Chapter 7
Functions

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§ 1. Two Models concerning Differential Calculus of Several Variables

The models shown in Photos 121 and 122 illustrate two "counterexamples" in the differential calculus of functions of several real variables. In the remarks that follow, we rely primarily on the 1884 book of GENOCCHI and PEANO [GP], and on the literature cited there.

1.1 Surfaces with $\partial^2 z/\partial x\, \partial y \neq \partial^2 z/\partial y\, \partial x$ (Photo 121)

The model shows the graph of a real-valued function $f(x, y)$ of two real variables which is remarkable for the following features: all four second-order partial derivatives exist everywhere and each is continuous at every point $(x, y) \neq (0, 0)$, but the mixed partial derivatives $\partial^2 f/\partial x\, \partial y$ and $\partial^2 f/\partial y\, \partial x$ are not continuous at $(0, 0)$, and their values there are not equal. (If they were continuous at $(0, 0)$, too, then these values would have to coincide – by a theorem of H. A. SCHWARZ [HAS].)

For $(x, y) \neq (0, 0)$ this function is given by

$$f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

and at the origin one sets $f(0, 0) = 0$. The first-order partial derivatives are

$$\frac{\partial f}{\partial x}(x, y) = y \frac{x^2 - y^2}{x^2 + y^2} + 4y \frac{x^2 y^2}{(x^2 + y^2)^2}$$

and

$$\frac{\partial f}{\partial y}(x, y) = x \frac{x^2 - y^2}{x^2 + y^2} + 4x \frac{x^2 y^2}{(x^2 + y^2)^2}$$

for $(x, y) \neq (0, 0)$, and both are 0 at the point $(0, 0)$. In particular,

$$\frac{\partial f}{\partial x}(0, y) = -y \quad \text{and} \quad \frac{\partial f}{\partial y}(x, 0) = x,$$

for all (x, y) and consequently

$$\frac{\partial^2 f}{\partial y\, \partial x}(0, 0) = -1, \quad \text{but} \quad \frac{\partial^2 f}{\partial x\, \partial y}(0, 0) = +1.$$

In the model, the (x, y)-plane (indicated by a line in the margin) is horizontal at an elevation about half-way up the model. The function values $f(x, y)$ are marked off in the vertical direction. Notice too that the function f has the form

$$f(r, \varphi) = \frac{r^2}{4} \sin 4\varphi$$

in polar coordinates, and from this the qualitative appearance of the surface is immediately clear. (Th. SCHMITT brought this to my attention.)

G. Fischer (Ed.), *Mathematical Models*, https://doi.org /10.1007/978-3-658-18865-8_14

The real "effect" that is at issue here (the inequality of the mixed partial derivatives) naturally cannot be perceived on the model with the naked eye, because this effect can be eliminated by means of arbitrarily slight deformations of the surface. Namely, if one selects a twice continuously differentiable function $\chi(x, y)$ with values between 0 and 1, which for a very small $\epsilon > 0$ vanishes whenever $x^2 + y^2 < \epsilon$ and has value 1 whenever $x^2 + y^2 > 2\epsilon$, then the graph of the function $\chi(x, y) f(x, y)$ is indistinguishable with the naked eye from the surface represented by the model.

Moreover, it is not difficult to show that the surface in the model is a C^1-submanifold of 3-dimensional space which is not a C^2-submanifold in the point $(0, 0)$. In all other points it is of course actually of class C^∞. In this sense, it thus has a singularity at the point $(0, 0)$.

It should be noted that this example can already be found in the 1884 book of GENOCCHI and PEANO [GP] (p. 161 of the 1899 German translation), and that ever since then it has been a staple in books on the differential calculus of several variables.

1.2 The Peano Surface (Photo 122)

In the second half of the last century, a number of erroneous views prevailed concerning sufficient conditions for the occurrence of a local maximum (or minimum) in a function of several real variables.

Thus, for example, PEANO 1884 wrote in the remarks to paragraphs 133–136 in [GP] (p. 332 in the 1899 German translation): "The proofs which most books give for the criteria for ascertaining maxima and minima of functions of several variables, rest on this assertion: In the Taylor expansion of a function the ratio of the remainder after any given term to that term itself converges to zero as the increment in the variables converges to zero. This assertion is generally false in cases where the given term is not expressible in some definite form in terms of the increments, and in those cases where it is so expressible the assertion still needs a proof.

