through simultaneously because the problem solver has developed the algorithm during an exploratory development of the code (see also the empirical study of Dilling & Vogler, 2022b). And an abstraction has taken place by introducing variables for every field to be able to create different cube buildings by entering the number of cubes on the corresponding fields. The motivation for the redesign of the first algorithm was the revision of the initial problem to: *A program is to be developed which creates a cube building according to* ***any*** *given construction plan*. This redesign can be intended by subtask (c) in which one should evaluate and possibly improve one's code. Thus, in terms of Pólya, the successive tasks encourage reflection procedures on one's own problem solution regarding the initial problem (Look back, see Fig. 1). The correctness of the algorithms can be checked by repeated rendering of the cube building model during the coding process. The problem solver receives visual feedback (rendered cube building on the right of the program's interface) if the algorithm leads to a desired result. The effort analysis leads the problem solver in this phase to the more sophisticated algorithm demonstrated above. According to the problem solving cycle of Müller and Weichert, the problem solver could ask himself now if he could improve his code by shorten the code, for example through adding loops and further variables. For this, the problem solver would process the whole problem solving cycle again.

4.2 Example 2: The building block generator

Develop a code which generates a building block by entering different values. Proceed as follows:

a) *Plan your approach by first making a sketch of a possible basic building block with appropriate dimensions/size specifications before you start coding.*
b) *Now develop a code in BlocksCAD that generates your basic building block.*
c) *Consider how you can develop the code into a real generator, which can generate different building blocks.*

d) *Consider if your code generates blocks which can be plugged together. Hint: Use the set operation block "Difference" to make cut-outs for a plug-in system on your block.*
e) *Evaluate if your code can be further optimized, for example, by loops.*
f) *Test your code and print two different building blocks with the 3D printer.*

The core problem in this task is to develop a code that generates building blocks like Lego or Duplo blocks. Subtask (a) is related to the second step of problem solving according to Pólya (Devise a plan, see Fig. 2). Here, the problem solver should plan his approach by first making a sketch of a possible basic building block with appropriate dimensions/size specifications. Then, when developing a code to generate the basic building block (subtask (b)), these initial considerations can play a key role. For example, one has to consider which solids can be used and if cube and cylinder are chosen, how the radius of the cylinder should be in ratio to the length of the sides of the cube. According to the solution shown in Fig. 6, the basic building block can be created by the union of a cube and cylinder. In addition, one can use the difference-block to create the difference of the described basic building block minus a cube which is a little smaller than the other one so that the building block can be plugged on another one at the end (compare subtask (d)). In subtask (c), the problem solver should consider how he can develop the code into a real generator which can create different building blocks. To realize that the solution in Fig. 6 shows a sophisticated code consisting of two loops. The algorithm to this code can be noted as follows:

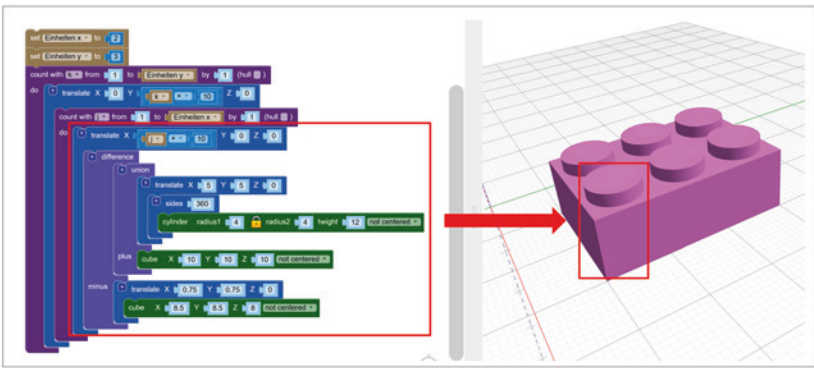


Fig. 6 Solution of Example Task 2

1. *Create a cuboid basic building block with the base area xy.*
2. *Multiply (1) with j and translate each time by jx.*
3. *Multiply (2) with k and translate each time by ky.*

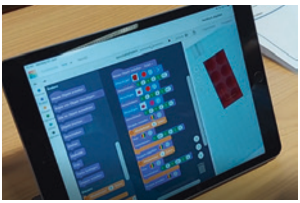
Step (2) und (3) of this algorithm can be transferred into a code by using loops. In Fig. 6, the variable "units x" (Einheiten x) is set to the value 2 and the variable "units y" (Einheiten y). These two variables play a decisive role in the two loops. The internal loop moves the basic block x times in the direction of the x-axis (red axis in the program). The external loop moves the resulting row of basic blocks y times in the direction of the y-axis (green axis in the program). Thus, the code generates a "2 times 3" building block. By entering different values for x and y, cuboid building blocks of any size can be generated with the code.

Furthermore, referring to this task, the findings of a case study investigating how the block coding in the 3D printing context can encourage knowledge development process of fourth grader in relation to algorithmic and spatial thinking and mathematical knowledge (Dilling & Vogler, 2022b) are shown in the following. To describe the findings in detail, a transcript except of a crucial scene is translated. In this scene, the student named Noel explains the teacher his code of a building block generator which he has developed in the scenes before.

At the beginning of the third scene (see Sect. 1, Table 1), Noel explains the plugging system of the block generated by his code. His references to specific elements of his code, such as the selection of drill solids (third blue block in his code), as well as the statements "Then I had to create this as a group so that this would actually be a drill hole" and "And then I had to repeat it 3 times so that 3 more of these would come" indicate that Noel used each block as well as the sequence of the blocks in a goal-oriented and thoughtful way to develop the building block generator. In Sect. 3 of the transcript excerpt, Noel explains, referring to the virtual object created by the code, that he took a similar approach in developing the code for the ice cube mold (task which he had previously worked on), and that one could "take the same thing that way." At this moment, Noel refers to the top view from below of the building block, which is similar to the structure of the mold for hemispherical ice cubes. Noel adds that the only difference to the Lego brick would be that such "blocks for the next Lego brick" would be added. It can be assumed that he means the attached spheres.

Further on, Noel is asked by the teacher to explain how one could change the code in order to get a brick with 3 times 3 "nubs". Whereupon Noel taps on the input field in the first repeat block of his code and explains that the displayed value is to be changed to 2. Then he looks at the program preview, which he does not start in this case, and shows with a finger movement what effect a repeat of "2 times"

Table 1 Transcript except of Dilling and Vogler (2022a, b)

1	Noel	 So first of all, I created a block here, which was supposed to be this shell, this one block. And then I just put here, what's up there (points to nubs of the building block on virtual work plane) on it, that if you would put another one under there, that you could then also assemble it together like Lego or something. (Noel rotates view on work plane) And I also made that if, that this also just fits exactly in there (finger movement from nub to hole). Then I had to create this as a group, so that this would actually be a drill hole. But here (pointing to the drill selection in the code) I had to make it really a hole. And then I had to repeat it 3 times to get 3 more of these. Then I had to create this again as a group. And then I had to repeat it again, that the next to it is also like this. Yes, and in the end I/
2	Teacher	Yes, and that what we did in the hours before, did that help you? You were really fast now. Does that have anything to do with what you have learned with the ice cube or the apartment block?
3	Noel	 Yes, that was just a bit similar (Noel rotates the view to a top view from vertically below the brick, see picture). You can also take that as the one with the spheres that I did yesterday. One can take it in such a way. Only that I have here then still above such blocks for the next Lego brick/

(continued)

Table 1 (continued)

4	Teacher	Yes
5	Noel	That I can then still put that into the next Lego brick
6	Teacher	Exactly, and is then also a real building block generator? Can I also enter other numbers somewhere, so that I then get, for example, a 3-times-3 brick with 3 times 3 nups?
7	Noel	Mhm
8	Teacher	Can I (points finger at code) change something in the code that I then get a brick that has 3 times 3 nubs?
9	Noel	Yes
10	Teacher	And how would I do that?
11	Noel	 Then I would have to change here the (points to input field with value 3 in the first repeat block) with 3 times, that it 2 times already is' or? (types 2) 2 times, so that this here also goes down only 2 times (makes finger movement, see picture) And then I have to, here × 20 is good, then I have to set this here (taps input field in second repeat block) again to, ehm, 2 as well. Then it should now actually be like ehm, yes 3 times 3 (starts program preview)
12	Noel	(Program preview ends) So, yes then I've got it

will have on the object. He completes this with the statement "2 times, so that this goes down only 2 times" (see Sect. 11, Tab. 1). Now he continues his explanation by noting that the value 20 is good as the x-coordinate in the move block within the first repeat loop, indicating that he sees no reason to change it. He then taps the input field of the second repeat block of his code and comments that he would also need to set this to 2. He concludes, "Then it should actually be like ehm […], yes so 3 times 3" and starts the program preview. When the program preview ends and displays the desired object (a brick with 3 times 3 nubs), Noel comments, "So, yes, then I've got it" (see Sect. 12, Table 1) (cf. Dilling & Vogler, 2022b).

The authors defined algorithmic thinking as the reflected development and description of algorithms using typical components such as the finite sequence of instructions or action steps, loops, variables etc. (cf. Dilling & Vogler, 2022b, translated by the authors). In the scene described above, the student Noel shows that he can identify the relevance of each individual step in his algorithm and the effect on the 3D object generated by the code. He demonstrates a sustainable understanding of the loop (orange in the code) and the related variable which defines the number of passes of the loop. Due to his description of the code, it can be assumed that he knows that the input of the value 2 in both loops leads to the addition of two copied objects of the same kind in x- and y-direction. Thus, he can correctly predict that his program will generate the intended building block with 3 times 3 nubs. This also indicates that Noel, in terms of the component of visualization (components of spatial ability according to Maier, 1998), has a dynamic mental concept of what 3D object his program models as it passes through the loop-block. Furthermore, his flexible use of the input variables in the loop-block shows that he has a strong understanding of the operations addition and multiplication—in sense of *such an object is added 2 times to the already existing object, thus I get 3 objects of the same kind in the end*—as (preexisting) mathematical knowledge which he activates in this situation. The detailed analysis of this scene and findings of the study illustrates that in such a learning environment, elements of spatial and algorithmic thinking can be activated and connected to each other (for more details see Dilling & Vogler, 2022b).

4.3 Example 3: The Spiral Staircase

Develop a code which generates a spiral staircase. Proceed as follows:

a) *Plan your approach by first making a sketch of a possible spiral staircase. What does a model of a spiral staircase look like, and which properties are crucial for the modeling in BlocksCAD?*
b) *Use a loop block to create the spiral staircase. Consider which element occurs several times? How should this element be transformed in each repetition to obtain a spiral staircase?*
c) *Render your 3D model and evaluate your code.*

Like the first problem presented, this task does not require advanced skills in mathematics or computer science and is therefore suitable for the lower secondary level to introduce BlocksCAD and/or an application for basic control structures. Although the problem can be solved by students with little prior knowledge, it does offer plenty of possibilities to focus on and introduce more advanced topics in computer science as well as in mathematics.

A spiral staircase is an object most if not all of the students know. Subtask (a) aims to encourage the students to have a closer look at a spiral staircase from a mathematical point of view and come up with a corresponding model. A first, simple and straight-forward way is to model a step of the staircase as a cuboid. As suggested in subtask (b), using several cuboids as steps and rotating and translating them results in a spiral staircase.

A potential solution for the problem is given in Fig. 7, left. By using distinct values for the width, height, and depth for the cube block in BlocksCAD, a cuboid can be created. This inaccuracy in labelling the geometrical figure should be discussed with the students.

The loop variable i is used inside the loop, as suggested in subtask (b), in order to rotate and translate each cuboid to its desired position. The resulting and rendered model can be seen in Fig. 7 on the right.

Following the problem solving cycles presented in Sect. 2.1, the algorithm, which can be seen as the solution according to Pólya (cf. Fig. 1), can be reflected and revised in relation to the initial problem. This procedure is similar to the revised cube building solutions but here the aim could also be to get an improved resulting model and not only a shortened code. Rounded steps, as present in generic spiral staircases, can be accomplished by using circular segments instead of cuboids for the steps. To obtain this shape, the intersection of a cuboid and a cylinder can be formed. To increase the tread area, the step width, i.e. the depth of the cuboid, and the distance between the steps, i.e. the angle of rotation, can be varied to obtain a model that is easier to walk on.

The presented problem can be regarded as an *Implementation* task (Milicic et al., 2020): given the description, the students are asked to develop an algorithm and implement it. In this article, BlocksCAD is used as the corresponding learning environment with its visual programming language. Based on the code in Fig. 7, left, other task formats can be used: *Parsons Puzzle*, *Analysis* and *Find the Error*. The difficulty of the tasks depends on the algorithm they are based on. However, empirical results (Lopez et al., 2008) indicate that *Parsons Puzzle* is the easiest and that *Analysis* and *Find the Error* tasks are more challenging for the students. The already discussed *Implementation* task is seen as particularly difficult (Venables et al., 2009).

By randomly arranging the blocks of a code, the input for the task type *Parsons Puzzle* is prepared. The students are asked to reposition the code blocks in order to get a working program for the requirements formulated in the description. Additional, but not required code blocks in the scripting area are called distractors and increase the complexity of the task.

By specifying the code blocks which should be used, an essential part of the original open problem solving process is already anticipated, namely the translation of the problem into the algorithm and the subsequent selection of the avail-

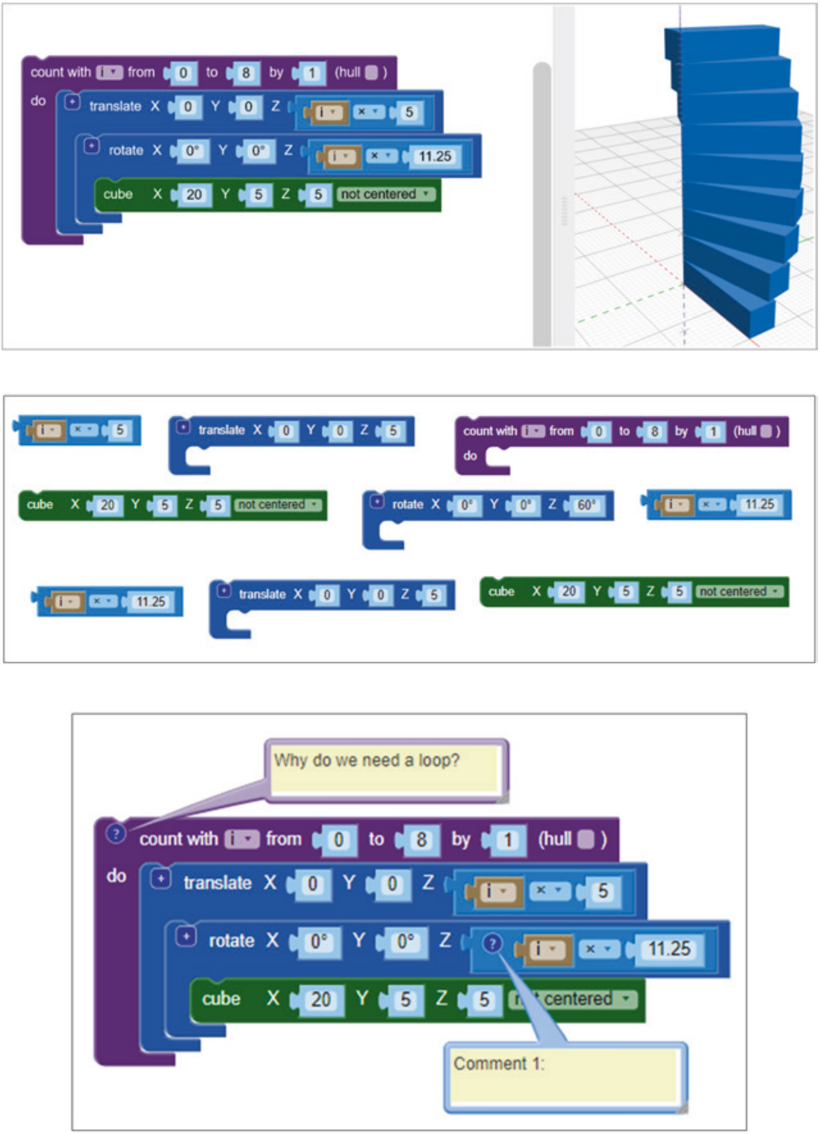


Fig. 7 Solution of Example Task 3. **a** Parsons Puzzle for the spiral staircase problem; **b** Input for the Analysis tasks type for the spiral staircase problem; **c** Input for the Find the Error tasks type for the spiral staircase problem

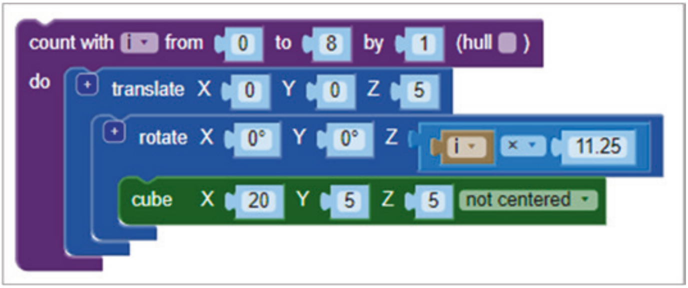


Fig. 7 (continued)

able code blocks. However, especially for more challenging problems, this can be desirable and provide students with a certain scaffold. In Fig. 7a the input for the task type *Parsons Puzzle* is shown. If the not required code blocks are also given, this is a Parsons puzzle with distractors.

In contrast to the described types *Implementation* and *Parsons Puzzle*, the goal of the task type *Analysis* is not the design and implementation of an algorithm, but the description of an already given code. The *Analysis* task type can be used, for example, to repeat or discuss already solved problems, to make sure that certain key concepts are understood (such as the usage of the loop variable inside the loop itself) or also to demonstrate new concepts. The teacher can set desired emphases on the code or algorithm by adding comments or posing questions at the corresponding positions, as can be seen in Fig. 7b.

The question attached to the loop is connected to subtask (b) of the original problem. The question aims to deepen the modeling competence of the students by repeating the idea of using rotated and translated cuboids to model the spiral staircase.

The comment attached to the usage of the loop variable can be seen as more open in comparison to the question. Commenting one's own code or understand a given code is an important task in software development. The students are asked to formulate a short, precise comment to the function of the corresponding code block.

In the *Find the error* type task, some errors are deliberately inserted in a correct algorithm or program to solve a problem. The students are then asked to find and correct the errors. The process of finding and correcting errors is often referred to as debugging in the context of programming and plays a major role in software development.

Ideally, the errors that are deliberately inserted are based on concepts already presented or even errors beforehand identified and addressed by the teacher, so that the task can initiate a renewed reflection process. This type of task is more extensive than *Analysis*, as all components of the erroneous code must be understood, analyzed, and subsequently corrected.

An input based on the presented spiral staircase problem is shown in Fig. 7c. As opposed to the original algorithm, the cuboids are translated in z-direction by a constant number. The cuboids will therefore be placed at the same height, not creating a staircase. Using the loop variable i twice inside the loop not only for the rotation but also for the translation can be a very challenging concept for the students. This *Find the Error* task addresses this concept.

5 Conclusion and Outlook

In this paper, problem solving at the intersection of mathematics and computer science in the context of 3D printing was considered. Three example tasks (creating a cube building, creating a building block generator, and creating a spiral staircase) were used to explain how mathematical and computer skills can be promoted in block-based coding environments. Coding 3D models using the software BlocksCAD provides an opportunity in math classrooms to engage students with a wide range of mathematical topics while promoting problem solving skills and algorithmic thinking. The detailed analysis of task 1 and possible solutions (Sect. 4.1) has indicated that such coding activities can encourage problem solving procedures, with reference to the problem solving cycles outlined in Sect. 2. Subsequently, the task analysis in connection with the detailed presentation of the results of a case study (Sect. 4.2) have shown that even elementary students are capable of independently developing and explaining an elaborate code for the extensive problem to create a building block generator. Finally, the third example task (Sect. 4.3), modeling a spiral staircase, was used to explain how the task formats for developing algorithms with block-based coding in CAD software can be adapted to different performance levels. In Dilling et al. (2022) further implementation scenarios for coding in the context of 3D printing as the modelling of a snail shell are illustrated.

In the sense of ongoing digitalization, further design-based research on suitable, authentic problems for coding in the context described in this article is required. Furthermore, there should be more studies investigating the extent to which the task formats presented activate problem solving strategies for students in order to show that coding can be considered a profitable intersection between mathematics and computer science. In addition, it could be empirically investigated to what extent this special use of CAD software can improve spatial ability.

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Modelling and 3D-Printing Architectural Models—a Way to Develop STEAM Projects for Mathematics Classrooms

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1 Introduction

In this chapter, we share ideas and experiences emerging from our work as teachers with the intention to integrate the use of technology in mathematics. We strongly believe that technology should be used to enable students with new skills and new ways of thinking. For this goal, we propose a STEAM (Science, Technology, Engineering, Arts and Mathematics) approach focusing on the mathematical aspects of modelling and the 3D-printing process.

It is important to put mathematics into practice and make it appealing for students. Our approach, using modelling to connect the concepts with other areas of knowledge could enrich teaching and learning and make it more engaging for students.

Research, in line with our experience, shows that incorporating technology in the classroom is time consuming and, in many cases, teachers feel they know less about the technology than the students. This creates an uncertainty, and our goal

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is to ensure that technology must be good enough to encourage teachers to overcome these difficulties.

We will start outlining the needs and opportunities in the Uruguayan context, then move towards the theoretical background, the genesis of the ideas, and the analysis of examples underlining the opportunities that this approach offers for the learning of meaningful mathematics.

1.1 The Uruguayan Context

In Uruguay, Plan Ceibal (PC)–a large-scale project created in 2007 for inclusion and equal opportunities with the aim of supporting Uruguayan educational policies with technology–have created an almost ideal situation for the fulfilment of innovative approaches for teaching. Since its implementation, every child who enters public education in any part of the country is given a computer for personal use with free Internet connection at school. In addition, PC provides teacher training and educational resources in the hope of transforming ways of teaching and learning in the country. As an example of this, while in 2006 only 28% of those under 18 years of age had a computer at home, in 2018, minors who have access to a computer at home represent 89% of the under 18 years old population. (Instituto Nacional de Estadística, 2020). Meanwhile, this represents a great opportunity but as shown in the work of Valiant et al. (2020) during the early years of PC's implementation, teachers noticed the need to change classroom practices, but apparently little change was done in the inclusion of technology. They also noticed that the magnitude of changes is in correlation to the socio-economic and cultural context of the educational centres, reinforcing the idea that technology availability is not sufficient for improving the educational practices.

