Historical Relations of Mathematics and Physics—an Overview and Implications for Teaching

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2.1 The Development of the Modern Sciences

If one studies the history of the modern sciences, it becomes clear that they were inextricably linked for a long period of time. The division into individual disciplines took place relatively late and further development was still characterized by mutual influence. If we look at physics as an example, it can be noted that humans have always dealt with the physical world. For a long time, however, this was done only by means of observation and in a philosophical way, which is why it was referred to as natural philosophy. At the same time, the ancient Greek natural philosophers were already using mathematics to describe nature. For the Pythagoreans,[1] the natural numbers were the reference point by which they oriented themselves. For example, the distances between the Earth and the Moon and the other planets were supposed to correspond to certain numerical ratios that were considered harmonious. Similar attempts can also be found in the early

[1] Pythagoreans are the followers of the teachings of Pythagoras of Samos (ca. 570–510 BCE), to whom significant contributions to mathematics are attributed.

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work of Johannes Kepler (1571–1630) to describe the structure of the solar system (cf. Galili, 2018). Of course, this is more a kind of numerical mysticism than a real mathematical approach.

Only after Galileo Galilei combined theoretical considerations, mathematical descriptions and systematic experimental investigation have we used the term physics in the modern sense.[2] However, the conceptual separation of the sciences was by no means accompanied by a clear separation between the actors in the two subjects. Thus, in the case of many historical personalities (e.g., Newton, Euler, Lagrange, Fourier, or the Bernoullis), it is not obvious whether they should be categorized as mathematicians or physicists (Uhden, 2012). This chapter attempts to show where the common roots of physics and mathematics lie and what pedagogical conclusions can be drawn from them. Since the interconnections are incredibly rich, however, we will limit ourselves to those subject areas that will be addressed later in the book, in the chapters comparing educational methods.

2.2 Scientific Problems as a Driving Force of Mathematics

One of the oldest branches of mathematics is geometry. Even from the name, which originates from the Greek for "earth measurement," it is clear that there is a close connection between mathematics and the physical world (Hischer, 2012, p. 1). It can also be shown for other fields of mathematics that they developed from the problem of concrete application. The statement that the driving force of mathematics has always been to solve problems can be traced back to David Hilbert (Kjeldsen, 2015). For the further development of mathematics, the formation of mathematical concepts was essential; however, they often do not originate from mathematics itself but represent external influences that can frequently be traced back to physics. If one follows individual concepts back to their origins, it also becomes clear why the mathematical description of nature is so successful. This phenomenon should by no means come as a surprise, as it follows directly from the historical development of mathematics (Kjeldsen, 2015). It should also be noted that only a fraction of the available mathematical concepts are used in physics at all, so that their successful application may seem even less mystical (Galili, 2018).

[2] See also Chapter "The Mathematization of Physics Throughout History" in Volume 1 (Tran et al., 2020).

The increasing mathematization of physics, however, also brought criticism. Representatives of the discipline saw themselves increasingly excluded from participating in its further development in the post-Newton era, which was also due to the fact that there was soon no room for an intuitive approach:

> The counter-intuitive effects of the mathematization of physical phenomena only began to be perceived with the development of dynamics, that is, the mathematization of the concept of force, as the cause of change in the state of motion. (Gingras, 2001)

Even in modern times, such turnarounds still took place, as seen in the example of Nobel Prize winner Johannes Stark, who after 1913 increasingly turned away from Planck's quantum hypothesis, of which he was initially one of the earliest supporters. Historians of science attribute this to the increasing mathematization of physics and not to the unusual new physics that disturbed Stark's experimental work[3] (Metzler, 2019).

2.3 Historical examples and educational remarks

The following examples of the common problem-oriented foundations of mathematics and physics were selected based on the school-relevant subjects in this volume. Here we want to give a basic overview of the historical development of the disciplines and offer some preliminary educational conclusions. A comprehensive analysis from a historical point of view should be left to special literature. A detailed didactic discussion from a modern point of view will be provided in Part B of this volume.