The criterion offered on p. 219 of SERRET's Calcul, namely 'le maximum ou le minimum a lieu si, pour les valeurs de $h, k, \ldots$ qui annulent $d^2 f$ et $d^3 f$, $d^4 f$ a constamment le signe – ou le signe +' is invalid."

More precisely formulated, this criterion of SERRET's runs as follows: Let f be a real-valued function which is defined and differentiable of every order at each point $\mathbf{x} = (x_1, \ldots, x_n)$ in real Euclidean space $\mathbb{R}^n$. Let $d^k f$ denote the k-homogeneous part of the Taylor development of f about the point $\mathbf{0} \in \mathbb{R}^n$, that is,

$$d^k f(\mathbf{x}) := \sum_{j_1 + \ldots + j_n = k} \frac{1}{j_1! \ldots j_n!} \frac{\partial^{j_1 + \ldots + j_n} f}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}} (\mathbf{0})\, x_1^{j_1} \ldots x_n^{j_n}.$$

It is hypothesized that

$$d^1 f(\mathbf{0}) = 0 \tag{1}$$

and that

$$d^2 f(\mathbf{x}) \leqslant 0 \quad (\text{resp., } d^2 f(\mathbf{x}) \geqslant 0) \tag{2}$$

holds for all $\mathbf{x} \in \mathbb{R}^n$. It is known that these are necessary conditions for the function f to experience a local maximum (resp., minimum) at the point $\mathbf{0}$, and one gets sufficient conditions by demanding that strict inequality prevail in (2) for every $\mathbf{x} \neq \mathbf{0}$.

SERRET's (false) criterion states that a local maximum (resp., minimum) is likewise present if in addition to (1) and (2) the strict inequality

$$d^4 f(\mathbf{x}) < 0 \quad (\text{resp., } d^4 f(\mathbf{x}) > 0) \tag{3}$$

prevails for all $\mathbf{x} \neq \mathbf{0}$ with $d^2 f(\mathbf{x}) = d^3 f(\mathbf{x}) = 0$.

Independently of PEANO, for example, LUDWIG SCHEEFFER grappled with these kinds of errors in 1885. In the resulting paper [Sch], which was only published after his death by A. MEYER, he wrote inter alia (p. 544): "It is customary to further conclude that if on each of the infinitely many lines through the origin the corresponding function of one variable has a maximum (resp., minimum) at 0, then the same must occur at the point $(0, 0)$ for the function $f(x, y)$ of the two independent variables x and y .... However, the preceding deduction contains an error."

PEANO gives (loc. cit.) an example to illustrate the incorrectness of SERRET's claim. As one can infer, e.g., from the foreword to the 1899 German translation of GENOCCHI and PEANO [GP], this example attracted considerable attention and became known as the "Peano example" or the "Peano surface." SCHEEFFER likewise used this example in [Sch] to prove the incorrectness of the inference he had criticized.

PEANO's example concerns the function

$$f(x, y) = (2x^2 - y)(y - x^2), \tag{4}$$

in which x and y are independent real variables. Throughout the (x, y)-region between the parabolas $y = 2x^2$ and $y = x^2$ this function assumes positive values, while at every point outside this region it assumes negative values

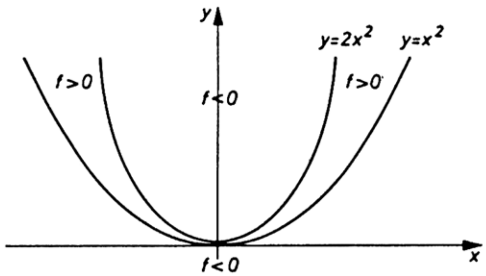


Fig. 7.1

It is therefore clear that this function does not experience a local maximum at the point (0, 0).

Nevertheless, SERRET's criterion indicates the presence of such a local maximum. In fact, from the slightly rewritten equation

$$f(x, y) = -y^2 + 3yx^2 - 2x^4$$

one reads off that $d^1 f(0, 0) = 0$ and $d^2 f(x, y) = -y^2 \leqslant 0$ (conditions (1) and (2) above) hold. If now $(x, y) \neq (0, 0)$ is a point with $d^2 f(x, y) = -y^2 = 0$ and $d^3 f(x, y) = 3yx^2 = 0$, then it follows that $y = 0$, and consequently $x \neq 0$; that is, $d^4 f(x, y) = -2x^4 < 0$ (condition (3)).

The function (4) also fulfills the above mentioned hypotheses of the inference scheme which SCHEEFFER criticized. This is so, because every straight line through the origin, as it leaves the origin, is initially in the region exterior to the two parabolas.