According to Testa (2013), the next step in PC's implementation was the support of teachers' pedagogical practices, organising workshops, designing classroom activities, and creating a structure of professional departments for giving pedagogical support to teachers in the implementation of these new technologies. The department responsible for the pedagogical implementation of 3D-printing technology is the laboratory of digital technologies (LabTeD), this "includes the technological innovation projects promoted by PC. In projects such as Sensors, Robotics, QR Codes, 3D-Printing, and video games. Mathematics is incorporated in a transversal way. Collaborative learning promotes the integration of technology and cognitive, stimulating logical-mathematical thinking, creativity, and collaboration, allowing the development of new learning." (Testa, 2013, p. 171).

Another revealing research on the way that teachers use these technologies is Testa and Téllez (2019) work about the elementary school teachers' use of PAM (Adaptative Mathematics Platform), their paper concludes that most teachers value the platform's help on the organisation process and their own practises but none of them mention the nature of the mathematical thinking generated by the tool. Also, the work of Vitabar (2011) underlines that many teachers use the technologies without considering the pedagogical dimension nor the mathematical thinking process of students.

Our experience also strengthens this vision on the need to produce and share an approach to teach better mathematics with technology. Since 2011, we are part of the Uruguayan's GeoGebra Laboratory (Aguilar et al., 2012) -a group of mathematics teachers and teacher educators that work around the meaningful implementation of GeoGebra in the classroom-, this group work under the premise that mathematical knowledge and the type of thinking to be developed are conditioned by the means used to represent, communicate, and produce mathematical ideas (Borba & Villarreal, 2005). Through this, we organise workshops and courses for teachers and students that allow us to develop a better understanding of the needs that teachers' community has, allowing us to start developing and share activities that have in mind these kinds of reflections.

1.2 Opportunities for Better Utilisation of Technologies

As mathematics experts, on many occasions we encounter something that relates maths with real life, often this has to do with architecture. We see a bridge, a building or some structure that makes us reflect about the mathematical properties of that object. We encourage the reader to bring that to class and make it visible for students, to make them wonder and try to understand how the formulas and functions we study in our classes are connected to real life. This may look simple for an expert eye, but it is not so simple for students. They also could learn to appreciate and encounter those bits of maths in the world around them.

For this purpose, students and teachers can bring some personal object to model or some building/structure that is part of their culture. According to El Bedewy et al. (2021) through this we can include many aspects like the cultural and historical background of the object. This presents an opportunity for us to include in the classes not only mathematics, technology and science, but also cultural aspects of the country or students' lives that can help link knowledge with personal experience which helps to situate the concepts or skills developed and "improve student's interest and motivation" (Duit et al., 2007, p. 120).

2 Theoretical Background

As a consequence of our particular context, interests, and academic background, we grow a view of 3D-printing technology as an opportunity for the development of interesting mathematical tasks for our high school classes and for the inclusion of knowledge from outside mathematics into it. From this perspective Inquiry-based learning arise as a student-centred approach that align with our previous views on mathematics education and technology, in the words of Lieban (2019, p. 20):

> Inquiry-based learning is a modality of active learning where students explore materials on their own and are guided by their curiosity and exploration. The process of learning can be triggered by posing questions, problems or scenarios. From an instructor's perspective, this learning style focuses on helping students jump from the curiosity stage into critical thinking and deeper levels of understanding. They guide students through the investigation process, encouraging them to ask questions through structured inquiry activities.

Working with 3D-printing in our classrooms started in a rather organic way, as students motivated us with inquiry-based tasks and we saw the potential for transforming such tasks, making students engage with critical thinking and deeper levels of mathematical understanding. As we started to work on these ideas the need for a theoretical frame emerged, thus we structured inquiry-based learning and technology enhanced classrooms as our theoretical framework, shown in Fig. 1.

We considered 3D-printing as an artefact in the sense of Artigue (2002) and through the modelling and printing process it became an instrument for learning. We made a distinction between artefact and instrument as described in the *Instrumental Genesis* (Trouche, 2005) theory. While an artefact is a device acting as a tool -in our case the printer, modelling software, or the 3D-printed object- an instrument includes an artefact and the mental schemes that users develop while using the artefact. This is important for the task-design's process and for the classroom implementation because we aim to identify what kind of tasks foster the instrumentation process and what kind of guidance is needed for students in the process of instrumentalization.

As we advance into a more systematic view of our work process for the task design, mathematical modelling appears strongly in the horizon as we believe those tasks have the potential for guiding the *Instrumental Genesis* process. They could allow students to construct a positive perception on school mathematics and acquire important mathematical competences. Accordingly, the design process of

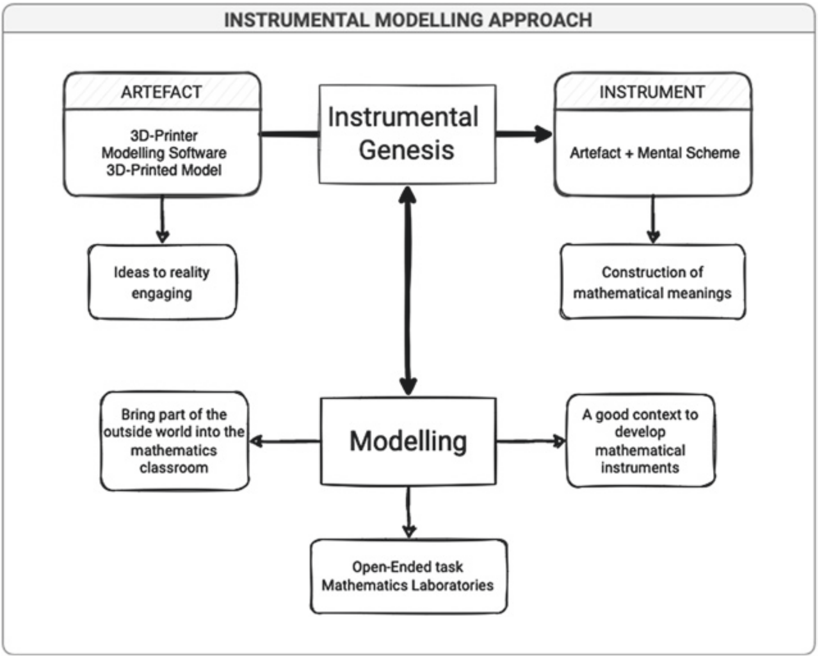


Fig. 1 Theoretical framework overview

architectural models is basically the process of modelling. This is "the process of translating between the real world and mathematics in both directions" (Blum & Borromeo, 2009, p. 45).

We aim to understand processes in which students go through as they get involved in such highly demanding cognitive tasks. We observe these processes through the lenses of the "modelling cycle" presented by Blum and Leiß (2007), see Fig. 2. This helped us in designing the tasks presented to students and analysing their productions as a part of the design cycle allowing us to link the real situation.

The main reason behind our special attention to modelling processes is stated by Blum and Borromeo (2009) "Mathematical models and modelling are everywhere around us, often in connection with powerful technological tools. Preparing students for responsible citizenship and for participation in societal developments requires them to build up modelling competency." (p. 47). Such modelling could assist students to understand the world supporting mathematics

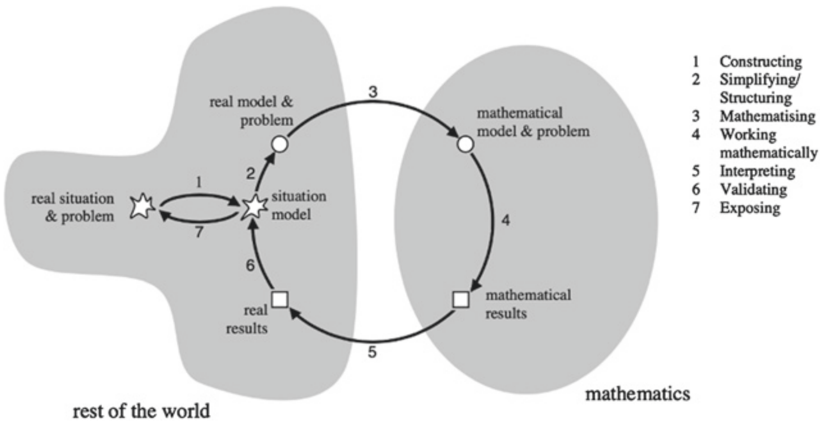


Fig. 2 Modelling cycle. Blum and Leiß (2007)

learning and increase their motivation, concept formation, retaining, and contributing to fostering students' mathematical competencies and attitudes.

The mathematization stages of this modelling process can be included in the *Parametric Design Method* (Alalouch, 2018). This is sometimes perceived as complex even for undergraduate architecture students, but from our point of view GeoGebra brings this kind of method closer to high school students as it facilitates the use through friendly commands and tools.

> The advantages of the parametric design method lie in the flexibility it offers in creating variation and change in design with minimum effort, as well as its ability to create new and complex forms. Another significant aspect of parametric design is that the definition of parameters determines the design goal/goals. To explain, parameters could be geometrical properties, architectural input, environmental factors, structural properties, or any combination of parameters that are of interest and relevance to the project. This allows design exploration based on modification of meaningful architectural features, such as structural, geometrical, environmental, functional, etc. (Alalouch, 2018, p. 164)

Parametric thinking requires more than merely the visual representation and modelling of the buildings. Also requires students to understand the design as a dynamic system of rules and relationships given through mathematical expressions, as seen in the example presented in Fig. 3. In our opinion, this kind of design is close to the mathematics in our precalculus and calculus courses, given a great importance to functional relationships and covariational thinking.

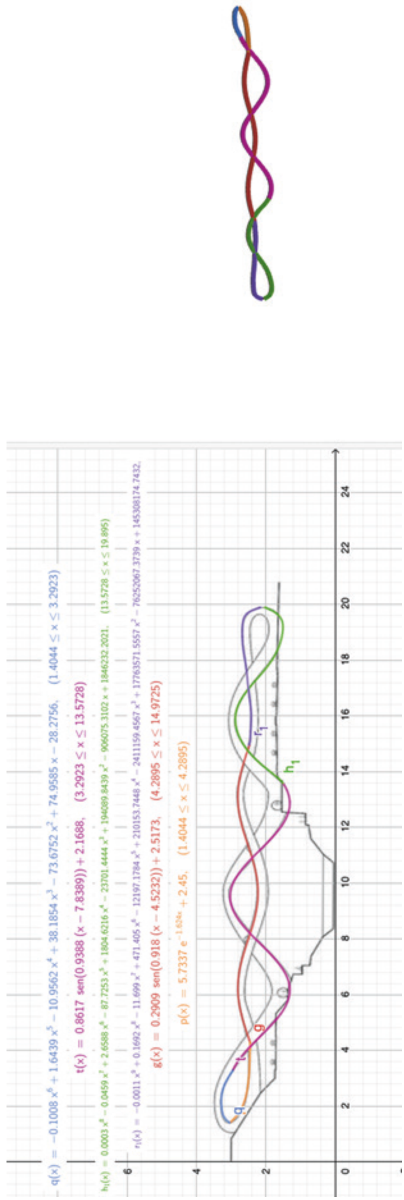


Fig. 3 Parametric design of Lucky Bridge Knot made by a students' team

This view on modelling and the way we aim to present tasks to students can be considered as Open-ended tasks. Accordingly, these tasks or problems allow a wide range of solutions, or multiple pathways to achieve the modelling goal. We want to underline that one of the main characteristics of this kind of assessment is the possibility to adapt difficulties of the problems depending on the skills of students and give tasks that comply with a *low threshold high ceiling* approach (NRICH, 2013). It is important to notice that this can help to differentiate the learning in the classroom or to create assessments for different grades and different classes based on students' mathematical levels and the intentions of teachers. As GeoGebra has built-in tools to create models, we can present a complicated structure and solve it partially using GeoGebra tools, partially with equations and function so we adapt these tasks to the level of mathematics we want to develop in that group. Also, there are some other ideas to include such as the use of integrals to calculate length of curves, integrals for areas or volumes that can be presented to the most advanced students.

3 The Proposed Work Process

Our proposed work process goes through three main stages that allow us to achieve the educational goals of the project. Firstly, teacher's topic introduction; Secondly student-guided work and thirdly reviewing learning outcomes. Every stage is divided into small tasks that can serve as a teacher guide for managing and evaluating the work process. It is important to let the students know in advance about the third part of the process so they can start planning the report and gathering all the elements from the mathematical modelling process. See Fig. 4.

Teacher's topic introduction to be able to carry out this work is highly important. This stage consists of a series of *short tasks* including pictures of buildings or structures with interesting mathematical elements as shown in Fig. 5, and the

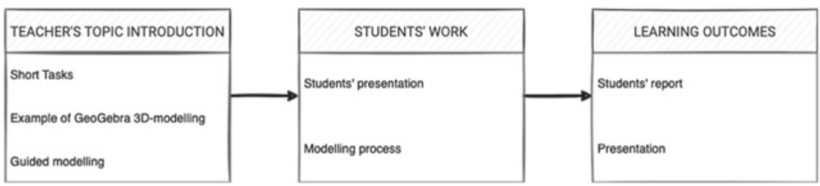


Fig. 4 Work process overview

Fig. 5 Example images for introductory tasks. Buildings designed by Oscar Niemeyer

questions *"What do you notice? What do you think about these pictures?"*. A good selection of pictures can help students to start a discussion about the elements of interest that can be mathematical, cultural, historical, or architectural among other things, and generates a climate of inquiry in the classroom (Rumack & Huinker, 2019). This can be repeated showing a selection of 3D-printed structures to increase motivation and start the connection between physical and digital worlds as this is suitable for starting the class' discussion about the modelling process on the computer. Then, a working example of the GeoGebra modelling process can be shown if the group does not have the technical expertise to do it independently, this has to be done with care because it can direct too much of the next stage of the student's work.

The next stage is the centre of the project's work and to start with students bringing to the classroom the buildings or structures they consider interesting for modelling and 3D-printing. It is important that space is given to students to present their selection and argument about the interest of the structure they intend to model. This is a good moment for the out of the field content to enter the classroom and enrich the project including other disciplines knowledge and skill students may have. Then the modelling process begins, and the work of the teacher should be carefully considered as this work is fundamental for the development of a good modelling situation.

> In this context, often strategic interventions are most adequate, that means interventions which gives hints to students on a meta-level ("Imagine the situation!", "What do you aim at?", "How far have you got?", "What is still missing?", "Does this result fit to the real situation?", etc.) (Blum & Borromeo, 2009, p. 52)

The last moment of our proposed work plan passes at a communication and a meta-cognition level. Students are requested in advance to prepare and present a document with the building information, a guide of the process of modelling, and a reflection about the knowledge acquired in the project.

3.1 A Working Example

To clarify the process presented in the previous section, we give an example to pave the way for other teachers that are willing to share these ideas with students. We aim to give teachers a glimpse of our work process in lesson development, implementation of the classroom work and the reflection this led us to do.

It is easy to talk about a journey when it is over, but the process is not so straightforward. The process of creating these assignments started with this example we are now going to write about. The original task was to create a model for the contour of this building. We were in a class of students aged 18 years old with little calculus background. And exploring opportunities to include real life situations and modelling in our classes to create better learning opportunities for the topics in our mathematics class, such as calculus and analytic geometry. So the task was settled, and students were looking for a model of the building known as the Gherkin. This building is in London at 30 St Mary Axe, see Fig. 6. Just by looking at it we can establish links between Architecture and Mathematics. So, when you see a connection like this an opportunity is presented. This starts Blum and Leiß (2007) modelling cycle by *constructing and simplifying* the real situation, as can be seen in Fig. 7.

Fig. 6 Gherkin in London, United Kingdom

Firstly, we loaded the image in GeoGebra and started looking for a coherent model for the contour of the building, starting this the *mathematising* moment of the modelling cycle. And then, decided to split the contour in two creating a model that fits quite well using a piecewise function, see Fig. 8. Using the mathematics that students already learned and *working mathematically* as described by Blum and Leiß (2007). At that moment ideas about parabolas, ellipses and functions arise as an opportunity to create an "articulated and integrated network of mathematical organisations that allow the development of a broad mathematical activity" (Bosch et al., 2006, p. 49). And allow us to talk about British culture, compare that building with the architecture of our country and state a relationship between mathematics and architecture, taking the cultural dimension of mathematical modelling into account. (Villa-Ochoa & Berrío, 2015).

This created a very good opportunity to use the knowledge students had about functions, transformations, models, equations, and many other subtopics in the field of calculus.

After doing this the class decided to construct a 3-dimensional model of the object to calculate the volume and to have a realistic model that could be used

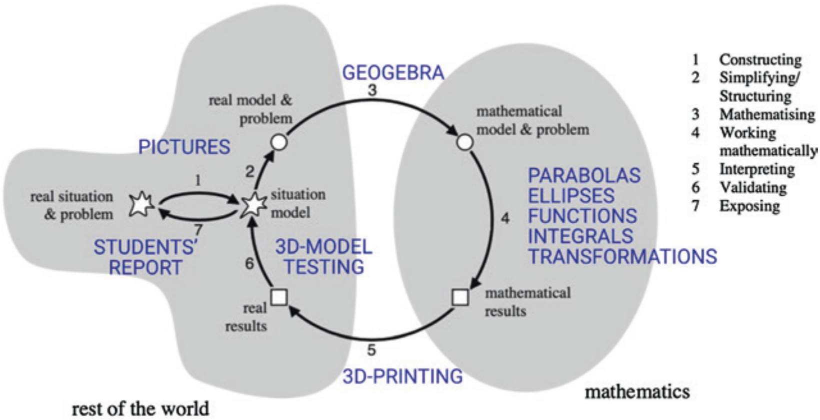


Fig. 7 Modelling cycle in this example

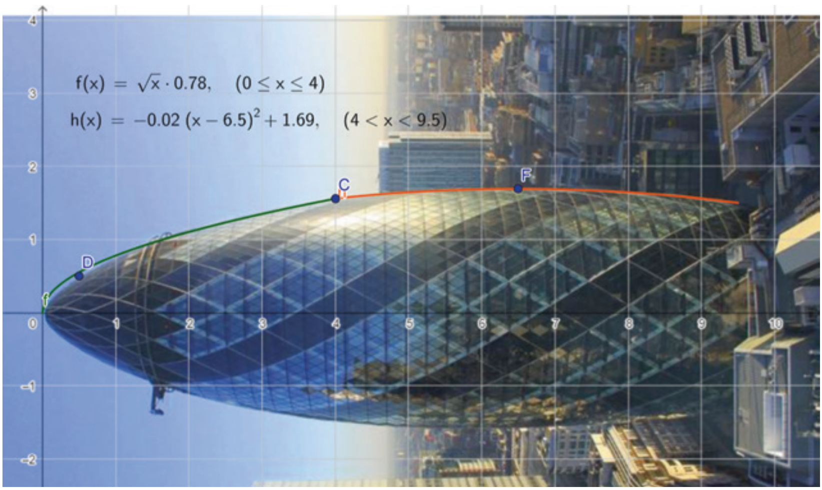


Fig. 8 Created by students in GeoGebra

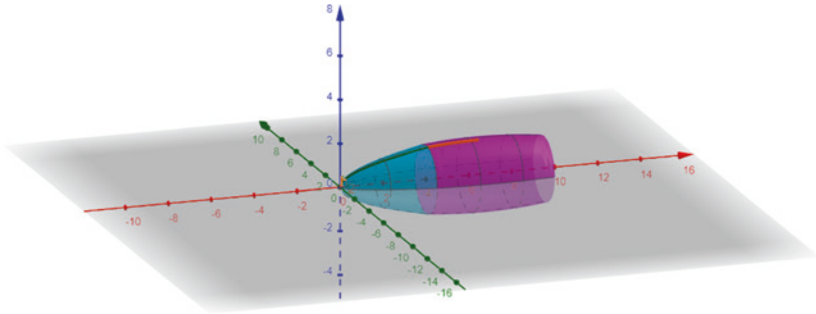


Fig. 9 Created by students in GeoGebra

as a gift or as part of a model of the city. This underlines the importance of the Open-ended tasks and the Inquiry-based learning approach because the student-guided inquiry led the class to a deeper level of mathematics. (Lieban, 2019).

We used the GeoGebra command *Surface*[1] to create a surface of revolution around the x-axis that could serve as a model of the building, see Fig. 9. This is an example of the artefact possibilities being integrated into a tool to achieve the task goal as in the *instrumentalization* process. This is so, because students found that the command given by GeoGebra was able to create what they were imagining, giving this to the artefact a practical and mathematical meaning in response to the task.

Then it was time to calculate the volume of the building, so we used the formula for the volume of a solid of revolution:

$$\pi \int_a^b f(x)^2 dx$$

This was a good opportunity to enrich our algebraic manipulation tools and to involve an integral in a "real life" context.

Finally, we had to find the scale of our model to change that result to units we can use in the real world (metres cubed), so we had a better idea of what the volume of the building was. Also, we can now compare that size with the size of our

[1] https://wiki.geogebra.org/en/Surface_Command

Fig. 10 Photo of the 3D-printed model

classroom to have a better idea of how large that building is, as seen in Fig. 10. At the end we used the 3D-printed model to test our result and find the volume using water. These magnitudes were found to have a better understanding about the size of the building, also to reflect about ratio/proportion. Lastly, this result can be considered one of the results of the task. In this work we found that the volume of the building was: 22,135 m cubed. Using water, we obtained a comparable result so we can say that the task was concluded in a satisfactory way, this helps students understand the notions about ratio and proportion in context and also can be used as a result of the task. We can clearly identify this with the *interpreting and validating* moments in the modelling cycle. (Blum & Leiß, 2007).