2.3.1 Numbers, Quantities, and Units

Counting objects and quantifying certain properties are basic practices that have been used by all cultures since the earliest times (Himbert, 2009). Practically inseparable is the concept of measurement, the objective of which is “the expression of characteristics of systems in terms of numbers” (Himbert, 2009, p. 25).

[3] Stark finally turned to so-called German physics, which was close to National Socialism, and is therefore one of the most controversial figures in physics in Germany (Hoffmann & Walker, 2006).

The earliest measurements were also based on pure counting in the form of whole numbers that referred to a certain unit. Then, there was the handling of ratios of these whole numbers and corresponding quantities. The extension of the rational numbers to the real numbers did not take place until the nineteenth century (cf. Jahnke, 2003).

As far as the reference units themselves are concerned, they have only been gradually standardized worldwide since the introduction of the SI units (e.g., the meter and the kilogram in 1889). In the process, an increasing degree of abstraction can be observed. For instance, a meter was originally defined in France in 1790 as one ten-millionth of the Earth's meridian quadrant (the distance between the equator and the pole along a meridian arc). A decoupling from this natural measure then occurred in 1889 with the introduction of the primordial meter, intended to make the base unit independent of the inaccuracies of measurement and take into account that different meridian arcs have different lengths due to the irregular shape of the Earth. In the definition valid since 2019, all SI units are now defined by physical constants (according to the current theory), which means that the meter is indirectly defined based on the distance light travels in a vacuum within a second.

However, for the determination of physical laws, as the relationship of various physical quantities, the available instruments were hardly suitable for most of history. An exception to this is classical astronomy, which, through the simple magnification of its instruments, achieved highly accurate angle measurements and precise models of the world in the mathematical sense even before the invention of the telescope.

An example that illustrates the struggle for a suitable instrument of measurement is Galileo Galilei's investigations of free fall.[4] Even after slowing down the process of falling by relocating the motion to the inclined plane, his time measurements were still inaccurate because he used his own pulse as a standard. Thus, it became necessary to determine the time differences and ratios via the detour of mass differences and ratios, which resulted from the uniform outflow from an elevated vessel.

Today, as well, the concept of quantities as "measurable natural phenomena" (Hischer, 2012, p. 138) is central in physics. Physical quantities in the classroom go far beyond those that are already familiar from everyday use, such as length or time. Research shows that the ability to estimate quantities is directly related to

[4] See also Volume 1, Chapter "The Mathematization of Physics Throughout History" (Tran, Nguyen, Krause, & Kraus, 2020).

whether the respective quantity can be perceived with the senses. Thus, not only can basic quantities such as mass, length, and temperature be estimated accurately through intuition, but so can derived quantities such as force and velocity. Complex quantities, on the other hand, for which sensory access is not possible, are predominantly overestimated—often by more than one order of magnitude. (Stinken, 2015; Stinken-Rösner, 2015).

Increasingly, even basic physical quantities are affected by a phenomenon that used to be limited to complex measurements: their determination is carried out with complex measuring instruments whose functions are almost impossible to understand. Digital instruments for voltage or electric current are long-established, as the measuring principle remains hidden for students in most cases. Due to the increasing use of electronic measuring tools, they are also found for basic quantities (e.g., electronic thermometers or calipers[5] with digital displays). These tools obscure the underlying process of comparison with a reference or at least direct the focus towards a purely numerical value (see also the chapter "Numbers, Quantities & Units").

While the use of a "black box variant" is avoidable for some instruments, this option is not available for other measuring instruments—a Geiger counter, for example. Pedagogical benefits can be gained by using a historical approach to reveal that each instrument once represented an "open box" in its early use in science (Pinch, 1985).