The model in Photo 122 shows the graph of the function (4) in which, however, the actual x, y, z coordinates have been stretched out somewhat to better display the essential features. [The surface shown is really that described by the equation $200z = (x^2 - 10y) \cdot (2y - x^2)$.] The (x, y)-plane is horizontal in the model and the point (0, 0) of it is the saddle-point of the surface being represented. The function values $z = f(x, y)$ are calibrated in the vertical direction. In Fig. 7.1, one can see very clearly the two parabolas which separate the points (x, y) with $f(x, y) > 0$ from those with $f(x, y) < 0$. The downward opening parabola $z = f(0, y)$ is labelled on the model above the y-axis.

§ 2. Models for Function Theory

Here the models shown in Photos 123 to 132 will be discussed. In this discussion, we will avail ourselves of the descriptions [MM1] (Photo 132) and [MM2] (Photos 123–126 and 128–131), which were provided for these models by their builders.

The models in Photos 123–126 and 128–131 are part of a series and exhibit a number of common features which will be discussed first, before we describe the individual models.

They all depict the graphs of the real or imaginary parts of (single- or multi-valued) analytic functions of a complex variable. In this way, for every analytic function one gets two surfaces in 3-dimensional space. These give, on the one hand, a complete picture of the function (in the case of a multi-valued function, the relationship between the sheets of the real and imaginary parts must also be indicated) which, on the other hand, is immediately accessible to our intuition (unlike the actual graph of the function, which is a surface in 4-dimensional space).

In all these models, the plane of the independent complex variable $z = x + iy$ is horizontal and passes through the center of the model, while the real and imaginary parts of the function values are laid off on the vertical axis. The models are inscribed on one side with an "R" or an "I" according to whether they illustrate the real or the imaginary part of the function. This letter is always on the side facing the observer, and the (positive) imaginary axis of the z-plane points away from this side. In the case of the function $w = (1 - z^2)^{1/4}$ (Photo 125), the real and imaginary parts happen to coincide, and consequently on the side of its model both letters "R" and "I" appear. In the case of Weierstraß' $\wp'$-function (Photo 131), the real and imaginary parts are just 90° rotations of one another, and so on its model the letters "R" and "I" appear on adjacent sides.

Each of the models shown in Photos 123–126 and 128–131 is marked with level lines representing successively 1 cm higher levels on it (1/2 cm in Photo 125, however) and some of the gradient lines, that is, trajectories orthogonal to the level lines, are also marked. This is done in such a way that on the surface of the real part (imaginary part) of f, the gradient lines shown are just those whose orthogonal projections in the z-plane coincide with the orthogonal projections in the z-plane of the level lines on the surface of the imaginary part (real part) of *f*. It is a property of analytic functions that the projections into the z-plane of the level lines of their real parts (imaginary parts) coincide with the z-plane projections of the gradient lines of their imaginary parts (real parts).

2.1 The Riemann Surface $w^2 = z^2 - 1$ (Photos 123 and 124)

These models illustrate the real part **(Photo 123)** and the imaginary part **(Photo 124)** of the two-valued function

$$w = \sqrt{z^2 - 1},$$

that is, the surfaces in 3-dimensional space described by the equations

$$u^4 - (x^2 - y^2 - 1)u^2 - x^2 y^2 = 0 \quad \text{(real part)}$$

and

$$v^4 + (x^2 - y^2 - 1)v^2 - x^2 y^2 = 0 \quad \text{(imaginary part)},$$

where

$$w = u + \mathrm{i}v \quad \text{and} \quad z = x + \mathrm{i}y.$$

On the model of the real part, one notices the imaginary axis $x = 0$ as well as the interval $\{y = 0, -1 < x < +1\}$ where both values of $\sqrt{z^2 - 1}$ are purely imaginary, that is, where the real part vanishes. Correspondingly, on the model of the imaginary part one notices the rays (half-lines) $\{y = 0, x < -1\}$ and $\{y = 0, x > +1\}$ where both values of $\sqrt{z^2 - 1}$ are real. Further, one sees on both models the branch-points $z = -1$ and $z = +1$.

As $z \to \infty$, the function $\sqrt{z^2 - 1}$ comes asymptotically into coincidence with the two-valued function $w = \pm z$, and this fact is already apparent in the finite portion of the graph shown in the model: The real parts (imaginary parts) of both values of $\sqrt{z^2 - 1}$ become asymptotic with $\pm x$ ($\pm y$).