At the end of the modelling process the students completed a document with an introduction, explaining what we were going to accomplish and what was the main goal. Also, showed all the steps of this work in a detailed way using the equation editor and citations of the data, pictures, definitions, and information taken from secondary sources were needed to avoid copyright issues providing students the opportunity of reflecting about good practices when using other people's work. This way students also practice mathematical vocabulary, terminology and using the corresponding mathematical notation and symbols. Finally, they obtained some conclusions regarding the use of models, mathematics, and architecture. This was done to give students the opportunity to express themselves and work on their reflections and communication skills, fitting this in the *exposing* moment of Blum and Leiß (2007) modelling cycle and marking the end of the task.

4 Results and Reflexion about the Students' Process

Students' reactions to this kind of task were varied. There were those who pay attention to each little detail of the structure and try to create a piecewise function to be as accurate as possible with the model and there were others that are happy with a shape that looks similar but did not fit really well. Here we observe dedication and understanding of the modelling process. Furthermore, there were students that work in a very autonomous way and advance on their own and others that required more help. This helped us to evaluate the mathematical insight of the students.

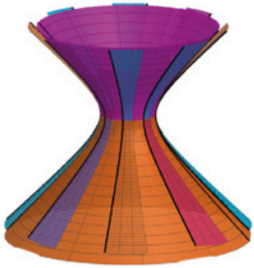
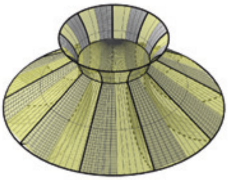
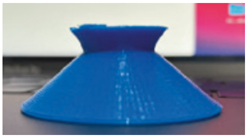
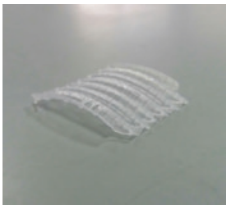
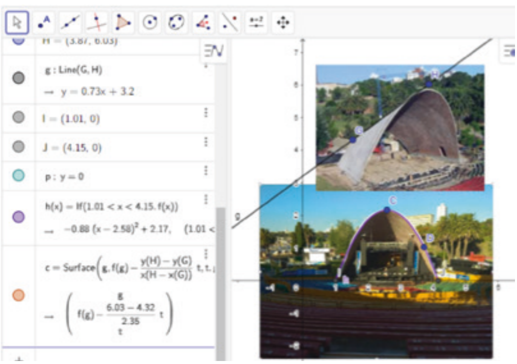
Some students encountered the possibility of printing an object motivating and many found the modelling part more interesting because they could relate what was learned in class with something in the real world. Through observation of the students' work process and spontaneous interviews with some of them it can be attested that this process was more motivating than the normal class. More systematic research is needed to further support this claim.

At the end of the process, students have the task to come up with a report explaining what they did and write down the formulas and equations they use. They are encouraged to include pictures, reflections and conclusions they found. This makes it easier for the teacher to notice students' self-organisation, communication skills, mathematical notation, use of mathematics and critical thinking. This is a very important step of the process because when students write down what they did, they could stop and think about what they have done, thus having the chance to reflect about metacognitive aspects of the task. It was an important element for us in the evaluation process because we were able to see how students accomplished their goals and how well they were communicating their ideas, which was essential for them to develop their understanding.

In Table 1, we will present some other examples and ideas about objects that we had modelled, to incorporate more examples and make the ideas clearer.

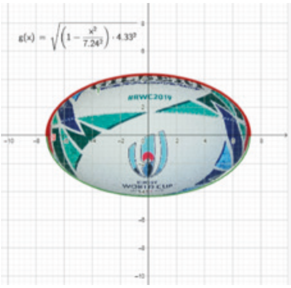
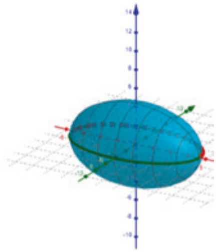
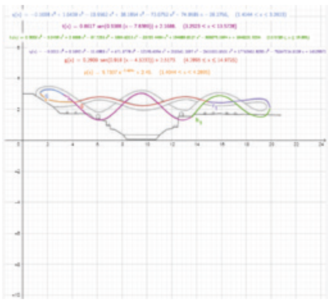
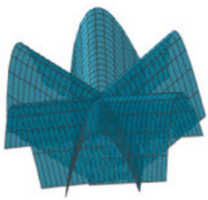
An important aspect that we noticed as a positive reaction to these kinds of tasks is, that they could assist students to bring their own objects and structures creating links to other areas of interest. An example of this can be the model of a rugby ball, in which many students that enjoy this sport feel motivated and willing to complete the tasks because it relates to another interest of theirs, as stated in Duit et al. (2007). Another motivating structure was the Summer Theatre of Montevideo where the "Carnaval" takes place, and many students enjoy.

Table 1 Examples of structures modelled and 3D-printed in our classrooms

		When modelling this structure of the "Catedral de Brasilia" we can enhance the knowledge about: Conics, Conic sections, Hiperbole, Geometric transformations
		
		In this model of "Cristo obrero Church" we revised knowledge about: Transformation on the general sine function, parabolas, two variable functions, equations involving the sine function
		To create this model of the "the Summer Theatre of Montevideo" we revisited knowledge about Parabolas, straight lines, proportion. Simultaneous equations. Also, we worked with integrals

(continued)

Table 1 (continued)

		When creating this model of a rugby ball we worked with ellipsoids, equations, integrals, solids of revolution and the formula for the volume of a solid of revolution
		In this model of the "Lucky bridge knot" we worked with polynomial functions, general sine functions, logarithmic functions, transformations, equations involving polynomials and sine functions. Also, we took advantage of the formula for calculating the length of a curve to calculate the length of the bridge
		To create this model of the "l'Oceanografic" many concepts about parabolas, simultaneous equations, straight lines, integrals, two variable functions and equations were revised. Also, some work was done using integrals to estimate how many square metres of glass was used

5 Discussion

As discussed earlier, the approaches we outlined offered us plenty of opportunities to the inducement of Mathematics in a real-world context and to be able to solve problems with a predetermined objective. This was the main motivation for our students and encouraged teachers as they could observe that students were engaged and using their mathematical knowledge with enthusiasm. Thus, this result was consistent with the ideas about the importance of real world and cultural connections in the modelling process presented in several research. (Blum & Borromeo, 2009; Blum & Leiß, 2007; Borba & Villarreal, 2005; Haas et al., 2020; Villa-Ochoa & Berrío, 2015).

Currently, we are looking for ways to implement such projects at a larger scale, creating modelling challenges and expositions that could serve to involve more teachers and more students. We believe this way of working is important for the development of specific mathematical skills and provide meaning to the mathematics class and develop soft skills in the students like communicational skills and teamwork. Also, by the development of STEAM projects based on these ideas, we could generate engaging tasks for both teachers and students of different courses which would be beneficial for all. This element is of great importance in the particular context of the experience because as we review, the teachers community in Uruguay is engaging in the use of technology but not yet reflecting systematically on the implications for mathematical learning. (Testa & Téllez, 2019; Vaillant et al., 2021; Vitabar, 2011). In this direction, the proposed working process for classroom implementation required teachers to work systematically in their project development. Our work still requires further research and evaluation to test our claims about the improvement of engagement and learning, but as we observed this approach offers confidence to teachers with basic technological skills to integrate technology in a meaningful and interesting way. To continue our research, a second cycle is going to be carried out with different teachers and questionnaires are being designed to test interest and self-assessed competence in mathematics/mathematics' teaching for both students and teachers.

It is also important to take into consideration the adaptability of the tasks. Assignments can be changed to include other cultural perspectives and different levels of algebraical skills and insights. This is a key element, because we can attend to the different skills/levels of students in the classroom in a way that is not too demanding for teachers. We could use different types of functions and equations to model the contour of a structure, or we can simply use the GeoGebra tools to achieve the model, depending on the skills/level of our students.

The use of GeoGebra as a modelling tool had been proved as very positive because of the explicit need for mathematics, no construction can be made without using at least some mathematical tools. In the case of our modelling task various mathematical constructs arise to transition GeoGebra from artefact to an instrument for learning (Artigue, 2002; Maschietto & Trouche, 2010). And the interconnected and dynamic nature of the GeoGebra representations fostered the understanding of the mathematical notions set in use for the modelling as reported by Tejera (2021) and Villa-Ochoa and Ruiz (2010).

The evaluation of feedback given by classroom observation and student's interviews indicated that the skills students gained in this type of assessment were enhanced because of this kind of work that generates high cognitive demand aligned with the results from Blum and Borromeo (2009). This means that many times in future classes we can make them remember that certain part of the syllabus was used in the project and they can bring it to class in a more vivid way, helping them remember and helping us construct new knowledge over the past one. This needs better testing and a more systematic investigation, but we are inclined to think that the skills they acquire in this type of project are related to many other aspects of their lives thus making it easier to recall it. These elements are central in a PhD study being carried out at JKU (Linz).

Another aspect that we need to test in a more systematic way, but we are inclined to believe, is that students learn from different perspectives, including cultural and artistic aspects that can help the knowledge be understood in a better way due to connections students created with their knowledge. Knowledge reaches beyond mathematics and offers enrichment for students. It is related to several aspects of the world around them, so it is easier for them to use similar knowledge in different contexts. According to the works of Borba and Villarreal (2005), El Bedewy et al. (2022), Lieban (2019) and Villa-Ochoa and Berrío (2015) this cultural connection is of great importance to the modelling process.

To sum up, we found numerous aspects of our work needed to be further investigated, but we believe that this kind of assignment could offer valuable directions for STEAM-based teaching approaches. And we hope this work can be a contribution for teachers willing to try new approaches in the inclusion of technology and serve to make the mathematics we teach more meaningful and engaging for students.

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Interfaces in Learning Mathematics—Challenging and Encouraging Visualizations Switching from 3D to 2D and 2D to 3D

Felicitas Pielsticker and Gero Stoffels

1 Introduction

Digital tools extend the range of possibilities of teaching and learning mathematics in educational contexts. Especially in geometry, the use of digital tools for teaching mathematics seems to be very popular (Sträßer, 2002; Abrahamson & Abdu, 2021). One reason for this can be found in the visualization and illustration of mathematical concepts in the classroom. Especially in elementary school, but also in lower secondary school, they play a fundamental role in concept formation processes in the field of geometry (Weigand, 2015).

Also, contemporary works deal with this topic. For example, at the recent GDM2020 conference, a keynote lecture was given by Prof. K. Jones, offering connections to "geometry and the imagination" (Jones, 2020). This talk explored some challenges in teaching and learning geometry and presented some opportunities for geometry in the classroom.

The goal of this paper is to consider interfaces in the transition from 3D to 2D in the area of prisms in the specific case of teaching mathematics to students in an 8th grade class, and to describe inferences that can be drawn for teacher education.

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Accordingly, it seems reasonable to allow prospective teachers—as in the case of the authors in a course on the elements of geometry—to have such experiences and to reflect on them, as shown in a short motivational anecdote. Of course, it can be assumed that pre-service teachers have made exactly such concept formation processes based on illustrations and their own intuition during their own school years (Stoffels, 2020). However, on the one hand, the pre-service teachers' and teachers' conceptions were formed long ago and on the other hand, they have to be remembered before they can be reflected. Especially from the theoretical standpoint of a developmental psychological approach such as theory theory (Gopnik & Meltzoff, 1997) it seems to make sense to give students the opportunity to train their visualization (again) in their first phase of education at the university. In the sense of theory theory, it can be assumed that basic learning and development processes of prospective teachers and students take place in a similar way (Gopnik & Meltzoff, 1997)—for example, in dealing with illustrations or in dealing with illustrative representations. In terms of mathematics courses, this means that prospective teachers should be given the opportunity to build and reflect on their knowledge experimentally by engaging with their environment themselves (Hoffart, 2015).

This idea is not new. Already Felix Klein, with special emphasis on a balanced relationship between logical development of mathematics and visualization in the field of mathematics (Klein, 1898), stated:

> that at least two categories of mathematical lectures should necessarily take their starting point from this view. These are first of all the elementary lectures, which introduce the beginners to higher mathematics in general […] furthermore those lectures, whose audience is dependent from the beginning on working very essentially with the view, i.e. the lectures for natural scientists and engineers (Klein, 1898, p. 127, translated by the authors).

From our point of view the same should apply to lectures given by (mathematics) teachers at universities, since the teachers are also supposed to stimulate students' conceptualization of mathematical content. A hint why this might be fruitful is given in our motivational part (cf. 2).

The paper is divided into five sections. First, there is an outline and motivation of the problem. This is followed by the theoretical perspective of the paper in Sect. 3. Here, we specifically address the concepts of visualization (Arcavi, 2003) and inscription (Roth & Mc Ginn, 1998). Subsequently, we present our research focus, before we discuss the design of the study in Sect. 5. Methodological steps are established, as well as the context of the study is described. Finally, Sect. 6 presents the results.

2 The Issue

Although the following analysis (parts 4–6) takes an in-depth look at an interaction in lower secondary, the following reflection assignment shows, just like the anecdotal sequence from university teaching, that questions about visualization between 2D and 3D have not been clarified during a first level as in van Hiele's model, but offer rich opportunities for reflection from elementary school to university. The issue under consideration here is pictures of 3D bodies on the paper plane. To give you, the readers the opportunity to reflect we describe two exemplary situation from teacher education and an anecdote, which shows difficulties of the interpretation of 2D representations even for university students.

The first situation for an initial perspective on the issue can be given, when we asked students which of the bodies in Fig. 1 was filled. We invite you as the reader to make your own decisions on this question, before you shall read the results of the students in Fig. 2.

It was interesting to note that most prospective teachers would consider geometric bodies h) and k) as "filled", rather than the other bodies. There are some differences in German and English language regarding "filled" and "hollow" bodies. What we call in Germany "Körper" (translated previously to "body"), is in

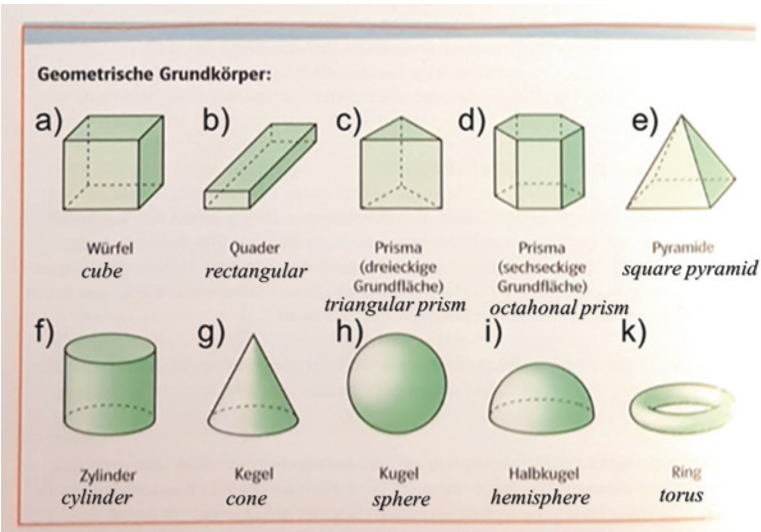


Fig. 1 Geometric bodies (Hußmann et al., 2005, p.146)

Fig. 2 Videos of “Cube A” (https://www.youtube.com/watch?v=3QXY6uuLp5Q) and “Cube B” (https://www.youtube.com/watch?v=rnKFweImlh0)

English “solid figure”. So, it seems reasonable to think a solid figure refers to the surface as well as the inner points of the “solid figure”. Nevertheless, it is often ambiguous if the inner points are meant referring to “cubes”, “parallelepipeds”, “prisms”, “pyramids”, “cylinders”, “cones”, “spheres” or “torus”. Also pictures of 3D solids in common teaching and learning materials (cf. Fig. 1), which are often referred to as 2D representations, do not clarify this issue on their own (Hallowell et al., 2015).

To this impression, we made a small research project with 40 prospective teachers. They were shown the video of two cubes (cf. Fig. 2). After that, the participants had to decide which video was a correct representation of a cube and what were the reasons for their judgment.

Before we discuss the results of this small study. The reader is again invited to reflect his or her own thoughts following the QR codes, or the links, and to think independently about which of the two representations shown, he or she would consider as correct.

Most participants voted for “both” (in red) or for “cube A” (in orange) (Fig. 3).

The individual reasons given by the participants were also interesting. We would like to highlight three justifications.

- *Participant 1: Object B does not appear to be a cube because three sides are dashed and thus should be hidden. However, they are still dashed even when shown [in the front, added by authors].*

Fig. 3 Query on “Which representation is correct” (cf. Fig. 2)

- *Participant 2: Both videos show the same thing: a cube rotating over two sides. A "real" cube does this over all 6 sides, so I would rate the simulations as equally good, but still in need of improvement to properly represent a cube rolling over all sides.*
- *Participant 3: Video A shows that the invisible surface is always changing and is therefore a realistic representation. But video B also shows the aspect in which position the surfaces are that we see from the initial position. Both videos show a different perspective on a cube and none of the videos is better because of this.*

Participant 1 focuses on the "dashed" lines of the cube B in his response. Participant 2 seems to think of a dice after watching the two videos (Fig. 2) and is bothered by the fact that the cube does not rotate over all 6 sides, so he is focusing on the motion. Participant 3 is intrigued with regard to the justification of a realism. The reasoning of participant 3 refers to the change of the invisible surface.

Both investigations were a result of an experience of the second author in his "Elements of Geometry", a lecture for teachers of elementary and lower secondary school. The chosen situation (or case) was crucial for the two authors to (re-) think about "challenges and chances of the transition from 3D to 2D and 2D to 3D".

The lecture held in winter semester 2019/20 was also designed to serve a contemporary teacher training for a competent use of digital media such as the 3D printing technology. In particular, spatial reasoning in the transition from 2D to 3D was taken into account. The content of the lecture changes in comparison to established introductions to the "Elements of Geometry" (Helmerich & Lengnink, 2016; Krauter & Bescherer, 2013) in that the creation of the three-dimensional object, e.g., a house-shaped prism (cf. Fig. 4), is addressed in the 3D CAD software and, through the presentation of the manufacturing process, which gives a natural example for a dynamic view of geometrical objects, by extending the height of the prism.

In addition to explicit addressing the 3D printing technology in the course, the prospective teachers are given the opportunity to use the Tinkercad tool to create a three-dimensional body from a given multiview projection (2D to 3D). In the lecture, the classical way of (3D to 2D) was chosen when introducing the multiview projection (cf. Fig. 5). New negotiation processes of the prospective teachers through suitable empirical objects of reference were made possible by these (new) tools. These different possibilities for discussion should give students the opportunity to experience projections and their connection with three-dimensional bodies more intensively. Additionally, the 3D geometry was used as the beginning of this elementary geometry course.

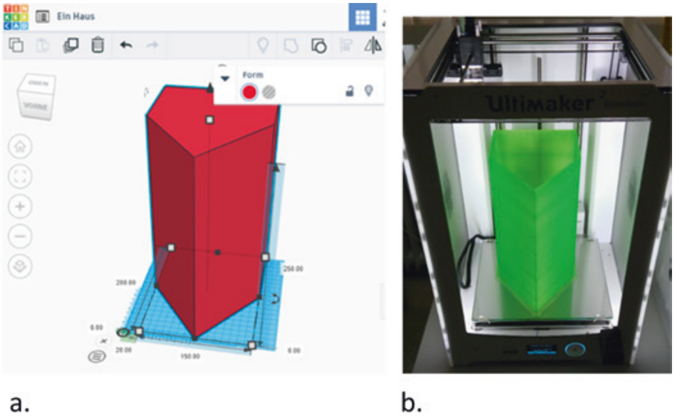


Fig. 4 (a, b) House-shaped prism in Tinkercad (https://www.tinkercad.com/) and in the 3D printer

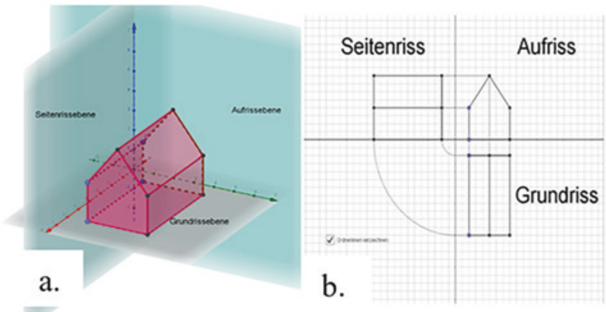


Fig. 5 (a, b) House-shaped prism in a 3D coordinate system and in the multiview projection (top, front and side view)

The following discussion which gives insights into the student's dealing with the transition from 2D to 3D was after the introduction of the partial ratio fidelity in the lecture. For this purpose, a military projection of the house (Fig. 6) was given and compared with the 3D-printed house (Fig. 4b).

Resulting formula: $\frac{|AT_1|}{|AB_1|} = \frac{|A'T_1'|}{|A'B_1'|}$

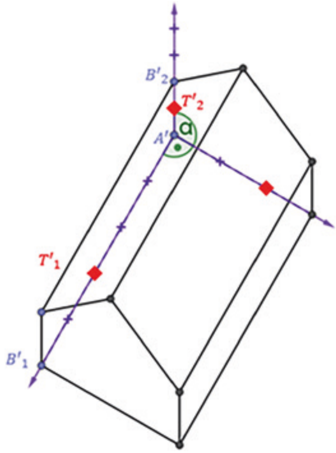


Fig. 6 Military projection of the house and the given formula for the partial ratio fidelity

On the blackboard it was illustrated that this special projection fulfills the following property: existing relations on straight lines of space are transferred into the image. In particular, the image of a bisector point of a line also divides the projection of the line into two parts of equal length. The distances of the original points A, B_1, B_2, T_1 and T_2 were measured on the 3D-printed house and their projected points in the image.

After the lecture, a pre-service teacher (short: pT) claims after the lecture that the original points have to be drawn into the picture (cf. 1 and 3, Table 1).