Also, the hasty introduction of abstract models, such as the interpretation of temperature as the mean kinetic energy of the particles of a system, can have a detrimental effect on the understanding of the quantity in question. Presenting the quantity in the form of a reconstruction based on history can help students understand both the quantity and the process of measurement itself. The intention is not to give a historically accurate presentation of the development of the concept of temperature, but a step-by-step approach, from a qualitative view, via quantitative experiments and laws to embedding in the network of physical theories (Mantyla, 2007).

2.3.2 Equations

Physics today is often associated with the use of formulas or equations, in addition to conducting experiments. However, the concept of the equation as we know

[5] A caliper is an instrument used to accurately measure the dimensions of an object, see: https://en.wikipedia.org/wiki/Calipers.

it today is much younger than the natural sciences; even modern physics, established by Galileo Galilei, could not initially make use of the language of formulas. The modern formula notation itself was first established by Leonhard Euler (1707–1783).

Despite the significant advantages of the new notation, it was not used widely in contemporary publications. For example, Isaac Newton explicitly refrained from using the new analytical method out of concern that it would unnecessarily hinder the dissemination of his discoveries among experts. Although the geometric methods were also difficult to understand, their use was widespread, so they at least had the advantage that people were used to them. With Euler, Lagrange, D'Alembert, and Laplace, the analytical approach we are accustomed to today found its way into physical publications (Kuhn, 2016). The original formulations of Newton's *Principia* seem unfamiliar from today's perspective (Fig. 2.1). The further development of mathematics thus caused the phenomenon of social selection, through which participants were excluded from the discourses of natural philosophy (Gingras, 2001).

Newton's work is also suitable for emphasizing the educational value of formula representation due to its compact presentation and easy applicability. For this purpose, a comparison between the (original) formulation and the representation using modern notation is suitable (Tab. 2.1).

Only from Euler onward are the great works accessible and familiar to today's scientists who do not have extensive specialist knowledge of their historical genesis. At this point, it should also be emphasized that Euler himself wrote secondary school textbooks to present the elementary basics of mathematics in a new way. Among the many innovations that Euler introduced into mathematics were the signs for differences Δ, sums Σ, binomial coefficients $\frac{p}{q}$, and the integration limits for the integral sign. Furthermore, the names of the sides of a triangle ABC with the corresponding minuscules a, b, and c, as well as the common notation for the angular functions sine, cosine, and tangent go back to him (Mattmüller, 2010, p. 178–180).

Euler successfully solved a number of complex physical problems with his formula notation. They range from the disturbance to the orbits of comets by the gravitational influence of the planets, the propagation of sound, and the speed of sound, to mechanics in general and shipbuilding in particular—to name but a few examples (cf. Gautschi, 2008).

This development in the representation of physical relationships points to one of the roles of mathematics in physics: mathematics functions here as the language of physics. In addition, Uhden (2012) points out two more roles of

Quod ſi pondus *p* ponderi *P* æquale partim ſuſpendatur filo *Np*, partim incumbat plano obliquo *pG*: agantur *pH*, *NH*, prior horizonti, poſterior plano *pG* perpendicularis; & ſi vis ponderis *p* deorſum tendens, exponatur per lineam *pH*, reſolvi poteſt hæc in vires *pN*, *HN*. Si filo *pN* perpendiculare eſſet planum aliquod *pQ*, ſecans planum alterum *pG* in linea ad horizontem parallela; & pondus *p* his planis *pQ*, *pG* ſolummodo incumberet; urgeret illud hæc plana viribus *pN*, *HN*, perpendiculariter nimirum planum *pQ* vi *pN*, & planum *pG* vi *HN*. Ideoque ſi tollatur planum *pQ*, ut pondus tendat filum; quoniam filum ſuſtinendo pondus jam vicem præſtat plani ſublati, tendetur illud eadem vi *pN*, qua planum antea urgebatur. Unde tenſio fili hujus obliqui erit ad tenſionem fili alterius perpendicularis *PN*, ut *pN* ad *pH*. Ideoque ſi pondus *p* ſit ad pondus *A* in ratione, quæ componitur ex ratione reciproca minimarum diſtantiarum filorum ſuorum *pN*, *AM* a centro rotæ, & ratione directa *pH* ad *pN*; pondera idem valebunt ad rotam movendam, atque ideo ſe mutuo ſuſtinebunt, ut quilibet experiri poteſt.