It should also be noted that, according to the description in [MM2], the relationship of the different branches on the respective models was originally indicated by coloring them correspondingly. But these colors did not endure the intervening 100 years, and today they are no longer visible.

2.2 The Riemann Surface $w^4 = 1 - z^2$ (Photo 125)

This model is supposed to illustrate the 4-valued function

$$w = \sqrt[4]{1 - z^2}. \tag{6}$$

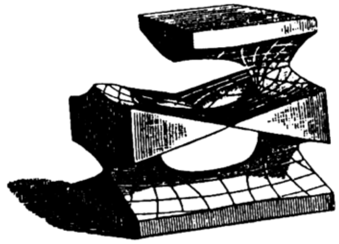


Fig. 7.2

If $z = x + \mathrm{i}y$ is an arbitrary complex number and $w = u + \mathrm{i}v$ a solution of (1), then

$$-w = -u - \mathrm{i}v, \quad \mathrm{i}w = -v + \mathrm{i}u, \quad \text{and} \quad -\mathrm{i}w = v - \mathrm{i}u$$

are the other three solutions. From this it follows that the real part and the imaginary part of the function (6) determine the same surface. According to [MM2], the equation of this surface (written in u) looks like

$$\begin{aligned}
&256u^{16} - 128u^{12}(-x^2 + y^2 + 1)\\
&- 16u^8[7(x^4 + y^4 - 2x^2 + 2y^2 + 1) + 20x^2y^2]\\
&- 8u^4[2(x^4 + y^4 - 2x^2 + 2y^2 + 1) + 3x^2y^2][-x^2 + y^2 + 1]\\
&+ x^4y^4 = 0.
\end{aligned}$$

Since the real and the imaginary part of the function (6) determine the same surface, only one model was made, the one shown in Photo 125. On the model one sees all four layers of the surface lying above one another with the branch points $z = \pm 1$. If one considers the part of the model which lies above (resp., below) the imaginary axis, one sees that for purely imaginary $z = \mathrm{i}y$, the equation (6) has two real and two purely imaginary solutions:

$$w = \pm\sqrt{1 + y^2} \quad \text{and} \quad w = \pm\mathrm{i}\sqrt{1 + y^2}.$$

According to [MM2], it was originally indicated with different colors (no longer visible) which values went together as the real and imaginary parts of the same solution of (6).

2.3 The Graph of $w = 1/z$ (Photo 126)

This model shows the graph of the real part of the function $w = 1/z$, that is, the surface

$$u = \frac{x}{x^2 + y^2} \tag{7}$$

where $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$. For purely imaginary z, $u = 0$, while for real z, the value of u tends to $\pm\infty$ if $x \to \pm 0$.

The imaginary part $v = -y/(x^2 + y^2)$ of $w = 1/z$, being just a $-90°$ rotation of the real part, was not represented by a model of its own.

The following general remark is perhaps appropriate at this point: If one considers the graph of the real or imaginary part of an analytic function in a neighborhood of a pole, then one can immediately ascertain the order of the pole as being the number of "peaks" and "pits" grouped around the pole. In the case of the function $u = 1/z$, (Photo 126) there is one "peak" and one "pit" because the pole at 0 is of the first order. In Photos 129 and 130, resp., 131, to be discussed later, one sees second order, resp., third order, poles. By contrast the order of a pole cannot be so easily recognized in the "landscape" of an analytic function (this is the term for the graph of its absolute value), because in this graph there is only one "peak".

The business with "peaks" and "pits" can be explained as follows: From (7) one reads off the fact that for each real $c \neq 0$ the curve

$$\operatorname{Re}\frac{1}{z} = c$$

is described by the equation

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2,$$

and so is a circle of radius $1/2|c|$ centered at the point $1/2c$ on the real axis. The horizontal cross-sections of