Often in parallel projections, the names of the original points of the solids are used for indicating points in their 3D representation. The difficulty of such representations is then, of course, to recognize that these are projections and should be "seen spatially". It seems as if this cognitive conflict is grounded in a belief about such representations and can only be reflected through different phenomenological experiences, which is indicated by the further sequence. The lecturer takes note of the student's difficulties and explains that the original points are located on the concrete object, in this case on the 3D-printed house (4, Table 1). He confirms this by the corresponding pointing gestures on the concrete object where he locates the original points on the 3D-printed house (4, Table 1). Based on the teacher's assessment that the student did not understand the location he presented (5, Table 1), he hands him the house over (cf. Fig. 3 b.) in order to enactively show the connection by pointing projection rays between the original points and the projected points (6, Table 1). In addition, he justifies the impossibility of the original image points in the drawing by the fact that a body, implicitly as a

Table 1 Memory protocol (Vogel & Funck, 2018) presented to and communicatively validated with the pre-service teacher

1	pT	Why did you not transfer the points of the left side of the equation into your image?
2	I	Because the original points cannot appear in the picture of the projection. They were mapped by projection rays from the space into the projection plane
3	pT	But they still have to be drawn into the picture
4	I	No, the original image points are located on the 3D house… (Takes 3D-printed house in hand) here, here and here
5	pT	(Looks confused on the house)
6	I	(Hands over the 3D house into the hands of the student and shows connecting lines from the original points on the 3D house to projection points in the projection on the board) Look, the original points are located on the house, this one is outside the projection plane because it is a solid
7	pT	(Looks several times from the 3D house to the image projected by the beamer) … Ahhh!

three-dimensional object, cannot be part of a plane. The conclusion of the interaction to clarify the question of the location of the original points is the interjection of the student, who is preceded by looking between the original points on the concrete 3D-printed object and the projected points in the image (7, Table 1).

This first outline of the problem may give some hints on further improvements for teacher training, which lays in the necessity of a deeper diagnosis of pre-service teachers' competencies in dealing with 2D representations of solids and of opportunities to experience this interface of 2D representations from the 3D solid in the mathematics classroom.

At this point the above presentation of the issue from a higher education standpoint, will be the starting point for considering the interfaces of visualization in the transition from 3D to 2D in the field of prisms in an 8th grade mathematics class. Before we start our intrinsic case study (cf. Sections 4 and 5), we will first discuss our theoretical perspective.

3 Theoretical Perspective

"Representation" is an important and often studied topic in mathematics education research. In particular, in geometric areas of mathematics, representations and related issues are frequently addressed. The changes from 3D to 2D and 2D to 3D are often associated with challenges in geometry lessons at school, but also give the opportunity discussing different modes for experiencing geometrical objects.

Presmeg (2014) states:

> there is much more to learn about the power and pitfalls of using the haptic system in teaching mathematics. How are touch and sight related in various relevant contexts? Despite the growing literature on gestures and embodied cognition, conceptual frameworks for research that links visual and haptic modalities are still in need of development (Presmeg, 2014, p. 156).

A connection of seeing and touching in the same context is especially possible using 3D printing technology (Witzke & Heitzer, 2019). For example, studies by Ng et al. (2018) show the benefits of 3D printing pen design outside of the drawing page—expressing a connection from 2D to 3D (Ng & Sinclair, 2018).

These different modes of experience can be described by the concept of domains of subjective experience (DSE), which at the same time offer an explanatory model for incongruences in the discourse of different discourse partners based on an interactionist constructivistic account. Bauersfeld (1983, IX) describes the "Domains of Subjective Experiences (DSE) as the basic issue for an interactive theory of mathematics learning and teaching", which he uses to model the particularity of experiences, their storage in disparate domains of knowledge and their non-hierarchical but competitive activation. Different DSE can offer different perspectives on mathematical objects. If intersubjective incongruence occurs in the interaction, according to this model, one cause is that different DSE of the interacting subjects are activated. This different activation can also occur when presumably identical objects are referred to in the interaction. Especially, the construction of super-ordinated DSE, which make it possible to connect, coordinate, draw a distinction and reflect different perspectives and functions of subordinated DSE makes the model a learning and teaching theory (Fetzer & Tiedemann, 2017; Stoffels, 2020). Thus, from a researchers' perspective, the identification of different perspectives may indicate different DSE. Two different perspectives on representations are given by the concepts of "visualization" and "inscription", which we apply in the following case study to reconstruct different DSE in students' interaction. To understand different perspectives on representation, we discuss the concepts "visualization" and "inscription", since the first concept also includes mental representations while the latter one focus on the materialized representations used in interaction.

As early as 1989, Bishop called for an increase in inquiry regarding questions about visualization in the field of mathematics (education) research. Bishop states in this regard:

> The aspect of visualization in Mathematics education has not attracted much research attention in the recent past. Nevertheless, it is felt by many Mathematics educators to be important in the education process (Bishop, 1989, p. 7).

Building on Bishop's claim, we can note that many studies have already addressed questions about and on visualization in a variety of ways (Presmeg, 1986; Hanna & Sidoli, 2007; Sinclair et al., 2016). Thereby.

> Mathematics education research [is…] far from consensus on the roles visualization can play in the teaching and learning of mathematics (Nardi, 2014, p. 193).

In this article we conceptually follow for the concept of visualization the definition of Arcavi (2003), who understands visualization.

> […] as both the product and the process of creation, interpretation and reflection upon pictures and images, (Arcavi, 2003, p. 215).

A visualization is thereby bound as a product to the context of its origin—a visualization of something (we expand on that in relation to our chosen case study in Sect. 5.).

With Roth and Mc Ginn (1998) we want to distinguish the concept of inscription from the concept of visualization. Since a description using the concept of inscriptions provides a (further) new perspective on questions of illustration in mathematical teaching–learning processes (Roth & Mc Ginn, 1998). Inscriptions are signs and/or symbols and therefore social objects of discourse.

> Inscriptions are signs that are materially embodied in some medium, such as paper or computer monitors. [...] Because of their material embodiment, inscriptions (in contrast to mental representations) are publicly and directly available, so that they are primarily social objects. Knowledgeability with respect to inscriptions is indicated by the degree to which individuals participate in purposive, authentic, inscription-related activities (Roth & Mc Ginn, 1998, p. 37).

For sharpen the distinction of both concepts we reconsider the military projection of Fig. 5. Understanding Fig. 5 from a "visualization" perspective it has similarities regarding shapes to the 3D Object, e.g. the the ground plan of the house is true to size according the original house and there are to pentagons who can be also found to the original house. During the discussion between the lecturer and the student the military projection works as an inscription, this is when both refer to the picture in their dialog, readdressing properties of mappings.

So, we can say every inscription can be a visualization, but not every visualization can be an inscription. Since an inscription needs to be materialized. Visualizations can be mental representations. For a discourse during learning processes, visualizations need to be addressable in a discourse, therefore they need to materialize and become inscriptions.

4 Research Question

In Sect. 6, a teaching–learning interview of two students of an 8th grade of a secondary school in North Rhine-Westphalia is presented (Pielsticker, 2020). The topic of the mathematics lesson under consideration is on the concept building of prisms. For our intrinsic case study, we focus on the discourse of the two students Paul and Max (names changed) and an interviewer. This intrinsic case study informs against a mathematics education background on the formation of (mental) representations in mathematics, more precise the following overarching question:

To what extent can emergent domains of subjective experience (short: DSE) of two students in the negotiation process on the interface of 2D and 3D in the context of prisms be reconstructed by an intrinsic case study using different perspectives on representation, which are visualization and inscription?

In the following section we will explain the design of the study and the methodological choices for description and analysis.

5 Design

5.1 Context and Situation

In our intrinsic case study according to Stake (1995) we address a learning situation of two students Paul and Max (13 years, names changed). The case is intrinsically motivated because Paul is dealing with different representations in a non-standard way, which shows on the one hand that he is able to change perspectives, which on the other hand hinders the interaction with his classmate. So, this case study informs us how students deal with representations on the interface between 2D and 3D and provides clues regarding the formation of mathematical understanding in an 8th grade mathematics classroom. Especially intrinsic case studies hold Stake's statement.

> Case studies are undertaken to make the case understandable, (Stake, 1995, p. 85).

The case itself and the explanation of the case are of primary interest for our research (Stake, 1995).

The data were collected in the mathematics classroom of an 8th grade in NRW, Germany, in 2019. The learning situation was part of the series of lessons on geometric solids in geometry, in particular on the definition of prisms. The negotiation situation on the definition of prisms, of the two students Max and Paul, was videographed and subsequently transcribed according to the rules of Meyer (2010).

In the series of lessons, the students have already learned a definition of prisms and should now also be able to calculate the volume and the surface area, for example (Fig. 7 and 8).

To motivate them in the series of lessons, the 8th grade students were first asked to design prisms (3-, 4-, 5- and 6-gonal prisms) independently using CAD software (Tinkercad) in math class and then print them out using the 3D printer.

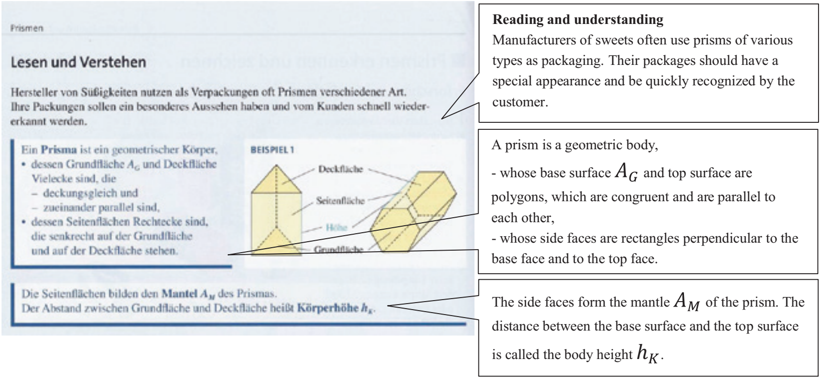
Max and Paul's group constructed a 5-gonal prism as shown in Fig. 9 and 10.

After printing their object—the 5-gonal prism—Paul and Max want to start into the work assignment (Fig. 9):

To create their explanatory video about the five-sided prism, Max and Paul discuss the definition of a prism again and include the 3D-printed object.

5.2 Methodological Decisions and Analysis

For the analysis of our case study, we use the concepts of visualization according to Arcavi (2003) and inscription according to Roth and Mc Ginn (1998) on description level 1 (see Sect. 3). If we can reconstruct both they are a hint of different DSE or a super-ordinated DSE, which contains both perspectives. On a second level of description, the negotiation situation of the two students Max and Paul is conducted using Mayring's (2015) procedure of explication in terms of a broad context analysis. By doing so, we aim to define the emergent DSE according to our research question of our case study. Mayring formulates a six-step process model for this method:


Prismen

Lesen und Verstehen

Hersteller von Süßigkeiten nutzen als Verpackungen oft Prismen verschiedener Art. Ihre Packungen sollen ein besonderes Aussehen haben und vom Kunden schnell wiedererkannt werden.

Ein **Prisma** ist ein geometrischer Körper,
- dessen Grundfläche A_G und Deckfläche Vielecke sind, die
  - deckungsgleich und
  - zueinander parallel sind,
- dessen Seitenflächen Rechtecke sind, die senkrecht auf der Grundfläche und auf der Deckfläche stehen.

Die Seitenflächen bilden den **Mantel** A_M des Prismas.
Der Abstand zwischen Grundfläche und Deckfläche heißt **Körperhöhe** h_K.

Reading and understanding
Manufacturers of sweets often use prisms of various types as packaging. Their packages should have a special appearance and be quickly recognized by the customer.

A prism is a geometric body,
- whose base surface A_G and top surface are polygons, which are congruent and are parallel to each other,
- whose side faces are rectangles perpendicular to the base face and to the top face.

The side faces form the mantle A_M of the prism. The distance between the base surface and the top surface is called the body height h_K.

Fig. 7 Definition of prisms in the students' textbook (Wennekers, 2015, p. 156)

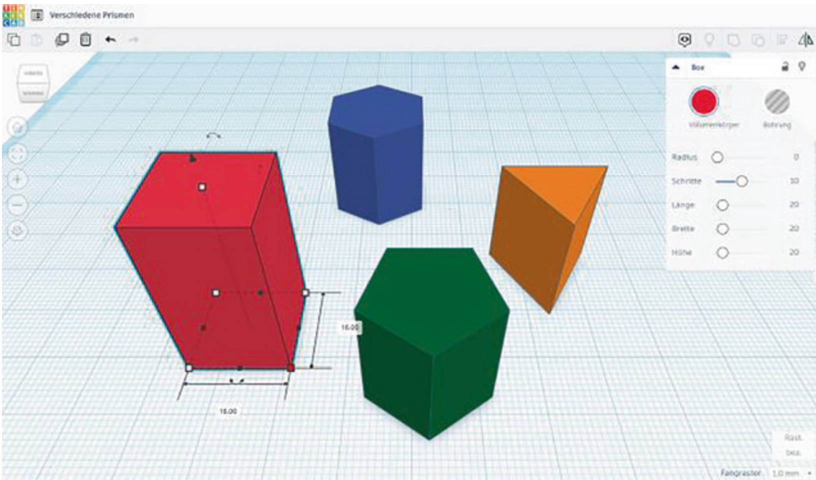


Fig. 8 Construction of various prisms in the CAD program Tinkercad

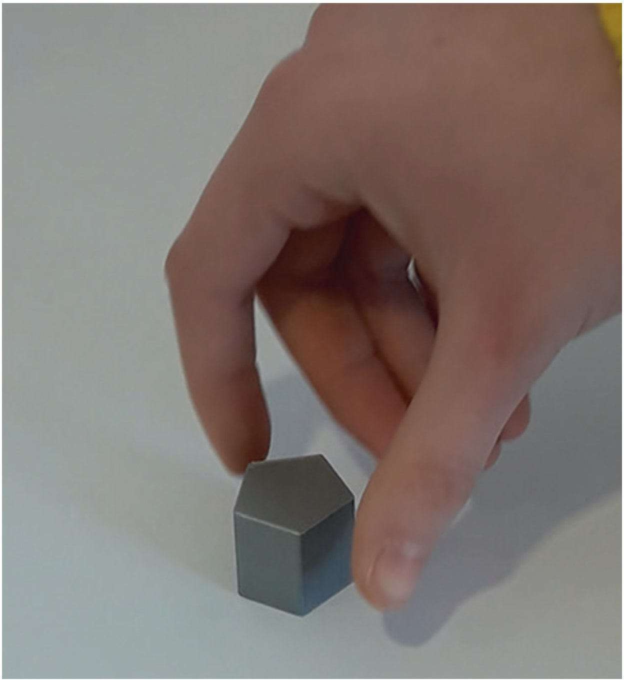
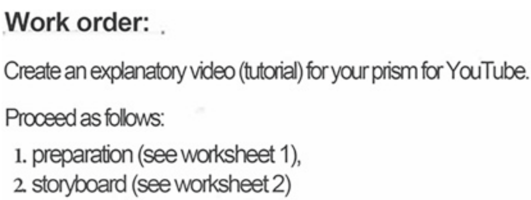


Fig. 9 5-gonal prism of Paul and Max

Fig. 10 Work order for the class



Work order:

Create an explanatory video (tutorial) for your prism for YouTube.

Proceed as follows:

1. preparation (see worksheet 1),
2. storyboard (see worksheet 2)

1. *Step: Determination of the text passage to be explicated.*
2. *Step: Lexical-grammatical definition of the text passage*
3. *Step: Determination of the permissible explicative material*
4. *Step: Collection of material (in case of a broad context analysis this is additional material, which also goes beyond the text)*
5. *Step: Formulation of the explicative paraphrases.*
6. *Step: Checking whether the explication is sufficient.*

In the following we look at the context of the negotiation of the two students in the learning situation in order to connect this with the conception of visualization and inscription.

6 Analysis

Our research question leading our analysis is to what extent can emergent DSE of two students in the negotiation process on the interface of 2D and 3D in the context of prisms be reconstructed by an intrinsic case study using different perspectives on representation, which are visualization and inscription. Our analysis of the data shows, that different DSE of the participants can be identified. Finding a case where different DSE can be identified easily is usually not simple. In fact, a distinction between DSE is only apparent when there is a discontinuity in behavior or interaction, or when the subject explicates a connection between DSE (Stoffels, 2020). For a comprehensive understanding of prisms, it can be assumed that super-ordinated DSE that enable different perspectives on representations are activated and thus give a comprehensive picture of (mental) representations.

6.1 Description Level 1—Visualization and Inscription

The following Fig. 11 is taken from the data collection situation with the two students Paul and Max. In connection with the transcript, we can describe the following inscriptions and visualizations.

We want to describe Paul's explanation "On the sheet the shape is shown on it, but in two dimensions—not three dimensions, so one cannot assume that there is a prism on the sheet." (Table 2, 0:01) to describe his assumption we use the term "visualization" of the prism on the sheet. Likewise, his explanation in 0:12 (Table 2) where Paul states, "There it is not a prism either. There it is only represented that it should look like a prism. But it isn't.". In the screen, a prism is "only represented" (0:12, Table 2). With Inscription, we describe Paul's reference to a materialized object with tactile properties, "There's that. You can hold that in your hand and turn it and feel it and stuff." (0:36, Table 2). Paul refers to the 3D-printed object on the table as the prism only (cf. Fig. 8). For him, it is important that this prism can be held in the hand and felt. In fact, the prism shown in the screen, the prism shown on the drawing sheet, and the 3D-printed prism can be described as three different mathematical entities.

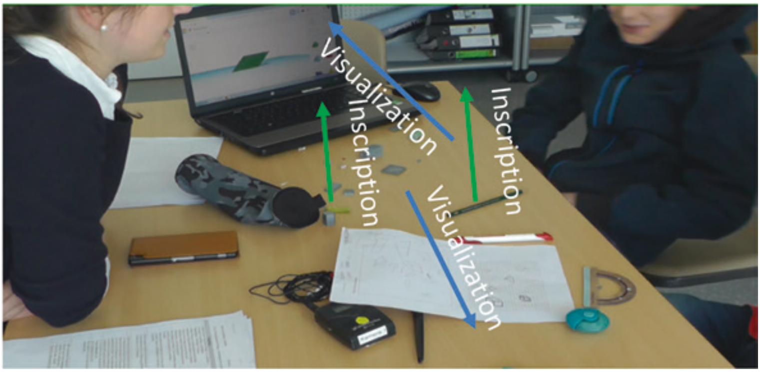


Fig. 11 Survey situation Paul and Max on prisms

As an inscription, we also describe Max's explanation, "To me, I mean, it's because even if you only see it on a two-dimensional plane, it still has the three-dimensional aspects" (1:12, Table 2) since it is again a materialized reference, which is an "inscription" in the sense of Roth and Mc Ginn (1998).

6.2 Description Level 2—Explicative Content Analysis

The basis for the following analysis of the negotiation process of the students is also the transcript excerpt (Table 2).

6.2.1 Analysis of the Teaching Situation

The following sections correspond to the steps of the process of an explicative content analysis proposed by Mayring (2015). Mayring formulates E1-E6 rules in addition to the steps explicitation rules (Mayring, 2015, p. 94). These also guided the analysis but are not explicitly mentioned here.

6.2.2 Determination of the Evaluation Unit

The evaluation unit to be explicated with the help of this narrow contextual analysis is the statement "Why does one see that in a two-dimensional thing?" (Table 2, 1:20) with a special focus on the "two-dimensional thing".

Table 2 Negotiation process between Max and Paul—research data for analysis

0:01	Paul	On the sheet the shape is shown on it, but in two dimensions—not three dimensions, so one cannot assume that there is a prism on the sheet
0:10	I	Ok. In the screen?
0:12	P	… There it is not a prism either. There it is only represented that it should look like a prism. But it isn't
0:18	I	Okay, what is one?
0:20	P	That's 120 frames per second
0:24	I	Okay, fine. How's that one there (points to the 3D-printed objects)
0:27	P	(picks up one of the 3D printed objects and looks at it up close) There it is, a little more frames per sec
0:32	Max	(laughs) a little more frames per sec
0:34	I	Why, how is that there (points to the 3D printed objects)
0:36	P	There's that. You can hold that in your hand and turn it and feel it and stuff
0:40	I	So, is that a prism for you? (Points to the 3D-printed objects)
0:42	P	Yes
0:42	I	Why?
0:43	M	Yes, because it's three-dimensional
0:45	P	Why, why? (4 s.) Yes, because it's three-dimensional
0:53	I	And what is your [to M.] opinion now?
1:12	M	To me, I mean, it's because even if you only see it on a two-dimensional plane, it still has the three-dimensional aspects
1:20	P	Why does one see it in a two-dimensional thing?

6.2.3 Lexical-grammatical Definition of the Text Passage

Some components are lexically grammatically well comprehensible in the text passage. Namely, the question structure "Why does one see that in a …?". The subject of the sentence "one" is an unspecified, possibly even general, observer. The object "that", which one can see, if necessary, is, as will be shown later in the interaction, a reference to the object, which is negotiated in the interaction, a "prism." So, Paul, who made the statement, wants to have a justification why one sees something in a certain local object. This local object is a "two-dimensional thing". This two-dimensional thing cannot be explained directly from this statement. It will be seen, however, that a close contextual analysis based on the text is sufficient to clarify the reference.

For a first analysis, a dictionary can be consulted. It gives a rather succinct meaning: "laid out or rendered in two dimensions, planar [appearing]." The corresponding example in the dictionaries online entry is "a two-dimensional image". Interestingly, the meaning as well as the epithet includes both aspects that become relevant later in the interaction, the conception of a two-dimensional object as well as a two-dimensional image of an object that is not necessarily two-dimensional.

Thus, by this lexical-grammatical definition we do not come closer to Paul's intended meaning, respectively we cannot narrow it down.

6.2.4 Determination of Acceptable Explicative Material and Collection of Material (Close Contextual Analysis)

When determining the permissible explicative material, according to Mayring, the first place to look is in the narrower context, i.e., in the text environment. This corresponds to the entire section from 0:01–1:20 (Table 2). Here it will quickly become clear that a reference to the reproduced verbal interaction is sufficient. It is even possible to allow an even narrower context for the explication. In this explication, the turns in the time 0:18 to 0:32 (Table 2) are omitted, since they provide interesting insights into the student theory of Paul's visual perception but are not necessary for the clarification of the explicative statement. Despite this omission, this explication also follows the temporal sequence of the scene. In particular one notices that without the omitted part, it was quasi an individual interview.

The starting point of the interaction is the student's observation that the shape of the prism is shown in two-dimensional form but not three-dimensional on the sheet. Paul's conclusion is that it therefore cannot be a prism (0:01, Table 2). The support of his statement is remarkably conceptual in nature with respect to two- and three-dimensionality.