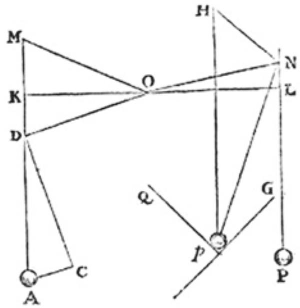

Fig. 2.1 Extract from Newton's Principia. Described and graphically represented is the decomposition of forces. The linguistic representation shown here, based on geometric considerations, is typical of Newton's approach. (Newton, 1726, p. 16, license: public domain)

Tab. 2.1 Overview of different ways of representing the second Newtonian axiom

Representation	Example
Original	Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur
Translation	The change of movement is proportional to the applied moving force and follows the direction of the straight line in which that force acts
Formula representation	$\dot{\vec{v}} \propto \vec{F}$

mathematics, through which the different uses and interdependencies of the disciplines can be described. The roles[6] he describes are:

- the pragmatic perspective
- the structural function.

Both functions occur particularly when dealing with equations, especially when considering the educational context. From the pragmatic perspective, mathematics serves as a tool for solving physical problems: often, to students, it is only a matter of finding the appropriate formula to reach a result as quickly as possible.

2.3.3 Functions

The first uses of functions or functional relations can be traced back 4000 years to the combinations of the squares and cubes of the natural numbers. One of their oldest uses is in astronomy, where tables were compiled with positional data that can be understood as a function: position as a function of time. In his astronomical work *Almagest*, Ptolemy also presented the chords of a circle as functions of angles. These tables were calculated and used as an element of the description of nature so that it is obvious to regard them as an empirical source of the function concept. However, the subsequent attribution of the concept of function to these early efforts is an anachronism, since no conceptual mathematical development took place. For Ptolemy, chords were simply lines in a circle, and the transition from angle to chord was not a mathematical process (Kjeldsen, 2015). In addition to tabulated values, early precursors of today's understanding of functions can also be found in the form of function graphs. The earliest representation is from the year 950 (Hischer, 2012, p. 131) and shows the celestial latitude of bodies like the Sun, the Moon, and the planet Saturn, as a function of time (Fig. 2.2). Contrary to modern representations, the horizontal time axis, which is divided 30 times, is not uniform for all celestial bodies; rather, each body is given its own time axis. This means that no temporal relations between the celestial latitudes of the bodies can be derived from the graph (Funkhouser, 1936).

[6] In addition, he mentions the “communicative perspective”, which we have already discussed under the keyword of language.

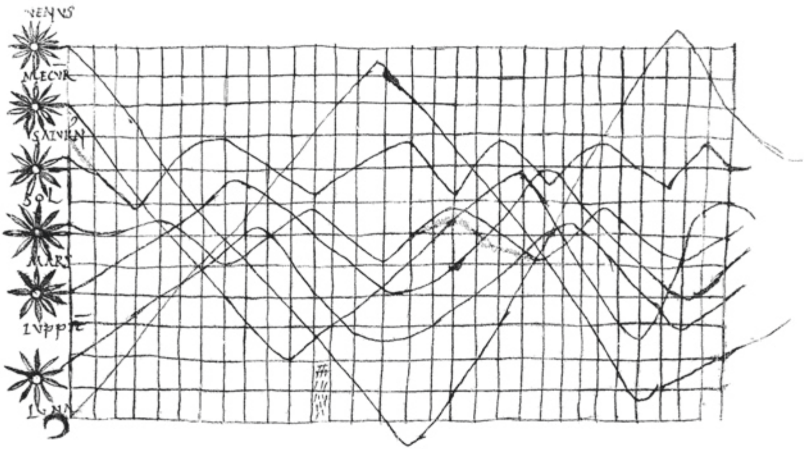

Fig. 2.2 The earliest known graphical representation of a function. It shows the celestial latitude of important celestial bodies as a function of time. (License: public domain)