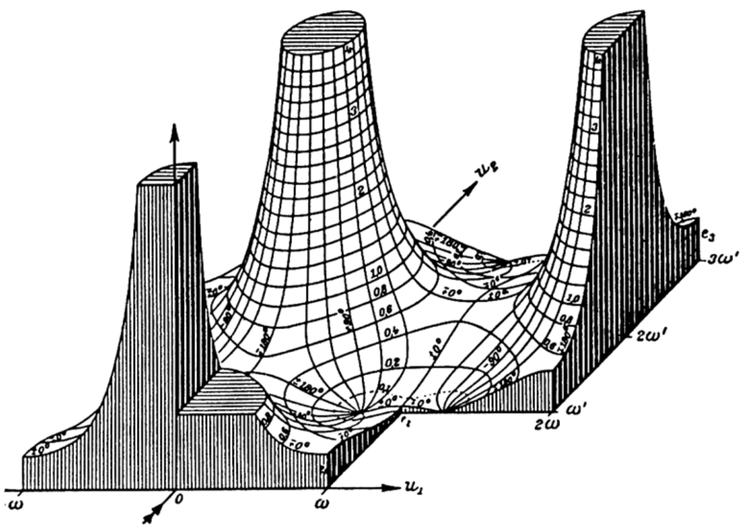


Fig. 7.3 Landscape of the ℘-function

the model in photograph 126 are therefore the corresponding circular discs. (The limiting case $c = 0$ can be incorporated into this pattern by thinking of the imaginary axis as a circle of infinite radius centered at the point ∞.) The behavior of an arbitrary analytic function near a first-order pole is "asymptotically" just like this.

In approaching an n^{th}-order pole ($n \geqslant 2$), the curves real part = const. and imaginary part = const. are curves of higher order, and the horizontal cross-sections of the corresponding models have qualitatively, near such a pole, the form of an n-petalled cloverleaf. For $n = 2$, this is Bernoulli's lemniscate (Photos 129 and 130). For n arbitrary, these curves are called "spirales sinusoides" in [GT], p. 259. In Photo 131 the case $n = 3$ is quite nicely visible.

2.4 The Graph of $w = e^z$ (Photo 127)

This model shows the graph of the real part of $w = e^z$, that is, the graph of the function

$$u = e^x \cos y,$$

where $z = x + iy$ and $w = u + iv$. This is a cosine curve running along the y-axis; its amplitude e^x converges to $+\infty$ (resp., 0) as $x \to +\infty$ (resp., $-\infty$). A translation of $\pi/2$ units along the imaginary axis transforms the real part into the imaginary part

$$v = e^x \sin y$$

of e^z.

2.5 The Graph of $6w = e^{1/6z}$ (Photo 128)

This model shows the real part of the function

$$w = \frac{1}{6} e^{1/6z},$$

that is, the graph of

$$u = \frac{1}{6} e^{x'} \cos y'$$

where $w = u + iv$, $z = x + iy$,

$$x' = \frac{x}{6(x^2 + y^2)} \quad \text{and} \quad y' = \frac{-y}{6(x^2 + y^2)}.$$

The imaginary part

$$v = \frac{1}{6} e^{x'} \sin y'$$

is essentially not very different, and so is not represented by a model of its own.

A function which is analytic in a neighborhood of a point z_0 from which z_0 itself has been deleted, is said to have an essential singularity at this point if z_0 is neither a pole nor a point over which f can be analytically continued (cf., e.g., [HC]). While it is possible to get quite a good overview of the behavior of an analytic function in the vicinity of a pole (cf. § 2.3), the behavior near an essential singularity is "very wild". Namely, every complex number, with the exception of at most one, is realized as a function value at least once in every neighborhood, however small, of z_0 (theorem of PICARD). The function $e^{1/z}$ (and so also $(1/6)e^{1/6z}$) has such an essential singularity at the point $z_0 = 0$, and indeed it is an especially simple manifestation of this phenomenon – e.g., the assertion in PICARD's theorem, which for the most general function is not simple to prove, can be confirmed rather quickly for $e^{1/z}$.

In order to describe the behavior of $\mathrm{Re}(e^{1/z})$ as $z \to 0$, we first consider $\mathrm{Re}(e^z)$ as $|z| \to \infty$. For real y let us denote by $L(y)$ the straight line parallel to the real axis which has imaginary part y, and by $L_+(y)$ (resp., $L_-(y)$) the half of it which lies to the right (resp., left) of the imaginary axis. Then from the equation