He consistently applies this reasoning in the same DSE that already expresses the same differentiation between paper and screen on one side and the 3D-printed prism on the other side (0:10, Table 2). This can be seen in the linguistic marker "either" (0:12, Table 2). Interestingly, Paul uses the description that it should look like a prism, but—as on paper—it is not. After the interviewer's stimulus, attention is drawn to a 3D-printed prism (0:34, Table 2). Here Paul changes the available perspectives. Whereas with paper and screen access is given through the process of seeing, this perspective is extended through tactile elements; one can "hold that in your hand and turn it and feel it and stuff." (0:36, Table 2).

In the final explanation of why this is a prism according to Paul, he returns to his initial assessment that it is a three-dimensional object, in contrast to the images on the page or on the screen (0:45, Table 2).

The teacher then asks Max (0:53, Table 2) what his opinion is on this topic. In the video it becomes clear that he often wanted to interject something and did so in 0:32 and 0:43 (Table 2). In his statement, Max makes it clear (1:12, Table 2) that even if you see it [a prism] on a two-dimensional plane, it still has three-dimensional aspects. This view is typical of reasoning based on (mental) representation or psychology of shapes.

This is followed by Paul's statement "Why does one see it in a two-dimensional thing" (1:20, Table 2), which we have chosen to explicate. According to this statement also the insertion in square brackets was made, for Paul it seems to be still about the prism in Max's utterance, which for him already from the beginning cannot exist in the two-dimensional. And therefore, from an analytical standpoint we can draw a clear distinction between Paul's and Max's DSE, which can explain the incongruence in their interaction.

6.2.5 Formulation of the Explicative Paraphrase

The explicative paraphrase of the statement "Why does one see it in a two-dimensional thing?" is now from the narrower contextual analysis "There can't actually be a prism in the two-dimensional. So why should someone see this in the two-dimensional?". Even in the further transcript there is no hint for solving or readdressing this major conflict in the interaction.

6.2.6 Evaluation of Explicative Paraphrase

The last step of an explicative content analysis is the examination whether the explication is now sufficient. The result of the explication shows that it is not the case that Paul does not speak previously of a "two-dimensional thing" because of a lack of technical language. From the beginning, a prism can only exist in three-dimensional space. This strong demarcation shows Paul's ability to make a clear and comprehensible distinction between two- and three-dimensional things on the basis of the objects that are located in the corresponding space or plane. Paul can thus be attested a viable view at the intersection of 3D to 2D and 2D to 3D. Also, first ideas about the concept of mapping become clear in 0:01 (Table 2).

7 Conclusion & Discussion

In addition to our results of the analysis, the quality of the given explicative context analysis should be discussed. The data basis is available in videographed form. The analysis was carried out by two evaluators, so there is intercoder reliability in a certain sense. However, due to the brevity of the material, this cannot be meaningfully determined quantitatively.

Again, the decision to discuss this example in this article lies in its paradigmatic nature and conciseness showing how the identification of different DSE are aligned to different perspectives which can be described by the concepts of inscription and visualization. Still it can not be decided whether Paul activates two distinct or one superordinated DSE, but it could be shown, that Max activated a different DSE. This differences result in an incongruent interaction. For teaching this different views needed to be addressed in the classroom, especially since both perspective are mathematically valid. If you are looking at a 2D representation of a prism as a visualization it helps to give some insights on the prism, e.g. how many edges a prism has. If you are looking at the 2D representation of a prism as an inscription it is a mere reference of a prism in the discourse, and so it doesn't need to have similar properties.

The interaction was chosen to cast perspectives on the 3D to 2D and 2D to 3D interface (intrinsic case). Interestingly, after a use of the 3D printing technologies, the student Paul distinguishes very accurately between two- and three-dimensional aspects (he activates a DSE that allows him to do so), in this case study dealing with geometric solids, prisms. We think this clear distinction is based on the interactive interaction with 2D in the CAD and 3D with the printed objects environment (Pielsticker, 2020). This interpretation is supported by the enactive handling of printed 3D objects and 2D representations in a CAD software, which addresses exactly this interface 3D to 2D and 2D to 3D.

Therefore, an intensive consideration through manifold experience in school (but also in teacher education, see Sect. 1) seems to be necessary. Especially since there is no obvious given connection between 3D objects and their standardized 2D representations, even from a mathematical standpoint since there are different projections possible and different 3D objects can be reconstructed from 2D images.

Additionally, this connection invites (re)thinking established standards and the way we are talking about them in mathematics education. How we introduce "standardized" 2D representations of 3D solids? In which ways we are talking about 2D representations of 3D objects? Should we emphasize the mathematical differences between representation and object in particular or should we better hide them implicitly in the discourse?

Some of this research questions the authors started to investigate via supervising bachelor thesis, which address this issues in the context of AR/VR Technology, which make it possible to interact visually with 3D objects without touch and observation in more realistic classroom settings. In the future it seems to be fruitful to collect data about teachers' views on the raised questions and compare them with the results of this investigation as well as the before mentioned studies in other settings.

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Mathematical Drawing Instruments and 3D Printing—(Re)designing and Using Pantographs and Integraphs in the Classroom

Frederik Dilling and Amelie Vogler

1 Introduction

Drawing instruments were of particular importance in the historical development of mathematics. Even today, they can be used beneficially in mathematics lessons by enriching mathematical knowledge development processes. 3D printing technology makes it possible to recreate instruments without great effort, some of which have already been forgotten or are no longer widely available. It is even possible for the students in mathematics classes to redevelop the instruments themselves. The aim of this article is to give a learning-theoretical foundation for mathematics learning based on drawing instruments and to describe two examples and corresponding ideas in detail.

In the following section of the article, the theoretical concept of empirical theories is first briefly outlined. On this basis, the use of historical mathematical drawing instruments in mathematics education is motivated and explained on a theoretical level. Especially, it is described, how technical and mathematical ideas

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of a drawing instrument can be connected. Sects. 3 and 4 follow with a description of the historical drawing instruments pantograph and integraph and their 3D printed recreations. A summary and an outlook are given in the fifth section.

2 Learning Mathematics with Drawing Instruments

2.1 Mathematical Knowledge as Empirical Theories

In the teaching of mathematics at school, mathematical knowledge is to a large extent developed in empirical contexts.

> Mathematical thinking builds initially through making sense of our perceptions and actions. (Tall, 2013, S. 139)

The basis for most mathematical investigations at school is the surrounding world. This also or especially applies when working with manipulatives from a 3D printer or with other digital technologies in general. The students therefore perceive mathematics as a kind of natural science with the aim of being able to adequately describe the phenomena they learn about with a focus on the visualization and the extra-mathematical application of the mathematical knowledge. We call this an empirical belief-system about mathematics (cf. Burscheid & Struve, 2020). A person's belief system about mathematics as a mental structure has a significant influence on the way he or she deals with mathematics and behaves in mathematical situations.

> Belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks. One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. Beliefs establish the context within which resources, heuristics and control operate. (Schoenfeld, 1985, S. 45)

The phenomena considered differ in mathematics and in the natural sciences—however, the approach, in particular the description by a theory and the ontological status of the concepts are to a large extent similar. For this reason, the knowledge developed in class can be described using scientific structuralism as the development of empirical theories about empirical phenomena and empirical objects, e.g., graphs of function, drawn geometrical figures or 3D printed manipulatives.

2.2 Mathematical Knowledge and Drawing Instruments

Drawing instruments play an important role in the historical development of mathematics. Greek mathematics was essentially based on the instruments compass and ruler. In the first book of Euclid's Elements, their use is defined as postulates and based on this, a theory is developed about the figures and curves that can be generated with these instruments. Also in subsequent times, instruments for drawing curves (e.g., parabolic compasses) or for precisely modifying geometric shapes (e.g., pantographs) were important in geometry. The development of drawing instruments extends into the early twentieth century, when, for example, integraphs were used in calculus for the precise drawing of graphs of antiderivatives for practical calculations. The importance of analog mathematical instruments decreased with the breakthrough of the computer in the second half of the twentieth century. In science as well as in fields of application of mathematics, they were replaced by digital instruments, which can fulfill corresponding requirements in a significantly shorter time and with an increased precision. Moreover, the digital instruments extend the analog ones by essential functions (cf. Dilling & Vogler, 2020).

In mathematics education at school, however, analog instruments continue to play an important role next to digital ones. The ruler, the protractor and the compass are part of the standard equipment of every student. They are used to operationally define important concepts of geometry such as distances, angles, or circles. Thus, when used and interpreted appropriately, they embody mathematical relationships. Especially more complex drawing instruments, whose usage is not immediately obvious, can arouse the interest of the students:

> If we see an instrument that is unfamiliar to us, we ask: What kind of instrument is this? What do you do with it? How do you handle it? Or perhaps even more challenging: Why does it work? On which idea is it based? (Vollrath, 2013, p. 5, translated by the author, emphasis in original).

Drawing devices thus have a high motivational effect. The mathematical drawing instruments represent authentic applications of the knowledge developed in class. They are no black boxes, but can be examined through targeted experimentation, so that the mode of operation can be recognized and related to mathematical concepts In this respect, Vollrath (2013) distinguishes the mathematical idea from the technical idea of a drawing instrument:

> Thus, one has to deal with mathematical ideas on the one hand and with technical ideas on the other hand. Both are usually closely connected with each other. And both should not be seen too narrowly. For example, mathematical ideas contain

> physical ideas such as movements. Technical ideas, on the other hand, can include ideas of handcraft, e.g., certain mechanisms for the transmission of forces. (S. 5, translated by the author)

Mathematical drawing instruments can be used in the classroom to introduce or justify a particular mathematical concept or relation. Randenborgh (2015) refers to this as a didactic idea within a process of instrumental development of knowledge. Using the concept of empirical theories, this process can be described as follows: The mechanisms implemented in the instrument (technical idea) and the resulting dependencies within the instrument or in relation to the constructed geometric objects can be experimentally investigated and thus justified by the students. The dependencies can be interpreted and related to the concepts of their empirical mathematical theory. In this way, building on the investigations of the drawing instrument, the mathematical theory can be further developed (mathematical idea). The mechanisms and drawings form the reference objects of the empirical concepts of the student's mathematical theory. The drawing instrument thus becomes an empirical setting that the teacher introduces into the learning process with a specific purpose and that the students use in relation to their empirical mathematical theories. Since drawing instruments can be applied to a variety of geometric objects in the same way, and since they are a tool not limited to a specific case, they offer a high potential for concept development processes (cf. Dilling & Vogler, 2020). A simplified schematic representation of such a knowledge development process can be find in Fig. 1.

3D printing makes it possible to integrate historical mathematical drawing instruments into the classroom. The instruments can be printed prior to the lesson and then used by the students to draw figures or graphs and systematically examine the mechanisms from a mathematical perspective (see Sect. 4 about the integraph). In this way, an active exploration of the instruments can be achieved. If the instruments are less complex, it can also be beneficial for the students to independently develop or redevelop the instruments directly in class (see Sect. 3 about the pantograph).

3 The Pantograph—Drawing Scaled Up and Scaled Down Figures

3.1 Functionality of a Pantograph

The pantograph is a mechanical instrument for copying, scaling up and scaling down drawings (see Fig. 2). The drawing instrument is constructed as follows (technical idea). The points named in the following description of the technical

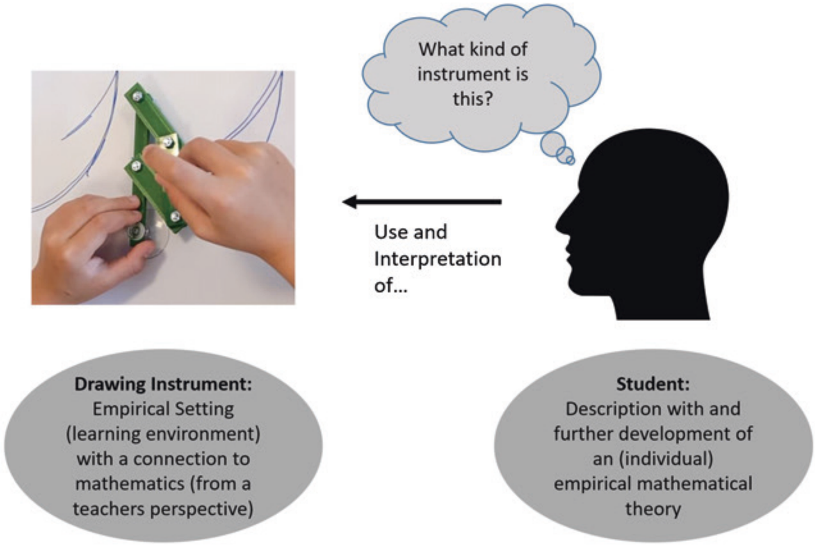


Fig. 1 Schematic representation of the interpretation of a mathematical drawing instrument

idea refer to the schematic representation in Fig. 3. The point Z, also called pole, is held by the user or fixed on the base. A tracing pen is attached to point B, with which a given figure on a drawing sheet can be traced.

The device moves around point Z. The figure traced with B is mapped onto point B', where a drawing pen is installed. The drawing pen draws an scaled up figure similar to the initial figure. The scale factor is given by the ratio $|\overline{ZB'}| : |\overline{ZB}|$. A change in this scale factor can be achieved by adjusting the rails at points A and C. If the tracing and drawing pens are switched, the pantograph scales down the figure.

The instrument is based on the mathematical principle (mathematicsl idea) of centric dilation. The construction consists of a movable frame with four rails. Figure 3 shows a schematic representation of a pantograph. The four rails correspond to the distances ZA', $A'B'$, AB as well as BC. The rails ZA' and $A'B'$ have the same length, so that $ZB'A'$ corresponds to an isosceles triangle. In addition, the rails with the distances AB and BC are bolted to the other rails in such a way that a movable joint parallelogram ($ABCA'$) is formed and Z, B and B' are collinear. Thus, the triangles ZBA and $ZB'A'$ are always similar to each other. The similarity

Fig. 2 Wooden pantograph made by the German manufacturer Rumold

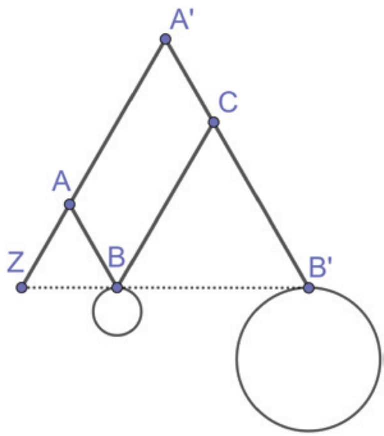


Fig. 3 Schematic representation of a pantograph (Drawn with ©GeoGebra)

mapping is a centric dilation with center Z. The same is true for the figures traced with B and B'.

The intercept theorems can be applied to the two triangles. According to the first intercept theorem, the ratio $\left|\overline{ZB'}\right| : \left|\overline{ZB}\right|$ corresponds to the ratio $\left|\overline{ZA'}\right| : \left|\overline{ZA}\right|$. This is equal to the scale factor of the drawn figures and can be denoted by k. For a length ratio of $\frac{|\overline{ZA'}|}{|\overline{ZA}|} = \frac{3}{1}$ the scale factor is $k = 3$ and thus the point B' shifts by three units when the point B is shifted by one unit. The figure is thus scaled up by a factor of $k = 3$. In mathematics lessons, it is a suitable approach to first examine the special case in which the joint of the pantograph is attached at points A and C exactly in the center of the rods. Thus $|\ ZA'\ | = |\overline{ZA}|$, resulting in a scale factor of $k = 2$. The drawn figure is doubled compared to the initial figure. If the tracing pen and the drawing pen are swapped, the new scale factor corresponds to the reciprocal of the previous factor—so the drawn figure is halved compared to the initial figure.

The geometrical and technical properties of the pantograph offer rich opportunities for use in mathematics teaching. Through instrumental activities with the drawing instrument, even elementary school children can derive initial properties of construction and function. In this context, it is important that the learning environment is designed appropriately according to the subject matter and the students. A decisive insight that can be gained by the students is that the points Z, B and B' are always collinear. Furthermore, they can discover that the scale factor can be determined not only from the drawing, but also from the instrument itself.

3.2 3D Printing of a Pantograph

Pantographs with a scale factor of 2 : 1 can be produced with little effort using a 3D printer. For this purpose, the four rods of the pantograph are created as flat cuboids in a CAD program. The two short rods have a length of about 8 cm and small holes with a diameter of 6 mm are inserted at both ends (see Fig. 4 bottom). The two large rails are about twice as long as the short rails, so three 6 mm diameter holes can be positioned at the same distance as on the small rail (see Fig. 4 top). The rods can be printed with a 3D printer and then joined with metric threaded bolts (see Fig. 5). The fixed point Z is a suction cup, the drawing pen is a simple felt-tip. By redesigning the rails of the pantograph and adding more holes, other scale factors can also be realized.

Dilling and Vogler (2020) investigated how elementary school students develop mathematical knowledge through the individual re-development of a pantograph. The study took place in the context of a two-days workshop in which the students first examined a wooden pantograph (see Fig. 2). Then they independently planned a pantograph in groups, constructed it with Tinkercad™, printed the components with a 3D printer, and finally assembled it with threaded bolts, a suction cup and a felt-tip. At the end of the workshop, different figures were drawn scaled up with the 3D printed pantographs.

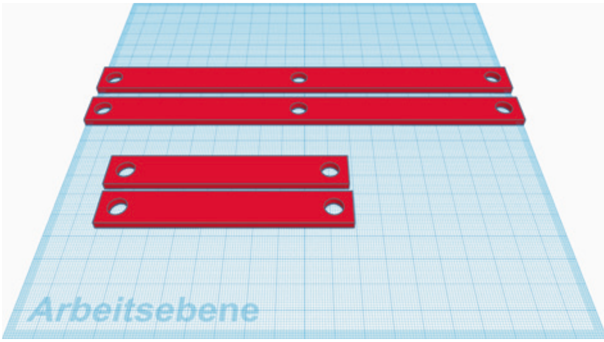
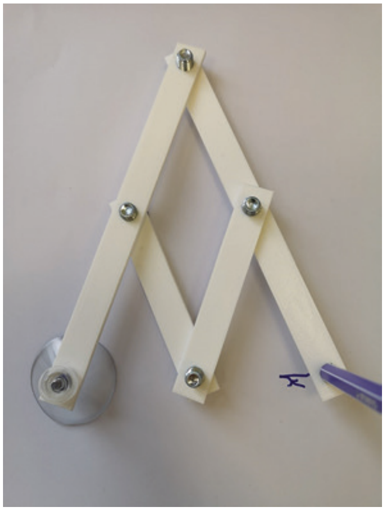


Fig. 4 Design of a pantograph with Tinkercad™

Fig. 5 3D printed pantograph



In summary, it should be mentioned about the knowledge development process of the students that each phase of the workshop seemed to contribute to the understanding of the pantograph as a mathematical drawing instrument. In the planning phase, the students developed a yet undifferentiated, general picture of the components needed to build a pantograph (see Fig. 6). In the construction phase with the software Tinkercad™, the students were challenged to deal with

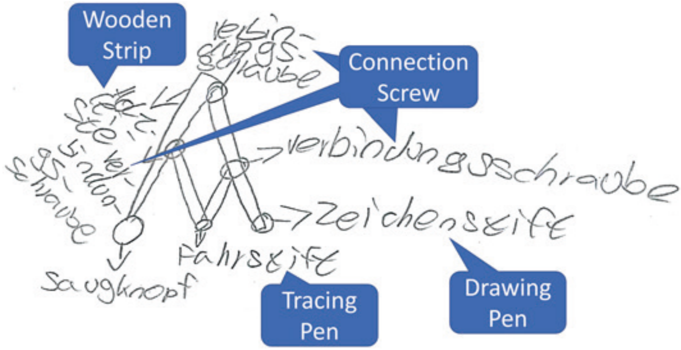


Fig. 6 Drawing in the research journal of Emma

the pantograph in greater detail. Previously imprecisely sketched components were then carefully constructed with attention to additional characteristics. The empirical objects used in the CAD software, such as cubes and cylinders, had a decisive influence on the students' knowledge development. The subsequent phase involved assembling the independently created components and drawing figures with the pantographs. In this context, one of the students described her insights and the experience as follows:

> Emma: Yes, a lot, ehm for example we learned, ehm, how to handle it and how to build it and when I built it, I really understood how it works (smiles). […] For example, where the different components (shows with hands) are inserted. (Translation by the authors)

From this moment on, the pantograph is no longer a black box for the students. When moving and testing the screwed components for the first time, the students gained deeper insights into the connection between the functionality of the pantograph and the specific parallelogram construction or the mechanism that enables the fluent tracing of different figures. In the interview, it also became clear that the students were now able to describe crucial features and the specific construction of the drawing device. In addition, they expressed initial explanations of why the pantograph scales up figures.

The mathematical knowledge developed by the group of students in the workshop can be reconstructed as empirical theories about the drawing instrument pantograph. The components represented in the program, i.e. the virtual objects such as the cubes or the cylinders, and the 3D printed pantographs themselves form the reference objects of the students' knowledge or theories. To describe the components of the pantograph, students use the geometric concepts of cuboids and cylinders. In addition, initial preformal notions of similarity play a role in the mathematical descriptions of the students.

4 The Integraph—Drawing Integral Functions

4.1 Functionality of an Integraph

An integraph is a device that mechanically draws the graph of the antiderivative for a given piecewise continuous function graph. Thus, the device can be used to solve indefinite integrals graphically. First concepts for an integraph go back to Leibniz in 1693 (cf. Leibniz, 1693). However, like many of the ideas for mechanical devices developed by Leibniz, his integraph was probably never realized. The probably first actually working integraph was developed by Abdank-Abakanowicz at the end of the nineteenth century (see Abdank-Abakanowicz, 1889). Further integraphs were developed up to the twentieth century. For the use of integraphs in schools, it is particularly important that the central mechanisms can be identified easily. This seems to be true in particular for the model developed by the company A. Ott at the beginning of the twentieth century (see Fig. 7, right).