Nicholas Oresme (1323–1382) also made important contributions to the representation of time-dependent quantities. In his time, Merton College at Oxford was discussing motion sequences and velocities, which in the case of uniformly accelerated motion ultimately led to the so-called Merton rule:

> If a body is uniformly accelerated in time t from the initial velocity v_0 to the final velocity v_1, then the distance travelled is: $s = \frac{v_0+v_2}{2}t$. (Quoted according to Hischer, 2012, p. 137, Authors' translation)

In contrast to earlier attempts to solve the problem, Oresme used a geometric approach. The visualization of his ideas reveals the functional thinking behind them. By plotting the time on the horizontal axis and the distance travelled by the body on the vertical axis, a Cartesian coordinate system is created here for the first time, and its representation also allows quantitative statements to be made. Historically, Orseme's contribution can be traced back to Galileo Galilei's investigations of free-falling bodies.

However, the real concept of function only came into existence with the invention of the mathematical operation known as differentiation. Thus, the development of the concept of function is directly linked to the development of calculus, which in turn was strongly influenced by physical applications. The term "function" also appears for the first time in a geometrical consideration of curves by

Gottfried Wilhelm Leibniz (1646–1716) (Kjeldsen, 2015). Kjeldsen (2015) summarizes the early development of the term as follows:

> To be sure, the calculus was invented primarily as a means to study curves. After the invention of analytic geometry, which attaches a curve to any equation in two variables, it became necessary to develop means for studying their properties, and so infinitesimal techniques were developed as early as the 1630s. But even in this early period of pre-calculus, curves were often connected to physical phenomena. Indeed, since ancient times some curves (like Archimedes' spiral) had been defined by a continuous motion and after Galileo's investigations of kinematics this became a standard way to think about curves. And as curves were considered from a kinematic point of view, the idea of a variable quantity became a central concept in the emerging field of analysis.

Curves in this context are, as in Euler's sense, lines drawn freehand or mechanically with special instruments, which has no relation to the modern, abstract concept of curves.

The concept of function finally broke away from its close connection to geometry, through the contributions of Johann Bernoulli, among others. Euler gave the following definition in 1748:

> A function of a variable quantity is an analytic expression which is composed in any manner of this variable and of numbers or constants. (quoted from Kjeldsen, 2015)

For Euler, a function was thus a formula to which algebraic operations could be applied, as could transcendental operations such as cosine or sine. Again, prompted by a physical problem—in this case, a vibrating string—Euler, in a discussion with D'Alembert, succeeded in further developing the concept of function (for details see Kjeldsen, 2015).

Of particular interest at this point is the dispute between the two scientists, which can be traced back to their different basic positions. While D'Alembert took the position of an internalist mathematical rigorist, Euler was an externalist for whom mathematics must be designed to solve physical problems.

Again, Kjeldsen and Lützen's (2015) statement on the interaction of mathematics and physics is significant for us:

> So physics forced Euler to extend the concept of function, and it is hard to imagine that such an extension could have been suggested by mathematics itself.

This demonstrates the long tradition of the strongly empirical, problem-solving-oriented use of functions. Hischer (2012, p. 158, Authors' translation) summarizes:

> What is remarkable about all these examples is that (from our point of view) these "functions" were not yet the subject of the respective investigation but were only a "means to a purpose". They were therefore not—as is the case today, for example, in analysis or function theory—the object of consideration, but only supports or even tools for considering a non-mathematical fact.