$$\mathrm{Re}(e^z) = e^x \cos y$$

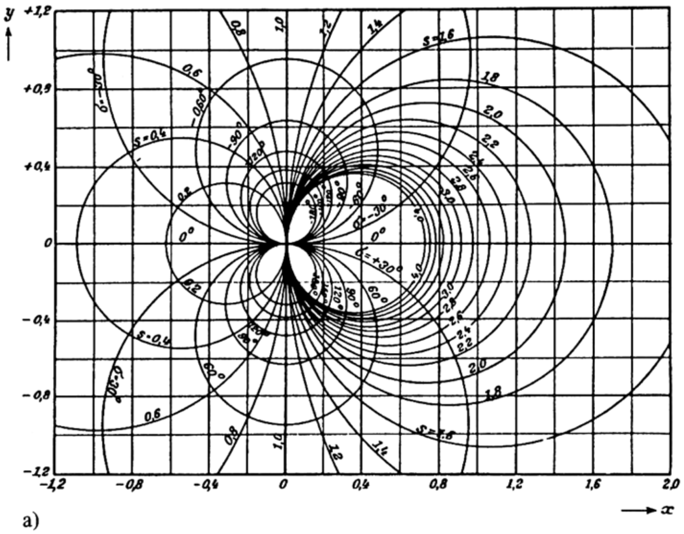
one reads off the following information: If $\cos y > 0$, then $\mathrm{Re}(e^z)$ approaches $+\infty$ when $z \in L_+(y)$ and $|z| \to \infty$, and it approaches 0 from above when $z \in L_-(y)$ and $|z| \to \infty$. If $\cos y < 0$, then $\mathrm{Re}(e^z)$ converges to $-\infty$ when $z \in L_+(y)$ and $|z| \to \infty$, and converges to 0 from below when $z \in L_-(y)$ and $|z| \to \infty$.

Now, as the equation

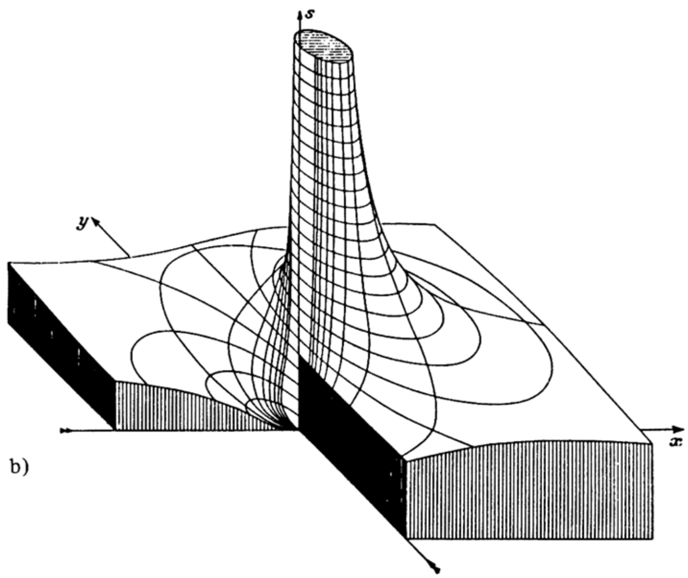
$$\left|\frac{1}{x+iy} - \frac{1}{2iy}\right| = \frac{1}{2|y|}$$

shows, the transformation $z \to 1/z$ carries the lines $L(y)$ into certain circles $C(y)$, if we again regard the real axis, which is carried into itself, as a circle. (More precisely, a full circle is only realized in the latter case when the point ∞ is adjoined to the line.) The centers $\frac{1}{2iy}$ of these circles all lie on the imaginary axis, and the half-lines $L_-(y)$, $L_+(y)$ are transformed into the semi-circles $C_-(y)$, $C_+(y)$ consisting of the part of $C(y)$ lying to the left or right, respectively, of the imaginary axis. It is further clear that $1/z$ lies in $C_\pm(y)$, and converges to 0 if $z \in L_\pm(y)$ and $|z| \to \infty$.

From the earlier description of the behavior of $\mathrm{Re}(e^z)$ on the lines $L(y)$, we are now in a position of infer the following: If $\cos y > 0$ (resp., < 0), then $\mathrm{Re}(e^{1/z})$ converges to $+\infty$ (resp., $-\infty$), when $z \in C_+(y)$ and $z \to 0$, and it converges to 0 from above (resp., from below) when $z \in C_-(y)$ and $z \to 0$.



a)



b)