The integraph of the company A. Ott consists of two perpendicular rails S_1 (y-parallel) and S_2 (x-parallel) (see schematic diagram in Fig. 7, left). The abscissa slide W_2 can be moved on the rail S_2. On this abscissa slide there is a rail parallel to S_1, on which the ordinate slide W_3 can be moved. At the end of the rail of the abscissa slide W_2 is the pivot Z_1; at the end of the ordinate slide W_3 initially at the same distance from the abscissa is the pivot Z_2. A directional ruler is defined by the pivots Z_1 and Z_2, which determines the orientation of a cutter wheel at the pivot Z_1. The cutter wheel rests on a plane located on the integrating slide W_1, which can be moved along the rail S_1 mentioned at the beginning. When the cutter wheel is moved in the x-direction, it correspondingly pushes the integrating slide W_1 in y-direction. Next to the pivot Z_1 there is a drawing pen ZS, which draws a line on the plane of W_1 when moved. At the bottom of the ordinate slide W_3 is a tracing pen F, which can be moved along any given curve (cf. Dilling, 2019; Willers, 1951).

The integral curve to a given curve can be generated with the integraph as follows. First, the x-distance of the pivots Z_1 and Z_2 is set to the unit of the curve.

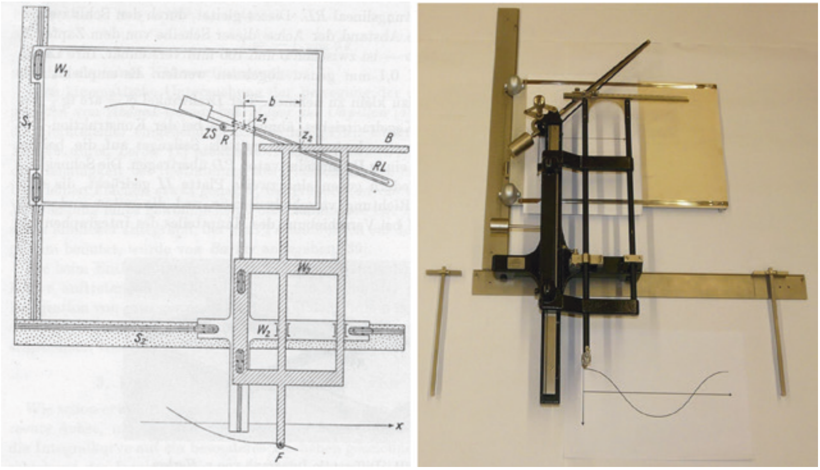


Fig. 7 Schematic illustration (left) (© Friedrich A. Willers, 1951, p. 240) and photography of the integraph A. Ott (right) (© Palm, 2013, p. 81)

For this purpose, the pivot Z_2 can be moved in the x-direction on the ordinate slide W_3 and fixed. In the starting position of the integrating graph, the tracing pen F is on the x-axis of the underlying curve. The direction ruler is parallel to the x-axis. If the tracing pen F is now raised by the height h, the y-distance h is generated between the pivots Z_1 and Z_2. Since the x-distance corresponds to the unit, the directional ruler has the slope h. If the tracing pen F is now dragged along a curve, the directional ruler and thus the cutter wheel have the changing y-values of the given curve as slope-values. By moving the drawing plane on the integrating slide W_3, the drawing pen ZS can draw a continuous curve with the function values of the given curve as its slope (cf. Dilling, 2019; Dröge & Metzler, 1983).

4.2 3D Printing of an Integraph

Using 3D printing technology, the historical integraphs can be rebuilt (e. g. Dilling & Witzke, 2019, Dilling, 2020). For this purpose, the basic mechanisms of the historical integraphs can be adopted, but also simplifications for didactic purposes can be carried out. This means that the multitude of adjustment possibilities of the precise instrument is deliberately omitted and thus, for example, the unit of the coordinate system cannot be adjusted in the 3D printed integraph. In this way, the focus can be placed on the essential functions of the instruments. The cutter wheel was replaced by a simple small rubber wheel. This transfers the function value of the initial curve to a movable drawing plane (see Fig. 8, left), a rotatable drawing cylinder (Fig. 8, bottom) or via a parallelogram construction to a sheet of paper (see Fig. 8, top). The drawing pen is a simple overhead marker.

The principle of the integraph is based on the mathematical relation that the derivative of an integral function is the original function, i.e., the integration and the differentiation are inverse processes. This statement is part of the fundamental theorem of calculus. For this reason, the integraph can be used in mathematics lessons to explain the fundamental theorem.

For this purpose, one considers the approximation of a function by a step function. For step functions the main theorem can be justified very easily by elementary geometry, e.g., with reference to the integraph. The approximation of a function by a step function is possible with arbitrary precision and is therefore also valid for the approximated function (see Blum, 1982 for further information).

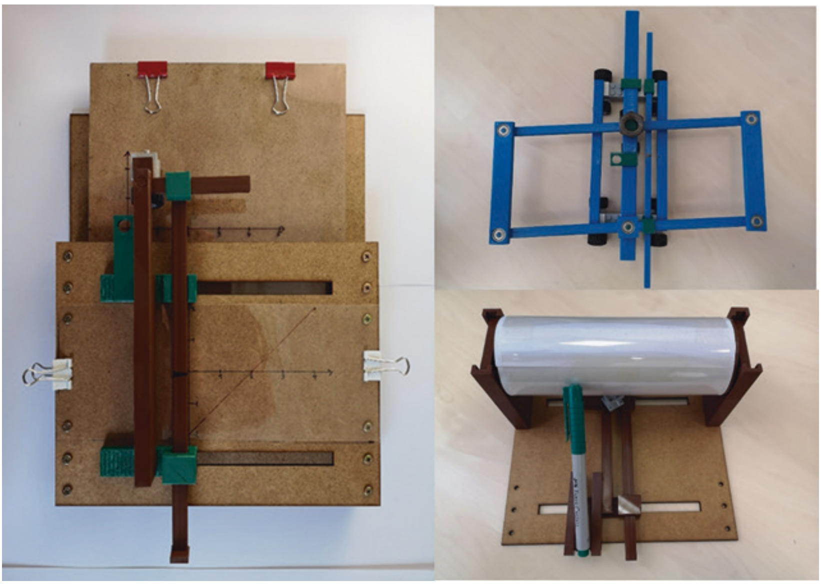


Fig. 8 3D printed integraphs with different mechanisms

The possibility to draw graphs of integral functions with the integraph and to determine the area under any given curve results in many further applications in the classroom. These include graphically solving higher degree algebraic equations, finding zeros of polynomials, or determining different parameters in technical applications.

Mathematical knowledge development with the drawing instrument integraph was the subject of a case study conducted by Dilling (2022). He observed three male students from secondary school with experience in integral calculus working independently with an integraph in clinical interviews. In summary, it must be mentioned that the students were not able to grasp the principle of the integraph immediately and that several impulses from the interviewer were necessary for this. However, with the help of the interviewer, the students were able to relate central concepts of their empirical mathematical theories to the integraph. Two of the students seemed to have uncovered the underlying mathematical principle of the integraph, expressed by the first part of the fundamental theorem of differential and integral calculus. They both think that the integraph can be used to graphically integrate any other graphs of functions. They can trace this relation between the initial curve and the curve drawn with the instrument back to the functionality of the instrument and theorems of their empirical theories (especially the fundamental theorem of calculus). The integraph and the curves drawn with the instrument serve in the arguments of the two students as generic examples of the process of generating antiderivative functions or antiderivative curves. The third student largely remains with arguments that refer to the concretely mapped function graphs. He describes the connection with the correspondence of extreme points and zero points—not with reference to the instrument. The student seems to think that it is sufficient to examine particular points on the graphs to conclude their relationship at each point. The student believes the integraph draws antiderivative functions, but he verifies this almost exclusively on the concrete graphs. The student explains the mathematical relation only marginally on the basis of his mathematical theory and that only due to strong impulses of the interviewer. The other two students also argue partly on the basis of the given and drawn function graphs, but this primarily serves to form hypothesis.

5 Conclusion and Outlook

In this paper, it could be shown that the 3D printing technology offers fruitful possibilities for bringing historical mathematical drawing instruments into the classroom. The 3D printed instruments can have great potential for fostering mathematical knowledge development processes. The mechanisms implemented in the instruments and the resulting dependencies within the instrument or within constructed geometric objects can be experimentally investigated and thus justified by the students. In a further step, those dependencies can be described from a mathematical perspective by relating them to the concepts of their empirical mathematical theory. In this way, the empirical mathematical theory can be further developed building on the investigations of the drawing instrument. By relating the technical and mathematical ideas, both the understanding of the drawing device and the empirical theory can be further developed.

The use of 3D printing in the context of mathematical drawing instruments can be done in two ways:

1. *The teacher designs a drawing instrument and prints it out before the mathematics lesson. Students then use and investigate the 3D printed drawing instruments in class.*
2. *Students design their own mathematical drawing instrument based on a given instrument using CAD software, print it with a 3D printer, assemble it, and use the 3D printed instruments to draw mathematical objects.*

Depending on the complexity of the drawing device and the available time in class, the first or the second form of use can be appropriate. For example, the pantograph (see Sect. 3) can be developed with comparatively little effort by the students. The integraph (see Sect. 4), in contrast, is comparatively complex and requires a lot of developing time.

Other mathematical drawing instruments are also suitable to be replicated with 3D printing, for example:

- *Parabolograph for drawing parabolas*
- *Ellipsographs for drawing ellipses*
- *Spirographs for drawing spirals*
- *Harmonographs for drawing periodic function graphs*
- *Rolling and parallel rulers for drawing parallel lines*
- *Line dividers for dividing lines into parts of equal length*
- …

Besides drawing instruments, also measuring instruments, or calculating instruments can be replicated with 3D printing.

3D printed mathematical instruments should be investigated more deeply in educational empirical studies, so that knowledge development processes with historical drawing instruments and the connection to digital technologies like 3D printing can be better understood. In addition, it is worthwhile to investigate what characteristics emerge when using digital simulations of drawing instruments compared to using the original analog instruments.

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3D-Printing in Calculus Education—Concrete Ideas for the Hands-On Learning of Derivatives and Integrals

Frederik Dilling

1 Introduction

The importance of subject-matter approaches to central topics in calculus education has been discussed for a long time. The increased use of digital technologies has brought subject-matter didactical considerations for the development of hands-on approaches back into focus in mathematics education. In this paper, 3D printing technology is investigated for the individual development of haptic working materials (manipulatives) for mathematics education.

To do so, the traditional German research approach of subject-matter didactics is described, with particular attention to the application of *Grundvorstellungen* of calculus. Then, several 3D-printed manipulatives for the hands-on learning of calculus concepts are presented and specific application scenarios for mathematics teaching are described in the context of the Grundvorstellungen approach. Finally, a conclusion is drawn and recommendations for future research are given.

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2 Subject-Matter Didactics—Concepts in Calculus Education

Analyzing mathematical content and contexts and applying them to practical concepts to teach mathematics in school are central to mathematics education research. In this field, there is a long tradition of research centering *Stoffdidaktik* or *subject-matter didactics* in German-speaking countries (cf. Hefendehl-Hebeker, 2016) that exerts a strong influence on research and teacher education today:

> Stoffdidaktik has been a dominant approach to mathematics education research within the German speaking countries, which puts the analysis of the mathematical subject matter at its heart. It has been the prominent approach to research until the 1980s. Nowadays it still influences research in mathematics education in German speaking countries. (Hußmann et al., 2016, p. 1–2)

The focus of classical subject-matter didactics is the mathematical content to be taught at school. The main objective is to facilitate access to and understanding of mathematics through mathematical content analysis. Subject-matter didactics investigates:

- Essential concepts, procedures and relationships including appropriate formulations, illustrations and arrangements for teaching
- Essential structures and domain-specific ways of thinking
- The inner network of paths by which the components are connected and possible learning paths throughout the domain. (Hefendehl-Hebeker et al., 2019, p. 26)

The concept of Grundvorstellungen (translation: basic ideas), according to vom Hofe (1995), is a classic approach within subject-matter didactics to facilitate the teaching of concepts. In general, Grundvorstellungen are used normatively and describe relevant aspects and ideas for students based on expert educators' (researchers' and teachers') evaluation of mathematical content. Vom Hofe (1995) describes Grundvorstellungen as follows (cited according to Vom Hofe & Blum, 2016):

> This concept describes the relationships between mathematical content and the phenomenon of individual concept formation. The numerous treatments that the GV [Grundvorstellungen] concept has received over time nonetheless focuses on three particular aspects of this phenomenon, albeit with different emphases among these treatments:

> The constitution of meaning of a mathematical concept by linking it back to a familiar knowledge or experiences, or back to (mentally) represented actions,
> The generation of a corresponding mental representation of that concept; that is, an “internalization”, which (following Piaget) enables operative action at the level of thought,
> The ability to apply a concept to real-life situations by recognizing a corresponding structure in subject-related contexts or by modelling a subject-related problem with the aid of mathematical structures. (p. 230)

Grundvorstellungen, which is popular in German-speaking countries, can be related to the internationally well-known ideas of *concept image* and *concept definition* proposed by Tall and Vinner (1981). Simplified, concept definition is “a form of words used to specify [the] concept” (p. 152) and emphasizes technical description. Concept image describes “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). Grundvorstellungen can be considered a mediator between the concept definition and students’ conceptions. These can be seen as normative categories (formulated through the empirical observation of students) that may guide teachers in the development of specific learning contexts (Dilling & Witzke, 2020).

Grundvorstellungen for several concepts in calculus education, e.g., functions, derivatives, and integrals, have already been formulated, explicated, and used to describe student conceptions in empirical studies.

The concept of functions forms the basis of calculus teaching at school. Vollrath (1989) describes three aspects of the concept of functions that are relevant in school. In recent literature, those aspects are also often referred to as the Grundvorstellungen of functions (see Greefrath et al., 2016a):

- Assignment: Exactly one f(x) is assigned to each x,
- Covariation: The change of f(x) when changing x,
- Object: The functional correlation is considered as a whole.

The *Assignment* Grundvorstellung emphasizes the definite assignment and dependence of quantities. It is present in the definition of a function via the function equation and notations such as “$f(x) =$” or “$x \to y$.” If a functional relation is noted in a horizontal table, then the vertical reading of that table underlines the Assignment Grundvorstellung.

Covariation as a Grundvorstellung is expressed in statements such as “The larger x becomes, the larger y becomes.” In a function noted as a horizontal table, covariation is emphasized by reading the table horizontally. Covariation is essential to consider slopes or curvatures in differential calculus.

When using the *Object* Grundvorstellung, one considers the whole set of value pairs instead of individual value pairs. Hence, the assignment is understood as a new object that becomes part of the operation. Thus, global properties become more important, e.g., zero points, monotonies, or extreme points. This Grundvorstellung is emphasized by graphical representations.

Greefrath et al., (2016a, b) distinguish, among others, the following three Grundvorstellungen of the concept of derivatives:

- Local Rate of Change
- Tangent Slope
- Local Linearity

The *Local Rate of Change* Grundvorstellung is based on the idea of instantaneous velocity in change processes (e.g., movement). It makes use of the notion of the slope of a curve at a point and the change of the dependent variable y in $\Delta y = f'(x) \cdot \Delta x$. The Grundvorstellung of the derivative as the *tangent slope* uses the students' experience with the slope of straight lines and the concept of geometrical tangents from middle school. An extension of these concepts is necessary for calculus. For example, the tangent is no longer defined as the straight line that touches the graph at exactly one point but rather the straight line that optimally adapts locally to the course of the graph. The Tangent Slope Grundvorstellung includes these ideas and interprets the derivative of a function at a point as the slope of the tangent at that point.

The *Local Linearity* Grundvorstellung uses the idea that, when looking closely (over a small interval) at the area surrounding a point on the graph of a differentiable function, it appears to be a straight line. For small changes in x-values, the function is almost linear and can be approximated by a linear function.

The concept of integrals has also been widely discussed from the perspective of subject-matter didactics. Greefrath et al., (2016a, b) describe the following four Grundvorstellungen of integrals:

- Area
- (Re-)construction
- Average Value
- Accumulation

The *Area* Grundvorstellung focuses on the determination of the areas of surfaces defined by curves. For this purpose, the integral must be understood as an oriented area with negative values for areas under the x-axis. The *(Re-)construction*

Grundvorstellung emphasizes the determination of a function from a given rate-of-change function or the determination of an antiderivative function to a given function. This Grundvorstellung is particularly present in the fundamental theorem of differential and integral calculus.

The *Average Value* Grundvorstellung emphasizes the possibility of determining the mean value of an interval of a function using the integral. This is based on the mean value theorem of integral calculus, according to which a mean function value for an interval always exists such that the integral is the product of the interval length and this mean function value.

The *Accumulation* Grundvorstellung is understood as the generation of the limits of product sums for the determination of the integral.

These Grundvorstellungen regarding concepts from calculus will be used in this article to describe the possible applications of 3D printing technology to calculus. In this context, the Grundvorstellungen are to be understood as one possible way to look on 3D printed material. According to a constructivist view of learning mathematics, students develop their own conceptions based on the material and incorporate the ideas into their own theories. There is no mathematical knowledge or a particular Grundvorstellung contained in the material that can be extracted. Instead, the development of knowledge is always an individual's constructive process (Dilling, 2022). In this context, Grundvorstellungen can represent the normative perspective—e. g., the perspective of the teacher or of teacher educators—that is one possible dimension of the objectives of calculus education (see Dilling et al., 2019; Dilling & Witzke, 2020).

3 3D Printing Technology in Calculus

3.1 Function Graphs

The OpenSCAD-based software "Graphendrucker"[1] enables the generation of 3D representations of the graphs of nearly every single-variable function (see Dilling, 2019). The user enters the equation describing a function and an interval and the software composes the model with cylinders along the defined spline. Additional elements can be inserted, e.g., a second graph, small coordinate axis, or a whole coordinate system. A screenshot of the Graphendrucker interface is shown in

[1] Download: https://www.uni-siegen.de/fb6/didaktik/personen/frederik-dilling/materialien.html?lang=d

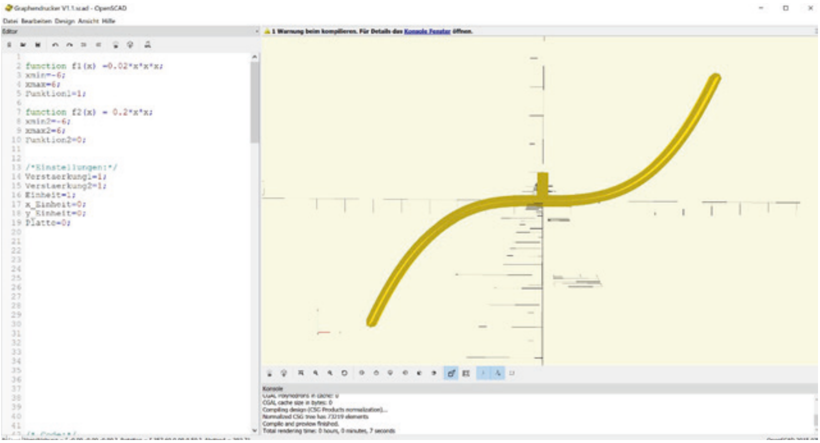

Fig. 1 Screenshot of the Graphendrucker software

Fig. 1. The 3D printed graphs of the sine function and a grade 4 polynomial are shown in Fig. 2.

Thus, the function graph becomes a physical object that can be experienced tactilely and so grasped qualitatively. In class, learners can use such models to develop initial notions of the central concepts of differential calculus. The "embodied approach" according to Tall (2013, p. 300 f.) offers interesting starting points for this development. It starts with mental actions on a drawn or mental graph "which we can trace with our finger and see as an object". This tracing of the function graph with a fingertip (Fig. 3, top left) can be performed with 3D printed models.

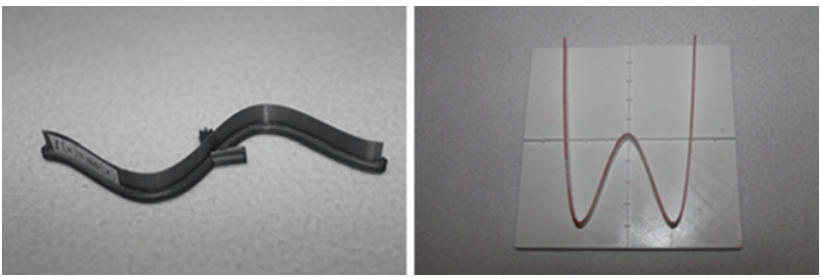

Fig. 2 3D-printed models of function graphs (left: sine, right: grade 4 polynomial)

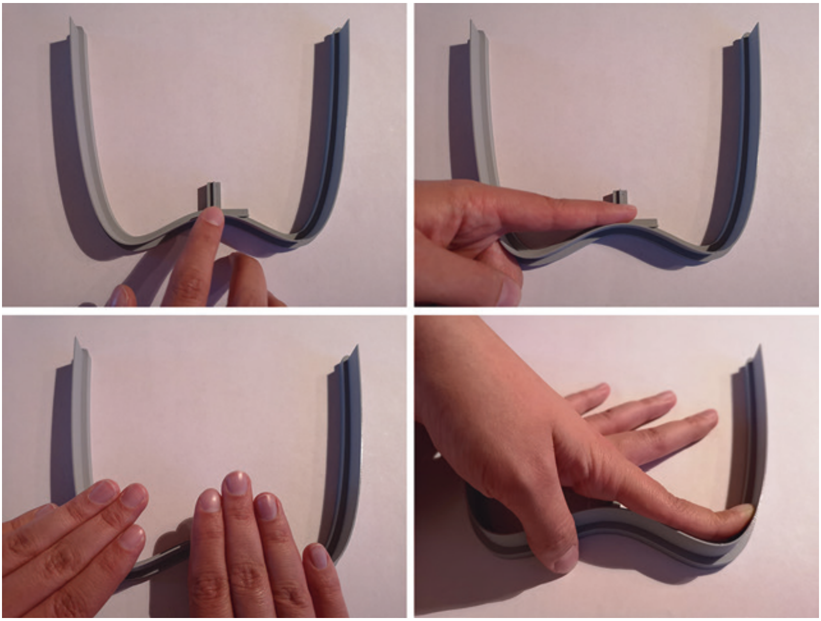

Fig. 3 Gestures with a 3D-printed model of a function graph

The continuity of the graph is perceived as "dynamic continuity". Tracing the model with an extended finger "to let the slope of the hand follow the changing slope" (Tall, 2013, p. 300) (Fig. 3, top right) embodies the changing slope. This experience directly connects with the Grundvorstellungen of the local rate of change and tangent slope.