Time-dependent plots, mainly of movements, were not only the first but are still the most common functional graphs in modern physics education. This is also important for assessing the general educational function of physics, as such time-dependent representations are among the most common forms of data visualization in the media (Hischer, p. 134). Over the course of the book (see the chapter "Functions") it will become clear how closely today's physics teaching is bound to these historical mathematical concepts and practices.

2.3.4 Vectors

The vector has a long tradition as a central mathematical concept. In its history, physical phenomena have played an important role. Thus, it was originally the goal of geometry to describe physical space as accurately as possible (see Struve, 1990). Until the seventeenth century, only scalar quantities were used for this purpose. Over the course of the seventeenth century, physics changed so that quantities such as velocity, force, momentum, and acceleration were used as directed and thus vectorial quantities. Later, the use also spread to electricity, magnetism, and optics (cf. Crowe, 1967).

In this way, the concept of vector developed from the interplay between mathematics and physics. In particular, physical problems played a significant role in terms of motivation:

> The vector calculation was developed in a long historical process, mainly due to the need for a geometric calculation and the requirements of physics. (Filler, 2011, p. 85, Authors' translation)

Historically, one of the most significant physical problems related to vectors was the combination of several forces and velocities acting in different directions. These considerations led to the development of the concept of vectors. At that time, however, the term did not refer to vectors in the modern sense as a class of arrows of equal length and direction—instead, vectors were situated in Euclidean space. The addition of two vectors with the same origin was defined as one vector with the same origin and extending to the opposite corners of the parallelogram defined by the two vectors. Simple ideas about the parallelograms of forces or velocities were employed in ancient Greece and were widely used in the sixteenth and seventeenth centuries, although they were not related to the concept of the vector. Vectors then helped to integrate this approach into a more global concept (cf. Crowe, 1967).

In the further history of the concept of vector in mathematics, ongoing exactifications were made. In particular, the situating of the arrows was omitted and vectors were considered from then on as equivalence classes of parallel arrows of the same length. However, this also had the consequence that certain directed quantities from physics, such as forces, could no longer be simply interpreted as vectors in the mathematical sense (cf. Wittmann, 1996). Nevertheless, further development and formalization up to the concept of vector space led to new fields of application (e.g., in computer science and economics) (cf. Filler, 2011).

The deep historical connections of mathematics and physics in the context of vectors should, according to Laugwitz, lead to integrated teaching:

> The age-honoured motivations for vector addition (parallelogram of forces), of the inner product (work), and of the vector product (moment) should of course be mentioned. They are still useful, if only for euclidean three-dimensional space. (Laugwitz, 1974, p. 245).

2.3.5 Derivations and Integrals

Differential and integral calculus is another mathematical domain that has a strong connection to physics in its historical development. It was developed almost simultaneously and independently by Gottfried Wilhelm Leibniz and Isaac Newton. In this section we want to deal in particular with Newton's theory—a historical excursus of the work of Leibniz is presented in the chapter "Differential Calculus Through Applications" of this volume.

In his work *De methodis serierum et fluxionum*, written in 1670/71, Newton develops an algorithm for quantities that flow over time. This theory, motivated by a physical idea, considers the trajectories of moving objects, such as the motion of a point, which results in a line or the motion of a line, which results in a surface. Newton calls the moving quantities "fluents." Their instantaneous velocities are called "luxions." The infinitely small increase in an infinitely small interval of time is called a "moment." According to Newton, moments behave like fluxions because fluxions would remain constant in an infinitely small interval of time; therefore, moment and fluxion are proportional (cf. Jahnke, 2003).

Thus, Newton's idea has a fundamentally physical character and refers to continuous motion. Furthermore, it has multiple applications in physics. For example, he used integral calculus to deduce Kepler's laws from his own laws of motion (Newton's laws). Leibniz's theory of calculus was also based on references to reality, although with fewer explicit references to classical physical problems.