Fig. 7.4 Altitudechart and landscape of $e^{1/z}$

In particular, therefore, $\mathrm{Re}(e^{1/z})$ converges to $+\infty$ when z converges to 0 along a semi-circle $C_+(y)$ with $\frac{-\pi}{2} < y < \frac{\pi}{2}$. On the model in Photo 128, this is manifested in the large spire with the teardrop-shaped horizontal cross-section near the middle of the model. The two long "icicles" lying symmetric to the real axis near this spire are caused by the fact that $\mathrm{Re}(e^{1/z})$ approaches $-\infty$ as z converges to 0 on any semi-circle $C_+(y)$ with $\frac{\pi}{2} < |y| < \frac{3\pi}{2}$. The fact that $\mathrm{Re}(e^{1/z})$ again approaches $+\infty$ if z approaches 0 along any semi-circle $C_+(y)$ with $\frac{3\pi}{2} < |y| < \frac{5\pi}{2}$ is also perceptible on the model, namely in the two steep spires with lune-shaped horizontal cross-sections which are situated symmetrically about the real axis. Finally, on the concave side of these steep spires two more icicles can be perceived; these correspond to the fact that $\mathrm{Re}(e^{1/z})$ converges to $-\infty$ if z converges to 0 on any semi-circle $C_+(y)$ with $\frac{5\pi}{2} < |y| < \frac{7\pi}{2}$.

It should be mentioned once again that the "landscape" of $e^{1/z}$ (Fig. 7.4), that is, the graph of $|e^{1/z}|$, does not manifest all the complexities of the singularity at 0 which were described above. One can easily get an idea of how it looks by recalling the relation

$$|e^{1/z}| = e^{\mathrm{Re}(1/z)}$$

and consulting the model of $\mathrm{Re}(1/z)$ (Photo 126).

2.6 The Weierstraß ℘-Function and the Weierstraß ℘′-Function (Photos 129–131)

A function f which is meromorphic in the whole complex plane is called elliptic (or doubly-periodic) if there are two complex numbers ω_1, ω_2 not lying on a common line through 0 such that $f(z+\omega_1) = f(z)$ and $f(z+\omega_2) = f(z)$, for all complex numbers z. The numbers ω_1, ω_2 are then called periods of f and every parallelogram of the form

$$\{z_0 + t_1\omega_1 + t_2\omega_2 : 0 \leqslant t_1, t_2 < 1\},$$

in which z_0 is a fixed but arbitrary complex number is called a period parallelogram (corresponding to the periods ω_1 and ω_2).

It is known (as a consequence of LIOUVILLE's theorem) that a non-constant elliptic function cannot be analytic in the whole plane and that it is also not possible that each period parallelogram contain exactly one pole if that pole is of order 1. In contrast to this latter property, however, there are elliptic functions which have exactly one pole and that of order 2 in each period parallelogram, and there are elliptic functions which have in each period parallelogram exactly two poles, each of order 1.

An important example of the first kind of elliptic function just mentioned is the Weierstraß ℘-function, which is uniquely determined by any eligible pair of complex numbers ω_1, ω_2. It has ω_1, ω_2 as periods and in every period parallelogram determined by them it has exactly one pole, that being of second order; the point 0 is always one of these poles.

This function $\wp(z)$ can be expressed in terms of the periods ω_1 and ω_2 in the following way

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega(\omega_1,\omega_2)\backslash 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right). \tag{8}$$

where $\Omega(\omega_1, \omega_2)$ designates the period group generated by ω_1 and ω_2, i.e., the group of all numbers of the form $n_1\omega_1 + n_2\omega_2$ for arbitrary integers n_1 and n_2.

The models shown in photographs 129 and 130 represent the real part **(Photo 129)** and the imaginary part **(Photo 130)** of the Weierstraß ℘-function with the periods

$$\omega_1 = 2{,}622 \quad \text{and} \quad \omega_2 = 2{,}622 \cdot \mathrm{i} \tag{9}$$

Altogether four period parallelograms (in this case squares of side-length 2.622) are shown; they illustrate the periodicity and among them contain four second-order poles.

The apparently bizarre choice of period (9) can be explained as follows: The Weierstraß ℘-function is indeed uniquely determined by the specification of its periods, but it is nevertheless possible that different pairs of periods give rise to the same ℘-function. However, from any two periods two other numbers

$$g_2 = 60 \sum_{\omega \in \Omega(\omega_1,\omega_2)\backslash 0} \frac{1}{\omega^4} \quad \text{and} \quad g_3 = 140 \sum_{\omega \in \Omega(\omega_1,\omega_2)\backslash 0} \frac{1}{\omega^6}$$

can be calculated, and these, it turns out, are determined by the ℘-function itself and are therefore designated as invariants of the ℘-function. The necessary and sufficient condition that two complex numbers g_2 and g_3 be the invariants (for appropriate periods ω_1, ω_2) of some $\wp$-function is that

$$g_2^3 - 27 g_3^2 \neq 0. \tag{10}$$

For the calculations needed in constructing the model, it is obviously easier to proceed from the invariants than from the periods. For the models shown, $g_2 = 4$ and $g_3 = 0$ were selected, and the periods (9) were computed from these. The actual calculations needed to build the model relied on [HASW] and are described in [MM2].