By covering parts of a graph with the hands, its local linearity can be illustrated (Fig. 3, bottom left, cf. Tall, 2010), invoking the Grundvorstellung of local linearity, and bending the finger to different degrees (Fig. 3, bottom right) demonstrates the graph's curvature.

The graph models can provide various opportunities for learners' meaning negotiation processes (Cobb & Bauersfeld, 1995) in the classroom. By verbalizing the described actions with the graph models, first notions can be formed that are then consolidated through communication with others. Grundvorstellungen of the concepts of calculus can thus be developed informally as the basis for a comprehensive understanding of calculus (see also the empirical studies in Dilling, 2019; Dilling et al., 2019; Dilling & Witzke, 2020).

The use of 3D-printed models of function graphs enables students to address the Grundvorstellungen of functions as well. The Covariation Grundvorstellung focuses on the change in $f(x)$ when changing x. This directly relates to the Local Rate of Change Grundvorstellung and is thus highly important for differential calculus (Hahn, 2005). The covariation of a function can be accentuated by sliding along the model with a finger. By not using a coordinate system for the 3D-printed models, the often-prevalent Assignment Grundvorstellung can be deliberately deemphasized to bring the covariation to the fore. The Object Grundvorstellung refers to the functional correlation as a whole. Characteristics such as monotonicity or symmetry are easy to recognize in graphical representations. This aspect is foregrounded by the model's physicality that allows students to work action-oriented with the graph as a whole.

This approach can also be applied to graphs of functions with two variables (see Fig. 4, Dilling, 2019). Although these are not part of the regular mathematics curriculum in German high schools, many of the functions students use in school are functions of two variables, e.g., formulas for surface and volume in geometry, arrays of functions, terms in algebra, etc. (Weigand & Flachsmeyer, 1997). Furthermore, functions of two variables can provide a new approach to arrays of functions (e.g., the function $f_a(x) = a \cdot x^2$). By using 3D models, the change in the graph when changing the parameters becomes visible. This change can also be sensed qualitatively by touching the objects. When starting to engage with such representations, it can make sense to start with discrete functions as shown in Fig. 4 (right) due to the complexity of the continuous representations.

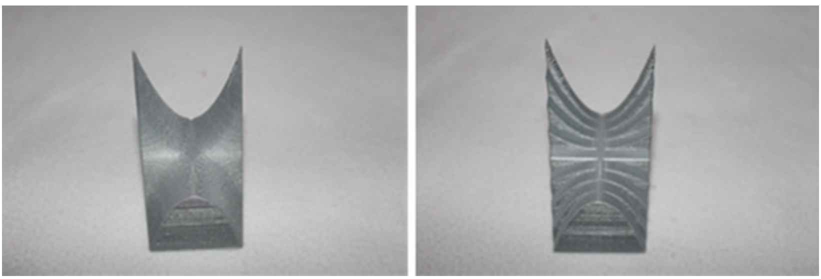

Fig. 4 3D printed graphs of the function $f(x, y) = 0{,}5x^2y$ (continuous, left, and with discrete y-values, right)

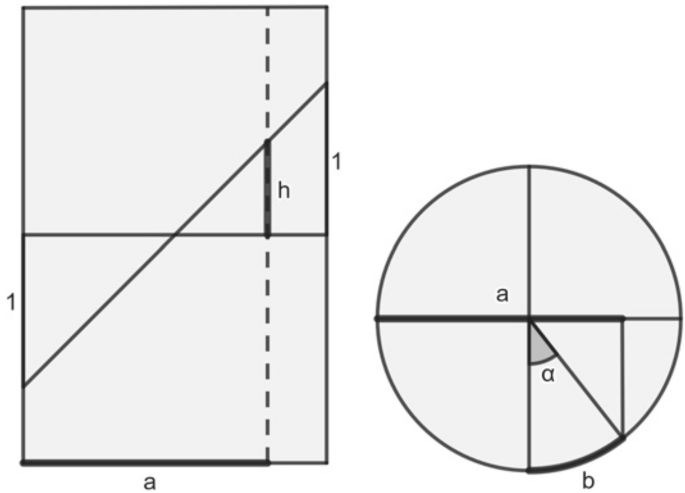

Fig. 5 Cross-sections of a cylinder (created with ©GeoGebra)

3.2 Periodic Functions

3D printing offers the possibility of developing manipulatives to teach periodic functions (cf. Dilling, 2019; Dilling et al., 2021). For this approach, we consider a cylinder with a radius of 1 that is intersected by a plane at a 45° angle. With suitable coordinatization, the intersection curve on the surface can be described as a sinusoidal curve. For this purpose, we will consider the two cross-sections of the cylinder shown in Fig. 5.

The height h of any point A on the intersection curve of the cylinder and the plane can be expressed as a function of the distance a to the side of the cylinder with the lowest point (due to the isosceles triangle that involves h) as follows:

$$\frac{h}{a-1} = \frac{1}{1} \iff h = a - 1$$

The arc length b and thus the angle in radians (radius equal to 1) can be related to the described distance a by trigonometry:

$$sin\,\mathrm{b} = a - 1 \iff a = sin\,\mathrm{b}\, + 1$$

This gives the height h as a function of the arc length b (cf. e.g., Henn & Filler, 2015):

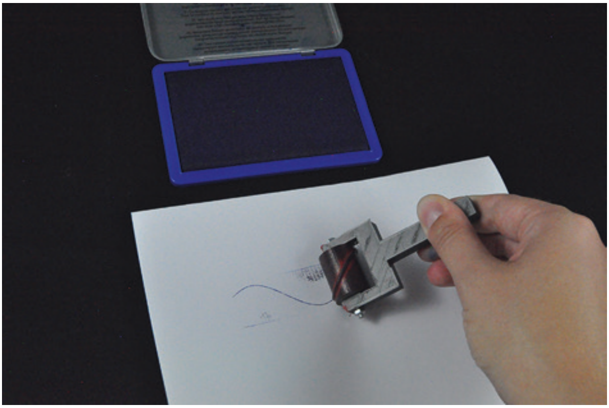

Fig. 6 3D-printed manipulative for periodic functions—the Sine Roller

$$h(b) = sin(b)$$

The described principle is exploited by the Sine Roller manipulative in Fig. 6. The intersection curve on the cylinder of the Sine Roller is raised $1mm$ relative to the surface of the cylinder, allowing the cylinder to stamp a periodic function graph on paper. By investigating this drawing device, students can learn about important properties of the sine and cosine functions and explain them using the 3D printed manipulative.

The students can work at different levels of representation, including (Bruner, 1974):

- Enactive: stamping on paper
- Iconic: curve on paper
- Symbolic: function term in the design of the roller and its algebraical description

This principle can be applied to other periodic functions as well and seems to encourage students to generate viable notions. In particular, students can develop initial notions about the concepts of period length and amplitude.

3.3 Upper and Lower Sums

The definition of the integral as the limit of upper and lower sums is a classic topic in calculus teaching. 3D-printed manipulatives can be developed that offer a qualitative approach to this definition of the integral (cf. Dilling, 2020). These materials consist of a coordinate system on which the graph of a function is drawn. Small tiles of different colors can be used to approximate the area between the graph of the function and the x-axis (Figs. 7 and 8).

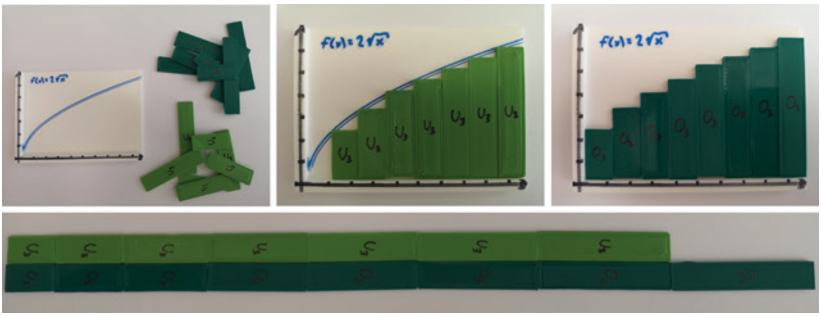

Fig. 7 Manipulative for the qualitative elaboration of upper and lower sums using the function $f(x) = 2\sqrt{x}$

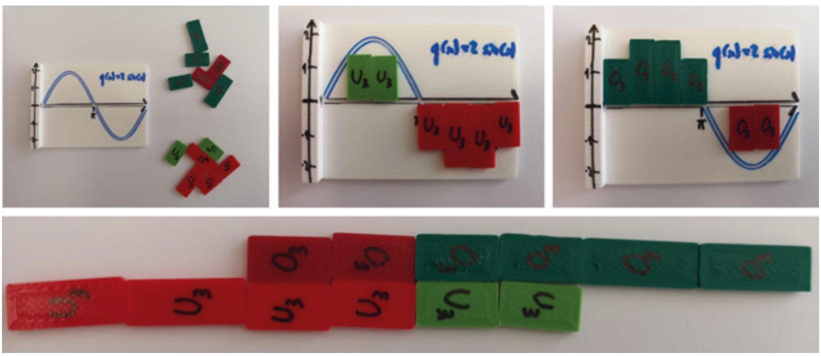

Fig. 8 Manipulative for the hands-on elaboration of upper and lower sums using the function $g(x) = 2sinx$

To do this, it is preferable to start with a function that has only positive function values in the considered interval, e.g., the function with the equation $f(x) = 2\sqrt{x}$ in the interval [0,8] (Fig. 7, left). The students should then be given tiles for a certain decomposition of the interval. The area of the tiles corresponds to the upper (dark green) or lower sum (light green). The area under the graph can be described with these tiles (Fig. 7, middle and right) and the difference between the upper and lower sums can be determined by comparing the rectangular areas (Fig. 7, bottom). This procedure can be repeated for different decompositions and the differences of the upper and lower sums of the different decompositions can be compared qualitatively. This allows students to see that the difference in sums decreases with finer decompositions.

Next, the principle can be transferred to functions with negative function values, e.g., the function with the equation $g(x) = 2sinx$ in the interval $[0,2\pi]$ (Fig. 8). The areas below the x-axis are described with red tiles to indicate a negatively valued area. The upper and lower sums are distinguished with dark red and light red tiles. For functions with negative function values, the difference of the upper and lower sums can be qualitatively determined for a certain decomposition by comparing the areas the tiles describe (Fig. 8, bottom).

Students can experimentally learn important aspects of the upper and lower sums using these materials. The qualitative comparison of the areas for different decompositions enables first experiences with limit processes and the Area Grundvorstellung. Through the consistent color concept, the concepts of upper and lower sums are cognitively linked to dark and light shades of the colors. Furthermore, green represents a positive area while red represents a negative area. Measuring the areas and writing down the values leads to the concept of product sums on a numerical level (Accumulation Grundvorstellung), which can then be generalized based on the material and formulated algebraically by applying the formula for the area of rectangles.

The benefit of the 3D printer is the high degree of product individuality. Any function graph and decomposition can be examined. Thus, the students can also develop tiles for specific decompositions themselves using CAD software. This permits an in-depth examination of the subject matter that can contribute to the development of the Grundvorstellungen of area and accumulation.

4 Integraph

An integraph is an instrument that draws the graph of the antiderivative for a given piecewise continuous function graph. Thus, the instrument can be used to solve indefinite integrals graphically. Integraphs were used by engineers in the early twentieth century in particular, but the development of computers put them out of use and they are now only found in a few museum collections. 3D printing technology enables the development of integraphs for educational use based on the original devices (cf. Dilling, 2019; Dilling & Witzke, 2018, 2019). Figure 9 shows an integraph developed based on an instrument from the A. Ott company.

The integraph allows the function value of a given graph to be described as the slope of a new graph. This is done with a cutting wheel that shifts the upper drawing plane to draw a continuous function graph. In a 3D printed integraph, this is made possible with a simple rubber wheel.

The integraph offers an approach to the concept of the integral that emphasizes the Area Grundvorstellung as well as the (Re-)construction Grundvorstellung.

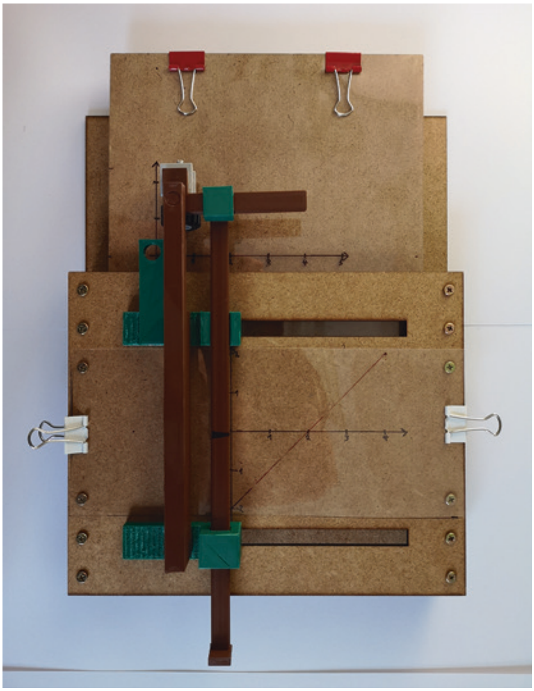

Fig. 9 3D-printed integraph

By approximating the function with staircase functions, the elementary area calculations of rectangles can be performed and the fundamental theorem of differential and integral calculus can be justified (cf. Blum, 1982). In addition, the integraph can be used for continuous graphical integration (and differentiation) instead of considering functions only at specific points on a graph. This emphasizes the functional character of the integral and addresses the (Re-)construction Grundvorstellung. Using integraphs in the classroom creates an active learning experience through which mechanisms and their underlying mathematical ideas can be explored by the students (for empirical studies, see Dilling, 2022, as well as the contribution by Frederik Dilling and Amelie Vogler in this book).

5 Solids of Revolution

Solids of revolution represent a common application of integral calculus in school. 3D printing technology can be used to design manipulatives for this concept as well (cf. Dilling, 2020). On one hand, this includes three-dimensional solids made of plastic that can be constructed with little effort using CAD software (Fig. 10, right) (see also Reichenberger et al., 2019). The design process already reveals the connection between the function defining the solid of revolution and the solid itself. The volume of the resulting solid can be determined experimentally by dipping it into a liquid-filled container. Alternatively, the connection between the defining function and the resulting solid of revolution can be illustrated with a 3D-printed tile affixed to a rapidly rotating motor (Fig. 10, left). Both approaches can contribute to the Area Grundvorstellung by showing how a solid of revolution is created from an integral area.

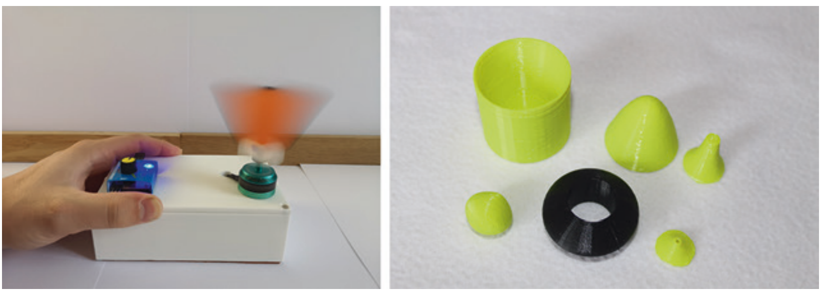

Fig. 10 Manipulatives for teaching solids of revolution

6 Mean Values

The mean values of functions are important in calculus teaching, especially for stochastics applications. 3D printing technology enables the development of new tools to address the Average Value Grundvorstellung (Figs. 11 and 12). Possible manipulatives for this include vertically oriented flat objects with a coordinate system and the graph of a function drawn on them (cf. Dilling, 2020). The objects should contain cavities that can be filled with a colored liquid and be printed with transparent material so the colored liquid is visible from the outside. The cavities inside the objects should be designed so that the area below the function graph is colored first (Fig. 11, left). When a plug is removed, a second cavity should open

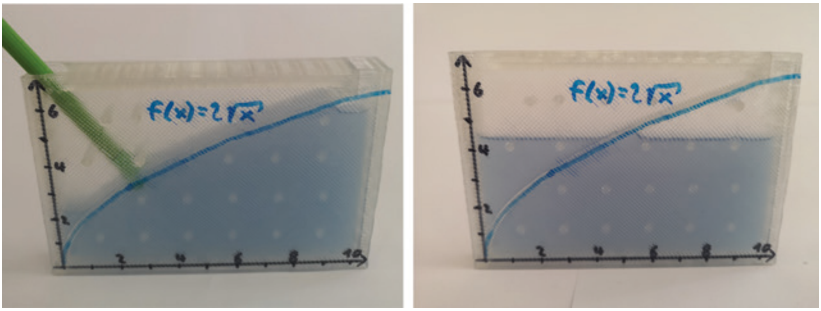

Fig. 11 Manipulative to address the Average Value Grundvorstellung using the function $f(x) = 2\sqrt{x}$

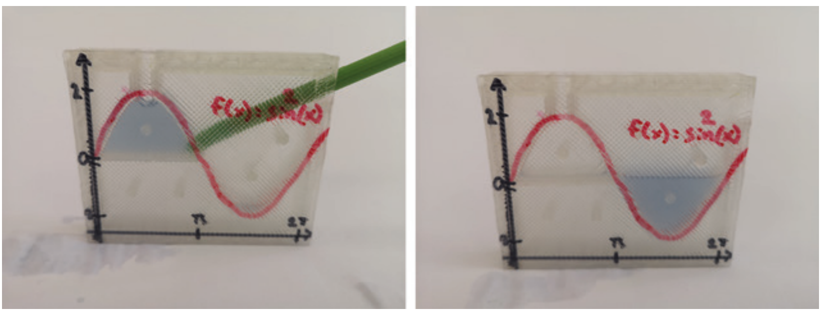

Fig. 12 Manipulative to address the Average Value Grundvorstellung using the function $g(x) = 2sin(x)$

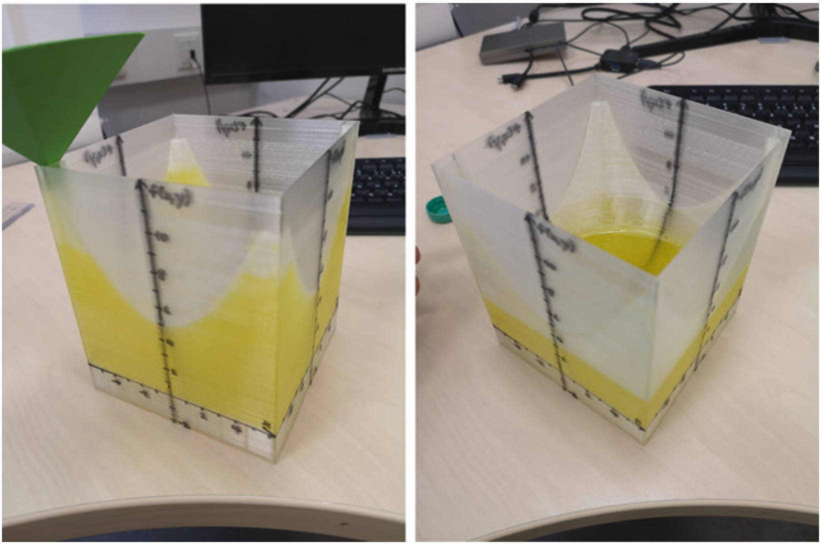

Fig. 13 Manipulative to address the Average Value Grundvorstellung for functions of two variables

and the water should reach equilibrium (Fig. 11, right). This will create a rectangle with the same area as the area under the graph.

The principle can be extended to functions with negative function values as well. A negatively oriented area, i.e., one that is evaluated negatively for the calculation of the integral, can be represented by a hollow into which the water flows before it can average over the entire interval (Fig. 12). The principle can also be applied to functions with two variables, to allow the developed notions to be applied beyond the mathematics learned at school (Fig. 13).

7 Conclusion and Outlook

This article used calculus examples to show how 3D printing technology can provide new approaches to promote the development of foundational mathematics concepts. The subject-matter didactical approach of Grundvorstellungen forms a good basis for the development of manipulatives employing 3D printing technology. The development of manipulatives with and for students and their subsequent use in hands-on learning environments can extend the existing purely analog or purely digital approaches in this field and enable new approaches to mathematical content. Scientific case studies in this context (e.g., Dilling, 2019, 2022; Dilling et al., 2019; Dilling & Witzke, 2020; Pielsticker, 2020) have shown that appropriately designed learning environments can lead to sustainable initial conceptions related to the Grundvorstellungen described in subject-matter didactics. Further controlled scientific research and intervention studies are needed to make reliable statements about the effectiveness of such learning environments.

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Maistaeder—on the Evolution of a Versatile Polyhedron

Robert Päßler

1 Material Mathematical Models

Before describing the development process of a new 3D-printed model, I would like to state my connection to mathematical models in general.

1.1 The Collection of Mathematical Models at the Tu Dresden

Many universities and colleges have collections of material mathematical models. Material mathematical models means often haptic, three-dimensional objects that visualize mathematical facts for cognitive support. They were mostly acquired as teaching aids many years ago. Some collections are presented externally in a vivid way. At the Technical University of Dresden—which has one of the largest collections of mathematical models in Germany with around 500 objects[1]—the

[1] https://tu-dresden.de/kustodie/sammlungen-kunstbesitz/mathematik-naturwissenschaften/sammlung-mathematische-modelle?set_language=en.

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models are presented in illuminated display cabinets in the corridors of the faculty building to arouse the interest of students and visitors. At other universities, such collections have unfortunately been lost in numerous cases or are stored in the basements of institutes and have not been used in academic teaching for some time.