In the time that followed, many physical theories were built on the basis of calculus. For example, Leonard Euler dealt extensively with physical questions and introduced many forms of notation that are still in use today (cf. Kjeldsen & Lützen, 2015). Other important milestones were the emergence of theoretical mechanics in the eighteenth century (particularly by Maupertuis, Euler, and Lagrange) with the principle of minimal effects (today, Hamilton's principle) and the study of boundary value problems in mathematical physics at the beginning of the nineteenth century (particularly by Green, Gauss, and Dirichlet).

The historical connection is still visible today in mathematics education. Physical applications from the field of kinematics can be found in many textbooks and the principle of continuous motion with instantaneous velocity is often used to introduce differential calculus.

2.3.6 Probability and Statistics

Physical measurements are, related to the corresponding mathematical theory, always affected by errors.[7] To move from experimental data alone to a physical theory is therefore impossible from a practical standpoint. Rather, a theory—mathematically formulated and enabling quantifying statements—must exist in advance in order to be able to classify the results of an experiment. We call this interplay of theory and experiment the "experimental method."[8]

[7] See also Volume 1, Chapter "On the Relationship between Mathematics and Physics according to Günther Ludwig" (Geppert et al., 2020).

[8] See also Volume 1, Chapter "The Mathematization of Physics Throughout History" (Tran et al., 2020).

However, even this methodology does not permit any quick conclusions with regard to disproving theories. The aforementioned fact that measured values are never exact makes direct conclusions impossible. The comparison between the predictions of the theory and the results of the experiment can take place only after the results have been analyzed with statistical methods. In contrast, Ernest Rutherford formulated his attitude toward the statistical analysis of measured data as follows: “If your experiment needs statistics, you ought to have done a better experiment.”

Especially in astronomy and cosmology, it has always been common to draw conclusions based on uncertain data—as it is necessary today, too, contrary to Rutherford's statement. As in many other fields, history shows that there has been a genuine interaction between astronomy and statistics that has fostered the further development of both disciplines (Coles, 2003).

The starting point of mathematical probability theory is often associated with Jacob Bernoulli, who devoted himself, among other things, to game theory. Thus, he worked out the basics of his theory for large numbers, as occur when throwing dice or randomly drawing different-colored balls from a bag. He was aware that the certainty of statistical statements would increase with the number of observations. However, Bernoulli also found it obvious that his observations on probability theory could not be applied to diseases or the weather, where the reasons for certain developments remained hidden (Stigler, 1986, pp. 63–65).

As early as Galileo, the first approaches to the weighting of measured values can be found, in which values with high uncertainties were given a lower weighting. Later, Daniel Bernoulli and John Michell, two astronomers, investigated whether certain celestial phenomena could be reconciled with random patterns. Bernoulli investigated this concerning the inclination of the orbital planes of the planets, while Michell asked the same question regarding the distribution of the stars on the celestial sphere (Sheynin, 1984; Coles, 2003).

Statistics, and subsequently astronomy, underwent a very significant further development with the application of the least-squares method by Carl Friedrich Gauss. This led to the rediscovery of the dwarf planet Ceres, of which only three individual observations had been previously made (Bruno & Baker, 1999). The connection between the least-squares method and astronomy (or the earth sciences) is described by Stigler (1986, p. 16–17) as follows:

> The development of the method of least squares was closely associated with three of the major scientific problems of the eighteen century: (1) to determine and represent mathematically the motion of the moon; (2) to account for an apparently secular (that is, nonperiodic) inequality that had been observed in the motion of the planets Jupiter and Saturn; and (3) to determine the shape or figure of the earth.

Also, it was Lambert Adolphe Jacques Quételet, a scientist who was also an astronomer, who organized the world's first conference on statistics in 1953 (Coles, 2003). Today's physics and astronomy cannot be imagined without the statistical consideration of measurement errors, which has also found its way into the classroom, as exemplified in the chapter "Lesson Plan on Statistics."