It is known that (10) is a necessary and sufficient condition for the cubic equation $4x^3 - g_2 x - g_3 = 0$ to have three distinct solutions. Furthermore, these solutions are the values of the appropriate $\wp$-function at the points $\omega_1/2$, $(\omega_1 + \omega_2)/2$ and $\omega_2/2$. Moreover, it is the case that up to periodicity, it is precisely at these points that the first derivative of the $\wp$-function vanishes; that is, the real and imaginary parts of the $\wp$-function have their saddle points just exactly at these points. These saddle points are clearly visible in photographs 129 and 130, and at them our function has the values -1, 0 and 1 (the solution of $4x^3 - 4x = 0$).

The first derivative of the Weierstraß $\wp$-function is called the Weierstraß $\wp'$-function and, on account of (8), has the series representation

$$\wp'(z) = -2 \sum_{\omega \in \Omega(\omega_1, \omega_2)} \frac{1}{(z-\omega)^3}.$$

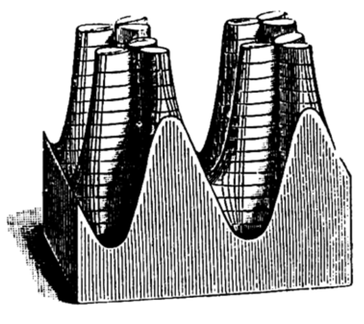


Fig. 7.5

The model in **Photo 131** shows the $\wp'$-function for the $\wp$-function which appears in Photos 129 and 130. Since the real and imaginary parts differ only by a $90°$ rotation, only one model, carrying the letters "R" and "I" on the appropriate sides, is presented. One sees immediately the four poles, each of order 3, of the $\wp'$-function which arise from the second-order poles of the $\wp$-function via differentiation. We also recognize that $\omega_1/2$, $(\omega_1 + \omega_2)/2$ and $\omega_2/2$ are the zeros of the $\wp'$-function.

The book of HURWITZ and COURANT [HC] is recommended to the reader who is interested in more detail about the theory of elliptic functions which we have broached in this section.

2.7 The Jacobi Amplitude (Photo 132)

This model shows a surface comprised of points (k, u, φ) in real 3-dimensional space whose coordinates are related to each other via the elliptic integral of the first kind

$$u = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}.$$

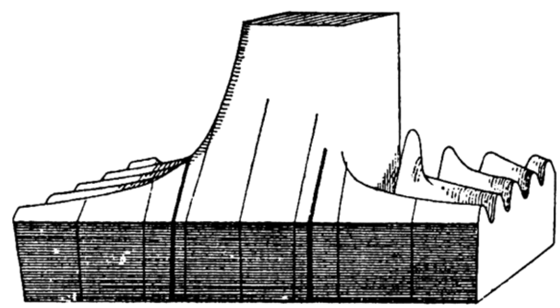


Fig. 7.6

The (k, u)-plane is horizontal, with the k-axis coinciding with the upper edge of the foremost side in the photograph. The φ-values are laid out in the vertical direction.

If we fix k, we can consider φ to then be a function of u; it is customary to designate this function with $\varphi = \mathrm{am}(u, k)$ and to call it the Jacobi amplitude function. The surface shown can now be described as follows (cf. [MM1], where this is done in greater detail): For all k, $\mathrm{am}(u, k) = \mathrm{am}(u, -k)$ and $\mathrm{am}(0, k) = 0$, accounting for the fact that the surface is symmetric with respect to the (u, φ)-plane and contains the k-axis. Moreover, for every k the function $\mathrm{am}(u, k)$ goes through $u = 0$ at an angle of $45°$. For $k = 0$ the function $\mathrm{am}(u, k)$ actually coincides with the line through 0 which bisects the first quadrant of the (u, φ)-plane. For $0 < k < 1$ $\operatorname{am}(u, k)$ is an increasing function of u for positive u, which somewhat like a sine curve wraps itself around an ascending straight line, the slope of which diminishes from 45° to 0° as k increases from 0 to 1. For $k = 1$ the curve $\operatorname{am}(u, k)$ initially increases and then asymptotically approaches the line $\varphi = \pi/2$. For $k > 1$, $\operatorname{am}(u, k)$ oscillates around the u-axis like a sine curve.

Regarding the concept of the elliptic integral, the book of HURWITZ and COURANT [HC] is again recommended to the interested reader.

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