The origin of material mathematical models can be traced back to France. Gaspard Monge (1746–1818) probably already created teaching models for higher mathematics at the end of the eighteenth century. Some institutions are still holding such old models of French origin. The University of Coimbra (Portugal), for example, has replicas of models by Théodore Olivier (1793–1853), a student of Gaspard Monge. However, the majority of the objects that can still be found in the collections today date from the period between 1875 and 1910. The German mathematicians Alexander Brill (1842–1935) and Felix Klein (1849–1925) wanted to do something about the lack of mathematical illustrative objects at the former Technical University in Munich. Under their guidance, the mathematics students created mathematical models in seminars, which were then initially distributed worldwide by the Ludwig Brill company in Darmstadt and later by Martin Schilling's company in Leipzig. Today their models can also be found in Japan or the United States of America.

Some collections were expanded by self-built models. In addition, universities also acquired models from teaching equipment companies. Examples of German model manufacturers from the middle of the twentieth century are the Günter Herrmann Lehrmittelfabrik,[2] which existed in Hofgeismar since 1950, or the Rudolf Stoll KG, which produced mathematical teaching models, among other things, in East Berlin from around the mid-1950s for about ten years.

1.2 Digital Archive of Mathematical Models

Since 2001, Prof. Daniel Lordick has been in charge of the Dresden collection, which is presented online with photos and accompanying texts. In addition, he began an in-depth indexing of the extensive holdings. A grant from the German Research Foundation (DFG) in the area of scientific literature supply and information systems made it possible to expand the web presence. The result was

[2] https://www.herrmann-lehrmittel.de.

DAMM—the Digital Archive of Mathematical Models.[3] The DAMM project began in 2012 and included.

- the digitizing and archiving of mathematical models for freely accessible multimedia presentation and flexible research via the Internet,
- a full inventory of the Mathematical Models Collection at the Institute of Geometry at TU Dresden,
- the development of a concept for visualizing the models, especially with 3-D methods, and
- a deepening of the scientific documentations.

The Saxon State Library—Dresden State and University Library (SLUB)[4] digitized the "grey literature" belonging to the Mathematical Models Collection in Dresden, i.e. catalogues, company publications and other publications on the models, which are now freely accessible in DAMM to complement the models.

At the end of the project, a supra-regional networking of selected data from DAMM was started, which continues to this day and has been expanded since then. For example, DAMM has an interface to the German Digital Library (DDB[5]) and all the data required for a collection can be compiled for transfer to the DDB in just a few steps.

The Geometric Modelling and Visualization working group (GMV[6]) and the DAMM project partner, Prof. Rainer Groh (Chair of Media Design at TU Dresden) are further developing DAMM with the aim of making it available internationally as a research infrastructure. For this purpose, not only models from the Mathematical Models Collection in Dresden, but also those from collections at other locations are being integrated into DAMM. Models from the collection of historical mathematical models at Martin Luther University Halle-Wittenberg[7] were already digitized during the DAMM project period in order to at least digitally close gaps in the collection in Dresden. The long-term goal is to map all material mathematical models in DAMM.

[3] https://www.mathematical-models.org.

[4] https://www.slub-dresden.de.

[5] https://www.deutsche-digitale-bibliothek.de.

[6] https://tu-dresden.de/mn/math/geometrie/lordick.

[7] https://www2.mathematik.uni-halle.de/modellsammlung.

1.3 Integration of Models into Teaching

The Geometric Modelling and Visualization working group and the Chair of Mathematics Didactics at the TU Dresden aim to develop and evaluate innovative and action-oriented teaching concepts for teaching mathematics to students by incorporating and further developing the collection of mathematical models.

Prof. Andrea Hoffkamp, responsible for teacher training and also one of the scientific directors of the Erlebnisland Mathematik in Dresden,[8] develops and evaluates learning environments and teaching concepts that are geared towards innovation. The learning workshop "Discovering Mathematics", for which Petra Woithe is responsible, works on teaching examples for problem- and activity-oriented work in mathematics lessons. The result is a collection of teaching materials and project ideas.

The goal is a dedicated use of the material, mathematical models for teaching and, beyond that, an expansion of the collection. The aim is to create a systematic link between object-related research and action-oriented didactics via the material models. A historical-genetic approach is particularly important for learning and teaching mathematics. In this way, it is shown that mathematics is a dynamic science with a long historical development, which is naturally also reflected in the historical collection of models.

School teaching is often accused of an excess of calculation ("calculating without understanding"). In the training of prospective teachers, we place a special emphasis on understanding-oriented approaches. Contributions are to be made above all in the subjects of geometry, didactics of geometry, didactics of linear algebra and the workshop "Discovering Mathematics". As a result, the design and production of own models will be encouraged, both in terms of handicraft and digitally-virtually. Based on this, teaching designs for the methodical revival of mathematics teaching according to understanding-oriented approaches are to be developed.

At the same time, an emancipated approach to digital media is to be taught. DAMM as a digital research infrastructure with 3-D visualization available online is an excellent test platform in the context of stronger and more meaningful integration of digital media in schools. Expanding DAMM as a tool for teaching is one of the long-term goals of the project. Digital resources have their origins in analogue approaches. A link between the historical collection and digital methods

[8] http://www.erlebnisland-mathematik.de.

for making and designing models is particularly valuable, as it simultaneously promotes awareness and understanding of the development of analogue media. In terms of new models, the use of modern technology, such as the 3-D printer or working with the laser cutter, is particularly appealing. Special (mathematical) methods (e.g. coordinate geometry) become the focus of interest. For this content, we are cooperating with the Makerspace of the SLUB.

2 Maistaeder

During a group meeting Margarete Ketelsen—a former student assistant of the GMV – told us about the website polytopia.eu,[9] where you can "adopt" a polyhedron. So, we wanted to adopt a polyhedron and were looking for a suitable one on politopia.eu. Polyhedra with eight or fewer vertices were taken. There are 2606 combinatorically different types of polyhedra with 9 vertices. What exactly this means follows shortly. We were looking for a "beautiful" polyhedron. Maybe there is a very symmetrical polyhedron that consists of many triangles? And yes! We found a polyhedron with the number 901638 – the meaning of this number is not clear. This polyhedron consists of a pentagon, which is axisymmetrical, a kite quadrilateral and otherwise only triangles. We dubbed it "Maistaeder" (Fig. 1).

The name "Maistaeder" refers phonetically to Gustl Bayrhammer's character "Meister Eder" in "Master Eder and his Pumuckl", a German children's series created by Ellis Kaut. To appeal to certain slang sounds, the name was modified a little, from Meistereder to Maistaeder. So, the name has no mathematical reference. Whether "Meister Eder" respectively Gustl Bayrhammer, Ellis Kaut or Pumuckl were good at mathematics is not known (Fig. 2).

3 Mathematics Behind Maistaeder

After the GMV working group had created a "new" polyhedron, we began to explore the Maistaeder. In the process, 3-D prints were made for some of the applications found, with which I would like to present both the development of the activities on the polyhedron and introductory mathematical considerations.

[9] https://www.polytopia.eu.

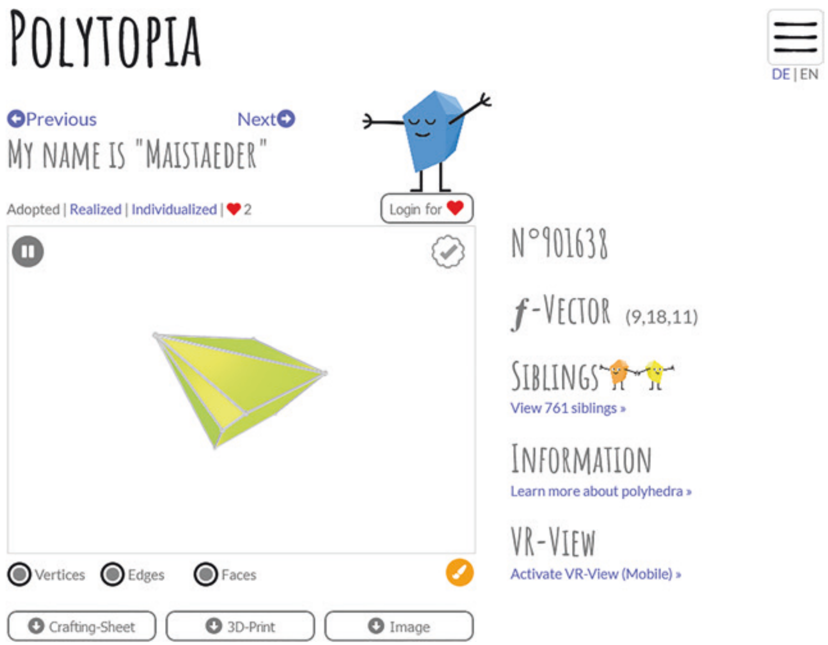


Fig. 1 Screenshot of the Maistaeder site of polytopia.eu

In a self-explanatory way, simple facts about polyhedrons – as defined by a finite number of planar surfaces – can be introduced with the Maistaeder in a playful way (Fig. 3).

Polytopia.eu provides an automatically generated craft sheet to cut out the polyhedron net and build them yourself (Fig. 4).

In this way, it is possible to intuitively learn about polyhedron nets. And of course, concepts of polyhedra can be introduced: Platonic solids, regular polyhedra, symmetry or Euler's polyhedral formula.

3.1 Combinatorics

The f-vector of the polyhedron indicates how many vertices, edges, and faces it has. Polyhedra are called "siblings" if they contain the same number of vertices,

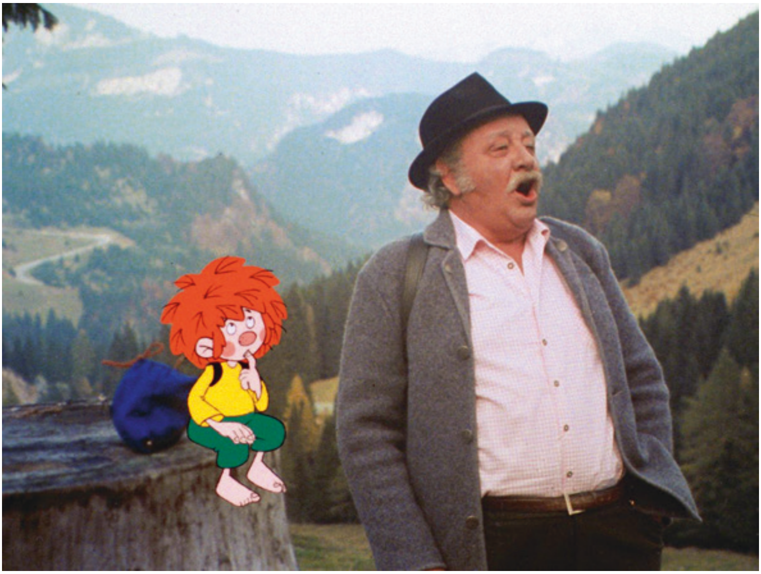
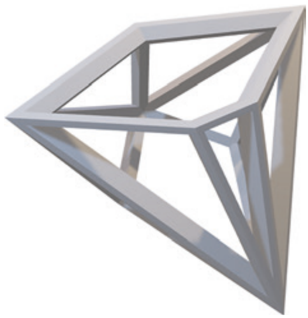


Fig. 2 Scene from Meister Eder and his Pumuckl (©Ellis Kaut, Barbara von Johnson, Infafilm Manfred Korytowski GmbH)

Fig. 3 Rendered edge model



edges and faces, hence the same f-vector. Similar to human siblings, some polyhedral siblings do look alike each other while others have a completely different form (Fig. 5).

The question can be asked: How many polyhedra with n vertices are there anyway? There is certainly no easy access to to the relation between n and the

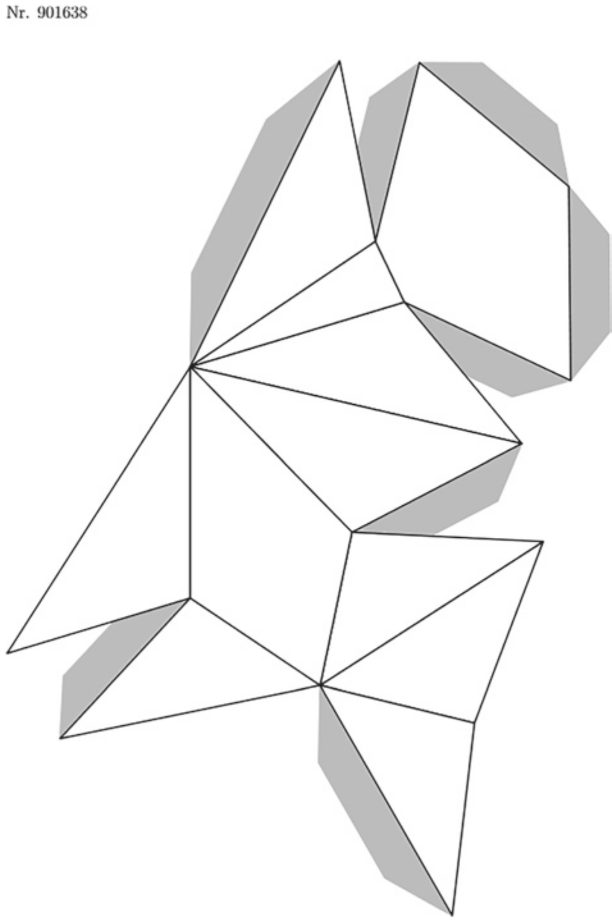


Fig. 4 Automatically generated craft sheet by polytopia.eu

number of polyhedra with n vertices. In higher mathematics, however, upper and lower bounds can be found. For lower mathematics, the question of the number of polyhedra with n vertices opens up the possibility of talking about exponential growth. By the way: 4 vertices – only the tetrahedron, 5 vertices – two possible polyhedral (pyramid with quadrangular base or double tetrahedron), with 6 vertices it becomes more difficult. In the following table, the number of possible

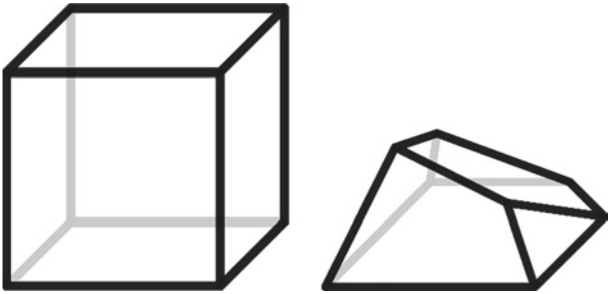


Fig. 5 A cube and a "sibling" (Source: https://www.polytopia.eu/glossar?id=901638#geschwister, called 25.01.2022)

polyhedra is given for the number of vertices. It is easy to see that the number of polyhedra grows rapidly.

Edges	Number of polyhedra
4	1
5	2
6	7
7	34
8	257
9	2.606
10	32.300
11	440.564
12	6.384.634
13	96.262.938
14	1.496.225.352
15	23.833.988.129
16	387.591.510.244
17	6.415.851.530.241
18	107.854.282.197.058
19	???

A model for teachers has to be more robust than a cardbord model, which can be made by polytopia.eu using craft instructions. So, we printed a plaster Maistaeder model. However, this model looks unappealing (Fig. 6).

In addition, three edge models were created. One model with wider edges and one model with narrow edges (Fig. 7).

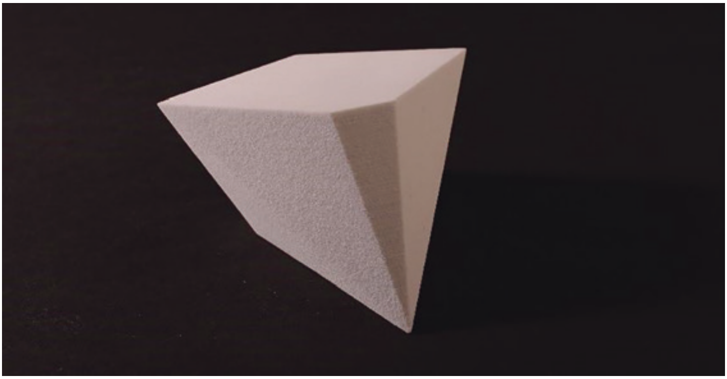


Fig. 6 The first plaster model

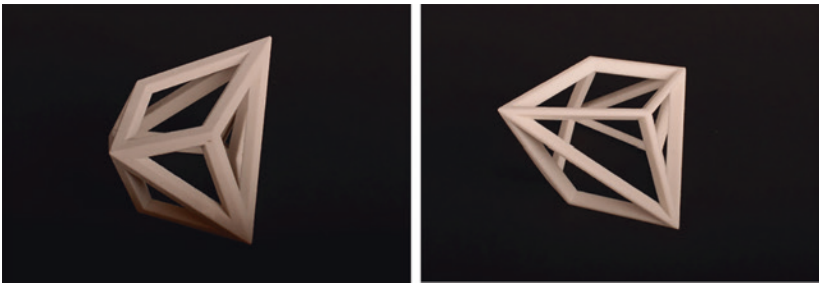


Fig. 7 Left: Maistaeder with wider edges. Right: Maistaeder with narrow edges

And one further model with deep edges (Fig. 8).

The deep edges model shows a similar smaller Maistaeder inside. This opens up conversations about combinatorial types.

Each polyhedron can be geometrically realized in different ways. It can be big or small, and its shape can also be changed, as long as the structure of the vertices, edges and surfaces remains the same. This structure, which is the number of edges meeting at the vertices, and the number of vertices belonging to each surface, is called the combinatorial type of a polyhedron. We call two polyhedra combinatorically equivalent if they possess the same combinatorial type.

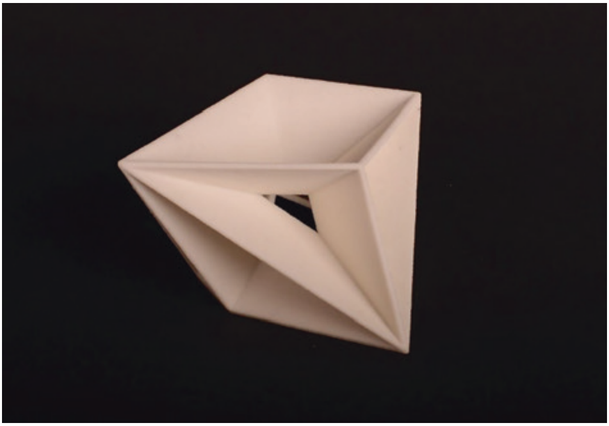


Fig. 8 Maistaeder with deep edges

3.2 Geometry

Every polyhedron has an infinite number of different geometric interpretations. The website polytopia.eu uses a unique realization. They give the canonical polyhedron form. This means that every polyhedron has a midsphere or intersphere, which is tangent to every edge of the polyhedron. Not every polyhedron has a midsphere, but for every polyhedron, there is a combinatorically equivalent polyhedron that does have a midsphere. Thus, in connection with the Maistaeder, the mathematical concept of incircle and circumcircle or their equivalents in space can be dealt with. And of course, also the concept of the midsphere can be investigated. The midsphere creates incircles with respect to the side faces of the polyhedron. Therefore, we created two models: one with narrow circles and one with wider circles (Fig. 9).

However, it is not yet possible to recognize the midsphere by using these models, the wide circles only can give an indication. Two plaster models were created: just the (cut-off) midsphere and the midsphere with the Maistaeder (Fig. 10).

3.3 Graph Theory

This realization by midsphere is called the Koebe-Andreev-Thurston realization, known for the circle packing theorem: “Every planar graph has a circle packing

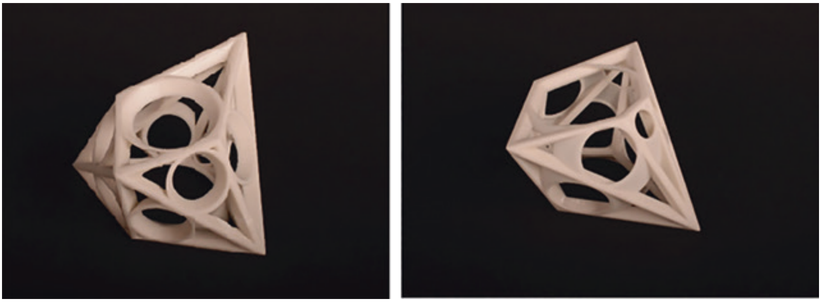


Fig. 9 Left: Maistaeder with narrow circles. Right: Maistaeder with wider circles

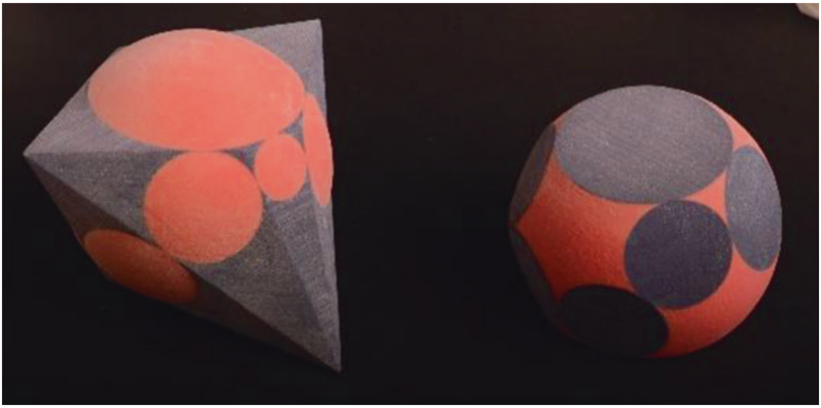


Fig. 10 Left: Maistaeder with midsphere. Right: (cut off) midsphere by Maistaeder

in the plane." One stronger form of the circle packing theorem, on representing planar graphs by systems of tangent circles, states that every polyhedral graph can be represented by a polyhedron with a midsphere. Other mathematical topics can therefore be motivated with the Maistaeder: planar graphs or the circle packing theorem.

4 Conclusion

We ourselves were very surprised that we could find so many applications after a few reflections on the Maistaeder. A momentum of its own developed: a new, printed model was tested for possible uses by various people independently of each other. And surely more uses can be found that I would like to hear about.

Unfortunately, we have not yet been able to evaluate the Maistaeder in school use. But we are confident that we will be able to do so.

In general, it can be said that the achievements with the Maistaeder are not an isolated case. Every model, no matter how historic, can still be used today in school mathematics as well as in higher mathematics. We will continue to work on making school lessons in particular more descriptive, which will also become easier and more specific with the help of 3D printing possibilities.