David Hilbert's call for an axiomatization of probability theory was equally closely linked to physics. Its temporal proximity to essential progress in the fields of statistical thermodynamics and mechanics allowed him to call probability theory itself a physical science.[9]

2.3.7 Geometrical Concepts

From the beginning, the mathematical subdiscipline of geometry was closely linked to the natural sciences—and physics in particular. For example, the goal of Euclidean geometry was to describe the construction of figures on a drawing sheet as part of reality. For a long time, Euclid's understanding of mathematics was the leading paradigm and both mathematical and scientific theories, such as Newton's mechanics, were constructed "*more geometrico*" (i.e., axiomatically according to Euclid's elements). This understanding of mathematics was extended in the sixteenth century with the development of projective geometry. Projective geometry had the goal of representing three-dimensional objects in perspective. Thus, the projective geometry of the time was still a theory for the explanation of empirical phenomena (cf. Struve, 1990). For this reason, geometry was often understood as part of the natural sciences, as described in the introduction of Moritz Pasch's *Vorlesungen über die neuere Geometrie*:

> Geometrical concepts form a special group within all concepts, which generally serve for the description of the external world; they refer to shape, measure and relative position of the bodies. Between the geometrical concepts, with the addition of numerical concepts, connections arise that are recognized by observation. With this, the point of view is given, which we intend to hold in the following, according to which we see in geometry a part of natural science. (Pasch, 1976, p. 3, Authors' translation)

[9] See the chapter "Stochastics with a Focus on Probability Theory".

With the *Foundations of Geometry* by David Hilbert, formalism was introduced into mathematics and the connection to reality was severed (cf. Freudenthal, 1961). The historical connection is nevertheless significant. For example, Hempel distinguishes between pure geometry and physical geometry. He describes physical geometry as follows:

> Historically speaking, at least, euclidean geometry has its origin in the generalization and systematisation of certain empirical discoveries which were made in connection with the measurement of areas and volumes, the practice of surveying, and the development of astronomy. Thus understood, geometry has factual import; it is an empirical science which might be called, in very general terms, the theory of the structure of physical space, or briefly, physical geometry. (Hempel, 1945, p. 12)

The contemporary geometry taught in schools largely corresponds to Euclidean geometry. The objects of mathematics instruction are empirical objects and students develop an empirical understanding of mathematics similar to that of a natural scientist (cf. Burscheid & Struve, 2009). For this reason, the connection to physics and the description of physical space should be targeted in geometry classes.

2.4 Conclusion

As has been shown in these excerpts, the development of mathematics and physics often ran in parallel. Moreover, for a long time, it was not possible to distinguish between physicists and mathematicians, and it was difficult to assign individual research areas to one subject. The sciences experienced a clear separation because of Hilbert's axiomatization, above all, and the scientists who followed. So today we find a separation that is expressed mainly in terms of the way the two disciplines are related to reality:

> Mathematics operates with abstract, strictly defined objects. The most fundamental of these have been inspired by reality, but are simplified and idealized. […]
>
> Physics, by contrast, deals with the real world of inanimate objects creating theories regarding the world order, its regularity and embedded causality. Where it can, physics tries to be as rigorous as mathematics, but quickly finds that this is often impossible and, in a sense, unnecessary. (Galili, 2018)

In the further course of this book, we want to show how this separation of the two subjects—which did not exist in the past—can be overcome in class without disregarding the characteristics of either discipline. For this purpose, however, it is first necessary to become aware of the different perspectives. Such comparisons will be made using the same concrete subject areas that have been discussed in this chapter, which constitute the essential connections between mathematics and physics in the classroom. The authors are guided by the following core idea:

> An implication of such comparison could upgrade the simplistic image: mathematics is not a toolkit or the language of physics, although it might serve as such. Nor does mathematics need to be isolated as a metaphor ignoring reality, although it can be. (Galili, 2018)

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